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CENTER AND ISOCHRONOUS CENTER AT INFINITY IN A CLASS OF PLANAR DIFFERENTIAL SYSTEMS

Haibo Chen[1], Yirong Liu[1] and Pei Yu[2]

[1]Mathematics school of Central South University, Changsha, 410075, China

[2]Department of Applied Mathematics, The University of Western Ontario, London, Ontario, N6A 5B7, Canada

Abstract. In this paper, the conditions of center and isochronous center at the infinity for a class of planar differential systems are studied. By a transformation, we first transform the infinity (the equator) of the systems into the origin. Then sufficient and necessary conditions for the infinity (the equator) of the systems being a center are obtained. A Construction Theorem of periodic constants is presented, which plays an important role in simplifying periodic constants. A complete classification of the sufficient and necessary conditions is given for the infinity of the systems being an isochronous center. All the computations for the quantities at infinity and periodic constants are performed using computer algebraic system – Mathematics 4.2, and the technique employed in this paper is different from others used in the literature.

Keywords. Planar system, infinity, center, isochronous center, focus value.

AMS (MOS) subject classification: 34C05.

1 Introduction

In the qualitative theory of planar differential systems, the problem to determine the conditions of center and isochronous center has received good attentions. In the case of the origin, many results have been obtained (see [27, 23, 25, 12, 2, 14]). Computation of focal values (Lyapunov constants) is one approach to study center conditions. For computing focal values, conventional methods include the method of Poincare return map and the method of Lyapunov coefficients (see [2]). In [17, 5], the authors gave a new computational method, which combined the calculation of focal values and saddle qualities into a unified calculation of singular point quantities.

For any center of a planar differential system, the largest neighborhood of the center, which is entirely covered by periodic orbits, is called the periodic annulus of the center. The function associated with the period of any periodic orbits in the periodic annulus is called periodic function. If the periodic function is a constant, the center is then said to be isochronous.

The problem of characterizing isochronous centers of the origin has attracted the attention of several authors (see [21, 6, 7, 8, 24]), and many results have been published. Quadratic systems were classified by Loud [19] and cubic systems with homogeneous nonlinearities by Pleshkan [22]. Kukles' systems were classified in [9]. The cubic time-reversible systems were classified in [20, 3]. A class of cubic complex polynomial systems were classified by Lin and Li [13]. For some other results on isochronicity at the origin we refer the reader to [4, 10, 26, 11] and references therein.

For the case of infinity, being difficult, there are very few results. As far as center conditions at infinity are concerned, several special systems have been studied: cubic system in [2, 14]; quintic system in [18]. But for the problem concerning the conditions of the infinity being an isochronous center, no result has been obtained except that given in [16]. In this paper, we study center and isochronous center at infinity for a class of planar systems described by analytic autonomous differential equations. We first transform the infinity of the systems into the origin. Then we will study the center conditions and isochronous center conditions at infinity by the methods for the origin.

Consider the following real planar autonomous differential system:

$$\begin{cases} \dfrac{dx}{dt} = -y + (x^2+y^2)^{-n} \displaystyle\sum_{k=1}^{n} X_k(x,y), \\ \dfrac{dy}{dt} = x + (x^2+y^2)^{-n} \displaystyle\sum_{k=1}^{n} Y_k(x,y), \end{cases} \tag{1.1}$$

where n is a positive integer, $X_k(x,y)$ and $Y_k(x,y)$ are homogeneous polynomials of degree k. System (1.1) can be transformed into the following autonomous polynomial differential equations in phase plane by a time transformation:

$$\begin{cases} \dfrac{dx}{dt} = -y(x^2+y^2)^{n} + \displaystyle\sum_{k=1}^{n} X_k(x,y) = X(x,y), \\ \dfrac{dy}{dt} = x(x^2+y^2)^{n} + \displaystyle\sum_{k=1}^{n} Y_k(x,y) = Y(x,y). \end{cases} \tag{1.2}$$

For the system (1.1) or (1.2), the equator Γ_∞ on the Poincaré closed sphere is a trajectory, having no real singular point. Γ_∞ is the equator cycle or the infinity (on Gauss sphere) of the system (1.1) or (1.2). The infinity of the system (1.1) or (1.2) is a center or a focus.

By means of Bendixson transformation:

$$x = \frac{\xi}{\xi^2+\eta^2}, \qquad y = \frac{\eta}{\xi^2+\eta^2}, \tag{1.3}$$

system (1.1) can be transformed into the following polynomial system:

$$\begin{cases} \dfrac{d\xi}{d\tau} = -\eta + \displaystyle\sum_{k=1}^{n}(\xi^2+\eta^2)^{n-k}[(\eta^2-\xi^2)X_k(\xi,\eta) - 2\xi\eta Y_k(\xi,\eta)], \\ \dfrac{d\eta}{d\tau} = \xi - \displaystyle\sum_{k=1}^{n}(\xi^2+\eta^2)^{n-k}[(\eta^2-\xi^2)Y_k(\xi,\eta) + 2\xi\eta Y_k(\xi,\eta)]. \end{cases} \tag{1.4}$$

Then the infinity of system (1.1) corresponds to the origin of system (1.4). The problem of center and isochronous center at infinity of system (1.1) are now transformed into, respectively, the ones at the origin of system (1.4).

With the following transformations:

$$z = \xi + i\eta, \quad w = \xi - i\eta, \quad T = i\tau, \quad i = \sqrt{-1}, \tag{1.5}$$

system (1.4) can be rewritten as

$$\begin{cases} \dfrac{dz}{dT} = z + z^2\displaystyle\sum_{k=1}^{n}(zw)^{n-k}W_k(z,w) = Z(z,w), \\ \dfrac{dw}{dT} = -w - w^2\displaystyle\sum_{k=0}^{n}(zw)^{n-k}Z_k(z,w) = -W(z,w), \end{cases} \tag{1.6}$$

where

$$\begin{cases} Z_k(\xi+i\eta,\xi-i\eta) = Y_k(\xi,\eta) - iX_k(\xi,\eta), \\ W_k(\xi+i\eta,\xi-i\eta) = Y_k(\xi,\eta) + iX_k(\xi,\eta). \end{cases} \tag{1.7}$$

Due to no time transformation in (1.3), we have the following lemma.

Lemma 1.1 The infinity of system (1.1) is a center (or an isochronous center) if and only if the origin of system (1.4) is a center (or an isochronous center).

System (1.2) can be transformed into system (1.4) by the transformations:

$$x = \frac{\xi}{\xi^2+\eta^2}, \quad y = \frac{\eta}{\xi^2+\eta^2}, \quad t_1 = (x^2+y^2)^{-n}t. \tag{1.8}$$

Definition 1.1[16] If the origin of system (1.4) is an isochronous center, then we say that the infinity of system (1.2) is a pseudo-isochronous center.

For center and isochronous center at infinity of system (1.1), according to the theory in [5, 15], we need only to study the complex center and complex isochronous center of complex system (1.6) instead of considering the real system (1.4).

In this paper, we study center and isochronous center at infinity for the case $n = 2$ in system (1.1), i.e., consider the following differential system

$$\begin{cases} \dfrac{dx}{dt} = -y + \dfrac{A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + A_{02}y^2}{(x^2+y^2)^2}, \\ \dfrac{dy}{dt} = x + \dfrac{B_{10}x + B_{01}y + B_{20}x^2 + B_{11}xy + B_{02}y^2}{(x^2+y^2)^2}. \end{cases} \tag{1.9}$$

By a time transformation, system (1.9) can be transformed to the following system:

$$\begin{cases} \dfrac{dx}{dt} = -y(x^2+y^2)^2 + A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + A_{02}y^2, \\ \dfrac{dy}{dt} = x(x^2+y^2)^2 + B_{10}x + B_{01}y + B_{20}x^2 + B_{11}xy + B_{02}y^2. \end{cases} \tag{1.10}$$

The rest of the paper is organized as follows. In Section 2 we calculate the singular point quantities of the infinity (the equator) of system (1.9). In Section 3 we study the integrability conditions of system (1.9) in the neighborhood of the equator. Sufficient and necessary conditions for the infinity to be a center are given and the potential systems with an isochronous center at infinity are presented. Finally in Section 4 we classify all the potential isochronous centers. The construction Theorem of periodic constants is presented, which plays an important role in simplifying the periodic constants. A complete classification of the sufficient and necessary conditions is given for the class of systems having an isochronous center at the infinity.

2 Singular Point Quantity

In this section, we first introduce the method for computing the singular point quantities of generalized planar polynomial differential systems (see [5] for details).

Consider the planar polynomial differential system:

$$\begin{cases} \dfrac{dx}{dt} = -y + \displaystyle\sum_{k=1}^{\infty} X_k(x,y), \\ \dfrac{dy}{dt} = x + \displaystyle\sum_{k=1}^{\infty} X_k(x,y), \end{cases} \tag{2.1}$$

where $X_k(x,y)$, $Y_k(x,y)$ are k-th degree homogeneous polynomials of x, y. The singular point $O(0,0)$ of system (2.1) is a focus or a center. Introduce the following transformations:

$$z = x + iy, \quad w = x - iy, \quad T = i\tau, \quad i = \sqrt{-1}, \tag{2.2}$$

into system (2.1) yields

$$\begin{cases} \dfrac{dz}{dt} = z + \displaystyle\sum_{k=2}^{\infty} Z_k(z,w) = Z(z,w), \\ \dfrac{dw}{dt} = -w - \displaystyle\sum_{k=2}^{\infty} W_k(z,w) = W(z,w), \end{cases} \tag{2.3}$$

where z, w, T are complex variables and

$$Z_k(z,w) = \sum_{\alpha+\beta=k} \alpha_{\alpha\beta} z^{\alpha} w^{\beta}, \quad W_k = \sum_{\alpha+\beta=k} b_{\alpha\beta} w^{\alpha} z^{\beta}. \tag{2.4}$$

Here, the two systems (2.1) and (2.3) are said to be concomitant.

If system (2.1) is a real planar differential system, then the coefficients of system (2.3) satisfy the following conjugate conditions:

$$\overline{a_{\alpha\beta}} = b_{\alpha\beta}, \quad \alpha \geq 0, \beta \geq 0, \ \alpha + \beta \geq 2. \tag{2.5}$$

By the transformations:

$$z = re^{i\theta}, \ w = re^{-i\theta}, \ T = it, \tag{2.6}$$

system (2.3) can be transformed into

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{ir}{2} \sum\limits_{m=1}^{\infty} \sum\limits_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} - b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m, \\ \dfrac{d\theta}{dt} = 1 + \dfrac{1}{2} \sum\limits_{m=1}^{\infty} \sum\limits_{\alpha+\beta=m+2} (a_{\alpha,\beta-1} + b_{\beta,\alpha-1}) e^{i(\alpha-\beta)\theta} r^m. \end{cases} \tag{2.7}$$

For a complex constant h, $|h| \ll 1$, we may write the solution of (2.7) satisfying the initial-value condition $r|_{\theta=0} = h$ as

$$r = \widetilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} v_k(\theta) h^k. \tag{2.8}$$

Evidently, if system (2.1) is a real system, $v_{2k+1}(2\pi)$, $k = 1, 2, \cdots$ are the k-th order focal value of the origin.

Lemma 2.1[1] For system (2.3), we can uniquely derive the following formal series:

$$\varphi(z, w) = z + \sum_{k+j=2}^{\infty} c_{k,j} z^k w^j, \quad \psi(z, w) = w + \sum_{k+j=2}^{\infty} d_{k,j} w^k z^j, \tag{2.9}$$

where

$$c_{k+1,k} = d_{k+1,k} = 0, \quad k = 1, 2, \cdots, \tag{2.10}$$

such that

$$\frac{d\varphi}{dT} = \varphi + \sum_{j=1}^{\infty} p_j \varphi^{j+1} \psi^j, \quad \frac{d\psi}{dT} = -\psi - \sum_{j=1}^{\infty} q_j \psi^{j+1} \varphi^j. \tag{2.11}$$

Let $\mu_0 = 0$, $\mu_k = p_k - q_k$, $k = 1, 2, \cdots$. In [5], μ_k is defined as the k-th order singular point quantity of the origin of system (2.3).

If $\mu_0 = \mu_1 = \cdots = \mu_{k-1} = 0$ but $\mu_k \neq 0$, then the origin of system (2.3) is called k-th order weak critical singular point. In other words, k is called the multiplicity of the origin of the system.

If $\mu_k = 0$ for $k = 1, 2, \cdots$, then the origin of system (2.3) is called an extended center (complex center).

If system (2.1) is a real autonomous differential system with the concomitant system (2.3), then for the origin, the k-th order focus quantity v_{2k+1} of system (2.1) and the k-th order quantity of the singular point of system (2.3) have the following relation [17]:

$$v_{2k+1} = i\mu_k \quad \text{for} \quad k = 1, 2 \cdots. \tag{2.12}$$

Theorem 2.1 (Theorem 2.2 in [5]) For system (2.3), $\forall \alpha, \beta, \alpha \neq \beta$, and $m \geq 1$, we have

$$C_{\alpha\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} \Big[(\alpha+1)a_{k,j-1} - (\beta+1)b_{j,k-1}\Big] C_{\alpha-k+1,\beta-j+1} \tag{2.13}$$

and

$$\mu_m = \sum_{k+j=3}^{2m+2} \Big(a_{k,j-1} - b_{j,k-1}\Big) C_{m-k+1,m-j+1}, \tag{2.14}$$

where $a_{kj} = b_{kj} = C_{kj} = 0$, for $k < 0$ or $j < 0$.

Now we turn to the computation of singular point quantities at the infinity of system (1.9). By means of transformations (1.3) and (1.5), system (1.9) becomes the following quintic complex system:

$$\begin{cases} \dfrac{dz}{dT} = z\Big[1 + b_{02}z^3 + (2a_{20} - b_{11})z^2w + b_{20}zw^2 + b_{01}z^3w + b_{10}z^2w^2\Big] \\ \qquad \triangleq Z(z,w), \\ \dfrac{dw}{dT} = -w\Big[1 + a_{02}w^3 + (2b_{20} - a_{11})w^2z + a_{20}z^2w + a_{01}w^3z + a_{10}w^2z^2\Big] \\ \qquad \triangleq -W(z,w), \end{cases} \tag{2.15}$$

where

$$\begin{aligned} a_{01} &= \frac{A_{01} + B_{10} - (A_{10} - B_{01})i}{2}, \\ a_{20} &= \frac{-A_{11} + B_{20} - B_{02} + (A_{02} - A_{20} - B_{11})i}{4}, \\ a_{11} &= \frac{-A_{11} - 2B_{02} + (B_{11} + 2A_{20})i}{2}, \\ a_{02} &= \frac{A_{11} - B_{02} + B_{20} + (A_{02} - A_{20} + B_{11})i}{4}, \\ b_{01} &= \overline{a}_{01}, \quad b_{20} = \overline{a}_{20}, \quad b_{11} = \overline{a}_{11}, \quad b_{02} = \overline{a}_{02}, \end{aligned}$$

and the infinity of system (1.9) is changed into the origin of system (2.15).

By (2.12), we will be able to reduce the problem of determining whether the infinity of system (1.9) is a center or a weak focus to the calculation of the singular point quantities μ_m at the origin of system (2.15). Since all $C_{\alpha\beta}$ and μ_m are polynomials with rational coefficients, the above formulas can be easily implemented using a computer algebra system such as Mathematica or Maple. In order to compute the quantities at infinity and simplify them quickly, as well as make use of the extended symmetric principle to obtain the conditions of integrability of system (1.9), we need to obtain all the Lie-invariants of (2.15). This can be achieved using the technique developed in [17].

Theorem 2.2 For system (2.15), there exist 32 elementary Lie-invariants, as listed in Table 1.

Table 1

a_{10}	b_{10},	$a_{20}a_{11}$	$b_{20}b_{11}$,	$a_{20}b_{20}$	$a_{11}b_{11}$
$a_{02}b_{02}$	$a_{01}b_{01}$,	$a_{20}^2a_{01}$	$b_{20}^2b_{01}$,	$a_{20}b_{11}a_{01}$	$b_{20}a_{11}b_{01}$
$b_{11}^2a_{01}$	$a_{11}^2b_{01}$,	$b_{01}^3a_{02}^2$	$a_{01}^2b_{02}^2$,	$a_{20}b_{01}a_{02}$	$b_{20}a_{01}b_{02}$
$b_{11}b_{01}a_{02}$	$a_{11}a_{01}b_{02}$,	$a_{20}a_{01}^2b_{02}$	$b_{20}b_{01}^2a_{02}$,	$b_{11}a_{01}^2b_{02}$	$a_{11}b_{01}^2a_{02}$
$a_{20}^3a_{02}$	$b_{20}^3b_{02}$,	$a_{20}^2b_{11}a_{02}$	$b_{20}^2a_{11}b_{02}$,	$a_{20}b_{11}^2a_{02}$	$b_{20}a_{11}^2b_{02}$
$b_{11}^3a_{02}$	$a_{11}^3a_{02}$				

Lemma 2.2 (Lemma 2.5 in [17]) For any positive integer m, the mth quantity of the singular point at the origin of system (2.15) is a linear combination having rational coefficients with respect to both of monomial Lie-invariant of this order and their inverse symmetric forms, i.e., μ_m has the following algebraic structure:

$$\mu_m = \sum_{k=1}^{s(m)} \gamma_{m,k}\,(g_{m,k} - g_{m,k}^*), \tag{2.16}$$

where $s(m)$ is a positive integer number, $\gamma_{m,k}$'s are rational numbers, and $g_{m,k}$'s and $g_{m,k}^*$'s are monomial Lie-invariants of order $m, k = 1, 2, \cdots, s(m)$.

By applying Theorems 2.1 and 2.2 to system (2.15), we compute the singular point quantities at the origin using the computer algebra system Mathematica, and simplify them using Lemma 2.2. We obtain the following theorem.

Theorem 2.3 The first 6 singular point quantities at the origin of system (2.15) are:

$$\begin{cases} \mu_1 = 0, \quad \mu_2 = b_{10} - a_{10}, \quad \mu_3 = b_{20}b_{11} - a_{20}a_{11}, \quad \mu_4 = 0, \\ \mu_5 = \dfrac{1}{6}\Big[(3a_{11}^2 - b_{11}a_{02} + 3a_{11}b_{20})b_{01} - (3b_{11}^2 - a_{11}b_{02} + 3b_{11}a_{20})a_{01}\Big], \\ \mu_6 = a_{20}b_{11}^2a_{02} - b_{20}a_{11}^2b_{02}, \end{cases} \tag{2.17}$$

in which we already let $\mu_l = 0$ for $l \leq k,\ k = 2, 3, \cdots, 6$.

Theorem 2.4 If all the first 6 singular point quantities at the origin of system (2.15) are zero, then there exist constants p, q, λ, s such that at least one of the following three conditions holds

$$\text{(i)}\quad a_{10}=b_{10}=\lambda,\quad a_{11}=b_{11}=0; \tag{2.18}$$

$$\text{(ii)}\quad \begin{cases} a_{10}=b_{10}=\lambda,\quad a_{20}=pb_{11},\quad b_{20}=pa_{11},\quad a_{02}=qa_{11}^3,\\ b_{02}=qb_{11}^3,\quad |a_{11}|+|b_{11}|\neq 0; \end{cases} \tag{2.19}$$

$$\text{(iii)}\quad \begin{cases} a_{10}=b_{10}=\lambda,\quad a_{20}=b_{20}=0,\quad a_{01}=s(b_{11}a_{02}-3a_{11}^2),\\ b_{01}=s(a_{11}b_{02}-3b_{11}^2),\quad b_{11}^3a_{02}-a_{11}^3b_{02}\neq 0,\quad |a_{11}|+|b_{11}|\neq 0. \end{cases} \tag{2.20}$$

Proof. Since $\mu_2=0$, there exists a constant λ such that $a_{10}=b_{10}=\lambda$. If $a_{11}=b_{11}=0$, then (2.18) holds. Now suppose $a_{11}b_{11}\neq 0$. When $\mu_3=0$, there exists a constant p such that $a_{20}=pb_{11}$, $b_{20}=pa_{11}$. It follows that $\mu_6=p(b_{11}^3a_{02}-a_{11}^3b_{02})$. Consider $\mu_6=0$. If $b_{11}^3a_{02}-a_{11}^3b_{02}=0$ then (2.19) holds; if $b_{11}^3a_{02}-a_{11}^3b_{02}\neq 0$, we have $p=0$ and then by $\mu_5=0$, (2.20) holds.

If (2.19) holds, then system (2.15) becomes

$$\begin{cases} \dfrac{dz}{dT}=z\left[1+b_{11}^3qz^3+(2p-1)b_{11}z^2w+a_{11}pzw^2+b_{01}z^3w+\lambda z^2w^2\right],\\ \dfrac{dw}{dT}=-w\left[1+a_{11}^3qw^3+(2p-1)a_{11}w^2z+b_{11}pz^2w+a_{01}w^3z+\lambda w^2z^2\right], \end{cases} \tag{2.21}$$

where

$$|a_{11}+b_{11}|\neq 0. \tag{2.22}$$

Theorem 2.5 The first 11 singular point quantities at the origin of system (2.21) are:

$$\begin{cases} \mu_1=\mu_2=\mu_3=\mu_4=0,\\ \mu_5=\dfrac{1}{6}(b_{11}^2a_{01}-a_{11}^2b_{01})(a_{11}b_{11}q-3p-3),\\ \mu_6=0,\\ \mu_7=-\dfrac{1}{12}(b_{11}^2a_{01}-a_{11}^2b_{01})[(b_{11}^2a_{01}+a_{11}^2b_{01})q-24\lambda],\\ \mu_8=\dfrac{1}{3}a_{11}b_{11}(b_{11}^2a_{01}-a_{11}^2b_{01})(p+2)(4p^2-5p+3),\\ \mu_9=\dfrac{2}{3}(b_{11}^2a_{01}-a_{11}^2b_{01})(a_{01}b_{01}+3\lambda^2),\\ \mu_{10}=\dfrac{1}{15}a_{11}b_{11}(b_{11}^2a_{01}-a_{11}^2b_{01})(55p^2-463p+126)\lambda,\\ \mu_{11}=\dfrac{21}{2}(b_{11}^2a_{01}-a_{11}^2b_{01})\lambda^3, \end{cases} \tag{2.23}$$

where we already let $\mu_l=0$ for $l\leq k,\ k=2,3,\cdots,11$.

If (2.20) is satisfied, then system (2.15) becomes

$$\begin{cases} \dfrac{dz}{dT} = z\left[1 + b_{02}z^3 - b_{11}z^2w + s(a_{11}b_{02} - 3b_{11}^2)z^3w + \lambda z^2w^2\right], \\ \dfrac{dw}{dT} = -w\left[1 + a_{02}w^3 - a_{11}w^2z + s(b_{11}a_{02} - 3a_{11}^2)w^3z + \lambda w^2z^2\right], \end{cases} \tag{2.24}$$

where

$$b_{11}a_{02} - a_{11}^3b_{02} \neq 0, \quad |a_{11}| + |b_{11}| \neq 0. \tag{2.25}$$

Theorem 2.6 The first 12 singular point quantities at the origin of system (2.24) are

$$\begin{cases} \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0, \\ \mu_7 = \dfrac{1}{12}(b_{11}^3a_{02} - a_{11}^3b_{02})(24r + 9a_{11}b_{11}s - a_{02}b_{02}s)s, \\ \mu_8 = -\dfrac{1}{36}(b_{11}^3a_{02} - a_{11}^3b_{02})(48r - 81a_{11}b_{11}s + a_{02}b_{02}s), \\ \mu_9 = -\dfrac{1}{8}(b_{11}^3a_{02} - a_{11}^3b_{02})\Big[5a_{11}b_{11} - 240a_{11}^2b_{11}^2s^3 - a_{02}b_{02} \\ \qquad +16(b_{11}^3a_{02} + a_{11}^3b_{02})s^3\Big], \\ \mu_{10} = -\dfrac{1}{5}(b_{11}^3a_{02} - a_{11}^3b_{02})\Big[160a_{11}^2b_{11}^2 - 13(b_{11}^3a_{02} + a_{11}^3b_{02}\Big]s^2, \\ \mu_{11} = -\dfrac{4139531}{49140}(b_{11}^3a_{02} - a_{11}^3b_{02})a_{11}^2b_{11}^2s, \\ \mu_{12} = \dfrac{35}{8}(b_{11}^3a_{02} - a_{11}^3b_{02})a_{11}^2b_{11}^2, \end{cases} \tag{2.26}$$

in which we again already let $\mu_l = 0$ for $l \leq k, k = 2, 3, \cdots, 12$.

3 Conditions of Center

From the discussion given in section 1, the conditions of center and isochronous center at the origin of system (2.15) are, respectively, the conditions of center and isochronous center (or pseudo-isochronous center) at the infinity of system (1.9) (or system (1.10)). In what follows, we give a complete classification of the sufficient and necessary conditions of the origin of system (2.15) being a center or an isochronous center. Correspondingly, the conditions of center and isochronous center (or pseudo-isochronous center) at the infinity of system (1.9) (or system (1.10)) are derived.

If (2.18) holds, then system (2.15) can be written as

$$\begin{cases} \dfrac{dz}{dT} = z\left(1 + b_{02}z^3 + 2a_{02}z^2w + b_{20}zw^2 + b_{01}z^3w + \lambda z^2w^2\right), \\ \dfrac{dw}{dT} = -w\left(1 + a_{02}w^3 + 2b_{02}w^2z + a_{20}z^2w + a_{01}w^3z + \lambda w^2z^2\right). \end{cases} \tag{3.1}$$

Theorem 3.1 System (3.1) has the first integral

$$\frac{z^3w^3}{2+2b_{02}z^3+6a_{20}z^2w+6b_{20}w^2z+2a_{02}w^3+(zw)(3b_{01}z^2+6\lambda zw+3a_{01}w^2)} = \text{const.} \tag{3.2}$$

From Theorem 2.5 we obtain the following result.

Theorem 3.2 For system (2.21), the first 11 singular point quantities at the origin are zero if and only if one of the following three conditions holds

(i) $b_{11}^2a_{01}-a_{11}^2b_{01}=0;$ (3.3)

(ii) $\lambda=q=p+1=b_{11}=a_{01}=0,\ a_{11}b_{01}\neq 0;$ (3.4)

(ii)* $\lambda=q=p+1=a_{11}=b_{01}=0,\ b_{11}a_{01}\neq 0.$ (3.5)

By the theory of Lie-invariant given in [17] or Lemma 1 in [26], we have the following theorem.

Theorem 3.3 If (3.3) holds, then the origin is a center of system (2.21), and if (3.4) or (3.5) holds, then system (2.21) has the integrating factor:

$$M_1=(zw)^{-7/2}(2+3b_{01}z^3w+3a_{01}w^3z)^{-1/6}. \tag{3.6}$$

Theorems 2.5, 3.2 and 3.3 imply the following assertion.

Theorem 3.4 For system (2.21), there exists a regular integral in the neighborhood of the origin if and only if the first 11 singular point quantities at the origin are zero or one of the three conditions of Theorem 3.2 holds. Relevantly, one of the three conditions of Theorem 3.2 is the center condition of the origin.

By Theorem 2.6, we have the following theorem.

Theorem 3.5 For system (2.24), the first 12 singular point quantities at the origin are zero if and only if one of the following two conditions holds

(i) $\lambda=s=b_{11}=a_{02}=0,\ a_{11}b_{02}\neq 0;$ (3.7)

(i)* $\lambda=s=a_{11}=b_{02}=0,\ b_{11}a_{02}\neq 0.$ (3.8)

Then the following theorem can be easily confirmed.

Theorem 3.6 If (3.7) or (3.8) holds, then system (2.24) has the integrating factor

$$M_2=(zw)^{-3}(1+a_{02}w^3)^{-1/3}(1+b_{02}z^3)^{-1/3}. \tag{3.9}$$

From Theorems 2.6, 3.5 and 3.6, we have the following result.

Theorem 3.7 For system (2.24), there exists a regular integral in the neighborhood of the origin if and only if the first 12 singular point quantities at the origin are zero or one of the two conditions in Theorem 3.5 holds. Relevantly, one of the two conditions of Theorem 3.5 is the center condition of the origin.

4 Conditions of Isochronous Center

In this section, we study the conditions of isochronous center at the infinity of the system. We first introduce the notions and definitions that will be used henceforth.

Denoting $\tau_0 = 0,\ \tau_k = p_k + q_k,\ k = 1, 2, \cdots$.

Definition 4.1 For any positive integer k, τ_k is called the k-th order periodic constant at the origin of complex systems (2.1) and (2.3).

Definition 4.2 If all the singular point quantities and periodic constants at the origin of system (2.3) are zero, then the origin of complex system (2.3) is called a complex isochronous center.

It is clear that the isochronous center in real field is a particular case of the complex isochronous center.

On the other hand, one can prove (see Theorem 2.3 in [17]) that p_k and q_k $(k = 1, 2, \cdots)$ are Lie-invariants of order k. Therefore, we have the following result.

Theorem 4.1 (Construction Theorem of Periodic Constant) For any positive integer k, the k-th order periodic constant τ_k of system (2.3) is a Lie-invariant of order k .

Theorem 4.2 If all the elementary Lie-invariants of system (2.3) are zero, then the origin of system (2.15) is a complex isochronous center.

The Construction Theorem of Periodic Constant shows that the essential property of periodic constants plays an important role in simplifying the period constants henceforth.

By Theorems 2.2 and 4.2, it is readily to obtain the following theorem.

Theorem 4.3 If a planar polynomial differential system can be brought to one of the following forms:

$$\begin{cases} \dfrac{dz}{dT} = z\left(1 + b_{20}zw^2\right), \\ \dfrac{dw}{dT} = -w\left[1 + a_{02}w^3 + (2b_{20} - a_{11})w^2 z + a_{01}w^3 z\right] \end{cases} \tag{4.1}$$

and

$$\begin{cases} \dfrac{dz}{dT} = z\left[1 + b_{02}z^3 + (2a_{20} - b_{11})z^2 w + b_{01}z^3 w\right], \\ \dfrac{dw}{dT} = -w\left(1 + a_{20}z^2 w\right), \end{cases} \tag{4.2}$$

then the origin of the system is a complex isochronous center.

Now, we consider the isochronous centers of system (3.1), (2.21) and (2.24), respectively.

4.1 Isochronicity of system (3.1)

In [15], a new recursive algorithm of periodic constants is given. By applying the recursive algorithm to system (3.1), we compute the periodic constants using the computer algebra system Mathematica, and simplify them using Theorem 4.1, leading to the following result.

Theorem 4.4 The first seven periodic constants at the origin of system (3.1) are given as follows:

$$\begin{cases} \tau_1 = 0, \quad \tau_2 = 2\lambda, \quad \tau_3 = 0, \quad \tau_4 = a_{01}b_{01}, \\ \tau_5 = -\dfrac{4}{3}\Big[3(a_{20}^2 a_{01} + b_{20}^2 b_{01}) + 2(a_{20}a_{02}b_{01} + b_{20}b_{02}a_{01})\Big], \\ \tau_6 = 0, \quad \tau_7 = -\dfrac{10}{3}\Big(a_{20}a_{01}^2 b_{02} + b_{20}b_{01}^2 a_{02}\Big), \end{cases} \tag{4.3}$$

where we already let $\tau_1 = \cdots = \tau_{k-1} = 0,\ k = 2, 3, \cdots, 7$.

From Theorem 4.4, we obtain the following theorem.

Theorem 4.5 For system (3.1), the first seven periodic constants are zero if and only if one of the following five conditions holds:

$$\text{(i)}\ \ \lambda = a_{01} = b_{01} = 0; \tag{4.4}$$

$$\text{(ii)}\ \ \lambda = a_{01} = b_{20} = a_{20} = 0, b_{01} \neq 0; \tag{4.5}$$

$$\text{(ii)}^*\ \ \lambda = b_{01} = a_{20} = b_{20} = 0, a_{01} \neq 0; \tag{4.6}$$

$$\text{(iii)}\ \ \lambda = b_{01} = a_{20} = b_{02} = 0, a_{01}b_{20} \neq 0; \tag{4.7}$$

$$\text{(iii)}^*\ \ \lambda = a_{01} = b_{20} = a_{02} = 0, b_{01}a_{20} \neq 0. \tag{4.8}$$

If (4.4) holds, system (3.1) becomes

$$\begin{cases} \dfrac{dz}{dT} = z\Big(1 + b_{02}z^3 + 2a_{20}z^2 w + b_{20}w^2 z\Big), \\ \dfrac{dw}{dT} = -w\Big(1 + a_{20}w^3 + 2b_{20}w^2 z + a_{20}z^2 w\Big). \end{cases} \tag{4.9}$$

Theorem 4.6 The origin of system (4.9) is an isochronous center.

Proof. By means of the transformations:

$$z = re^{i\theta},\ w = re^{-i\theta},\ t = -iT, \tag{4.10}$$

system (4.9) can be written as

$$\begin{cases} \dfrac{dr}{dt} = \dfrac{ir^4}{2}\Big[(a_{20}e^{i\theta} - b_{20}e^{-i\theta}) + (b_{02}e^{3i\theta} - a_{02}e^{-3i\theta})\Big], \\ \dfrac{d\theta}{dt} = 1 + g(\theta)r^3, \end{cases} \tag{4.11}$$

where

$$g(\theta) = \frac{1}{2}\Big[3(a_{20}e^{i\theta} + b_{20}e^{-i\theta}) + (b_{02}e^{3i\theta} + a_{02}e^{-3i\theta}\Big]. \tag{4.12}$$

The first integral of system (4.11) is

$$\frac{r^6}{1+2g(\theta)r^3} = \text{const.} \tag{4.13}$$

From (4.13), the solution of system (4.11) satisfying the initial-value condition $r|_{\theta=0} = h$ is

$$r^3 = \frac{g(\theta)h^6 + h^3\sqrt{1+2g(0)h^3+g^2(\theta)h^6}}{1+2g(0)h^3}. \tag{4.14}$$

Substituting (4.14) into equation (4.11), we obtain

$$\frac{dt}{d\theta} = 1 - \frac{g(\theta)h^3}{\sqrt{1+2g(0)h^3+g^2(\theta)h^6}}. \tag{4.15}$$

For $g(\theta)$ satisfying $g(\theta+\pi) = -g(\theta)$, we have

$$\int_0^{2\pi} \frac{dt}{d\theta} d\theta \equiv 2\pi. \tag{4.16}$$

This completes the proof.

If (4.5) holds, system (3.1) can be brought into

$$\begin{cases} \dfrac{dz}{dT} = z\left(1 + b_{02}z^3 + b_{01}z^3 w\right), \\ \dfrac{dw}{dT} = -w\left(1 + a_{02}w^3\right). \end{cases} \tag{4.17}$$

Theorem 4.7 The origin of system (4.17) is an isochronous center.

Proof. The analytic change of coordinates given by

$$u = \frac{z(1+a_{02}w^3)^{1/3}}{(1+b_{02}z^3+a_{02}w^3+\frac{3}{2}b_{01}z^3w)^{1/3}}, \quad v = \frac{w}{(1+a_{02}w^3)^{1/3}}, \tag{4.18}$$

brings system (4.17) into the form:

$$\frac{du}{dT} = u, \quad \frac{dv}{dT} = -v. \tag{4.19}$$

Therefore, the origin is a complex isochronous center.

Similar to the proof for Theorem 4.6, we can show that if (4.6) holds, then the origin of system (3.1) is a complex isochronous center.

Taking into account Theorem 4.3 we have the following result.

Theorem 4.8 For system (3.1), if (4.7) or (4.8) holds, then the origin of the system is an isochronous center.

From Theorems 4.4-4.8, we have the following theorem.

Theorem 4.9 For system (3.1), the origin is an isochronous center if and only if the first seven period constants are zero or one of the five conditions in Theorem 4.5 holds.

4.2 Isochronicity of system (2.21)

For $|a_{11}|+|b_{11}| \neq 0$, if (3.3) in Theorem 3.2 holds, then there exists a constant d such that $a_{01} = da_{11}^2,\ b_{01} = db_{11}^2$, and it brings system (2.21) into the form:

$$\begin{cases} \dfrac{dz}{dT} = z\left[1 + b_{11}^3 qz^3 + (2p-1)b_{11}z^2w + a_{11}pzw^2 + db_{11}^2 z^3 w + \lambda z^2 w^2\right], \\ \dfrac{dw}{dT} = -w\left[1 + a_{11}^3 qw^3 + (2p-1)a_{11}w^2 z + b_{11}pz^2 w + da_{11}^2 w^3 z + \lambda w^2 z^2\right]. \end{cases} \tag{4.20}$$

Theorem 4.10 For system (4.20), the first nine periodic constants of the origin are given as follows:

$$\begin{cases} \tau_1 = 0, & \tau_2 = 2\lambda, & \tau_3 = 2(1-3p)a_{11}^2 b_{11}^2, \\ \tau_4 = a_{11}^2 b_{11}^2 d^2, & \tau_5 = 0, & \tau_6 = \dfrac{1}{2}a_{11}^3 b_{11}^3 (a_{11}b_{11}q - 4)q, \\ \tau_7 = 0, & \tau_8 = 0, & \tau_9 = \dfrac{640}{21}a_{11}^4 b_{11}^2 q, \end{cases} \tag{4.21}$$

where we already let $\tau_1 = \cdots = \tau_{k-1} = 0,\ k = 2, 3, \cdots, 9$.

Theorem 4.11 For system (4.20), the first nine periodic constants are zero if and only if one of the following three conditions holds:

(i) $\lambda = 1 - 3p = d = q = 0,\ a_{11}b_{11} \neq 0;$ (4.22)

(ii) $\lambda = b_{11} = 0,\ a_{11} \neq 0;$ (4.23)

(ii)* $\lambda = a_{11} = 0,\ b_{11} \neq 0.$ (4.24)

If (4.22) holds, system (4.20) becomes

$$\begin{cases} \dfrac{dz}{dT} = z\left(1 - \dfrac{1}{3}b_{11}z^2 w + \dfrac{1}{3}a_{11}zw^2\right), \\ \dfrac{dw}{dT} = -w\left(1 - \dfrac{1}{3}a_{11}w^2 z + \dfrac{1}{3}b_{11}wz^2\right). \end{cases} \tag{4.25}$$

Theorem 4.12 The origin of system (4.25) is an isochronous center.

Proof. The transformation (4.10) brings system (4.25) into the apparently trivial form:

$$\frac{d\theta}{dt} = 1. \tag{4.26}$$

If (4.23) or (4.24) holds, then system (4.20) becomes the particular case of system (4.1) or (4.2) in Theorem 4.3, respectively. Therefore, we have the following result.

Theorem 4.13 For system (4.20), if (4.23) or (4.24) holds, then the origin of the system is an isochronous center.

From Theorems 4.10–4.13, we obtain the following theorem.

Theorem 4.14 For system (4.20), the origin is a complex isochronous center if and only if the first nine periodic constants are zero or one of the three conditions in Theorem 4.11 holds.

If (3.4) in Theorem 3.2 holds, then system (2.21) becomes

$$\begin{cases} \dfrac{dz}{dT} = z\left(1 - a_{11}zw^2 + b_{01}z^3w\right), \\ \dfrac{dw}{dT} = -w\left(1 - 3a_{11}w^2z\right). \end{cases} \tag{4.27}$$

Theorem 4.15 The origin of system (4.27) is an isochronous center.

Proof. Noting that the origin of system (4.27) is a center, there exists an analytical integral factor in the neighborhood of the origin satisfying

$$\left.\frac{dM_3}{dt}\right|_{(4.27)} = -zw\left(7a_{11}w + 4b_{01}z^2\right)M_3, \quad M_3(0,0) = 1. \tag{4.28}$$

The following transformations

$$u = z\left(1 + \frac{3}{2}b_{01}z^3w\right)^{-11/21} M_3^{-1/7}, \quad v = w\left(1 + \frac{3}{2}b_{01}z^3w\right)^{4/7} M_3^{3/7}, \tag{4.29}$$

bring system (4.28) into the form:

$$\frac{du}{dT} = u, \quad \frac{dv}{dT} = -v. \tag{4.30}$$

This completes the proof.

By the same technique, we have the following result.

Theorem 4.16 For system (2.21), if (3.5) in Theorem 3.2 holds, then the origin is an isochronous center.

4.3 Isochronicity of system (2.24)

If (3.7) in Theorem3.5 holds, system (2.24) becomes

$$\begin{cases} \dfrac{dz}{dT} = z\left(1 + b_{02}z^3\right), \\ \dfrac{dw}{dT} = -w\left(1 - a_{11}w^2z\right). \end{cases} \tag{4.31}$$

Theorem 4.17 The origin of system (4.31) is an isochronous center.

Proof. Noting that the origin of system (4.27) is a center, there exists an analytical integral factor in the neighborhood of the origin satisfying

$$\left.\frac{dM_4}{dT}\right|_{(4.27)} = -4z\left(a_{11}w^3 + b_{02}z^2\right)M_4, \quad M_4(0,0) = 1. \tag{4.32}$$

Under the following transformations:

$$u = z\left(1 + b_{02}z^3\right)^{-1/3}, \quad v = w\left(1 + b_{02}z^3\right)^{1/3} M_4^{1/4}, \tag{4.33}$$

system (4.31) can be rewritten as

$$\frac{du}{dT} = u, \quad \frac{dv}{dT} = -v. \tag{4.34}$$

The proof is complete.

By the same technique, we have the following theorem.

Theorem 4.18 For system (2.24), if (3.8) in Theorem 3.5 holds, then the origin is an isochronous center.

All the results obtained in this section completes the classification of isochronous center at the infinity of system (1.9) (or pseudo-isochronous center at the infinity of system (1.10)).

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