Nilpotent center conditions in cubic switching polynomial Liénard systems by higher-order analysis

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Abstract

The aim of this paper is to investigate two important problems related to nilpotent center conditions and bifurcation of limit cycles in switching polynomial systems. Due to the difficulty in calculating the Lyapunov constants of switching polynomial systems at non-elementary singular points, it is extremely difficult to use the existing Poincaré-Lyapunov method to study these two problems. In this paper, we develop a higher-order Poincaré-Lyapunov method to consider the nilpotent center problem in switching polynomial systems, with particular attention focused on cubic switching Liénard systems. With proper perturbations, explicit center conditions are derived for switching Liénard systems at a nilpotent center. Moreover, with Bogdanov-Takens bifurcation theory, the existence of five limit cycles around the nilpotent center is proved for a class of switching Liénard systems, which is a new lower bound of cyclicity for such polynomial systems around a nilpotent center.
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1. Introduction

During the past several decades, a large number of works has been focused on the study of the so-called Liénard equation,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

which frequently appears in many disciplines and applications [16]. The dot in (1) denotes derivative with respect to time t. By introducing $y = F(x) + \dot{x} = \int_0^x f(x)dx + \dot{x}$, the equation (1) can be brought to the following planar first-order differential system,

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -g(x), \end{aligned} \tag{2}$$

which is the so-called generalized Liénard system.

For the Liénard system (2), three important problems related to qualitative dynamical behaviors are classified as the center conditions, the number of limit cycles and the global phase portraits, see [3,9,29,45]. Christopher [10] introduced an algebraic approach to classify linear-type centers in smooth polynomial Liénard systems, which are called elementary singular points characterized by a pair of purely imaginary eigenvalues. Further, such a center is considered as global if the whole vector field of the system is filled with periodic orbits except for this point. Later, Llibre and Valls [29] studied all types of generalized Liénard systems having a global linear-type center.

Assume that the polynomials $F(x)$ and $g(x)$ are given in the form of $F(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=0}^{m} b_i x^i$. The problem of the number of bifurcating limit cycles and their relative positions for the classical Liénard system (i.e., $g(x) = x$) is the well-known Smale's 13th problem [2,35]. The authors of [27] conjectured that the classical Liénard system can have at most $[\frac{n-1}{2}]$ limit cycles, where $[\cdot]$ denotes the integer function. Maesschalck and Dumortier [32] proved that some classical Liénard system can have at least $[\frac{n-1}{2}] + 2$ limit cycles for $n \geq 6$.

It has been noted that much attention for the generalized Liénard system was paid to consider the maximum number of limit cycles bifurcating from a monodromic singular point, which is classified as either a center or a focus, usually called Hopf cyclicity, see for example [11,22,23,38]. For the Hopf cyclicity at the origin of (2), Han [19] proved it to be $[\frac{n-1}{2}] + 2$ when $g(-x) = -g(x)$; Tian and Han [38] showed it to be $[\frac{2n-1}{3}]$ when $\deg(g) = 2$; and Tian *et al.* [39] proved it to be $[\frac{3n-1}{4}]$ when $\deg(g) = 3$. Chen *el at.* [4,5] proved the existence of two limit cycles in two classes of cubic Liénard systems, and analyzed their global dynamics. However, to the best of our knowledge, the three problems for the general Liénard system are still open.

In recent years, increasing attention has been attracted to the research in dynamics of non-smooth systems since non-smoothness has been included more and more in models describing problems arising in engineering [6], epidemiology [37] and electronics [13], and in particular in the generalized Liénard system [1,12,28,33,34]. For the non-smooth Liénard system (2), $F(x)$ and $g(x)$ are usually assumed to be piecewise smooth. For example, Chen *el at.* [6] studied the global dynamics of a mechanical system with dry friction, which can be transformed to a piecewise smooth Liénard system. However, the center and Hopf cyclicity problems become extremely complicated in switching systems. To overcome the difficulty, the authors of [17] developed a useful approach for computing the Lyapunov constants for the switching polynomial systems with an elementary singular point. With this method, they solved the linear-type center

Table 1

The local qualitative properties on the nilpotent singular point $(0, 0)$ of (2).

Conditions				Type of the origin
$a_n = 0, b_m \neq 0$	*m* odd, $b_m > 0$			a center or a focus
	m odd, $b_m < 0$			a saddle
	m even			a cusp
$a_n \neq 0, \ b_m \neq 0$, $\Delta = -4nb_m + n^2 a_n^2$	$m = 2k+1$	$b_m < 0$		a saddle
		$b_m > 0$	$k > n - 1$, or $k = n - 1$ and $\Delta \geq 0$, *n* even	consisting of one hyperbolic sector and one elliptic sector
			$n < k - 1$, or $k = n - 1$ and $\Delta \geq 0$, *n* odd	a node
			$n > k - 1$, or $k = n - 1$ and $\Delta < 0$	a center or a focus
	$m = 2k$	$k > n - 1$		a saddle-node
		$k \leq n - 1$		a cusp

problem for a class of switching Liénard systems. In [38,39], the Hopf cyclicities are obtained for system (2) when $F(x)$ is a piecewise polynomial having a switching manifold at $x = 0$ with $\deg(g) = 2, 3$.

It should be pointed out that although many research results have been obtained on the center problem and bifurcation of small-amplitude limit cycles for switching Liénard systems associated with elementary singular points, no attention has been paid to switching Liénard systems associated with an isolated nilpotent singular point. By an isolated nilpotent singular point in planar polynomial systems, it means that the two eigenvalues of the Jacobian matrix of the system, evaluated at the singular point, are zero but the Jacobian matrix is not null, more details can be found in [15,18,25,26,30,31,36,42,44]. From Theorem 3.5 in [14] we know that if m is the smallest integer satisfying $b_m \neq 0$, the multiplicity of the nilpotent singular point of system (2) is exactly m. Further, let n be the smallest number for which $a_n \neq 0$. Then, the local qualitative properties on the nilpotent origin $(0, 0)$ of (2) are summarized in Table 1.

Regarding Table 1, it is worth to mention that the origin of (2) is a monodromic singular point and a cusp when the parameters satisfy

$$\Xi_1 = \left\{ (a_n, b_m) \in \mathbb{R}^2 : a_n = 0, \ b_{2k+1} > 0 \right\} \bigcup \left\{ (a_n, b_m) \in \mathbb{R}^2 : a_n \neq 0, \ b_{2k+1} > 0, \ k > n - 1 \right\}$$

$$\bigcup \left\{ (a_n, b_m) \in \mathbb{R}^2 : a_n \neq 0, \ b_{2k+1} > 0, \ k = n - 1, \ \Delta < 0 \right\} \tag{3}$$

and

$$\Xi_2 = \left\{ (a_n, b_m) \in \mathbb{R}^2 : a_n = 0, \ b_{2k} \neq 0 \right\} \bigcup \left\{ (a_n, b_m) \in \mathbb{R}^2 : a_n \neq 0, \ b_{2k} \neq 0, \ k \leq n - 1 \right\}, \tag{4}$$

respectively.

In this paper, we will develop a higher-order Poincaré-Lyapunov method to determine the nilpotent center conditions and bifurcation of small-amplitude limit cycles in switching polynomial systems. It is a more challenging and interesting compared to the works for switching systems with elementary singular points. We will apply this method to investigate a class of cubic

switching polynomial Liénard systems with a nilpotent singular point. Without loss of generality, the cubic switching Liénard systems can be written in the form of the differential equations,

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} =
\begin{cases}
\begin{pmatrix} y - (a_0^+ + a_1^+ x + a_2^+ x^2 + a_3^+ x^3) \\ -(b_0^+ + b_1^+ x + b_2^+ x^2 + b_3^+ x^3) \end{pmatrix}, & \text{if } x \geq 0, \\[2ex]
\begin{pmatrix} y - (a_0^- + a_1^- x + a_2^- x^2 + a_3^- x^3) \\ -(b_0^- + b_1^- x + b_2^- x^2 + b_3^- x^3) \end{pmatrix}, & \text{if } x < 0,
\end{cases}
\tag{5}
$$

where $x = 0$ is the unique switching manifold, and $\lambda = (a_i^\pm, b_j^\pm) \in \mathbf{R}^{16}$, $i = 0, 1, 2, 3$, represents the parameter vector. For the convenience in the following analysis, we call the system with "+" sign "the first system" and the system with "−" sign "the second system".

Assume that $(0,0)$ is a singular point of (5). Then, we have $a_0^\pm = b_0^\pm = 0$. Thus, the Jacobian matrices of the first and the second systems of (5) evaluated at the origin are given by

$$
J^\pm = \begin{bmatrix} -a_1^\pm & 1 \\ -b_1^\pm & 0 \end{bmatrix}.
\tag{6}
$$

The necessary and sufficient conditions for the origin of (5) to be an isolated nilpotent singular point are $\mathrm{Tr}(J^\pm) = \det(J^\pm) = 0$, with $J^\pm$ not being identically zero. It is easy to obtain that $a_1^\pm = b_1^\pm = 0$. Then, (5) is reduced to

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} =
\begin{cases}
\begin{pmatrix} y - (a_2^+ x^2 + a_3^+ x^3) \\ -(b_2^+ x^2 + b_3^+ x^3) \end{pmatrix}, & \text{if } x \geq 0, \\[2ex]
\begin{pmatrix} y - (a_2^- x^2 + a_3^- x^3) \\ -(b_2^- x^2 + b_3^- x^3) \end{pmatrix}, & \text{if } x < 0.
\end{cases}
\tag{7}
$$

It follows from Table 1 that different types of the nilpotent singular point can generate much more rich dynamics than that of the elementary one, leading to that the determination of the center conditions of (7) becomes more involved. In fact, the center of system (7) on the switching manifold can be classified as two monodromic singular points, or a cusp and a monodromic singular point, or two cusps. Due to this complexity it is extremely difficult to consider the nilpotent center problem in the switching polynomial Liénard systems of general degree n. Here, we derive the necessary and sufficient conditions for the center problem associated with the nilpotent origin of the cubic switching polynomial Liénard system (7). We have the following result.

Theorem 1.1. *Assume $b_2^+ \geq 0$ and $b_2^- \leq 0$. The origin of the cubic switching Liénard system (7) is a nilpotent center if one of the following conditions holds:*

$$\mathrm{I}: a_2^\pm = a_3^\pm = b_2^- = 0,\ b_2^+ > 0,\ b_3^- > 0;$$

$$\mathrm{II}: a_2^\pm = a_3^\pm = 0,\ b_2^+ > 0,\ b_2^- < 0;$$

$$\mathrm{III}: a_2^\pm = b_3^\pm = a_3^+ b_2^- - a_3^- b_2^+ = 0,\ a_3^+(b_2^+ + b_2^-) \neq 0,\ b_2^+ > 0,\ b_2^- < 0;$$

$$\text{IV}: a_2^- = a_2^+, \ a_3^- = -a_3^+, \ b_2^- = -b_2^+ < 0, \ b_3^- = b_3^+; \tag{8}$$
$$\text{V}: a_2^\pm = a_3^\pm = b_2^+ = 0, \ b_2^- < 0, \ b_3^+ > 0;$$
$$\text{VI}: a_2^\pm = b_2^\pm = 0, \ a_3^- = -a_3^+, \ b_3^- = b_3^+ > \tfrac{3}{4}(a_3^+)^2;$$
$$\text{VII}: a_2^\pm = a_3^\pm = b_2^\pm = 0, \ b_3^+ > 0, \ b_3^- > 0;$$
$$\text{VIII}: a_2^- = a_2^+ \neq 0, \ a_3^- = -a_3^+, \ b_2^\pm = 0, \ b_3^- = b_3^+ > \tfrac{1}{2}(a_2^+)^2.$$

The next result further characterizes the nilpotent center $(0,0)$ of the cubic switching Liénard system (7) to be global.

Theorem 1.2. *Assume $b_2^+ \geq 0$ and $b_2^- \leq 0$. The origin of the cubic switching Liénard system* (7) *is a nilpotent global center if one of the following conditions holds:*

$$\text{G}_1: a_2^\pm = a_3^\pm = 0, \ b_3^+ > 0, \ b_3^- > 0;$$
$$\text{G}_2: a_2^- = a_2^+, \ a_3^\pm = 0, \ b_2^- = -b_2^+, \ b_3^- = b_3^+ > \tfrac{1}{2}(a_2^+)^2. \tag{9}$$

Moreover, regarding the bifurcation of limit cycles around the origin of (7), we construct a perturbed system using the center condition IV and obtain that the maximal number of bifurcating limit cycles from the nilpotent origin of system (7) is 5. This is a new lower bound on the limit cycles for such cubic switching Liénard systems around a nilpotent singular point.

Theorem 1.3. *With the nilpotent center condition* IV *in Theorem* 1.1, *the cubic switching Liénard system* (7) *can have at least* 5 *limit cycles bifurcating from the origin by higher-order cubic perturbations.*

The rest of the paper is organized as follows. In the next section, we present our high-order Poincaré-Lyapunov method with some formulas which are needed in Section 3. Section 3 is devoted to derive the nilpotent center conditions at the origin of system (7). The conditions on the global nilpotent center of system (7) are obtained in Section 4. In Section 5, a perturbed system of (7) is constructed to show the bifurcation of 5 limit cycles from the origin of (7). Finally, conclusion is drawn in Section 6.

2. The high-order Poincaré-Lyapunov method

We consider the following switching nilpotent systems divided by the y-axis,

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y + \displaystyle\sum_{i+j=2}^{n} A_{ij}^+ x^i y^j, \ \sum_{i+j=2}^{n} B_{ij}^+ x^i y^j \right), & \text{if } x \geq 0, \\[4mm] \left(y + \displaystyle\sum_{i+j=2}^{n} A_{ij}^- x^i y^i, \ \sum_{i+j=2}^{n} B_{ij}^- x^i y^j \right), & \text{if } x < 0, \end{cases} \tag{10}$$

where $A_{ij}^\pm$ and $B_{ij}^\pm$ are parameters. We have the following proposition for proving the nilpotent center conditions at the origin $(0,0)$, which is the common nilpotent singular point in both the first and the second systems of (10).

Proposition 2.1. *Assume that the origin* $(0, 0)$ *of the switching nilpotent system* (10) *is monodromic. If the first and second systems in* (10) *have respectively the first integrals* $I^+(x, y)$ *and* $I^-(x, y)$ *near the origin, and either both* $I^+(0, y)$ *and* $I^-(0, y)$ *are even functions in* x *or* $I^+(0, y) \equiv I^-(0, y)$, *then* $(0, 0)$ *is a nilpotent center.*

See [17] for more details about Proposition 2.1. In [24], the authors redefined the symmetry of switching systems. By modifying the conditions, we obtain the following result for proving the nilpotent center conditions of (10) at the origin.

Proposition 2.2. *Assume that the nilpotent origin of the switching nilpotent system* (10) *is monodromic. If the systems in* (10) *are symmetric with respect to the x-axis, i.e., the parameters on the right-hand side of* (10) *satisfy*

$$A_{i,2k}^\pm = B_{i,2k+1}^\pm = 0, \tag{11}$$

or the systems in (10) *are symmetric with respect to the y-axis, i.e., the parameters on the right-hand side of* (10) *satisfy*

$$A_{2k+1,j}^+ = -A_{2k+1,j}^-, \quad A_{2k,j}^+ = A_{2k,j}^-, \quad B_{2k+1,j}^+ = B_{2k+1,j}^-, \quad B_{2k,j}^+ = -B_{2k,j}^-, \tag{12}$$

then the origin of (10) *is a nilpotent center.*

Now we introduce our higher-order Poincaré-Lyapunov method to study the switching system (10), which establishes the relation between the unperturbed systems and the perturbed systems based on Bogdanov-Takens (B-T) bifurcation theory. Hence, we consider the following perturbed system of (10),

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} =
\begin{cases}
\begin{pmatrix} \vartheta_1 \varepsilon x + y + \sum\limits_{i+j=2}^{n} A_{ij}^+ x^i y^j + \sum\limits_{k=1,i+j=2}^{n} P_{kij}^+ \varepsilon^k x^i y^j \\ -\varepsilon^2 x + \vartheta_2 \varepsilon y + \sum\limits_{i+j=2}^{n} B_{ij}^+ x^i y^j + \sum\limits_{k=1,i+j=2}^{n} Q_{kij}^+ \varepsilon^k x^i y^j \end{pmatrix}, & \text{if } x \geq 0, \\[3em]
\begin{pmatrix} \vartheta_1 \varepsilon x + y + \sum\limits_{i+j=2}^{n} A_{ij}^- x^i y^i + \sum\limits_{k=1,i+j=2}^{n} P_{kij}^- \varepsilon^k x^i y^j \\ -\varepsilon^2 x + \vartheta_2 \varepsilon y + \sum\limits_{i+j=2}^{n} B_{ij}^- x^i y^j + \sum\limits_{k=1,i+j=2}^{n} Q_{kij}^- \varepsilon^k x^i y^j \end{pmatrix}, & \text{if } x < 0,
\end{cases} \tag{13}
$$

where $-\varepsilon^2 x$ is called unfolding with sufficiently small $\varepsilon > 0$, $\lambda^\pm = (\vartheta_{1,2}, A_{ij}^\pm, B_{ij}^\pm, P_{ij}^\pm, Q_{ij}^\pm)$ represent two parameter vectors. For convenience, we denote that (10) is the *limit* system of (13).

Note that the linear perturbation terms used in (13) are $\varepsilon^2 x$ rather than εx can avoid the $\sqrt{\varepsilon}$ and $\frac{1}{\varepsilon^k}$ perturbation terms ($k \in \mathbb{Q}^+$) in later transformed systems. Based on the relation established for these two systems (10) and (13), and the B-T bifurcation theory, we directly have the following lemma. More detailed discussions on this subject can be found in [7].

Lemma 2.3. *Assume that the origin of* (10) *is monodromic, and that the two systems in* (10) *are limit systems of* (13). *If the linear-type elementary singular point at the origin of* (13) *becomes a linear-type center, then the origin of* (10) *is a nilpotent center.*

We give the following example to illustrate our basic idea, as it is known that the nilpotent polynomial smooth systems can be transformed to the normal form (see [20]),

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= a_k\big(1 + h(x)\big)x^k + b_k y\big(1 + g(x)\big)x^{k-1} + y^2 p(x, y),
\end{aligned}
\tag{14}
$$

where $k \geq 2$, $h(x)$, $g(x)$ and $p(x, y)$ are real analytic functions satisfying $h(0) = g(0) = 0$. For example, we consider a codimension-2 symmetric B-T bifurcation of the cubic normal form (14), given in the following form [21]:

$$
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= \varepsilon_1 x + \varepsilon_2 y + a_3 x^3 + b_3 x^2 y,
\end{aligned}
\tag{15}
$$

where the terms $\varepsilon_1 x + \varepsilon_2 y$ are called unfolding with small ε_1 and ε_2. Note that if $\varepsilon_1 < 0$, the system (15) can have Hopf bifurcation near the origin from the bifurcation line $\varepsilon_2 = 0$. It is easy to check that the elementary origin of (15) is a center if and only if $b_3 = 0$. The isolated elementary origin of (15) is reduced to a nilpotent point when $\varepsilon_1 \to 0^-$ and $\varepsilon_2 \to 0$. Furthermore, it can be verified that the nilpotent monodromic origin (when $a_3 < 0$) of system (15) without unfolding is also a center if $b_3 = 0$.

The difficulty arising from the problem of distinguishing a center from a focus in switching nilpotent system (10) is that the problem may be not algebraically solvable. That is, it does not have an infinite sequence of independent polynomials involving the coefficients of the systems such that the Lyapunov constants vanish simultaneously, which guarantees the existence of a center. We will show that the higher-order Poincaré-Lyapunov method we develop for the switching polynomial system (13) to determine nilpotent center can overcome the difficulty. More details are described below.

To achieve this, introducing the transformation $(x, y, t) \to (\varepsilon^3 x, \varepsilon^2 y, \frac{t}{\varepsilon})$ into (13), we obtain

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} =
\begin{cases}
\begin{pmatrix}
\vartheta_2 x - y + \displaystyle\sum_{k=1, i+j=2}^{n} \widetilde{A_{kij}}^{+}(\lambda^{+})\varepsilon^k x^i y^j \\[2mm]
x + \vartheta_1 y + \displaystyle\sum_{k-1, i+j=2}^{n} \widetilde{B_{kij}}^{+}(\lambda^{+})\varepsilon^k x^i y^j
\end{pmatrix}, & \text{if } y \geq 0, \\[12mm]
\begin{pmatrix}
\vartheta_2 x - y + \displaystyle\sum_{k=1, i+j=2}^{n} \widetilde{A_{kij}}^{-}(\lambda^{-})\varepsilon^k x^i y^j \\[2mm]
x + \vartheta_1 y + \displaystyle\sum_{k=1, i+j=2}^{n} \widetilde{B_{kij}}^{-}(\lambda^{-})\varepsilon^k x^i y^j
\end{pmatrix}, & \text{if } y < 0,
\end{cases}
\tag{16}
$$

where $\widetilde{A_{kij}}^{\pm}(\lambda^{\pm})$ and $\widetilde{B_{kij}}^{\pm}(\lambda^{\pm})$ are the polynomials in the parameter vector $\lambda^{\pm}$.

With the polar coordinates transformation: $x = r\cos\theta$ and $y = r\sin\theta$, the perturbation system (16) becomes the form,

$$\frac{dr}{d\theta} = \begin{cases} \dfrac{\displaystyle\sum_{k=0,i+j=1}^{n} c_{kij}^{+}(\lambda^{+})\cos\theta^{i}\sin\theta^{j}\varepsilon^{k}r^{i+j}}{1+\displaystyle\sum_{k=0,i+j=1}^{n} d_{kij}^{+}(\lambda^{+})\cos\theta^{i}\sin\theta^{j}\varepsilon^{k}r^{i+j-1}}, & \text{if } \theta\in[0,\pi], \\[2em] \dfrac{\displaystyle\sum_{k=0,i+j=1}^{n} c_{kij}^{-}(\lambda^{-})\cos\theta^{i}\sin\theta^{j}\varepsilon^{k}r^{i+j}}{1+\displaystyle\sum_{k=0,i+j=1}^{n} d_{ij}^{-}(\lambda^{-})\cos\theta^{i}\sin\theta^{j}\varepsilon^{k}r^{i+j-1}}, & \text{if } \theta\in(\pi,2\pi), \end{cases} \tag{17}$$

where $c_{kij}{}^{\pm}(\lambda^{\pm})$ and $\widetilde{d_{kij}}{}^{\pm}(\lambda^{\pm})$ are the polynomials in the parameter vector $\lambda^{\pm}$.

Let $r^{+}(\xi,\lambda^{+},\varepsilon,\theta) = \sum_{k\geq1} v_{k}^{+}(\lambda^{+},\varepsilon,\theta)\xi^{k}$ and $r^{-}(\xi,\lambda^{-},\varepsilon,\theta) = \sum_{k\geq1} v_{k}^{-}(\lambda^{-},\varepsilon,\theta)\xi^{k}$ be the solutions of the first and second systems of (17) associated with the initial conditions $r^{+}(\xi,\lambda^{+},\varepsilon,0) = r^{-}(\xi,\lambda^{-},\varepsilon,\pi) = \xi$. We denote by

$$\Pi^{+}(\xi) = r^{+}(\xi,\lambda^{+},\varepsilon,\pi) = \sum_{k\geq1} v_{k}^{+}(\lambda^{+},\varepsilon,\pi)\xi^{k}$$

and

$$\Pi^{-}(\xi) = r^{-}(\xi,\lambda^{+},\varepsilon,2\pi) = \sum_{k\geq1} v_{k}^{-}(\lambda^{-},\varepsilon,\pi)\xi^{k},$$

the first half-return map $\Pi^{+}(\xi)$ and the second half-return map $\Pi^{-}(\xi)$, respectively, where $v_{k}^{\pm}(\lambda^{\pm},\varepsilon,\pi)$ are the coefficients in Taylor expansions. However, it is extremely difficult to composite these two maps to compute the displacement map of (17). We may follow the procedure in [17] and introduce the transformation $(x,y,t) \rightarrow (x,-y,-t)$ into the piecewise smooth system (16) to yield

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -\vartheta_{2}x - y - \displaystyle\sum_{k=1,i+j=2}^{n} \widetilde{A_{kij}}^{-}(\lambda^{-})\varepsilon^{k}x^{i}(-y)^{j} \\ x - \vartheta_{1}y + \displaystyle\sum_{k-1,i+j=2}^{n} \widetilde{B_{kij}}^{-}(\lambda^{-})\varepsilon^{k}x^{i}(-y)^{j} \end{pmatrix}, & \text{if } y>0, \\[2.5em] \begin{pmatrix} -\vartheta_{2}x - y + \displaystyle\sum_{k=1,i+j=2}^{n} \widetilde{A_{kij}}^{+}(\lambda^{+})\varepsilon^{k}x^{i}(-y)^{j} \\ x - \vartheta_{1}y + \displaystyle\sum_{k=1,i+j=2}^{n} \widetilde{B_{kij}}^{+}(\lambda^{+})\varepsilon^{k}x^{i}y^{j} \end{pmatrix}, & \text{if } y\leq0. \end{cases} \tag{18}$$

Then, the displacement function can be written as

$$d(\xi) = \Pi^+(\xi) - (\Pi^-)^{-1}(\xi) = \Pi^+(\xi) - \Pi^+_-(\xi) = \sum_{j \geq 1} V_j(\lambda^\pm, \varepsilon, \pi)\xi^j, \qquad (19)$$

where $\Pi^+_-(\xi)$ is the first half-return map of (18), and

$$V_j(\lambda^\pm, \varepsilon, \pi) = \sum_{k=1}^\infty V_{jk}\,\varepsilon^k, \quad j = 1, 2, \ldots, \qquad (20)$$

in which V_{jk} denotes the jth ε^k-order Lyapunov constant, see [7] for more details about the computation of the generalized Lyapunov constants. Hence, we can derive the center conditions of system (13) by vanishing the ε terms in these generalized Lyapunov constants, and then by Lemma 2.3 we derive the algebraic conditions characterizing the nilpotent center of system (10). Thus, we prove that these conditions are necessary for $(0, 0)$ of (10) be a nilpotent center. In general, these nilpotent center conditions can be satisfied by appropriately choosing the perturbation coefficients $P^\pm_{kij}$ and $Q^\pm_{kij}$.

Further, we generalize our method to study the bifurcation of limit cycles from the nilpotent center of the switching system (10). Actually, by B-T bifurcation theory, we may add the linear perturbation term $-\varepsilon^2 x$ to the switching system (10) such that the system has a linear-type center at the origin. Then we perturb such system by adding higher ε-order terms, and compute the generalized Lyapunov constants of the perturbed system. Finally, by using the higher ε-order Lyapunov constants [41,43], we find the bifurcation of limit cycles from the center as many as possible. More precisely, with the result in [40], we derive the following lemma giving the sufficient conditions for the existence of small-amplitude limit cycles around the origin of (10).

Lemma 2.4 *(Lemma 4, [40]). If there exists a critical point $\lambda_* = (a_{1c}, a_{2c}, \cdots, a_{nc})$ such that a set of ε^k-order Lyapunov constants satisfies $V_{1k}(\lambda_*) = V_{2k}(\lambda_*) = \cdots = V_{nk}(\lambda_*) = 0$, $V_{n+1,k}(\lambda_*) \neq 0$ and*

$$\det\left[\frac{\partial(V_{1k}, V_{2k}, \cdots, V_{nk})}{\partial(a_{1c}, a_{2c}, \cdots, a_{nc})}(\lambda_*)\right] \neq 0, \qquad (21)$$

then small appropriate perturbations about $\lambda = \lambda_*$ lead to that the switching system (16) has exactly n limit cycles bifurcating from $(0, 0)$.

3. The proof of Theorem 1.1

From the results given in the precious section, we know that the nilpotent center of (7) at the origin can be combined by a monodromic singular point or a cusp, i.e. the parameters satisfy the condition Ξ_1 or Ξ_2. Firstly, assume that the origin in the first smooth system of (7) is a cusp, then we have $b_2^+ \neq 0$. If $b_2^- \neq 0$, then the origin in the second system of (7) is also a cusp with multiplicity two. If $b_2^- = 0$, we have the following statement: The origin of the second system of (7) is a monodromic singular point with multiplicity three if and only if one of the following conditions holds:
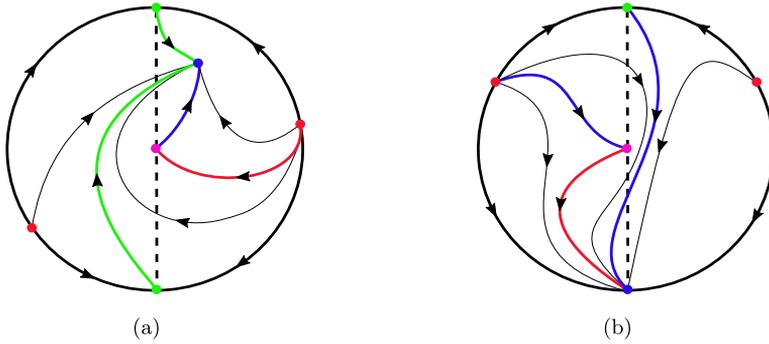
Fig. 1. Two phase portraits of the cubic switching Liénard system (7) showing that the origin is a cusp for (a) $a_2^\pm = b_3^\pm = 1$, $a_3^\pm = 2$, $b_2^+ = -1$ and $b_2^- = -2$; and (b) $a_2^\pm = a_3^\pm = b_3^\pm = 1$, $b_3^- = -1$, $b_2^+ = 2$ and $b_2^- = 3$.

$$a_2^- = b_2^- = 0, \quad b_3^- > \frac{3}{4}(a_3^-)^2;$$
$$b_2^- = 0, \quad a_2^- \neq 0, \quad b_3^- > \frac{1}{2}(a_2^-)^2. \tag{22}$$

We only consider $b_2^+ > 0$ and $b_2^- \leq 0$ because the origin of (7) cannot be monodromic when either $b_2^+ < 0$ or $b_2^- > 0$, which are illustrated in the two phase portraits of system (7), as depicted in Fig. 1. It is shown that when $b_2^+ < 0$ the first system of (7) has two seperatrices connecting the origin and two singular points in the right half Poincaré disc, see Fig. 1(a). Similarly when $b_2^- > 0$ the second system of (7) has two seperatrices connecting the origin and the other two ones in the left half Poincaré disc, see Fig. 1(b). Thus, the origin cannot be a center when either $b_2^+ < 0$ or $b_2^- > 0$.

Example 3.1. Two different phase portraits of the cubic switching Liénard system (7), as depicted in the Poincaré disc, show that the origin is a cusp for $b_2^+ = -1$ (see Fig. 1(a)) or $b_2^- = 3$ (see Fig. 1(b)).

Next, we discuss how to apply the higher-order Poincaré-Lyapunov method to derive the nilpotent center conditions for the origin of system (7). With perturbations added to system (7), we obtain the following perturbed systems,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y - (a_2^+ x^2 + a_3^+ x^3) + \sum\limits_{k=1} \varepsilon^k P_k^+(x) \\ -\varepsilon^2 x - (b_2^+ x^2 + b_3^+ x^3) + \sum\limits_{k=1} \varepsilon^k Q_k^+(x) \end{pmatrix}, & \text{if } x \geq 0, \\ \begin{pmatrix} y - (a_2^- x^2 + a_3^- x^3) + \sum\limits_{k=1} \varepsilon^k P_k^-(x) \\ -\varepsilon^2 x - (b_2^- x^2 + b_3^- x^3) + \sum\limits_{k=1} \varepsilon^k Q_k^-(x) \end{pmatrix}, & \text{if } x < 0, \end{cases} \tag{23}$$

where

$$P_k^\pm(x) = p_{k2}^\pm x^2 + p_{k3}^\pm x^3, \quad Q_k^\pm(x) = q_{k2}^\pm x^2 + q_{k3}^\pm x^3,$$

in which $p_{ki}^\pm$, $q_{ki}^\pm$ are real parameters. It is very difficult for computing the generalized Lyapuov constants of (23) with large number of parameters. So we only consider the ε^2-order perturbations. Further, introducing the transformation $(x, y, t) \to (\varepsilon^3 x, \varepsilon^2 y, \frac{t}{\varepsilon})$ into system (23), we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y - (b_2^+ + q_{22}^+\varepsilon^2)y^2 - (b_3^+\varepsilon^2 + q_{23}^+\varepsilon^4)y^3 \\ x - (a_2^+\varepsilon + p_{22}^+\varepsilon^3)y^2 - (a_3^+\varepsilon^3 + p_{23}^+\varepsilon^5)y^3 \end{pmatrix}, & \text{if } y \geq 0, \\[4mm] \begin{pmatrix} -y - (b_2^- + q_{22}^-\varepsilon^2)y^2 - (b_3^-\varepsilon^2 + q_{23}^-\varepsilon^4)y^3 \\ x - (a_2^-\varepsilon + p_{22}^-\varepsilon^3)y^2 - (a_3^-\varepsilon^3 + p_{23}^-\varepsilon^5)y^3 \end{pmatrix}, & \text{if } y < 0. \end{cases} \tag{24}$$

Then, we consider two cases: $b_2^- = 0$ and $b_2^- < 0$ to obtain the nilpotent center conditions by considering each ith ε^k-order Lyapunov constant.

3.1. Case 1: $b_2^- = 0$

We use the higher-order Poincaré-Lyapunov method to compute the generalized Lyapunov constants associated with the origin of (24). The first two generalized Lyapunov constants are $V_1(\varepsilon) = 0$ and

$$V_2(\varepsilon) = \frac{4}{3}\big[(a_2^+ - a_2^-) + (p_{22}^+ - p_{22}^-)\varepsilon^2\big]\varepsilon. \tag{25}$$

By (22) we consider the two subcases: (i) $a_2^- = 0$, $b_3^- > \frac{3}{4}(a_3^-)^2$, and (ii) $a_2^- \neq 0$, $b_3^- > \frac{1}{2}(a_2^-)^2$.

(i) $a_2^- = 0$, $b_3^- > \frac{3}{4}(a_3^-)^2$. Setting the ε-order and ε^3-order Lyapunov constants in $V_2(\varepsilon)$ zero yields the necessary center conditions $a_2^+ = 0$ and $p_{22}^- = p_{22}^+$. Then, the 3rd generalized Lyapunov constant is obtained as

$$V_3(\varepsilon) = -\frac{\pi}{8}\big[3a_3^+ + 3a_3^- - 2b_2^+p_{22}^+ + (3p_{23}^+ + 3p_{23}^- - 2p_{22}^+q_{22}^+ - 2p_{22}^+q_{22}^-)\varepsilon^2\big]\varepsilon^3.$$

Letting the ε^3-order and ε^5-order terms in $V_3(\varepsilon)$ equal zero we obtain the following conditions,

$$p_{22}^+ = \frac{3(a_3^+ + a_3^-)}{2b_2^+},$$

$$p_{23}^- = \frac{1}{b_2^+}\big(-b_2^+p_{23}^+ + a_3^+q_{22}^+ + a_3^-q_{22}^+ + a_3^+q_{22}^- + a_3^-q_{22}^-\big).$$

Then, we have the 4th generalized Lyapunov constant, given by

$$V_4(\varepsilon) = \frac{4}{15b_2^+}\big[4a_3^-(b_2^+)^2 - M_1\varepsilon^2 + M_2\varepsilon^4\big]\varepsilon^3,$$

where

$$M_1 = 3a_3^+ b_3^+ + 3a_3^- b_3^+ - 3a_3^+ b_3^- - 3a_3^- b_3^- - 4(b_2^+)^2 p_{23}^- - 4a_3^- b_2^+ q_{22}^+ + 4a_3^+ b_2^+ q_{22}^-,$$

$$M_2 = 4b_2^+ p_{23}^- q_{22}^+ + 4b_2^+ p_{23}^- q_{22}^- - 4a_3^+ q_{22}^+ q_{22}^- - 4a_3^- q_{22}^- q_{22}^- - 4a_3^+ (q_{22}^-)^2 - 4a_3^- (q_{22}^-)^2$$
$$- 3a_3^+ q_{23}^+ - 3a_3^- q_{23}^+ + 3a_3^+ q_{23}^- + 3a_3^- q_{23}^-.$$

Letting each ε term in $V_4(\varepsilon)$ be zero, we obtain the conditions:

$$a_3^- = 0,$$

$$p_{23}^- = \frac{3a_3^+ b_3^+ - 3a_3^+ b_3^- + 4a_3^+ b_2^+ q_{22}^-}{4(b_2^+)^2},$$

$$q_{23}^- = \frac{1}{b_2^+}\left(-b_3^+ q_{22}^+ + b_3^- q_{22}^+ - b_3^+ q_{22}^- + b_3^- q_{22}^- + b_2^+ q_{23}^+\right).$$

Then, we have

$$V_5(\varepsilon) = -\frac{5\pi}{256(b_2^+)^3} a_3^+ (b_2^+ + q_{22}^+\varepsilon^2 + q_{22}^-\varepsilon^2)\left[2(b_2^+)^2(b_3^+ + 7b_3^-) + M_3\varepsilon^2\right]\varepsilon^5,$$

where

$$M_3 = 9(b_3^+)^2 - 18b_3^+ b_3^- + 9(b_3^-)^2 - 14b_2^+ b_3^+ q_{22}^+ + 14b_2^+ b_3^- q_{22}^+ - 2b_2^+ b_3^+ q_{22}^-$$
$$+ 2b_2^+ b_3^- q_{22}^- + 16(b_2^+)^2 q_{23}^+.$$

$a_3^+ = 0$ (leading to $V_5(\varepsilon) = 0$) gives the condition I. Otherwise, if $a_3^+ \neq 0$, setting the other ε terms in $V_5(\varepsilon)$ zero we obtain

$$b_3^+ = -7b_3^-, \quad q_{23}^+ = -\frac{1}{(b_2^+)^2}\left[36(b_3^-)^2 + 7b_2^+ b_3^- q_{22}^+ + b_2^+ b_3^- q_{22}^-\right].$$

Further, under the above conditions, we have the 6th generalized Lyapunov constant given by

$$V_6(\varepsilon) = -\frac{128}{315(b_2^+)^2} a_3^+ b_3^- (b_2^+ + q_{22}^+\varepsilon^2 + q_{22}^-\varepsilon^2)^3 \varepsilon^5 \neq 0$$

when $a_3^+ \neq 0$.

(ii) $a_2^- \neq 0$, $b_3^- > \frac{1}{2}(a_2^-)^2$. Setting the ε-order and ε^3-order terms in $V_2(\varepsilon)$ zero yields the necessary center conditions $a_2^- = a_2^+ \neq 0$ and $p_{22}^- = p_{22}^+$. Then, the 3rd generalized Lyapunov constant is given by

$$V_3(\varepsilon) = \frac{\pi}{8}\left[2a_2^- b_2^+ - (3a_3^+ + 3a_3^- - 2b_2^+ p_{22}^+ - 2a_2^+ q_{22}^+ - 2a_2^+ q_{22}^-)\varepsilon^2 - (3p_{23}^+ + 3p_{23}^-\right.$$
$$\left. - 2p_{22}^+ q_{22}^+ - 2p_{22}^+ q_{22}^-)\varepsilon^4\right]\varepsilon \neq 0.$$

3.2. Case 2: $b_2^- < 0$

Similarly, setting each ε term in $V_2(\varepsilon)$ zero yields the necessary center conditions $a_2^- = a_2^+$ and $p_{22}^- = p_{22}^+$. Then, we obtain the 3rd generalized Lyapunov constant,

$$V_3(\varepsilon) = \frac{\pi}{8}\big[2a_2^+(b_2^+ + b_2^-) - (3a_3^+ + 3a_3^- - 2b_2^+ p_{22}^+ - 2b_2^- p_{22}^+ - 2a_2^+ q_{22}^+ - 2a_2^+ q_{22}^-)\varepsilon^2$$
$$- (3p_{23}^+ + 3p_{23}^- - 2p_{22}^+ q_{22}^+ - 2p_{22}^+ q_{22}^-)\varepsilon^4\big]\varepsilon.$$

Considering the ε-order term in $V_3(\varepsilon)$, we have two subcases: (i) $a_2^+ = 0$, and (ii) $b_2^+ + b_2^- = 0$.

(i) Assume that $a_2^+ = 0$ and $b_2^+ + b_2^- \neq 0$. We let the ε^3-order and ε^5-order terms in $V_3(\varepsilon)$ equal zero to obtain the conditions,

$$p_{22}^+ = \frac{3(a_3^+ + a_3^-)}{2(b_2^+ + b_2^-)},$$
$$p_{23}^- = \frac{1}{b_2^+ + b_2^-}\big(b_2^+ p_{23}^+ + b_2^- p_{23}^+ - a_3^+ q_{22}^+ - a_3^- q_{22}^+ - a_3^+ q_{22}^- - a_3^- q_{22}^-\big).$$

Then, the 4th generalized Lyapunov constant becomes

$$V_4(\varepsilon) = \frac{-4}{15(b_2^+ + b_2^-)}\big[4(b_2^+ + b_2^-)(a_3^- b_2^+ - a_3^+ b_2^-) - M_4\varepsilon^2 - M_5\varepsilon^4\big]\varepsilon^3,$$

where

$$M_4 = 3a_3^+ b_3^+ + 3a_3^- b_3^+ - 3a_3^+ b_3^- - 3a_3^- b_3^- + 4(b_2^+)^2 p_{23}^+ + 8b_2^+ b_2^- p_{23}^+ + 4(b_2^-)^2 p_{23}^+$$
$$- 4a_3^+ b_2^+ q_{22}^+ - 8a_3^- b_2^+ q_{22}^+ - 4a_3^- b_2^- q_{22}^+ - 4a_3^- b_2^+ q_{22}^- + 4a_3^+ b_2^- q_{22}^-,$$
$$M_5 = 4b_2^+ p_{23}^+ q_{22}^+ + 4b_2^- p_{23}^+ q_{22}^+ - 4a_3^+(q_{22}^+)^2 - 4a_3^-(q_{22}^+)^2 + 4b_2^+ p_{23}^+ q_{22}^- + 4b_2^- p_{23}^+ q_{22}^-$$
$$- 4a_3^+ q_{22}^+ q_{22}^- - 4a_3^- q_{22}^+ q_{22}^- + 3a_3^+ q_{23}^+ + 3a_3^- q_{23}^+ - 3a_3^+ q_{23}^- - 3a_3^- q_{23}^-.$$

Thus, we have

$$a_3^- = \frac{a_3^+ b_2^-}{b_2^+}, \qquad p_{23}^+ = \frac{-3a_3^+ b_2^+ + 3a_3^+ b_3^- + 4a_3^+ b_2^+ q_{22}^+ + 4a_3^+ b_2^- q_{22}^+}{4b_2^+(b_2^+ + b_2^-)}$$

by setting the ε^3-order and ε^5-order terms in $V_4(\varepsilon)$ zero. Then, M_5 is simplified as

$$M_5 = -\frac{3}{b_2^+}a_3^+\big(b_3^+ q_{22}^- - b_3^- q_{22}^+ + b_3^+ q_{22}^- - b_3^- q_{22}^- - b_2^+ q_{23}^- - b_2^- q_{23}^- + b_2^+ q_{23}^- + b_2^- q_{23}^-\big).$$

(i.1) If $a_3^+ = 0$, we obtain $M_5 = 0$, leading to a condition included in the condition II.

(i.2) If $a_3^+ \neq 0$, we have

$$q_{23}^+ = \frac{1}{b_2^+ + b_2^-}(b_3^+ q_{22}^+ - b_3^- q_{22}^+ + b_3^+ q_{22}^- - b_3^- q_{22}^- + b_2^+ q_{23}^- + b_2^- q_{23}^-)$$

by setting $M_5 = 0$. Further, we obtain the 5th generalized Lyapunov constant,

$$V_5(\varepsilon) = -\frac{5a_3^+ \pi}{256 b_2^+ (b_2^+ + b_2^-)^2}\left[b_2^+ + b_2^- + (q_{22}^+ + q_{22}^-)\varepsilon^2\right]\left[2(b_2^+ + b_2^-)M_6 + M_7 \varepsilon^2\right]\varepsilon^5$$

where

$$
\begin{aligned}
M_6 &= b_3^+(b_2^+ + 7b_2^-) + b_3^-(7b_2^+ + b_2^-),\\
M_7 &= 9(b_3^+)^2 - 18 b_3^+ b_3^- + 9(b_3^-)^2 + 2b_2^+ b_3^+ q_{22}^+ + 2b_2^- b_3^+ q_{22}^+ - 2b_2^+ b_3^- q_{22}^+ - 2b_2^- b_3^- q_{22}^+\\
&\quad + 14 b_2^+ b_3^+ q_{22}^- + 14 b_2^- b_3^+ q_{22}^- - 14 b_2^+ b_3^- q_{22}^- - 14 b_2^- b_3^- q_{22}^- + 16(b_2^+)^2 q_{23}^-\\
&\quad + 32 b_2^+ b_2^- q_{23}^- + 16(b_2^-)^2 q_{23}^-.
\end{aligned}
$$

Since $a_3^+(b_2^+ + b_2^-) \neq 0$, the only possibility for $V_5(\varepsilon) = 0$ is $M_6 = M_7 = 0$. Considering $M_6 = 0$, if $b_2^+ + 7b_2^- \neq 0$, we have $b_3^+ = \frac{-(7b_2^+ + b_2^-)b_3^-}{b_2^+ + 7b_2^-}$; and if $b_2^+ = -7b_2^-$, we obtain $M_6 = -48 b_2^- b_3^- = 0$ which yields $b_3^- = 0$ due to $b_2^- < 0$. Thus, we consider the following two subcases.

(i.2.1) If $b_3^+ = \frac{-(7b_2^+ + b_2^-)b_3^-}{b_2^+ + 7b_2^-}$, setting $M_7 = 0$ yields

$$q_{23}^- = \frac{1}{(b_2^+ + 7b_2^-)^2}[-36(b_3^-)^2 + b_2^+ b_3^- q_{22}^+ + 7b_2^- b_3^- q_{22}^+ + 7b_2^+ b_3^- q_{22}^- + 49 b_2^- b_3^- q_{22}^-].$$

Then, we obtain the 6th generalized Lyapunov constant,

$$V_6(\varepsilon) = -\frac{128}{315 b_2^+ (b_2^+ + 7b_2^-)}a_3^+ b_3^-\left[b_2^+ + b_2^- + (q_{22}^+ + q_{22}^-)\varepsilon^2\right]^3 \varepsilon^5.$$

If $b_3^- = 0$, we have $V_6(\varepsilon) = 0$, and this necessary condition is included in the condition III. Otherwise, we have $V_6(\varepsilon) \neq 0$ when $b_2^+ + b_2^- \neq 0$.

(i.2.2) If $b_3^- = b_2^+ + 7b_2^- = 0$ we have

$$q_{23}^- = \frac{1}{192(b_2^-)^2}[-3(b_3^+)^2 + 4b_2^- b_3^+ q_{22}^- + 28 b_2^- b_3^+ q_{22}^-]$$

from $M_7 = 0$. Then the 6th generalized Lyapunov constant has the following form

$$V_6(\varepsilon) = -\frac{8}{6615(b_2^-)^2}a_3^+ b_3^+\left[6b_2^- + (q_{22}^+ + q_{22}^-)\varepsilon^2\right]^3 \varepsilon^5 \neq 0.$$

Assume that $b_3^+ = 0$, we have $V_6(\varepsilon) = 0$. Combining the necessary conditions from (i.2.1) and (i.2.2), we obtain the condition III. Otherwise we have $V_6(\varepsilon) \neq 0$ when $a_3 \neq 0$.

(ii) Assume that $b_2^- + b_2^+ = 0$. Consider two subcases: (ii.1) $a_2^+ = 0$ and (ii.2) $a_2^+ \neq 0$.
(ii.1) If $a_2^+ = 0$, setting each ε term in $V_3(\varepsilon)$ zero we have

$$a_3^- = -a_3^+,$$
$$p_{23}^- = \frac{1}{3}(-3p_{23}^+ + 2p_{22}^+q_{22}^+ + 2p_{22}^+q_{22}^-).$$

Then, the 4th generalized Lyapunov constant is given by

$$V_4(\varepsilon) = -\frac{8}{45}\left(M_8 + M_9\varepsilon^2\right)\varepsilon^5,$$

where

$$M_8 = -3b_3^+ p_{22}^+ + 3b_3^- p_{22}^+ - 6a_3^+ q_{22}^+ + 4b_2^+ p_{22}^+q_{22}^+ - 6a_3^+ q_{22}^- + 4b_2^+ p_{22}^+q_{22}^-,$$
$$M_9 = 6p_{23}^+q_{22}^+ - 4p_{22}^+(q_{22}^+)^2 + 6p_{23}^+q_{22}^- - 4p_{22}^+q_{22}^+q_{22}^- + 3p_{22}^+q_{23}^+ - 3p_{22}^+q_{23}^-.$$

(ii.1.1) If $q_{22}^- = -q_{22}^+$, it follows from $M_8 = M_9 = 0$ that either $b_3^- = b_3^+$, $q_{23}^- = q_{23}^+$, or $p_{22} = 0$. The first choice leads to a necessity condition included in the condition IV. For the second choice, we have the 5th generalized Lyapunov constant,

$$V_5(\varepsilon) = \frac{15}{64}\left(a_3^+ + p_{23}^+\varepsilon^2\right)\left(b_3^+ - b_3^- + q_{23}^+\varepsilon^2 - q_{23}^-\varepsilon^2\right)\varepsilon^5,$$

which yields $a_3^+ = p_{23}^+ = 0$ by setting $V_5(\varepsilon) = 0$, leading to a condition included in the condition II. Combining the subcases (i.1) and (ii.1.1) we have the condition II.
(ii.1.2) If $q_{22}^- \neq -q_{22}^+$, we have

$$a_3^+ = \frac{1}{6(q_{22}^+ + q_{22}^-)}(-3b_3^+ p_{22}^+ + 3b_3^- p_{22}^+ + 4b_2^+ p_{22}^+q_{22}^+ + 4b_2^+ p_{22}^+q_{22}^-),$$
$$p_{23}^+ = \frac{1}{6(q_{22}^+ + q_{22}^-)}\left[4p_{22}^+(q_{22}^+)^2 + 4p_{22}^+q_{22}^+q_{22}^- - 3p_{22}^+q_{23}^+ + 3p_{22}^+q_{23}^-\right]$$

by setting $M_8 = M_9 = 0$. Then, we obtain the 5th generalized Lyapunov constant,

$$V_5(\varepsilon) = -\frac{5\pi}{384(q_{22}^+ + q_{22}^-)}p_{22}^+\left(M_{10} + 2M_{11}\varepsilon^2 + M_{12}\varepsilon^2\right)\varepsilon^5,$$

where

$$M_{10} = 3(b_3^+ - b_3^-)(3b_3^+ - 3b_3^- - 4b_2^+ q_{22}^+ - 4b_2^+ q_{22}^-),$$

$$M_{11} = b_3^+ (q_{22}^+)^2 + 7b_3^- (q_{22}^+)^2 + 8b_3^+ q_{22}^+ q_{22}^- + 8b_3^- q_{22}^+ q_{22}^- + 7b_3^+ (q_{22}^-)^2 + b_3^- (q_{22}^-)^2$$
$$+ 9b_3^+ q_{23}^+ - 9b_3^- q_{23}^+ - 6b_2^+ q_{22}^+ q_{23}^+ - 6b_2^+ q_{22}^- q_{23}^+ - 9b_3^+ q_{23}^- + 9b_3^- q_{23}^- + 6b_2^+ q_{22}^+ q_{23}^-$$
$$+ 6b_2^+ q_{22}^- q_{23}^-,$$

$$M_{12} = 2(q_{22}^+)^2 q_{23}^+ + 16 q_{22}^+ q_{22}^- q_{23}^+ + 14(q_{22}^-)^2 q_{23}^+ + 9(q_{23}^+)^2 + 14(q_{22}^+)^2 q_{23}^- + 16 q_{22}^+ q_{22}^- q_{23}^-$$
$$+ 2(q_{22}^-)^2 q_{23}^- - 18 q_{23}^+ q_{23}^- + 9(q_{23}^-)^2.$$

If $p_{22}^+ = 0$ or $b_2^+ = \frac{3(b_3^- - b_3^+)}{4(q_{22}^+ + q_{22}^-)}$, we have $a_3^+ = 0$, which is included in the condition II. Otherwise, we have $b_3^- = b_3^+$ from M_{10}, which leads to a necessity condition included in the condition IV.

(ii.2) If $a_2^+ \neq 0$, letting the ε^3-order and ε^5-order terms in $V_3(\varepsilon)$ equal zero we obtain the conditions:

$$q_{22}^- = \frac{3a_3^+ + 3a_3^- - 2a_2^+ q_{22}^+}{2a_2^+},$$

$$p_{23}^- = -\frac{a_3^+ p_{22}^+ + a_3^- p_{22}^+ - a_2^+ p_{23}^+}{a_2^+}.$$

Then, we have the 4th generalized Lyapunov constant, given by

$$V_4(\varepsilon) = \frac{8}{15 a_2^+} \left(a_2^+ M_{13} + M_{14} \varepsilon^2 + M_{15} \varepsilon^4 \right) \varepsilon^3,$$

where

$$M_{13} = -2a_3^+ b_2^+ - 2a_3^- b_2^+ + a_2^+ b_3^+ - a_2^+ b_3^-,$$

$$M_{14} = 3(a_3^+)^2 + 3a_3^+ a_3^- - 2a_3^+ b_2^+ p_{22}^+ - 2a_3^- b_2^+ p_{22}^+ + a_2^+ b_3^+ p_{22}^+ - a_2^+ b_3^- p_{22}^+$$
$$- 2a_2^+ a_3^+ q_{22}^+ - 2a_2^+ a_3^- q_{22}^+ + (a_2^+)^2 q_{23}^+ - (a_2^+)^2 q_{23}^-,$$

$$M_{15} = 3a_3^+ p_{23}^+ + 3a_3^- p_{23}^+ - 2a_3^+ p_{22}^+ q_{22}^+ - 2a_3^- p_{22}^+ q_{22}^+ + a_2^+ p_{22}^+ q_{23}^+ - a_2^+ p_{22}^+ q_{23}^-.$$

Setting $M_{13} = M_{14} = 0$, we obtain the conditions:

$$b_3^- = \frac{1}{a_2^+}(-2a_3^+ b_2^+ - 2a_3^- b_2^+ + a_2^+ b_3^+),$$

$$q_{23}^- = \frac{1}{(a_2^+)^2} \left[3(a_3^+)^2 + 3a_3^+ a_3^- - 2a_2^+ a_3^+ q_{22}^+ - 2a_2^+ a_3^- q_{22}^+ + (a_2^+)^2 q_{23}^+ \right].$$

Then, two subcases follow from $M_{15} = 0$: (ii.2.1) $a_3^+ + a_3^- = 0$ and (ii.2.2) $p_{23}^+ = \frac{a_3^+ p_{22}^+}{a_2^+}$.

(ii.2.1) If $a_3^+ + a_3^- = 0$, which is combined with the necessary conditions from (ii.1.1) to yield the condition IV.

(ii.2.2) If $p_{23}^+ = \frac{a_3^+ p_{22}^+}{a_2^+}$, then we have $V_4(\varepsilon) = 0$ and obtain the 5th generalized Lyapunov constant,

$$V_5(\varepsilon) = -\frac{5\pi}{128(a_2^+)^3}(a_3^+ + a_3^-)(a_2^+ + p_{22}^+\varepsilon^2)[2a_2^+ M_{16} + M_{17}\varepsilon^2]\varepsilon^5,$$

where

$$M_{16} = -7a_3^+ b_2^+ - a_3^- b_2^+ + 4a_2^+ b_3^+,$$
$$M_{17} = 21(a_3^+)^2 + 3a_3^+ a_3^- - 14a_2^+ a_3^+ q_{22} - 2a_2^+ a_3^- q_{22}^+ + 8(a_2^+)^2 q_{23}^+.$$

Setting $M_{16} = M_{17} = 0$ results in

$$b_3^+ = \frac{7a_3^+ b_2^+ + a_3^- b_2^+}{4a_2^+},$$

$$q_{23}^+ = -\frac{1}{q_{23}^+}[21(a_3^+)^2 + 3a_3^+ a_3^- - 14a_2^+ a_3^+ q_{22}^+ - 2a_2^+ a_3^- q_{22}^+],$$

under which the 6th generalized Lyapunov constant becomes

$$V_6(\varepsilon) = \frac{8}{105(a_2^+)^4}(a_3^+ + a_3^-)^3(a_2^+ + p_{22}^+\varepsilon^2)(2a_2^+ b_2^+ - 3a_3^+\varepsilon^2 + 2a_2^+ q_{22}^+\varepsilon^2)\varepsilon^7 \neq 0$$

with $a_3^+ + a_3^- \neq 0$.

The above results give the necessary nilpotent center conditions for system (7) at the origin when the origin of the first system of (7) is a cusp. Now we consider the center conditions associated with the origin of the first system of (7) being a monodromic singular point with multiplicity three if and only if one of the following holds:

$$a_2^+ = b_2^+ = 0, \quad b_3^+ > \frac{3}{4}(a_3^+)^2; \tag{26}$$
$$b_2^+ = 0, \quad a_2^+ \neq 0, \quad b_3^+ > \frac{1}{2}(a_2^+)^2.$$

If $b_2^- < 0$, the origin in the second system of (7) is a cusp, which is a similar case to that discussed above, leading to the condition V in Theorem 1.1. Hence, we assume that one of the conditions in (22) holds. Then, the origin in the second system of (7) is also a monodromic singular point.

Setting the second generalized Lyapunov constant (25) zero, we find two cases: either $a_2^+ = a_2^- = 0$ or $a_2^+ a_2^- \neq 0$ when the conditions in (22) and (26) are satisfied. The detailed analysis for the two cases are given below.

3.3. Case 3: $a_2^+ = a_2^- = 0$

The second generalized Lyapunov constant becomes

$$V_2(\varepsilon) = \frac{4}{3}\,(p_{22}^+ - p_{22}^-)\,\varepsilon^3.$$

Setting $V_2(\varepsilon) = 0$ yields the condition $p_{22}^- = p_{22}^+$. Then, the 3rd generalized Lyapunov constant is given by

$$V_3(\varepsilon) = -\frac{\pi}{8}\big[3(a_3^+ + a_3^-) + (3p_{23}^+ + 3p_{23}^- - 2p_{22}^+ q_{22}^+ - 2p_{22}^+ q_{22}^-)\varepsilon^2\big]\varepsilon^3.$$

Letting the ε^3-order and ε^5-order terms in $V_3(\varepsilon)$ equal zero we obtain the conditions,

$$a_3^- = -a_3^+,$$
$$p_{23}^- = \frac{1}{3}\big(-3p_{23}^+ + 2p_{22}^+ q_{22}^+ + 2p_{22}^+ q_{22}^-\big).$$

Then, we have the 4th generalized Lyapunov constant, given by

$$V_4(\varepsilon) = \frac{8}{45}\big[3(b_3^+ + b_3^-)p_{22}^+ + 6a_3^+(q_{22}^+ + q_{22}^-) + M_{18}\varepsilon^2\big]\varepsilon^5,$$

where

$$M_{18} = 6p_{23}^+ q_{22}^+ - 4p_{22}^+(q_{22}^+)^2 + 6p_{23}^+ q_{22}^- - 4p_{22}^+ q_{22}^+ q_{22}^- + 3p_{22}^+ q_{23}^+ - 3p_{22}^+ q_{23}^-.$$

Setting the ε^5-order term in $V_4(\varepsilon)$ zero yields three necessary center conditions: (i) $a_3^+ = 0$ and $b_3^- = b_3^+$, which leads to conditions included in the condition VI; (ii) $a_3^+ = p_{22}^+ = 0$, giving the condition VII; and (iii) $a_3^+ \neq 0$ and $q_{22}^- = \frac{1}{2a_3^+}(-b_3^+ p_{22}^+ + b_3^- p_{22}^+ - 2a_3^+ q_{22}^+)$.

For the case (iii), by $M_{18} = 0$ we have

$$q_{23}^+ = \frac{1}{3a_3^+}(3b_3^+ p_{23}^+ - 3b_3^- p_{23}^+ - 2b_3^+ p_{22}^+ q_{22}^+ + 2b_3^- p_{22}^+ q_{22}^+ + 3a_3^+ q_{23}^-).$$

Then, we have the 5th generalized Lyapunov constant, given by

$$V_5(\varepsilon) = -\frac{5\pi}{1152(a_3^+)^2}(b_3^+ - b_3^-)\big[54(a_3^+)^3 - a_3^+ M_{19}\varepsilon^2 - M_{20}\varepsilon^4\big]\varepsilon^5,$$

where

$$M_{19} = -7b_3^+(p_{22}^+)^2 - b_3^-(p_{22}^+)^2 - 36a_3^+ p_{23}^+ + 12a_3^+ p_{22}^+ q_{22}^+,$$
$$M_{20} = -21b_3^+(p_{22}^+)^2 p_{23}^+ + 21b_3^-(p_{22}^+)^2 p_{23}^+ - 54a_3^+(p_{23}^+)^2 + 14b_3^+(p_{22}^+)^3 q_{22}^+$$
$$\qquad - 14b_3^-(p_{22}^+)^3 q_{22}^+ + 36a_3^+ p_{22}^+ p_{23}^+ q_{22}^+ - 24a_3^+(p_{22}^+)^2 q_{23}^-.$$

Since $a_3^+ \neq 0$, we have $b_3^+ = b_3^-$ from $V_5(\varepsilon) = 0$. Combining the conditions from the subcases (i) and (iii), we have the condition VI.

3.4. Case 4: $a_2^+ a_2^- \neq 0$

Setting each ε^k-order term in $V_2(\varepsilon)$ zero yields the necessary center conditions $a_2^- = a_2^+$ and $p_{22}^- = p_{22}^+$. Then, the 3rd generalized Lyapunov constant is given by

$$V_3(\varepsilon) = -\frac{\pi}{8}\left[3a_3^+ + 3a_3^- - 2a_2^+ q_{22}^+ - 2a_2^+ q_{22}^- + (3p_{23}^+ + 3p_{23}^- - 2p_{22}^+ q_{22}^+ - 2p_{22}^+ q_{22}^-)\varepsilon^2\right]\varepsilon^3.$$

Letting the ε^3-order and ε^5-order terms in $V_3(\varepsilon)$ equal zero we obtain the conditions,

$$a_3^- = \frac{1}{3}(-3a_3^+ + 2a_2^+ q_{22}^+ + 2a_2^+ q_{22}^-),$$

$$p_{23}^- = \frac{1}{3}(3p_{23}^+ + 2p_{22}^+ q_{22}^+ + 2p_{22}^+ q_{22}^-).$$

Then, 4th generalized Lyapunov constant becomes

$$V_4(\varepsilon) = \frac{1}{90}\left[48a_2^+(b_3^+ + b_3^-) + 16M_{21}\varepsilon^2 + 225a_2^+ p_{23}^+\pi\varepsilon^3 + 16M_{22}\varepsilon^4 + 225p_{22}^+ p_{23}^+\pi\varepsilon^5\right]\varepsilon^3,$$

where

$$M_{21} = 3b_3^+ p_{22}^+ - 3b_3^- p_{22}^+ + 6a_3^+ q_{22}^+ - 4a_2^+(q_{22}^+)^2 + 6a_3^+ q_{22}^- - 4a_2^+ q_{22}^+ q_{22}^- + 3a_2^+ q_{23}^+ - 3a_2^+ q_{23}^-,$$
$$M_{22} = 6p_{23}^+ q_{22}^+ - 4p_{22}^+(q_{22}^+)^2 - 6p_{23}^+ q_{22}^- - 4p_{22}^+ q_{22}^+ q_{22}^- + 3p_{22}^+ q_{23}^+ - 3p_{22}^+ q_{23}^-.$$

Letting the ε^k-order terms ($k = 3, 5, 6, 7$) in $V_4(\varepsilon)$ equal zero we obtain the following conditions:

$$b_3^- = b_3^+, \quad p_{23}^+ = 0, \quad q_{23}^- = \frac{1}{3a_2^+}\left[6a_3^+ q_{22}^+ - 4a_2^+(q_{22}^+)^2 + 6a_3^+ q_{22}^- - 4a_2^+ q_{22}^+ q_{22}^- + 3a_2^+ q_{23}^+\right].$$

Then, M_{22} is reduced to

$$M_{22} = -\frac{6}{a_2^+}a_3^+ p_{22}^+(q_{22}^+ + q_{22}^-).$$

If $q_{22}^- = -q_{22}^+$, we obtain $a_3^- = -a_3^+$, yielding the condition VIII. Otherwise, we obtain the 5th generalized Lyapunov constant, given by

$$V_5(\varepsilon) = -\frac{1}{1440a_2^+}(q_{22}^+ + q_{22}^-)\left[300(a_2^+)^2 b_3^+\pi + 25\pi M_{23}\varepsilon^2 - 6144a_2^+ a_3^+ p_{22}^+\varepsilon^3 \right.$$
$$\left. - 50p_{22}^+\pi M_{24}\varepsilon^4 - 6144a_3^+(p_{22}^+)^2\pi\varepsilon^5\right]\varepsilon^5,$$

where

$$M_{23} = 27(a_3^+)^2 + 12a_2^+ b_3^+ p_{22}^+ - 15a_2^+ a_3^+ q_{22}^+ - 2(a_2^+)^2 (q_{22}^+)^2 + 3a_2^+ a_3^+ q_{22}^-$$
$$- 2(a_2^+)^2 q_{22}^+ q_{22}^- + 12(a_2^+)^2 q_{23}^+,$$
$$M_{24} = 9a_3^+ q_{22}^+ + a_2^+ (q_{22}^+)^2 - 12a_3^+ q_{22}^- + a_2^+ q_{22}^+ q_{22}^- - 6a_2^+ q_{23}^+.$$

With $b_3^+ > \frac{1}{2}(a_2^+)^2$, we have $V_5(\varepsilon) \neq 0$ when $a_2^+ p_{22}^+ = 0$ and $q_{22}^+ + q_{22}^- \neq 0$.

To this end, we finish the proof for the necessity of the conditions in Theorem 1.1. Next, we prove the sufficiency of these eight nilpotent center conditions.

If the condition I in Theorem 1.1 holds, (7) is reduced to

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y, \ -(b_2^+ x^2 + b_3^+ x^3) \right), & \text{if } x \geq 0, \\ \left(y, \ -b_3^- x^3 \right), & \text{if } x < 0. \end{cases} \tag{27}$$

The two systems in (27) are Hamiltonian systems, having respectively the Hamiltonian functions,

$$I^+(x, y) = \frac{1}{2}y^2 + \frac{1}{3}b_2^+ x^3 + \frac{1}{4}b_3^+ x^4 \quad \text{and} \quad I^-(x, y) = \frac{1}{2}y^2 + \frac{1}{4}b_3^- x^4,$$

which implies that the condition $I^+(0, y) \equiv I^-(0, y)$ in Proposition 2.1 is satisfied. So the origin of system (27) is a nilpotent center.

Similar to the proof for the condition I, we can derive the Hamiltonian functions (which are polynomials similar to the above $I^{\pm}(x, y)$) for system (7) with the condition II, or V or VII to show that the origin of (7) is a nilpotent center for these three cases.

If the condition III in Theorem 1.1 holds, (7) becomes

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y - a_3^+ x^3, \ -b_2^+ x^2 \right), & \text{if } x \geq 0, \\ \left(y - a_3^- x^3, \ -b_2^- x^2 \right), & \text{if } x < 0, \end{cases} \tag{28}$$

with $-\frac{a_3^-}{a_3^+} = -\frac{b_2^-}{b_2^+} \equiv h > 0, \neq 1$. Then, system (28) can be rewritten as

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y - a_3^+ x^3, \ -b_2^+ x^2 \right), & \text{if } x \geq 0, \\ \left(y + h a_3^+ x^3, \ h b_2^+ x^2 \right), & \text{if } x < 0. \end{cases} \tag{29}$$

Then, we apply the integrating factors:

$$\mu^+ = \frac{9(a_3^+)^2}{3(a_3^+)^2 x^3 - 3a_3^+ y - b_2^+} \quad \text{and} \quad \mu^- = \frac{9(a_3^+)^2}{-3(a_3^+)^2 h x^3 - 3a_3^+ y - b_2^+}$$

respectively to the first and second systems of (29) to obtain the first integrals:

$$\begin{cases} I^+(x, y) = -3a_3^+ y + b_2^+ \ln |-3(a_3^+)^2 x^3 + 3a_3^+ y + b_2^+|, & \text{for } x \geq 0, \\ I^-(x, y) = -3a_3^+ y + b_2^+ \ln |3h(a_3^+)^2 x^3 + 3a_3^+ y + b_2^+|, & \text{for } x < 0, \end{cases}$$
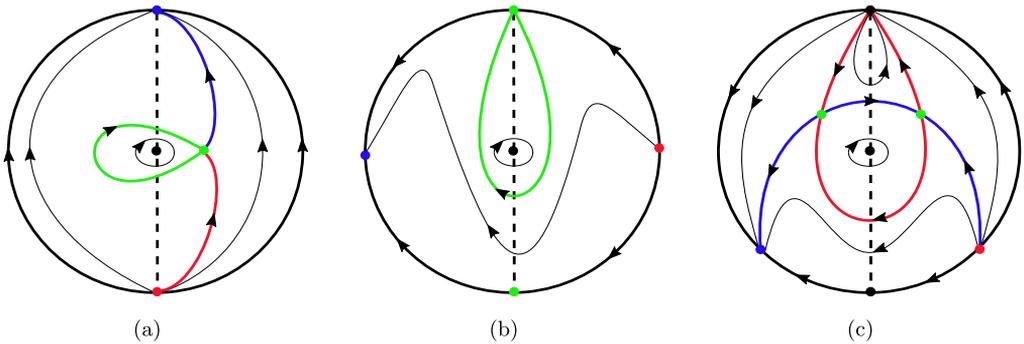
Fig. 2. Phase portraits for (a) system (27) with $b_2^+ = 1$, $b_3^+ = -2$, $b_3^- = 1$, (b) system (28) with $a_3^+ = 1$, $b_2^+ = 2$, $a_3^- = -2$, $b_2^- = -4$, and (c) system (30) with $a_2^+ = 1$, $a_3^+ = 2$, $b_2^+ = 1$, $b_3^+ = -2$.

which clearly shows that $I^+(0, y) \equiv I^-(0, y)$, implying that the origin of the system (29) and so the system (28) is a center.

If the condition IV in Theorem 1.1 holds, (7) can be rewritten as

$$(\dot{x}, \dot{y}) = \begin{cases} \left(y - (a_2^+ x^2 + a_3^+ x^3), \ -(b_2^+ x^2 + b_3^+ x^3)\right), & \text{if } x \geq 0, \\ \left(y - (a_2^+ x^2 - a_3^+ x^3), \ -(-b_2^+ x^2 + b_3^+ x^3)\right), & \text{if } x < 0, \end{cases} \tag{30}$$

which is symmetric with respect to the y-axis, and by Proposition 2.2 we know that the origin of (30) is a nilpotent center. Similarly, since (7) is symmetric with respect to the y-axis when the condition VI or VIII holds, the origin of system (7) is a nilpotent center.

In the following, we present examples for the above systems (27), (28) and (30) to show global phase portraits with a nilpotent center at the origin.

Example 3.2. The global phase portraits of the switching Liénard systems (27), (28) and (30) with three sets of parameter values show a nilpotent center at the origin, as illustrated in the Poincaré disc of Fig. 2.

4. The proof of Theorem 1.2

We recall the notion of Poincaré compactification of switching differential systems. This compactification identifies $\mathbb{R}^2$ with the interior of the closed unit Poincaré disc $\mathbb{D}^2$ centered at the origin of coordinates, and extends the switching differential systems to the boundary of $\mathbb{D}^2$, which is called the circle of the infinity. The switching manifold is indicated by the dash line "$- - -$" in $\mathbb{D}^2$. The singular points on the boundary of Poincaré disc are called *infinite singular points*. More details about the Poincaré compactification for switching polynomial systems can be found in [8].

With the results in [29], we have the proposition which characterizes the global center of switching polynomial systems.

Proposition 4.1. [Propositions 5, [29]] *Consider the switching polynomial system* (10) *which has a unique finite center at the singular point. Assume that the infinity of* (10) *is not filled of*

singular points. Then, the center is global if and only if all the infinite singular points in $\mathbb{D}^2$, if they exist, are such that the local phase portrait of each infinite singular point is consisting of two hyperbolic sectors, where the two separatrices are on the circle of the infinity.

Suppose the origin of the switching Liénard system (7) is the unique monodromic finite singular point. Then, the remaining finite singular points, given by

$$e^+ = (x^+, y^+) = \left(-\frac{b_2^+}{b_3^+}, -\frac{(b_2^+)^2(a_3^+ b_2^+ - a_2^+ b_3^+)}{(b_3^+)^3} \right)$$

and

$$e^- = (x^-, y^-) = \left(-\frac{b_2^-}{b_3^-}, -\frac{(b_2^-)^2(a_3^- b_2^- - a_2^- b_3^-)}{(b_3^-)^3} \right).$$

in the first and the second systems of (7) must be virtual, i.e., $x^+ < 0$ and $x^- > 0$. Hence, we have the necessity conditions:

$$\begin{aligned} G_1^+ &= \{b_2^+ = 0, \ b_3^+ \neq 0\}, \\ G_2^+ &= \{b_2^+ > 0, \ b_3^+ \geq 0\} \end{aligned} \tag{31}$$

and

$$\begin{aligned} G_1^- &= \{b_2^- = 0, \ b_3^- \neq 0\}, \\ G_2^- &= \{b_2^- < 0, \ b_3^- \geq 0\}. \end{aligned} \tag{32}$$

Next, we consider the infinite singular point of (7). For studying the dynamical behaviors in the circle of the infinity, we use the local charts: $U_k = \{(z_1, z_2) \in \mathbb{D}^2 : z_k > 0\}$ and $W_k = \{(z_1, z_2) \in \mathbb{D}^2 : z_k < 0\}$, $k = 1, 2$, with the corresponding diffeomorphisms,

$$\phi_k : U_k \to \mathbb{R}^2, \qquad \psi_k : W_k \to \mathbb{R}^2, \tag{33}$$

defined by $\phi_k(z_1, z_2) = \left(\frac{z_2}{z_1}, \frac{1}{z_1} \right) = (u, w)$ and $\psi_k(z_1, z_2) = \left(\frac{z_1}{z_2}, \frac{1}{z_2} \right) = (u, w)$. Here, the coordinates (u, w) play different roles in the distinct local charts. Thus, the corresponding vector fields of (7) in the local charts U_1 and W_1 are given by

$$\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -b_3^+ + a_3^+ u - b_2^+ w + a_2^+ uw - u^2 w^2 \\ w(a_3^+ + a_2^+ w - uw^2) \end{pmatrix} \tag{34}$$

and

$$\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -b_3^- + a_3^- u - b_2^- w + a_2^- uw - u^2 w^2 \\ w(a_3^- + a_2^- w - uw^2) \end{pmatrix}, \tag{35}$$

respectively. We derive the possible infinite singular points $\left(\frac{b_3^\pm}{a_3^\pm}, 0\right)$ or $\left(\frac{b_2^\pm}{a_2^\pm}, 0\right)$ which are nodes. It follows from Proposition 4.1 that the origin of (7) cannot be a global center in these cases. Hence, the systems (34) and (35) cannot have these infinite singular points when the origin of (7) is a global center. Therefore, the conditions (31) and (32) become

$$
\begin{aligned}
G_1^+ &= \{a_2^+ = a_3^+ = b_3^+ = 0, \ b_2^+ > 0\}, \\
G_2^+ &= \{a_3^+ = 0, \ b_2^+ > 0, \ b_3^+ > 0\}, \\
G_3^+ &= \{a_3^+ = 0, \ b_2^+ = 0, \ b_3^+ \neq 0\}
\end{aligned}
\tag{36}
$$

and

$$
\begin{aligned}
G_1^- &= \{a_2^- = a_3^- = b_3^- = 0, \ b_2^- < 0\}, \\
G_2^- &= \{a_3^- = 0, \ b_2^- < 0, \ b_3^- > 0\}, \\
G_3^- &= \{a_3^- = 0, \ b_2^- = 0, \ b_3^- \neq 0\}.
\end{aligned}
\tag{37}
$$

Then we consider the following cases.

(i) Firstly, we consider the case $G_1^+ \cap G_1^-$, for which system (7) becomes a quadratic switching Liénard system. Since the expression of the quadratic switching system (7) in W_2 is the one in U_2 multiplied by -1, the flows in the local phase portrait around the origin of W_2 have the opposite direction compared to the ones in U_2. Thus, we only need to study the dynamics around the origin in U_2. Then, in the local chart U_2 we have the following system,

$$
\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = \begin{cases} \begin{pmatrix} w + b_2^+ u^3 \\ b_2^+ u^2 w \end{pmatrix}, & \text{if } u \geq 0, \\[2ex] \begin{pmatrix} w + b_2^- u^3 \\ b_2^- u^2 w \end{pmatrix}, & \text{if } u < 0, \end{cases}
\tag{38}
$$

where both the two origins of the first and the second system of (38) are nilpotent singular points. Note that if the origin of U_2 is an infinite singular point, it must come from the combination of the two origins of the first and the second systems of (38). Since $b_2^+ > 0$ and $b_2^- < 0$, it is easy to verify that the origin of the first system of (38) is an unstable node while the origin of the second one of (38) is a stable node. Thus, it follows from Proposition 4.1 that the origin of (7) cannot be a global center.

The above analysis shows that the infinite singular point of the quadratic switching Liénard system (7) in U_2 does not consist of two hyperbolic sectors. This implies that three is the lowest degree of the first and the second systems of the switching Liénard system (7) to have a global nilpotent center. Thus, we only need to consider the case $G_i^+ \cap G_j^-$, $i, j = 2, 3$.

(ii) For the case $G_i^+ \cap G_j^-$, $i, j = 2, 3$, we obtain the following system from (7),

$$
\begin{pmatrix} \dot{u} \\ \dot{w} \end{pmatrix} = \begin{cases} \begin{pmatrix} b_3^+ u^4 - a_2^+ u^2 w + b_2^+ u^3 w + w^2 \\ u^2 w (b_3^+ u + b_2^+ w) \end{pmatrix}, & \text{if } u \geq 0, \\[2mm] \begin{pmatrix} b_3^- u^4 - a_2^- u^2 w + b_2^- u^3 w + w^2 \\ u^2 w (b_3^- u + b_2^- w) \end{pmatrix}, & \text{if } u < 0, \end{cases}
\tag{39}
$$

in U_2 with $u \geq 0$ and $u < 0$, respectively. Since system (7) is a cubic switching system, the local qualitative property of the origin of W_2 has the same sense with respect to the one in U_2. Obviously, the origins of the first and second system in (39) in U_2 are singular points which are linearly zero. In order to understand the local qualitative properties around these two singular points, we apply a direct blow-up $(u, v) \rightarrow (u, w)$ with $v = \frac{w}{u}$ to eliminate the common factor u, yielding

$$
\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{cases} \begin{pmatrix} u(b_3^+ u^2 - a_2^+ uv + b_2^+ u^2 v + v^2) \\ (a_2^+ u - v)v^2 \end{pmatrix}, & \text{if } u \geq 0, \\[2mm] \begin{pmatrix} u(b_3^- u^2 - a_2^- uv + b_2^- u^2 v + v^2) \\ (a_2^- u - v)v^2 \end{pmatrix}, & \text{if } u < 0, \end{cases}
\tag{40}
$$

where the two origins are singular points with linearly zero. We do a further blow-up $(u, v) \rightarrow (u, V)$ with $V = \frac{v}{u}$, to obtain the following system,

$$
\begin{pmatrix} \dot{u} \\ \dot{V} \end{pmatrix} = \begin{cases} \begin{pmatrix} u(b_3^+ - a_2^+ V + b_2^+ uV + V^2) \\ -V(b_3^+ - 2a_2^+ V + b_2^+ uV + 2V^2) \end{pmatrix}, & \text{if } u \geq 0, \\[2mm] \begin{pmatrix} u(b_3^- - a_2^- V + b_2^- uV + V^2) \\ -V(b_3^- - 2a_2^- V + b_2^- uV + 2V^2) \end{pmatrix}, & \text{if } u < 0, \end{cases}
\tag{41}
$$

where the common factor u^2 has been eliminated.

When $(a_2^+)^2 - 2b_3^+ < 0$, for $u = 0$ the first system of (41) has a saddle at $E_1 = (0, 0)$. Going back through the change of variables we get that the local phase portrait of the right origin in U_2 has two hyperbolic sectors and one parabolic sector, see Fig. 3. When $(a_2^+)^2 = 2b_3^+ \neq 0$, for $u = 0$ the first system of (41) has two singular points at $E_1 = (0, 0)$ (a saddle) and at $E_2 = (\frac{a_2^+}{2}, 0)$ (a saddle-node). Again, going back through the change of variables we find that the local phase portrait of the right origin in U_2 has two hyperbolic and two parabolic sectors, see Fig. 4. When $(a_2^+)^2 - 2b_3^+ > 0$, for $u = 0$ the first system of (41) has three singular points at $E_1 = (0, 0)$ (a saddle), $E_{2,3} = (\frac{1}{2}(a_2^+ \pm ((a_2^+)^2 - 2b_3^+)^{1/2}, 0)$ (a saddle and a node). Going back through the change of variables we obtain that the local qualitative property of the right origin in U_2 has two hyperbolic and two parabolic sectors, see Fig. 5. In the last two cases the local phase portrait of the origin in U_2 can only consist of two hyperbolic sectors.

Similarly, if $(a_2^-)^2 - 2b_3^- < 0$, we derive that the local qualitative property around the left origin in the second system of (39) has two hyperbolic sectors if and only if $(a_2^-)^2 - 2b_3^- < 0$.
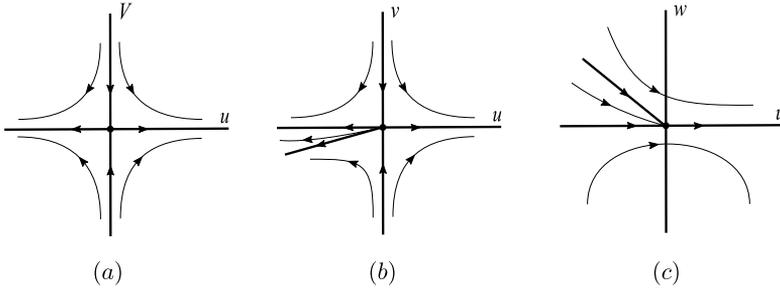
Fig. 3. Local phase portraits around the origin of the local chart U_2 with $(a_2^+)^2 - 2b_3^+ < 0$ for (a) the first system of (41), (b) the first system of (40), and (c) the first system of (39).
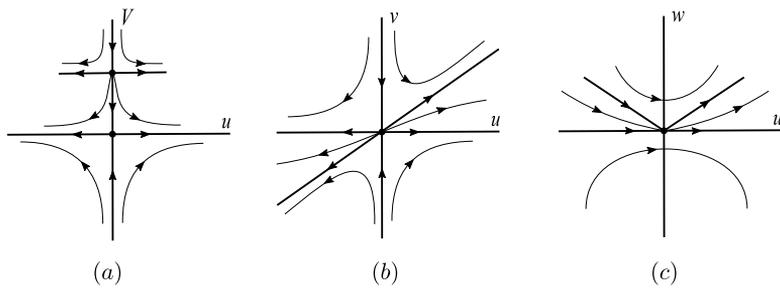
Fig. 4. Local phase portraits around the origin of the local chart U_2 with $(a_2^+)^2 - 2b_3^+ = 0$ for (a) the first system of (41), (b) the first system of (40), and (c) the first system of (39).
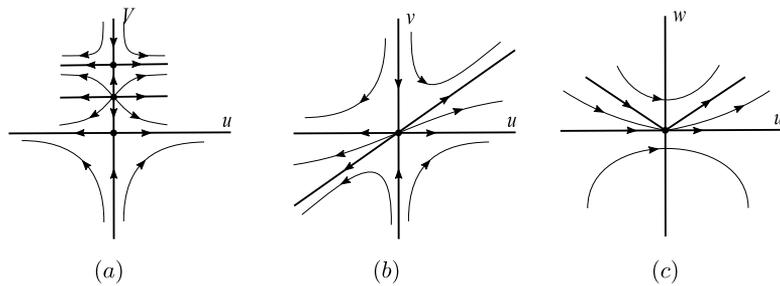
Fig. 5. Local phase portraits at the origin of the local chart U_2 with $(a_2^+)^2 - 2b_3^+ > 0$ for (a) the first system of (41), (b) the first system (40), and (c) the first system of (39).

By combining the right of origin in the first system of (39) and the left of the origin in the second system of (39), the local phase portrait of the origin in U_2 consists of two hyperbolic sectors, where the two separatrices are on the circle of the infinity. Thus, the switching Liénard system (7) can only have a global center at the origin if condition

$$G = \{a_3^{\pm} = 0, \ b_2^+ \geq 0, \ b_2^- \leq 0, \ (a_2^+)^2 - 2b_3^+ < 0, \ (a_2^-)^2 - 2b_3^- < 0\} \tag{42}$$

holds. Thus, by Theorem 1.1 we have proved Theorem 1.2.

In fact, combining the conditions I, II, V and VII of Theorem 1.1 and the above condition G, we obtain the condition G_1 of Theorem 1.2. From the condition III of Theorem 1.1, we have

$a_3^+ a_3^- \neq 0$ which contradicts the condition G, and thus the origin of system (7) cannot be a global center. From the conditions IV, VI and VIII of Theorem 1.1 together with the condition G, we obtain the condition G_2 of Theorem 1.2.

5. The proof of Theorem 1.3

In this section, we prove the existence of 5 small-amplitude limit cycles by perturbing the nilpotent origin of the cubic switching Liénard system (7) under the center condition IV. More precisely, with the center condition IV, we perturb system (7) to obtain

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y - (a_2^+ x^2 + a_3^+ x^3) + d\varepsilon^9 + \sum_{k=1}^{6} \varepsilon^{k+1}\delta_k x + \sum_{k=1}^{5} \varepsilon^k (p_{k2}^+ x^2 + p_{k3}^+ x^3) \\ -\varepsilon^2 x - (b_2^+ x^2 + b_3^+ x^3) + \sum_{k=1}^{5} \varepsilon^k (q_{k2}^+ x^2 + q_{k3}^+ x^3) \end{pmatrix}, & \text{if } x \geq 0, \\[4ex] \begin{pmatrix} y - (-a_2^+ x^2 + a_3^+ x^3) + \sum_{k=1}^{6} \varepsilon^{k+1}\delta_k x + \sum_{k=1}^{5} \varepsilon^k (p_{k2}^- x^2 + p_{k3}^- x^3) \\ -\varepsilon^2 x - (b_2^+ x^2 - b_3^+ x^3) + \sum_{k=1}^{5} \varepsilon^k (q_{k2}^- x^2 + q_{k3}^- x^3) \end{pmatrix}, & \text{if } x < 0. \end{cases}
$$

(43)

Denote by one parameter vector

$$
\lambda^* = (a_2^+, a_3^+, b_2^+, b_3^+, d, \delta_i, p_{j2}^{\pm}, p_{j3}^{\pm}, q_{j2}^{\pm}, q_{j3}^{\pm}),
$$

$i = 1, 2, ..., 6$, $j = 1, 2, ..., 5$. Further, introducing the scaling $(x, y, t) \rightarrow \left(\varepsilon^2 y, \varepsilon^3 x, \frac{t}{\varepsilon}\right)$ into (43), we obtain a system up to ε^6-order terms,

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y - b_2^+ y^2 - b_3^+ \varepsilon^2 y^3 - \sum_{k=1}^{5} \varepsilon^k q_{k2}^+ y^2 - \sum_{k=1}^{5} \varepsilon^{k+2} q_{k3}^+ y^3 \\ x + \sum_{k=1}^{6} \varepsilon^k \delta_k y - (a_2^+ \varepsilon y^2 + a_3^+ \varepsilon^3 y^3) - \sum_{k=1}^{5} \varepsilon^{k+1} p_{k2}^+ y^2 - \sum_{k=1}^{5} \varepsilon^{k+3} p_{k3}^+ y^3 \end{pmatrix}, \\ \quad \text{if } y \geq 0, \\[4ex] \begin{pmatrix} -y + b_2^+ y^2 - b_3^+ \varepsilon^2 y^3 - \sum_{k=1}^{5} \varepsilon^k q_{k2}^- y^2 - \sum_{k=1}^{5} \varepsilon^{k+2} q_{k3}^- y^3 \\ x + \sum_{k=1}^{6} \varepsilon^k \delta_k y - (a_2^+ \varepsilon y^2 - a_3^+ \varepsilon^3 y^3) + d\varepsilon^6 - \sum_{k=1}^{5} \varepsilon^{k+1} p_{k2}^- y^2 - \sum_{k=1}^{5} \varepsilon^{k+3} p_{k3}^- y^3 \end{pmatrix}, \\ \quad \text{if } y < 0. \end{cases}
$$

(44)

Note that we let the constant term be zero, i.e., $d = 0$, when we compute ε^k-order Lyapunov constants. By using the generalized Lyapunov constants, we start from the ε-order terms to determine the number of limit cycles. Thus, we need to find ε^k-order center conditions such that all the ε^k-order Lyapunov constants being zero when we want to obtain more limit cycles by using ε^{k+1}-order terms. That is to say that the origin of the system is a center up to ε^k-order, see [41]. Finally, we perturb the coefficient d of the constant term to obtain one more small-amplitude limit cycle.

More precisely, to prove the existence of small-amplitude limit cycles around the origin, we compute the higher-order Lyapunov constants V_{jk} for $j = 1, 2, \ldots, 5$ and $k = 1, 2, \ldots, 6$. We start from $k = 1$. For a fixed k, we choose appropriate parameter values such that as many as possible higher-order Lyapunov constants vanish.

First, for $k = 1$, we obtain that all ε-order Lyapunov constants are zero when $\delta_1 = 0$, i.e., $V_{j1} = 0$ for all j.

Next, for $k = 2$, we have $V_{12} = \delta_2 \pi = 0$ when $\delta_2 = 0$. Then, we solve $V_{22} = \frac{4}{3}(p_{12}^- - p_{12}^+) = 0$ to obtain

$$p_{12}^- = p_{12}^+,$$

under which the 3rd ε^2-order Lyapunov constant becomes

$$V_{32} = \frac{\pi}{4} a_2^+ (q_{12}^- + q_{12}^+).$$

Setting $q_{12}^- = -q_{12}^+$ yields $V_{j2} = 0$ for all j. So, for $k = 2$, we can have 2 limit cycles by perturbing δ_2 and p_{12}^- with $V_{32} \neq 0$.

Now, consider $k = 3$. We have $V_{13} = 0$ with $\delta_3 = 0$ and $V_{23} = \frac{4}{3}(p_{22}^- - p_{22}^+)$. Solving $V_{23} = 0$ we have

$$p_{22}^- = p_{22}^+.$$

Further, the 3rd ε^3-order Lyapunov constant is obtained as

$$V_{33} = \frac{\pi}{4} a_2^+ (q_{22}^- + q_{22}^+).$$

Setting $q_{22}^- = -q_{22}^+$ yields $V_{j3} = 0$ for all j, which indicates that for $k = 3$ we can have 2 limit cycles if choosing $V_{33} \neq 0$ and perturbing δ_3 and p_{22}^-.

For $k = 4$, solving $V_{24} = V_{34} = 0$ we have

$$p_{32}^- = p_{32}^+ \quad \text{and} \quad p_{13}^- = \frac{2}{3} a_2^+ q_{32}^- + \frac{2}{3} a_2^+ q_{32}^+ - p_{13}^+,$$

under which the 4th ε^4-order Lyapunov constant becomes

$$V_{44} = -\frac{8}{45} a_2^+ (4 q_{32}^- b_2^+ + 4 b_2^+ q_{32}^+ + 3 q_{13}^- - 3 q_{13}^+).$$

Setting $V_{44} = 0$ we have

$$q_{13}^- = -\frac{4}{3}b_2^+ q_{32}^- - \frac{4}{3}b_2^+ q_{32}^+ - q_{13}^+,$$

leading to $V_{j4} = 0$ for all j. So, for $k = 4$ we can have 3 limit cycles by choosing $V_{44} \neq 0$ and perturbing δ_4, p_{32}^- and p_{13}^-.

For $k = 5$, we similarly obtain

$$p_{42}^- = p_{42}^+$$

by solving $V_{25} = 0$, and

$$p_{23}^- = \frac{2}{3}q_{32}^- p_{12}^+ + \frac{2}{3}q_{42}^- a_2^+ + \frac{2}{3}a_2^+ q_{42}^+ + \frac{2}{3}p_{12}^+ q_{32}^+ - p_{23}^+$$

by solving $V_{35} = 0$. Then, the 4th ε^5-order Lyapunov constant is given by

$$V_{45} = -\frac{8}{45}a_2^+ (4q_{32}^- q_{12}^+ + 4q_{42}^- b_2^+ + 4b_2^+ q_{42}^+ + 4q_{12}^+ q_{32}^+ + 3q_{23}^- - 3q_{23}^+).$$

Setting

$$q_{23}^- = -\frac{4}{3}q_{12}^+ q_{32}^- - \frac{4}{3}q_{42}^- b_2^+ - \frac{4}{3}b_2^+ q_{42}^+ - \frac{4}{3}q_{12}^+ q_{32}^+ + q_{23}^+,$$

under which $V_{j5} = 0$ for all j. This implies that 3 limit cycles can be obtained for $k = 4$ if choosing $V_{45} \neq 0$ and perturbing δ_5, p_{42}^- and p_{23}^-.

Finally, for $k = 6$, similarly by solving $V_{16} = V_{26} = V_{36} = 0$ we obtain

$$p_{52}^- = p_{52}^+,$$
$$p_{33}^- = \tfrac{2}{3}q_{32}^- p_{22}^+ + \tfrac{2}{3}q_{42}^- p_{12}^+ + \tfrac{2}{3}q_{52}^- a_2^+ + \tfrac{2}{3}a_2^+ q_{52}^+ + \tfrac{2}{3}p_{12}^+ q_{42}^+ + \tfrac{2}{3}p_{22}^+ q_{32}^+ - p_{33}^+.$$

Then, we solve $V_{46} = 0$ for q_{33}^- to obtain

$$\begin{aligned}
q_{33}^- = -&\frac{4}{3a_2^+}(q_{32}^- a_2^+ q_{22}^+ + 4q_{42}^- a_2^+ q_{12}^+ + 4q_{52}^- a_2^+ b_2^+ + 4a_2^+ b_2^+ q_{52}^+ \\
&+ 4a_2^+ q_{12}^+ q_{42}^+ + 4a_2^+ q_{22}^+ q_{32}^+ - 6q_{32}^- a_3^+ - 3a_2^+ q_{33}^+ - 6a_3^+ q_{32}^+),
\end{aligned}$$

under which the 5th ε^6-order Lyapunov constant is reduced to

$$V_{56} = -\frac{5\pi}{48}(2a_2^+ b_3^+ - 3a_3^+ b_2^+)(q_{32}^- + q_{32}^+).$$

So letting

$$(2a_2^+ b_3^+ - 3a_3^+ b_2^+)(q_{32}^- + q_{32}^+) = 0$$

yields $V_{j6} = 0$ for all j. Therefore, for $k = 6$, we can obtain 4 limit cycles for choosing $V_{56} \neq 0$.

Summarizing the above results, 4 small-amplitude limit cycles can be obtained from the ε^6-order Lyapunov constants. Assume that $p_{22}^+ \neq 0$, a direct computation shows that

$$\det \left[\frac{\partial(V_{16}, V_{26}, V_{36}, V_{46})}{\partial(\delta_6, p_{52}^-, q_{32}^+, q_{33}^+)} \right] = \frac{8\pi^2}{45} a_2^+ p_{22}^+ \neq 0,$$

which implies that there exists 4 small-amplitude limit cycles by perturbing the parameters, in backward, δ_6, p_{52}^-, q_{32}^+ and q_{33}^+.

One more small-amplitude limit cycle is obtained by perturbing d, leading to a total 5 small-amplitude limit cycles. For d and ε sufficiently small, the switching system (44) has a small sliding segment on the switching manifold $y = 0$ with the end points at $(0, 0)$ and $(x_\varepsilon, 0)$, where $x_\varepsilon = -d\varepsilon^6$ is the unique root of the equation $\dot{y} = 0$ in the second system of (44). Thus, the sliding segment shrinks to $(0, 0)$ when ε goes to zero.

For the point $(x_e, 0)$ on the switching manifold $y = 0$ with $x_\varepsilon < x_e \ll 1$, we define a bifurcation function,

$$d(x_e, \varepsilon) = \Pi^+(x_e, \varepsilon) - \Pi_-^+(x_e, \varepsilon), \tag{45}$$

for which has two small positive constants ε_1 and ε_2 such that $\Pi^+(x_e, \varepsilon) : (0, \varepsilon_1) \to (-\varepsilon_1, 0)$ is the first half-bifurcation function of (44), and $\Pi_-^+(x_e, \varepsilon) : (x_\varepsilon, x_\varepsilon + \varepsilon_2) \to (x_\varepsilon - \varepsilon_2, x_\varepsilon)$ is the second half-bifurcation function derived from the second system of (44) by the transformation $(x, y, t) \to (x, -y, -t)$. By the polar coordinate transformation $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\Pi^+(x_e, \varepsilon) = V_1^+(\varepsilon)x_e + O(x_e^2) \quad \text{and} \quad \Pi_-^+(x_e, \varepsilon) = V_0^-(\varepsilon) + V_1^-(\varepsilon)x_e + O(x_e^2), \tag{46}$$

where

$$V_0^-(\varepsilon) = 2d\varepsilon^6 + O(\varepsilon^7) \quad \text{and} \quad V_1^\pm(\varepsilon) = \pm\frac{1}{2}\delta_6\pi\varepsilon^6 + O(\varepsilon^7), \tag{47}$$

It follows from (45) and (46) that $V_{06} = -2d$ and $V_{16}|_{d=0} = \delta_6\pi$. Hence, we obtain the ε^6-order Lyapunov constants V_{06}, V_{16}, V_{26}, V_{36} and V_{46}, which are independent of the parameter vector λ^*. Therefore, we have proved the existence of 5 small-amplitude crossing limit cycles near the origin of cubic switching Liénard system (7).

To end this section, we provide an example with exact critical parameter vector λ^* to demonstrate the existence of 5 small-amplitude limit cycles. To achieve this, we need to find 5 positive roots which are solved from the displacement equation (19):

$$d(\xi) = V_0(\lambda^*) + V_1(\lambda^*)\xi + \cdots + V_4(\lambda^*)\xi^4 + V_5(\lambda^*)\xi^5 + O(\xi^6) = 0. \tag{48}$$

As discussed above, we have $V_j(\lambda^*) = V_{j6}\varepsilon^6 + O(\varepsilon^6)$, $j = 0, 1, \ldots, 5$. Hence, (48) becomes

$$d(\xi) = (V_{06} + V_{16}\xi + \cdots + V_{46}\xi^4 + V_{56}\xi^5)\varepsilon^6 + O(\varepsilon^7) = 0. \tag{49}$$

In general, it is a very challenging task to find the perturbed parameter vector to give a numerical realization. However, for our system the parameters in the displacement equation are linearly

decoupled. Thus, we can obtain these values by perturbing exactly one parameter for one higher-order Lyapunov constant such that equation (48) has 5 positive roots.

In the following, we construct a concrete example. First, note that the critical parameter vector λ^* given above satisfies that $V_{j6} = 0$, $j = 0, 1, \ldots, 4$, but $V_{56} \neq 0$, and V_{56} does not contain any non-zero lower-order ε terms. Thus, we can take perturbations in the backward order: on q_{32}^- for V_{56}, on q_{33}^- for V_{46}, on p_{33}^- for V_{36}, on p_{52}^- for V_{26}, on δ_6 for V_{16}, and on d for V_{06}. In particular, with the help of Maple taking the accuracy up to 60 decimal points, we set the free parameters to take the values:

$$a_2^+ = b_3^+ = p_{12}^+ = p_{22}^+ = p_{33}^+ = p_{52}^+ = q_{12}^+ = q_{22}^+ = q_{32}^+ = q_{33}^+ = q_{42}^\pm = q_{52}^\pm = 1,$$
$$a_3^+ - 3 = b_2^+ + 2 = q_{32}^- + 2 = 0,$$

(50)

and choose the following values for the perturbed parameters:

$$
\begin{aligned}
d^* &= d - 5 \times 10^{-31}, & \delta_6^* &= \delta_6 - 3.2 \times 10^{-21}, \\
p_{52}^{-*} &= p_{52}^+ + 1.5 \times 10^{-13}, & p_{33}^{-*} &= p_{33}^- - 8.5 \times 10^{-8}, \\
q_{33}^{-*} &= q_{33}^- + 0.0188.
\end{aligned}
$$

(51)

With the above perturbed parameter values, we obtain the perturbed ε^6-order Lyapunov constants as follows:

$$
\begin{aligned}
V_{06} &= -1.0 \times 10^{-30}, & V_{16} &\approx 1.0053096491 \times 10^{-20}, \\
V_{26} &\approx -2.0006218904 \times 10^{-13}, & V_{36} &\approx 1.0013826592 \times 10^{-7}, \\
V_{46} &\approx -0.0100266667, & V_{56} &\approx 6.5449846949,
\end{aligned}
$$

(52)

for which the displacement equation (49) has 5 positive roots:

$$
\begin{aligned}
\xi_1 &\approx 0.0015219219, & \xi_2 &\approx 7.3120695527 \times 10^{-6}, & \xi_3 &\approx 2.6762403416 \times 10^{-6}, \\
\xi_4 &\approx 5.1472016794 \times 10^{-8}, & \xi_5 &\approx 9.9669521943 \times 10^{-11},
\end{aligned}
$$

(53)

which are approximations of the amplitudes for the bifurcating 5 small-amplitude limit cycles.

6. Conclusion

In this paper, we have studied the nilpotent center problem and the limit cycle bifurcation problem for switching nilpotent systems in $\mathbb{R}^2$. We have developed a higher-order Poincaré-Lyapunov method to compute the generalized Lyapunov constants for switching nilpotent systems. By using this method, we derive the nilpotent center conditions for the origin of the switching cubic polynomial Liénard systems. Further, we characterize all the switching cubic polynomial Liénard systems to have a global nilpotent center. Finally, we construct a perturbed system with one of the center conditions to show the existence of 5 small-amplitude limit cycles bifurcating from the nilpotent center, which is a new lower bound of the maximal number of limit cycles in such switching cubic Liénard systems with a nilpotent singular point. The methodology developed in this paper can be applied to investigate complex dynamics of other nonlinear systems with nilpotent singular points.

Data availability

No data was used for the research described in the article.

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