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# Double Hopf bifurcation in a container crane model with delayed position feedback

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ABSTRACT

In this paper, we study dynamics in a container crane model with delayed position feedback, with particular attention focused on non-resonant double Hopf bifurcation. By using multiple time scales and center manifold reduction methods, we obtain the equivalent normal forms near a double Hopf critical point in this neutral delayed differential system. Moreover, bifurcations are classified in a two-dimensional parameter space near the critical point. Numerical simulations are presented to demonstrate the applicability of the theoretical results.



## 1. Introduction

Recently, much attention has been focused on the study of delayed differential equations, since they may exhibit complex dynamical behaviors [1–3]. Some delayed differential equations were proposed via delayed feedback scheme, such as [4–7]. In many practical problems, the changing rates of some state variables not only depend on the state values at present and earlier instances, but also are influenced by the changing rates of the state variables in the past. Thus, neutral delayed differential equations (NDDE) have been proposed in the study of population dynamics, neural network, engineering problems, etc. [8–10]. Since then, NDDE models have attracted attentions of researchers, and some results have been obtained with focus on local stability and global asymptotic behaviors of trivial solutions [11–15]. Several interesting articles on the bifurcation theory of NDDE, such as normal form of Hopf bifurcation, global existence of periodic solutions, equivariant Hopf bifurcation theory, have been published [16–23]. A few papers considered the existence of positive periodic solutions in neutral delayed ecological models by using a continuation theorem based on coincidence degree theory or other analytical techniques [24–26].

As we all know, it is important to compute normal forms of differential equations in the study of nonlinear dynamical systems [27,28]. Multiple time scales (MTS) [29,30] and center manifold reduction (CMR) [27,28] are two useful tools for computing the normal forms of differential equations. Multiple time scales method is systematic and can be directly applied to the original nonlinear dynamical system, not only to ordinary differential equations (ODE) but also to delayed differential equations (DDE), without application of center manifold theory. In fact, this approach combines the two steps involved in using center manifold theory and normal form theory into one unified procedure to obtain the normal form and nonlinear transformation simultaneously [31,32]. Moreover, MTS method only contains algebraic manipulation, which greatly facilitate computer implement in symbolic computations. By comparison, the CMR method is more complex, especially in DDE, which requires integration in computation. For a given ODE, the basic idea of the center manifold theory is to apply successive coordinate transformations to systematically construct a simpler system which has less dimension compared

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to the original system, and thus greatly simplifies the dynamical analysis of the system. For DDE, however, one needs to first change the DDE to a operator differential equation, and then decompose the solution space of their linearized form into stable and center manifolds. Next, with adjoint operator equations, one computes the center manifold by projecting the whole space to the center manifold, and finally calculates the normal form restricted to the center manifold (e.g. see [33–36]). Nayfeh [32] used these two approaches to derive equivalent normal forms of Hopf bifurcation in delayed nonlinear dynamical systems. Ding et al. [37] applied the two methods to obtain the normal forms near a double Hopf critical point in a DDE, and showed the equivalence of the two normal forms.

Some results have been obtained for the normal forms of NDDE. Nayfeh and Baumann [4] developed a nonlinear NDDE model, which describes controlled container crane system, by modeling the crane-hoist-payload assembly as a double pendulum, and derived the normal form of Hopf bifurcation by using the MTS method. Wang and Wei [19] extended the computation of Hopf bifurcation properties (such as the direction of bifurcation and the stability of bifurcating periodic solutions) introduced by Kazarinoff et al. [38] to NDDE by using the normal form theory and center manifold theorem. Weedermann [16] computed the normal forms of NDDE by using the CMR method introduced by Faria and Magalhães [33] for DDE. Then, Weedermann [17], Wang and Wei [18] extended the idea in [34] to the NDDE with parameters. To our best knowledge, MTS and CMR methods have not been used to consider the normal forms associated with high co-dimensional bifurcations in NDDE. In this paper, we will derive the normal form of double Hopf bifurcation in a container crane model with delayed position feedback [4] by using the two methods, and further show that the MTS is simpler than the CMR, though the results from the two methods are equivalent.

The rest of the paper is organized as follows. In Section 2, we consider the existence of non-resonant double Hopf bifurcation in a container crane model with delayed position feedback, and use two methods to derive the normal form associated with double Hopf bifurcation. Then, bifurcation analysis and numerical simulations are presented in Section 3. Finally, conclusion is drawn in Section 4.

## 2. Analytical study

In this section, we consider a container crane model with delayed position feedback [4], and use the MTS and CMR methods to derive the normal form of double Hopf bifurcation. The equation of the model is described by

$$\begin{aligned}\ddot{\phi}(t)+\alpha_1\phi(t)+2\mu\dot{\phi}(t)+k\ddot{\phi}(t-\tau) &= -\epsilon\alpha_3\phi^3(t)-\epsilon\alpha_4\phi(t)\dot{\phi}^2(t)-\epsilon\alpha_4\phi^2(t)\ddot{\phi}(t)-\epsilon k\phi(t-\tau)\dot{\phi}^2(t-\tau)\\ &\quad -\epsilon k\alpha_5\phi^2(t)\ddot{\phi}(t-\tau)-\frac{1}{2}\epsilon k\phi^2(t-\tau)\ddot{\phi}(t-\tau), \qquad (1)\end{aligned}$$

where $\phi$ is the oscillating angle, $\tau$ the time delay, $\mu$ the inherent damping coefficient, $k=-\frac{\hat{k}}{b-aR}$, where $\hat{k}$ is the feedback gain (we also call $k$ the feedback gain in this paper), $\epsilon$ is a small dimensionless parameter and $\alpha_i's(i=1,3,4,5)$ are scaled parameters in terms of known constants, given by

$$\alpha_1=\frac{g\hat{\alpha}_1}{4b(b-aR)^2},\quad \alpha_3=\frac{4g\hat{\alpha}_3}{(b-aR)^2},\quad \alpha_4=\frac{\hat{\alpha}_1^2+96(b-aR)\hat{\alpha}_5}{16b^2(b-aR)^2},\quad \alpha_5=\frac{3\hat{\alpha}_5}{b-aR},$$

with

$$\begin{aligned}&a=\frac{d-c}{c},\quad b=\sqrt{L^2-\frac{1}{4}a^2c^2},\quad \hat{\alpha}_1=4b^2+4a^2bR+a^2(1+a)c^2,\\ &\hat{\alpha}_3=\frac{16b^4+16a^2(8+12a+3a^2)b^3R+4a^2(2+14a+15a^2+3a^2)b^2c^2+3a^4(1+a)^2c^4}{96b^3},\\ &\hat{\alpha}_5=\frac{4b^2+4a(2+3a)bR+3a^2(1+a)c^2}{8b},\end{aligned}$$

where $g$ is gravitational acceleration, $d$ is the length of trolley, $c$ is the length of spreader bar, $L$ is the connection length between trolley and spreader bar, and $R$ is the distance between the center of gravity of trolley and spreader bar. Further, the physical measurements (see [4]) implies $a>0$, $b>0$, $b-aR>0$. Thus, all the parameters take positive values. The details of the modeling process and derivation of the full nonlinear Eq. (1) can be found in [4], and thus we omit the derivation of the model and the lengthy expressions of the parameters here for brevity.

### 2.1. System formulation

Let $\phi(t)=x_1(t)$ and $\dot{\phi}(t)=x_2(t)$. Then, rescale the time by $t\mapsto(t/\tau)$ to normalize the delay so that system (1) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}Dx_t=L_0x_t+F(x_t)+G(x_t,\dot{x}_t), \qquad (2)$$

where

$$Dx_t = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix},$$

$$L_0x_t = \tau\begin{pmatrix} 0 & 1 \\ -\alpha_1 & -2\mu \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

$$F(x_t) = \tau\begin{pmatrix} 0 \\ -\epsilon\alpha_3x_1^3(t) - \epsilon\alpha_4x_1(t)x_2^2(t) - \epsilon k x_1(t-1)x_2^2(t-1) \end{pmatrix}$$

and

$$G(x_t, \dot{x}_t) = \begin{pmatrix} 0 \\ -\epsilon\alpha_4x_1^2(t)\dot{x}_2(t) - \epsilon k\alpha_5x_1^2(t)\dot{x}_2(t-1) - \frac{1}{2}\epsilon k x_1^2(t-1)\dot{x}_2(t-1) \end{pmatrix}.$$

The characteristic equation of the linearized equation of (2), evaluated at the trivial equilibrium $x_1 = x_2 = 0$, is given by:

$$(1 + ke^{-\lambda})\lambda^2 + 2\mu\tau\lambda + \alpha_1\tau^2 = 0. \tag{3}$$

To find possible periodic solutions, which may bifurcate from a Hopf or double Hopf critical point, let $\lambda = i\omega\tau(i^2 = -1, \omega > 0)$ be a root of (3). Substituting $\lambda = i\omega\tau$ into (3) and separating the real and imaginary parts yields

$$\begin{aligned} &-\omega^2 - \omega^2 k\cos(\omega\tau) + \alpha_1 = 0, \\ &k\omega^2\sin(\omega\tau) + 2\mu\omega = 0, \end{aligned} \tag{4}$$

from which we obtain the solution for $\omega^2$ as

$$\omega^2 = \frac{\alpha_1 - 2\mu^2 \pm \sqrt{(\alpha_1 - 2\mu^2)^2 - \alpha_1^2(1-k^2)}}{1-k^2},$$

It is easy to see from the above equation that the existence of two positive solutions for $\omega^2$ requires the following conditions

$$1 - k^2 > 0 \quad \text{and } \alpha_1 - 2\mu^2 > \alpha_1\sqrt{1-k^2}, \tag{5}$$

to be satisfied, under which we have

$$\omega_{1,2} = \sqrt{\frac{\alpha_1 - 2\mu^2 \pm \sqrt{4\mu^4 + \alpha_1^2k^2 - 4\alpha_1\mu^2}}{1-k^2}},$$

giving rise to double-Hopf bifurcation. It should be noted that the first inequality in (5) implies that the feedback gain should be chosen small, $k < 1$, while the second inequality means that the damping coefficient $\mu$ should not be taken too large, as expected.

Further, it follows from (4) that

$$\tau_{1,2}^{(j)} = \begin{cases} \frac{1}{\omega_{1,2}}\left[\arccos\left(\frac{\alpha_1-\omega_{1,2}^2}{k\omega_{1,2}^2}\right) + 2j\pi\right], & \text{for } k\mu < 0, \\ \frac{1}{\omega_{1,2}}\left[2(j+1)\pi - \arccos\left(\frac{\alpha_1-\omega_{1,2}^2}{k\omega_{1,2}^2}\right)\right], & \text{for } k\mu > 0, \end{cases}$$

where $j = 0, 1, 2, \ldots$. The transversality conditions are obtained as

$$\mathrm{Re}\left(\frac{d\tau}{d\lambda}\right)\Big|_{\tau=\tau_{1,2}^{(j)}} = \frac{\pm\sqrt{4\mu^4 + \alpha_1^2k^2 - 4\alpha_1\mu^2}}{k^2\omega_{1,2}^2}, j = 0, 1, 2, \ldots. \tag{6}$$

Thus, a possible double Hopf bifurcation occurs when two such families of surfaces intersect, with $\tau_c = \tau_1^{(j)} = \tau_2^{(l)}$, where $j, l = 0, 1, 2, \ldots$. The equality $\tau_c = \tau_1^{(j)} = \tau_2^{(l)}$ implies that the linearized system on the trivial equilibrium has two pairs of purely imaginary eigenvalues $\pm i\omega_1\tau$ and $\pm i\omega_2\tau$. Assume $\omega_1 : \omega_2 = n_1 : n_2$. Then a possible double Hopf bifurcation with the ratio $n_1 : n_2$ appears. When $n_1, n_2 \in Z_+$, it is called an $n_1 : n_2$ resonant double Hopf bifurcation; otherwise, it is called a non-resonant double Hopf bifurcation. In this paper, we only consider the non-resonant double Hopf bifurcation.

## 2.2. Multiple time scales

We consider the influence of feedback gain $k$ and time delay $\tau$ on the controlled system (2), and thus treat $k$ and $\tau$ as two bifurcation parameters. From $\tau_c = \tau_1^{(j)} = \tau_2^{(l)}, l, j = 0, 1, 2, \ldots$, we get the critical value $k_c$. We take perturbations as $k = k_c + \epsilon k_\epsilon$ and $\tau = \tau_c + \epsilon\tau_\epsilon$ in (2). Suppose system (2) undergoes a double Hopf bifurcation from the trivial equilibrium at the critical point: $k = k_c$, $\tau = \tau_c$. Further, by the MTS, the solution of (2) is assumed to take the form:

$$\begin{aligned} x_1(t) &= x_{1,0}(T_0, T_1, \ldots) + \epsilon x_{1,1}(T_0, T_1, \ldots) + \cdots, \\ x_2(t) &= x_{2,0}(T_0, T_1, \ldots) + \epsilon x_{2,1}(T_0, T_1, \ldots) + \cdots, \end{aligned} \tag{7}$$

where $T_j = \epsilon^j t,\ j = 0, 1, 2, \ldots$. The derivative with respect to $t$ is now transformed into

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \cdots = \mathrm{D}_0 + \epsilon\mathrm{D}_1 + \cdots,$$

where the differential operator $\mathrm{D}_i = \frac{\partial}{\partial T_i}, i = 0, 1, 2, \ldots$.

Suppose the characteristic equation of the linear differential system, $\frac{\mathrm{d}[Dx_t]}{\mathrm{d}t} = L_0 x_t$, has two pairs of purely imaginary roots $\pm\mathrm{i}\omega_1\tau$ and $\pm\mathrm{i}\omega_2\tau$, and no other roots with zero real part. Let $h_1 = (h_{11}, h_{12})^{\mathrm{T}}$ and $h_2 = (h_{21}, h_{22})^{\mathrm{T}}$ be the eigenvectors of the characteristic matrix, corresponding to the eigenvalues $\mathrm{i}\omega_1\tau$ and $\mathrm{i}\omega_2\tau$, respectively. Further, let $h_1^* = (h_{11}^*, h_{12}^*)^{\mathrm{T}}$ and $h_2^* = (h_{21}^*, h_{22}^*)^{\mathrm{T}}$ be the normalized eigenvectors of the adjoint matrix of the characteristic matrix, corresponding to the eigenvalues $-\mathrm{i}\omega_1\tau$ and $-\mathrm{i}\omega_2\tau$, respectively, satisfying the inner product,

$$\langle h_j^*, h_j\rangle = \bar{h}_j^{*\mathrm{T}} h_j = 1, \quad j = 1, 2.$$

By a simple calculation, we have

$$\begin{aligned} h_j &= (h_{j1}, h_{j2})^{\mathrm{T}} = (1, \mathrm{i}\omega_j)^{\mathrm{T}}, \\ h_j^* &= (h_{j1}^*, h_{j2}^*)^{\mathrm{T}} = \left(\frac{\alpha_1}{\alpha_1 + \omega_j^2}, \frac{\mathrm{i}\omega_j}{\alpha_1 + \omega_j^2}\right)^{\mathrm{T}}, \quad j = 1, 2. \end{aligned} \tag{8}$$

To deal with the delayed terms, we expand $x_{i,j}(T_0 - 1, T_1 - \epsilon(t-1), \ldots)$ at $x_{i,j}(T_0 - 1, T_1)$ for $i = 1, 2; j = 0, 1, 2, \ldots$. Then, substituting the solutions with the multiple scales into (2) and balancing the coefficients of $\epsilon^n (n = 0, 1, \cdots)$ yields a set of ordered linear differential equations.

First, for the $\epsilon^0$–order terms, we have

$$\begin{aligned} &\mathrm{D}_0 x_{1,0} - \tau_c x_{2,0} = 0, \\ &\mathrm{D}_0 x_{2,0} + k_c \mathrm{D}_0 x_{2,0,\tau} + \alpha_1\tau_c x_{1,0} + 2\mu\tau_c x_{2,0} = 0, \end{aligned} \tag{9}$$

where $x_{j,0} = x_{j,0}(T_0, T_1, T_2)$ and $x_{j,0,\tau} = x_{j,0}(T_0 - 1, T_1, T_2),\ j = 1, 2$. Since $\pm\mathrm{i}\omega_1\tau$ and $\pm\mathrm{i}\omega_2\tau$ are the eigenvalues of the linear operator $L_0$, the solution of (9) can be expressed in the form of

$$\begin{pmatrix} x_{1,0}(T_0, T_1) \\ x_{2,0}(T_0, T_1) \end{pmatrix} = G_1(T_1) h_1 \mathrm{e}^{\mathrm{i}\omega_1\tau T_0} + \overline{G}_1(T_1)\bar{h}_1 \mathrm{e}^{-\mathrm{i}\omega_1\tau T_0} + G_2(T_1) h_2 \mathrm{e}^{\mathrm{i}\omega_2\tau T_0} + \overline{G}_2(T_1)\bar{h}_2 \mathrm{e}^{-\mathrm{i}\omega_2\tau T_0}, \tag{10}$$

where $h_j (j = 1, 2)$ is given by (8).

Next, for the $\epsilon^1$–order terms of (2), we obtain

$$\begin{aligned} &\mathrm{D}_0 x_{1,1} - \tau_c x_{2,1} = -\mathrm{D}_1 x_{1,0} + \tau_\epsilon x_{2,0}, \\ &\mathrm{D}_0 x_{2,1} + k_c \mathrm{D}_0 x_{2,1,\tau} + \alpha_1\tau_c x_{1,1} + 2\mu\tau_c x_{2,1} = -\mathrm{D}_1 x_{2,0} - k_c \mathrm{D}_1 x_{2,0,\tau} + k_c \mathrm{D}_0 \mathrm{D}_1 x_{2,0,\tau} \\ &\qquad - \alpha_1\tau_\epsilon x_{1,0} - 2\mu\tau_\epsilon x_{2,0} - k_\epsilon \mathrm{D}_0 x_{2,0,\tau} - \tau_c\alpha_3 x_{1,0}^3 - \tau_c\alpha_4 x_{1,0} x_{2,0}^2 - k_c\tau_c x_{1,0,\tau} x_{2,0,\tau}^2 \\ &\qquad - \alpha_4 x_{1,0}^2 \mathrm{D}_0 x_{2,0} - k_c\alpha_5 x_{1,0}^2 \mathrm{D}_0 x_{2,0,\tau} - \frac{1}{2} k_c x_{1,0,\tau}^2 \mathrm{D}_0 x_{2,0,\tau}, \end{aligned} \tag{11}$$

where $x_{j,1} = x_{j,0}(T_0, T_1, T_2)$ and $x_{j,1,\tau} = x_{j,0}(T_0 - 1, T_1, T_2), j = 1, 2$. Substituting solution (10) into (11) and simplifying, we can obtain linear nonhomogeneous equations for $x_{1,1}$ and $x_{2,1}$, which have a solution if and only if the so-called solvability condition is satisfied [30]. That is, the right-hand side of nonhomogeneous equation is orthogonal to every solution of the adjoint homogeneous problem. Then $\frac{\partial G_1}{\partial T_1}$ and $\frac{\partial G_2}{\partial T_1}$ are solved to yield

$$\begin{aligned} \frac{\partial G_1}{\partial T_1} &= \beta_1 G_1 + P_1 G_1^2 \overline{G}_1 + P_2 G_1 G_2 \overline{G}_2, \\ \frac{\partial G_2}{\partial T_1} &= \beta_2 G_2 + P_3 G_2^2 \overline{G}_2 + P_4 G_1 \overline{G}_1 G_2, \end{aligned} \tag{12}$$

where

$$\begin{aligned}
\beta_j &= \frac{2(\mathrm{i}\omega_j\alpha_1 - \omega_j^2\mu)\tau_\epsilon - \mathrm{i}\omega_j^3\tau_c \mathrm{e}^{-\mathrm{i}\omega_j\tau_c}k_\epsilon}{\alpha_1 + \omega_j^2 + \omega_j^2 k_c \mathrm{e}^{-\mathrm{i}\omega_j\tau_c} - \mathrm{i}\omega_j^3 k_c\tau_c \mathrm{e}^{-\mathrm{i}\omega_j\tau_c}}, \quad j = 1,2,\\
P_1 &= -\frac{\mathrm{i}\omega_1\tau_c(2\mathrm{e}^{\mathrm{i}\omega_1\tau_c}\omega_1^2 k_c\alpha_5 - 6\alpha_3 + 4\omega_1^2\alpha_4 + \mathrm{e}^{-\mathrm{i}\omega_1\tau_c}\omega_1^2 k_c + 4\mathrm{e}^{-\mathrm{i}\omega_1\tau_c}\omega_1^2 k_c\alpha_5)}{2(\alpha_1 + \omega_1^2 + \omega_1^2 k_c \mathrm{e}^{-\mathrm{i}\omega_1\tau_c} - \mathrm{i}\omega_1^3 k_c\tau_c \mathrm{e}^{-\mathrm{i}\omega_1\tau_c})},\\
P_2 &= -\frac{\mathrm{i}\omega_1\tau_c[4k_c\alpha_5\omega_2^2\cos(\omega_2\tau_c) + \mathrm{e}^{-\mathrm{i}\omega_1\tau_c}\omega_1^2 k_c + 2\alpha_4\omega_2^2 - 6\alpha_3 + 2\omega_1^2\alpha_4 + 2\mathrm{e}^{-\mathrm{i}\omega_1\tau_c}\omega_1^2 k_c\alpha_5]}{\alpha_1 + \omega_1^2 + \omega_1^2 k_c \mathrm{e}^{-\mathrm{i}\omega_1\tau_c} - \mathrm{i}\omega_1^3 k_c\tau_c \mathrm{e}^{-\mathrm{i}\omega_1\tau_c}},\\
P_3 &= -\frac{\mathrm{i}\omega_2\tau_c(2\mathrm{e}^{\mathrm{i}\omega_2\tau_c}\omega_2^2 k_c\alpha_5 - 6\alpha_3 + 4\omega_2^2\alpha_4 + \mathrm{e}^{-\mathrm{i}\omega_2\tau_c}\omega_2^2 k_c + 4\mathrm{e}^{-\mathrm{i}\omega_2\tau_c}\omega_2^2 k_c\alpha_5)}{2(\alpha_1 + \omega_2^2 + \omega_2^2 k_c \mathrm{e}^{-\mathrm{i}\omega_2\tau_c} - \mathrm{i}\omega_2^3 k_c\tau_c \mathrm{e}^{-\mathrm{i}\omega_2\tau_c})},\\
P_4 &= -\frac{\mathrm{i}\omega_2\tau_c[4k_c\alpha_5\omega_1^2\cos(\omega_1\tau_c) + \mathrm{e}^{-\mathrm{i}\omega_2\tau_c}\omega_2^2 k_c + 2\alpha_4\omega_1^2 - 6\alpha_3 + 2\omega_2^2\alpha_4 + 2\mathrm{e}^{-\mathrm{i}\omega_2\tau_c}\omega_2^2 k_c\alpha_5]}{\alpha_1 + \omega_2^2 + \omega_2^2 k_c \mathrm{e}^{-\mathrm{i}\omega_2\tau_c} - \mathrm{i}\omega_2^3 k_c\tau_c \mathrm{e}^{-\mathrm{i}\omega_2\tau_c}}.
\end{aligned} \tag{13}$$

With the polar coordinates: $G_1 = R_1\mathrm{e}^{\mathrm{i}\rho_1}$, $G_2 = R_2\mathrm{e}^{\mathrm{i}\rho_2}$, letting $\omega_{jc} = \omega_j\tau_c$ and $\Theta_j = \omega_{jc}t + \rho_j (j = 1,2)$, we finally obtain the amplitude and phase equations of (12) on the center manifold as

$$\begin{aligned}
\frac{\mathrm{d}R_1}{\mathrm{d}t} &= \mathrm{Re}(\beta_1)R_1 + \mathrm{Re}(P_1)R_1^3 + \mathrm{Re}(P_2)R_1R_2^2,\\
\frac{\mathrm{d}R_2}{\mathrm{d}t} &= \mathrm{Re}(\beta_2)R_2 + \mathrm{Re}(P_3)R_2^3 + \mathrm{Re}(P_4)R_1^2R_2,\\
\frac{\mathrm{d}\Theta_1}{\mathrm{d}t} &= \omega_{1c} + \frac{\mathrm{d}\rho_1}{\mathrm{d}t} = \omega_{1c} + \mathrm{Im}(\beta_1) + \mathrm{Im}(P_1)R_1^2 + \mathrm{Im}(P_2)R_2^2,\\
\frac{\mathrm{d}\Theta_2}{\mathrm{d}t} &= \omega_{2c} + \frac{\mathrm{d}\rho_2}{\mathrm{d}t} = \omega_{2c} + \mathrm{Im}(\beta_2) + \mathrm{Im}(P_3)R_2^2 + \mathrm{Im}(P_4)R_1^2.
\end{aligned} \tag{14}$$

### 2.3. Center manifold reduction

In this section, we compute the normal form near the double Hopf bifurcation critical point $(k_c, \tau_c)$ using the CMR method. The trivial equilibrium of (2) is $x_1 = x_2 = 0$. At the critical point $(k, \tau) = (k_c, \tau_c)$, we choose

$$\xi(\theta) = \begin{cases} N_0, & \theta = -1,\\ 0, & \theta \in (-1, 0], \end{cases} \qquad \eta(\theta) = \begin{cases} \tau_c N_1, & \theta = 0,\\ 0, & \theta \in [-1, 0), \end{cases}$$

with

$$N_0 = \begin{pmatrix} 0 & 0\\ 0 & k_c \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 1\\ -\alpha_1 & -2\mu \end{pmatrix}.$$

Then, the linearized equation of (2) at the trivial equilibrium is

$$\frac{\mathrm{d}}{\mathrm{d}t}[Dx_t] = L_0 x_t,$$

where $D\varphi = \varphi(0) - \int_{-1}^{0}\mathrm{d}\xi(\theta)\varphi(\theta)$, $L_0\varphi = \int_{-1}^{0}\mathrm{d}\eta(\theta)\varphi(\theta), \varphi \in \mathrm{C} = \mathrm{C}([-1, 0], \mathrm{R}^2)$, and the bilinear form on $\mathrm{C}^* \times \mathrm{C}$ ($*$ stands for adjoint) is

$$\langle \psi(s), \varphi(\theta)\rangle = \psi(0)\varphi(0) - \int_{-1}^{0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left[\int_0^{\varsigma}\psi(s-\varsigma)\mathrm{d}\xi(\theta)\varphi(s)\mathrm{d}s\right] - \int_{-1}^{0}\int_0^{\theta}\psi(s-\theta)\mathrm{d}\eta(\theta)\varphi(s)\mathrm{d}s,$$

in which $\varphi \in \mathrm{C}, \psi \in \mathrm{C}^*$. Then, the phase space C is decomposed by $\Lambda = \{\pm\mathrm{i}\omega_1\tau_c, \pm\mathrm{i}\omega_2\tau_c\}$ as $\mathrm{C} = P \oplus Q$, where $Q = \{\tilde{\varphi} \in \mathrm{C} : (\psi, \tilde{\varphi}) = 0, \text{for all} \psi \in P^*\}$, and the bases for $P$ and its adjoint $P^*$ are given respectively by

$$\Phi(\theta) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\omega_1\tau_c\theta} & \mathrm{e}^{-\mathrm{i}\omega_1\tau_c\theta} & \mathrm{e}^{\mathrm{i}\omega_2\tau_c\theta} & \mathrm{e}^{-\mathrm{i}\omega_2\tau_c\theta}\\ \mathrm{i}\omega_1\mathrm{e}^{\mathrm{i}\omega_1\tau_c\theta} & -\mathrm{i}\omega_1\mathrm{e}^{-\mathrm{i}\omega_1\tau_c\theta} & \mathrm{i}\omega_2\mathrm{e}^{\mathrm{i}\omega_2\tau_c\theta} & -\mathrm{i}\omega_2\mathrm{e}^{\mathrm{i}\omega_2\tau_c\theta} \end{pmatrix}$$

and

$$\Psi(s) = \begin{pmatrix} d_1\mathrm{e}^{-\mathrm{i}\omega_1\tau_c s} & -\frac{\mathrm{i}\omega_1 d_1}{\alpha_1}\mathrm{e}^{-\mathrm{i}\omega_1\tau_c s}\\ \bar{d}_1\mathrm{e}^{\mathrm{i}\omega_1\tau_c s} & \frac{\mathrm{i}\omega_1\bar{d}_1}{\alpha_1}\mathrm{e}^{\mathrm{i}\omega_1\tau_c s}\\ d_2\mathrm{e}^{-\mathrm{i}\omega_2\tau_c s} & -\frac{\mathrm{i}\omega_2 d_2}{\alpha_1}\mathrm{e}^{-\mathrm{i}\omega_2\tau_c s}\\ \bar{d}_2\mathrm{e}^{\mathrm{i}\omega_2\tau_c s} & \frac{\mathrm{i}\omega_2\bar{d}_2}{\alpha_1}\mathrm{e}^{\mathrm{i}\omega_2\tau_c s} \end{pmatrix},$$

where $d_j = \frac{\alpha_1}{\alpha_1 + \omega_j^2 + \omega_j^2 k_c\mathrm{e}^{-\mathrm{i}\omega_j\tau_c} - \mathrm{i}\omega_j^3 k_c\tau_c\mathrm{e}^{-\mathrm{i}\omega_j\tau_c}}, j = 1, 2.$

We also use the same bifurcation parameters given by $k = k_c + k_\varepsilon$ and $\tau = \tau_c + \tau_\varepsilon$ in (2), where $k_\varepsilon$ and $\tau_\varepsilon$ are perturbation parameters, and denote $\varepsilon = (k_\varepsilon, \tau_\varepsilon)$. Rescale $x_i \to \frac{1}{\sqrt{\epsilon}} x_i (i = 1, 2)$, then (2) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}[Dx_t] = L_\varepsilon x_t + F(x_t, \varepsilon) + G(x_t, \dot{x}_t, \varepsilon), \tag{15}$$

where

$$L_\varepsilon x_t = \begin{pmatrix} (\tau_c + \tau_\varepsilon)x_2(t) \\ -\alpha_1(\tau_c + \tau_\varepsilon)x_1(t) - 2\mu(\tau_c + \tau_\varepsilon)x_2(t) \end{pmatrix},$$

$$F(x_t, \varepsilon) = \begin{pmatrix} 0 \\ -(\tau_c + \tau_\varepsilon)[\alpha_3 x_1^3(t) + \alpha_4 x_1(t)x_2^2(t) + (k_c + k_\varepsilon)x_1(t-1)x_2^2(t-1)] \end{pmatrix}$$

and

$$G(x_t, \dot{x}(t), \varepsilon) = \begin{pmatrix} 0 \\ -\alpha_4 x_1^2(t)\dot{x}_2(t) - [k_\varepsilon + (k_c + k_\varepsilon)\alpha_5 x_1^2(t) + \frac{1}{2}(k_c + k_\varepsilon)x_1^2(t-1)]\dot{x}_2(t-1) \end{pmatrix}.$$

**Remark 1.** For the MTS method, the solution of (2) with $\epsilon$ (noticing that $F$ and $G$ contain $\epsilon$) is assumed to take form (7), and the solution of (15) without $\epsilon$ is assumed to take the form:

$$x_1(t) = \epsilon x_{1,0}(T_0, T_1, \ldots) + \epsilon^2 x_{1,1}(T_0, T_1, \ldots) + \cdots,$$
$$x_2(t) = \epsilon x_{2,0}(T_0, T_1, \cdots) + \epsilon^2 x_{2,1}(T_0, T_1, \cdots) + \cdots.$$

Under the two forms, we can get the same normal form by using MTS method. Therefore, it is fine if we use the CMR method to consider Eq. (15).

We now consider the enlarged phase space $BC$ of functions from $[-1, 0]$ to $\mathrm{R}^2$, which are continuous on $[-1, 0)$ with a possible jump discontinuity at zero. This space can be identified as $\mathrm{C} \times \mathrm{R}^2$. Thus, its elements can be written in the form $\hat{\varphi} = \varphi + X_0 c$, where $\varphi \in \mathrm{C}$, $c \in \mathrm{R}^2$ and $X_0$ is a $2 \times 2$ matrix-valued function, defined by $X_0(\theta) = 0$ for $\theta \in [-1, 0)$ and $X_0(0) = \mathrm{I}$. In the $BC$, (15) becomes an abstract ODE,

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = Ax_t + X_0\tilde{F}(x_t, \varepsilon), \tag{16}$$

where $x_t \in \mathrm{C}$, and $A$ is defined by

$$A : \mathrm{C}^1 \to BC, Ax_t = x_t'(\theta) + X_0[L_0 x_t - Dx_t']$$

and

$$\tilde{F}(x_t, \varepsilon) = [L_\varepsilon - L_0]x_t + F(x_t, \varepsilon) + G(x_t, \dot{x}_t, \varepsilon).$$

By the continuous projection $\pi : BC \mapsto P$, $\pi(\varphi + X_0 c) = \Phi[(\Psi, \varphi) + \Psi(0)c]$, we can decompose the enlarged phase space by $\Lambda = \{\pm i\omega_1\tau_c, \pm i\omega_2\tau_c\}$ as $BC = P \oplus \mathrm{Ker}^\pi$, where $\mathrm{Ker}^\pi = \{\varphi + X_0 c : \pi(\varphi + X_0 c) = 0\}$, denoting the Kernel under the projection $\pi$. Let $u = (u_1, \bar{u}_1, u_2, \bar{u}_2)^\mathrm{T}$, $v_t \in Q^1 := Q \cap \mathrm{C}^1 \subset \mathrm{Ker}^\pi$, and $A_{Q^1}$ the restriction of $A$ as an operator from $Q^1$ to the Banach space $\mathrm{Ker}^\pi$. Further, denote $x_t = \Phi u + v_t$. Then, Eq. (16) is decomposed as

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}t} &= Bu + \Psi(0)\widetilde{F}(\Phi u + v_t, \varepsilon), \\ \frac{\mathrm{d}v_t}{\mathrm{d}t} &= A_{Q^1} v_t + (\mathrm{I} - \pi)X_0\widetilde{F}(\Phi u + v_t, \varepsilon), \end{aligned} \tag{17}$$

where $B = \mathrm{diag}\{i\omega_1\tau_c, -i\omega_1\tau_c, i\omega_2\tau_c, -i\omega_2\tau_c\}$.

Next, let $M_2^1$ denote the operator defined in $V_2^6(\mathrm{C}^4 \times Ker^\pi)$, with

$$M_2^1 : V_2^6(\mathrm{C}^4) \mapsto V_2^6(\mathrm{C}^4), (M_2^1 p)(u, \varepsilon) = D_u p(u, \varepsilon)Bu - Bp(u, \varepsilon),$$

where $V_2^6(\mathrm{C}^4)$ represents the linear space of the second order homogeneous polynomials in six variables $(u_1, \bar{u}_1, u_2, \bar{u}_2, k_\varepsilon, \tau_\varepsilon)$ with coefficients in $\mathrm{C}^4$. Then, it is easy to verify that one may choose the decomposition $V_2^6(\mathrm{C}^4) = \mathrm{Im}(M_2^1) \oplus \mathrm{Im}(M_2^1)^c$with complementary space $\mathrm{Im}(M_2^1)^c$ spanned by the elements $k_\varepsilon u_1 e_1, \tau_\varepsilon u_1 e_1$, $k_\varepsilon \bar{u}_1 e_2, \tau_\varepsilon \bar{u}_1 e_2$, $k_\varepsilon u_2 e_3, \tau_\varepsilon u_2 e_3$, $k_\varepsilon \bar{u}_2 e_4, \tau_\varepsilon \bar{u}_2 e_4$, where $e_i's (i = 1, 2, 3, 4)$ are unit vectors.

Consequently, the normal form of (15) on the center manifold associated with the equilibrium $(0, 0)$ near $k_\varepsilon = 0, \tau_\varepsilon = 0$ has the form

$$\frac{du}{dt} = Bu + \frac{1}{2}g_2^1(u,0,\varepsilon) + \text{h.o.t.},$$

where $g_2^1$ is the function giving the quadratic terms in $(u,\varepsilon)$ for $v_t = 0$, and is determined by $g_2^1(u,0,\varepsilon) = Proj_{(\mathrm{Im}(M_2^1))^c} \times f_2^1(u,0,\varepsilon)$, where $f_2^1(u,0,\varepsilon)$ is the function giving the quadratic terms in $(u,\varepsilon)$ for $v_t = 0$ defined by the first equation of (17). Thus, the normal form, truncated at the quadratic order terms, is given by

$$\begin{aligned}
\frac{du_1}{dt} &= i\omega_1\tau_c u_1 + 2d_1(i\omega_1 - \frac{\mu\omega_1^2}{\alpha_1})\tau_\varepsilon u_1 - \frac{i\omega_1^3 d_1\tau_c e^{-i\omega_1\tau_c}k_\varepsilon u_1}{\alpha_1},\\
\frac{du_2}{dt} &= i\omega_2\tau_c u_2 + 2d_2(i\omega_2 - \frac{\mu\omega_2^2}{\alpha_1})\tau_\varepsilon u_2 - \frac{i\omega_2^3 d_2\tau_c e^{-i\omega_2\tau_c}k_\varepsilon u_2}{\alpha_1},
\end{aligned} \tag{18}$$

where $d_j = \frac{\alpha_1}{\alpha_1+\omega_j^2+\omega_j^2k_ce^{-i\omega_j\tau_c}-i\omega_j^3k_c\tau_ce^{-i\omega_j\tau_c}}, j = 1,2.$

To find the normal form up to third order, similarly, let $M_3^1$ denote the operator defined in $V_3^4(\mathrm{C}^4 \times Ker^\pi)$, with

$$M_3^1 : V_3^4(\mathrm{C}^4) \mapsto V_3^4(\mathrm{C}^4), \quad (M_3^1p)(u,\varepsilon) = D_up(u,\varepsilon)Bu - Bp(u,\varepsilon),$$

where $V_3^4(\mathrm{C}^4)$ denotes the linear space of the third-order homogeneous polynomials in four variables $(u_1,\bar{u}_1,u_2,\bar{u}_2)$ with coefficients in $\mathrm{C}^4$. Then, one may choose the decomposition $V_3^4(\mathrm{C}^4) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$with complementary space $\mathrm{Im}(M_3^1)^c$ spanned by the elements $u_1^2\bar{u}_1e_1, u_1u_2\bar{u}_2e_1$, $u_1\bar{u}_1^2e_2, \bar{u}_1u_2\bar{u}_2e_2$, $u_2^2\bar{u}_2e_3, u_1\bar{u}_1u_2e_3, u_2\bar{u}_2^2e_4$, $u_1\bar{u}_1\bar{u}_2e_4$, where $e\prime_is(i = 1,2,3,4)$ are unit vectors.

Therefore, the normal form up to third-order terms is given by

$$\frac{du}{dt} = Bu + \frac{1}{2!}g_2^1(u,0,\varepsilon) + \frac{1}{3!}g_3^1(u,0,\varepsilon) + \text{h.o.t.}, \tag{19}$$

where

$$\frac{1}{3!}g_3^1(u,0,0) = \frac{1}{3!}(I - P_{I,3}^1)f_3^1(u,0,0),$$

and $f_3^1(u,0,0)$ is the function giving the cubic terms in $(u, v_t, \varepsilon)$ for $\varepsilon = 0$, and $v_t = 0$ is defined by the first equation of (17). Finally, the normal form on the center manifold arising from (17) becomes

$$\begin{aligned}
\dot{u}_1 &= i\omega_1\tau_cu_1 + \beta_1u_1 + P_1u_1^2\bar{u}_1 + P_2u_1u_2\bar{u}_2,\\
\dot{u}_2 &= i\omega_2\tau_cu_2 + \beta_2u_2 + P_3u_2^2\bar{u}_2 + P_4u_1\bar{u}_1u_2.
\end{aligned} \tag{20}$$

where $\beta_j(j = 1,2)$ and $P_j(j = 1,2,3,4)$ are given by (13).

With the polar coordinates: $u_1 = R_1e^{i\Theta_1}$, $u_2 = R_2e^{i\Theta_2}$, we finally obtain the amplitude and phase equations of (20) on the center manifold as

$$\begin{aligned}
\frac{dR_1}{dt} &= \mathrm{Re}(\beta_1)R_1 + \mathrm{Re}(P_1)R_1^3 + \mathrm{Re}(P_2)R_1R_2^2,\\
\frac{dR_2}{dt} &= \mathrm{Re}(\beta_2)R_2 + \mathrm{Re}(P_3)R_2^3 + \mathrm{Re}(P_4)R_1^2R_2,\\
\frac{d\Theta_1}{dt} &= \omega_{1c} + \mathrm{Im}(\beta_1) + \mathrm{Im}(P_1)R_1^2 + \mathrm{Im}(P_2)R_2^2,\\
\frac{d\Theta_2}{dt} &= \omega_{2c} + \mathrm{Im}(\beta_2) + \mathrm{Im}(P_3)R_2^2 + \mathrm{Im}(P_4)R_1^2.
\end{aligned} \tag{21}$$

It is seen from (14) and (21) that the two normal forms obtained by using the MTS method and the CMR method are identical.

## 3. Bifurcation analysis and numerical simulation

In this section, we consider the following equations:

$$\begin{aligned}
\dot{R}_1 &= \delta_1R_1 + Q_1R_1^3 + Q_2R_1R_2^2,\\
\dot{R}_2 &= \delta_2R_2 + Q_3R_2^3 + Q_4R_1^2R_2,
\end{aligned} \tag{22}$$

where $\delta_j = \mathrm{Re}(\beta_j)(j = 1,2)$ and $Q_j = \mathrm{Re}(P_j)(j = 1,2,3,4)$. Eq. (22) actually consists of the first two equations of (21). We first give a bifurcation analysis for (22), and then present some numerical simulation results.

### 3.1. Bifurcation analysis

For the normal form (22), according to the signs of $Q_1Q_3$, there exist two different cases, i.e. "simple case" (with no periodic solutions) and "difficult case" (with periodic solutions) [28]. Here, we are interested in the "difficult case", i.e., when $Q_1Q_3 < 0$. Without loss of generality, we assume $Q_1 > 0$ and $Q_3 < 0$. Let $r_1 = Q_1R_1^2, r_2 = |Q_3|R_2^2$ and $\tilde{t} = 2t$. Then, we have the following planar system in terms of $r_1$ and $r_2$:

$$\begin{aligned} \dot{r_1} &= r_1(\delta_1 + r_1 - \kappa r_2), \\ \dot{r_2} &= r_2(\delta_2 + \chi r_1 - r_2), \end{aligned} \tag{23}$$

where $\kappa = \frac{Q_2}{Q_3}, \chi = \frac{Q_4}{Q_1}$.

Now, we consider the equilibria and bifurcations in the $(\delta_1, \delta_2)$ parameter space. First, note that $M_0 = (0,0)$ is always an equilibrium of (23). The two semi-trivial equilibria given in terms of perturbation parameters are $M_1 = (-\delta_1, 0)$ and $M_2 = (0, \delta_2)$, which bifurcate from the origin on the bifurcation lines $L_1 : \delta_1 = 0$ and $L_2 : \delta_2 = 0$, respectively. There may also exist a nontrivial equilibrium $M_3 = (\frac{\delta_1 - \kappa\delta_2}{\kappa\chi - 1}, \frac{\delta_2 - \chi\delta_1}{1 - \kappa\chi})$. For this equilibrium to exist, it needs $\kappa\chi \neq 1$. The nontrivial equilibrium $M_3$ collides with a semi-trivial one on the bifurcation line $L_3 : \delta_1 = \kappa\delta_2$ or $L_4 : \delta_2 = \chi\delta_1$. If $(1-\chi)\delta_1 + (1-\kappa)\delta_2 < 0$, then $M_3$ represents a sink, otherwise $M_3$ is a source. Therefore, we further need consider the bifurcation line $L_5 : (1-\chi)\delta_1 + (1-\kappa)\delta_2 = 0$.

In order to give a more clear bifurcation picture, we choose the parameter values used in [31]: $g = 9.8\text{m/s}^2$, $L = 30\text{m}, d = 2\text{m}, c = 1\text{m}, R = 2.5\text{m}$ and $\mu = 0.001$. Thus, $\alpha_1 = 0.4214467448$, $\alpha_3 = 0.7585650638, \alpha_4 = 1.403351954$, $\alpha_5 = 2.321033878$. Let $k_c = 0.38646254$ and $\tau_c = 11.37917097$. Then, the characteristic Eq. (3) has two pairs of purely imaginary eigenvalues $\Lambda = \{\pm i\omega_1\tau_c, \pm i\omega_2\tau_c\}$, and no other eigenvalues with zero real part. Assume that system (15) undergoes a double Hopf bifurcation from the equilibrium $(0,0)$. By a simple calculation, we obtain

$$\begin{aligned} &\omega_1 = 0.8287969484, \tau_1^{(0)} = 3.798079696, \tau_1^{(1)} = 11.37917097, \mathrm{Re}\left(\frac{d\lambda_1}{d\tau_1^{(0)}}\right) > 0, \\ &\omega_2 = 0.5513405668, \tau_2^{(0)} = 11.37917097, \mathrm{Re}\left(\frac{d\lambda_2}{d\tau_2^{(0)}}\right) < 0. \end{aligned} \tag{24}$$

Thus, $\delta_1 = 0.2493799487\tau_\varepsilon + 2.319040951k_\varepsilon$, $\delta_2 = -0.271026524\tau_\varepsilon + 1.122528609k_\varepsilon$, $Q_1 = 7.85128853$, $Q_2 = 7.346313018, Q_3 = -1.980998341$, $Q_4 = -13.02378723, \kappa = -3.708389283$ and $\chi = -1.658808892$.

Therefore, the critical bifurcation lines become: $L_1 : \tau_\varepsilon = -9.299227797k_\varepsilon$, $L_2 : \tau_\varepsilon = 4.4141766615k_\varepsilon$, $L_3 : \tau_\varepsilon = 8.577323571k_\varepsilon$, $L_4 : \tau_\varepsilon = -34.8362831k_\varepsilon$, $L_5 : \tau_\varepsilon = 18.67920452k_\varepsilon$, as shown in the bifurcation diagram (see Fig. 1). Note that for convenience the bifurcation diagram is shown in the $(k_\varepsilon, \tau_\varepsilon)$ parameter space, rather than the $(\delta_1, \delta_2)$ parameter space.

Since there does not exist unstable manifold containing the equilibrium, according to the center manifold theory, the solutions on the center manifold determine the asymptotic behavior of the solutions of the original system (2). Therefore, if Eq. (23) has one or two asymptotically stable (unstable) semi-trivial equilibria $M_1$ and $M_2$, then (2) has one or two asymptotically stable (unstable) periodic solutions in the neighborhood of the trivial equilibrium. If Eq. (23) has an asymptotically stable (unstable) equilibrium $M_3$, then (2) has an asymptotically stable (unstable) quasi-periodic solution in the neighborhood of $(0,0)$. So, we shall call the periodic solution the source (respectively, saddle, sink) periodic solution of (2) if the semi-trivial equilibrium of (23) is a source (respectively, saddle, sink), and call the quasi-periodic solution the source

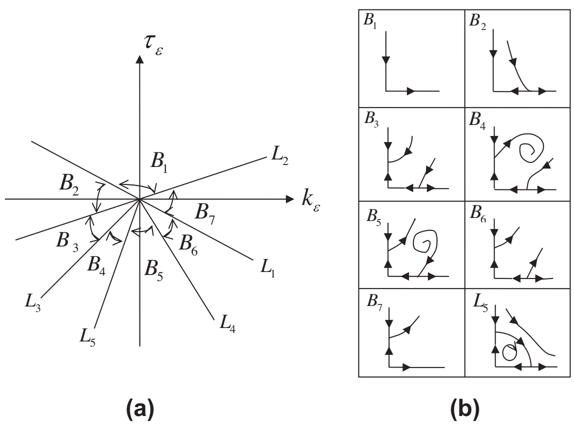


**Fig. 1.** (*a*) critical bifurcation lines in the $(k_\varepsilon, \tau_\varepsilon)$ parameter space near $(k_c, \tau_c)$; and (*b*) the corresponding phase portraits in the $(r_1, r_2)$ plane.

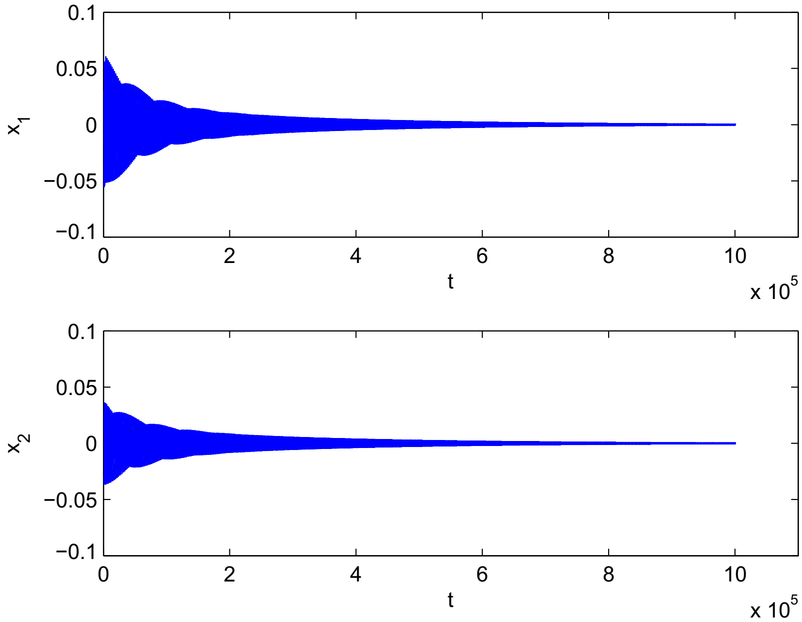


**Fig. 2.** Simulated solution of system (15) for $(k_\varepsilon, \tau_\varepsilon) = (-0.01, 0.06)$, showing a stable fixed point.

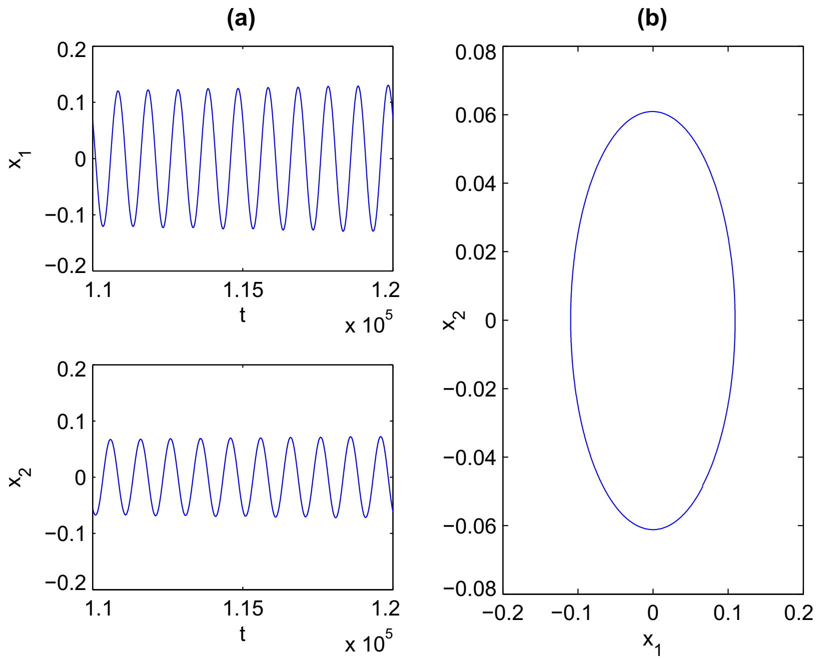


**Fig. 3.** Simulated solution of system (15) for $(k_\varepsilon, \tau_\varepsilon) = (-0.01, -0.06)$: (a) the time history; and (b) the phase portrait, showing a stable periodic solution.

(respectively, saddle, sink) quasi-periodic solution of (2) when the nontrivial equilibrium of (23) is a source (respectively, saddle, sink).

For the bifurcation behaviors of the original system (15) in the neighborhood of the trivial equilibrium, the above critical bifurcation boundaries divide the $(k_\varepsilon, \tau_\varepsilon)$ parameter plane into seven regions (see Fig. 1). We explain the bifurcations in the counterclockwise direction, starting from $B_1$ and ending at $B_1$. First, in region $B_1$, there is only one trivial equilibrium which is a saddle. When the parameters are varied across the line $L_1$ from region $B_1$ to $B_2$, an unstable periodic solution $O_1$ (saddle)

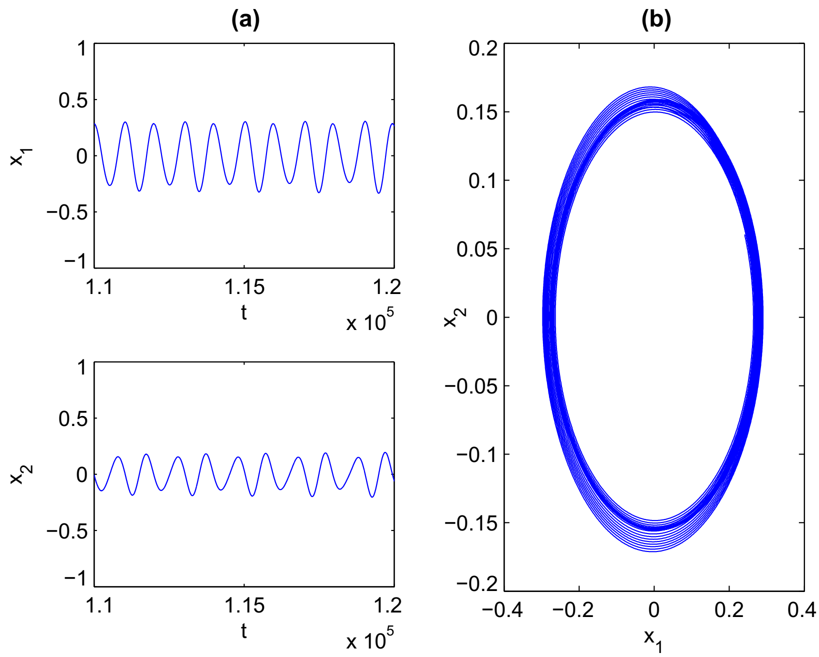


**Fig. 4.** Simulated solution of system (15) for $(k_\varepsilon, \tau_\varepsilon) = (-0.01, -0.15)$: (a) the time history; and (b) the phase portrait, showing a stable quasi-periodic solution.

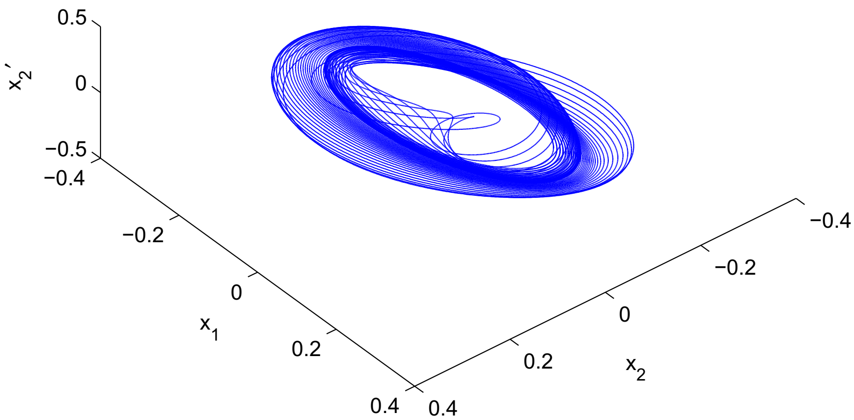


**Fig. 5.** A simulated 2-D torus corresponding to Fig. 4(b), shown in the three dimensional $x_1$-$x_2$-$x_2'$ space.

appears from the trivial solution due to a Hopf bifurcation, and the trivial equilibrium becomes a sink. Similarly, when the parameters are changed from region $B_2$ to $B_3$, another periodic solution $O_2$ (sink) occurs from the trivial solution due to a Hopf bifurcation while the trivial equilibrium becomes a saddle. In region $B_4$, a quasi-periodic solution (stable focus) occurs from the periodic solution $O_2$ due to a Neimark–Sacker bifurcation, and $O_2$ is changed to a saddle from a sink. Further, the quasi-periodic solution becomes an unstable focus when the parameters are varied across line $L_5$ from region $B_4$ to $B_5$, and when the parameters are further changed from region $B_5$ to $B_6$ crossing the line $L_4$, the quasi-periodic solution collides with the periodic solution $O_1$ and then disappears, and $O_1$ becomes a source. When the parameters are further varied across line $L_1$

from region $B_6$ to $B_7$, the periodic solution $O_1$ collides with the trivial solution and then disappears, and the trivial solution becomes a source from a saddle. Finally, when the parameters are varied across line $L_2$ from region $B_7$ to $B_1$, the periodic solution $O_2$ (saddle) collides with the trivial solution and then disappears, and the trivial solution becomes a saddle from a source.

### 3.2. Numerical simulation

To demonstrate the analytic results obtained in Section (3.1) , here we present some numerical simulation results. We choose three groups of perturbation parameter values: $(k_\varepsilon, \tau_\varepsilon) = (-0.01, 0.06), (-0.01, -0.06)$ and $(-0.01, -0.15)$, belonging to the regions $B_2, B_3$ and $B_4$, corresponding to a stable fixed point shown in Fig. 2, a stable periodic solution as depicted in Fig. 3, a stable quasi-periodic solution, see Fig. 4, with a corresponding 2-D torus displayed in the three dimensional $x_1$-$x_2$-$x_{2\prime}$ space (see Fig. 5). It is clear that the numerical simulations agree very well with the analytical predictions.

## 4. Conclusion

In this paper, we have studied double Hopf bifurcation in a container crane model with delayed position feedback. We derived the equivalent normal forms of double Hopf bifurcation by using multiple time scales and center manifold reduction methods. The MTS and CMR methods are the first time to be used to derive the normal forms of high co-dimensional bifurcations in neutral delay differential equations. It is further shown that the results from the two methods are equivalent, but the MTS method is simpler than the CMR method. Moreover, bifurcation analysis near the double Hopf critical point is given, showing that the system may exhibit a stable fixed point, stable periodic solutions, and stable quasi-periodic solutions in the neighborhood of the critical point. Numerical simulations are presented to verify the analytical predictions.

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## References

[1] C. Zhang, B. Zheng, L. Wang, Multiple Hopf bifurcations of three coupled van der Pol oscillators with delay, Appl. Math. Comput. 217 (2011) 7155–7166.
[2] X. Mao, Stability switches, bifurcation, and multi-stability of coupled networks with time delays, Appl. Math. Comput. 218 (2012) 6263–6274.
[3] Y. Song, Y. Peng, Stability and bifurcation analysis on a logistic model with discrete and distributed delays, Appl. Math. Comput. 181 (2006) 1745–1757.
[4] N.A. Nayfeh, W.T. Baumann, Nonlinear analysis of time-delay position feedback control of container cranes, Nonlinear Dyn. 53 (2008) 75–88.
[5] S.M. Lee, S.J. Choi, D.H. Ji, J.H. Park, S.C. Won, Synchronization for chaotic Lur'e systems with sector restricted nonlinearities via delayed feedback control, Nonlinear Dyn. 53 (2010) 277–288.
[6] O.M. Kwon, J.H. Park, S.M. Lee, Secure communication based on chaotic synchronization via interval time-varying delay feedback control, Nonlinear Dyn. 63 (2011) 239–252.
[7] S.M. Lee, O.M. Kwon, J.H. Park, Synchronization of chaotic Lur'e systems with delayed feedback control using deadzone nonlinearity, Chin. Phys. B 20 (2011) 010506.
[8] Y. Kuang, On neutral delay logistic Gause-type predator–prey systems, Dyn. Stab. Syst. 6 (1991) 173–189.
[9] H.A. El-Morshedy, K. Gopalsamy, Nonoscillation, oscillation and convergence of a class of neutral equations, Nonlinear Anal. 40 (2000) 173–183.
[10] R.K. Brayton, Nonlinear oscillations in a distributed network, Quart. J. Appl. Math. 24 (1967) 289–301.
[11] P.T. Nam, V.N. Phat, An improved stability criterion for a class of neutral differential equations, Appl. Math. Lett. 22 (2009) 31–35.
[12] J.H. Park, Delay-dependent criterion for asymptotic stability of a class of neutral equations, Appl. Math. Lett. 17 (2004) 1203–1206.
[13] Y.G. Sun, L. Wang, Note on asymptotic stability of a class of neutral differential equations, Appl. Math. Lett. 19 (2006) 949–953.
[14] Z. Zhang, W. Liu, D. Zhou, Global asymptotic stability to a generalized Cohen–Grossberg BAM neural networks of neutral type delays, Neural Netw. 25 (2012) 94–105.
[15] A. Ardjouni, A. Djoudi, Fixed points and stability in linear neutral differential equations with variable delays, Nonlinear Anal. 74 (2011) 2062–2070.
[16] M. Weedermann, Normal forms for neutral functional differential equations, Fields Inst. Commun. 29 (2001) 361–368.
[17] M. Weedermann, Hopf bifurcation calculations for scalar delay differential equations, Nonlinearity 19 (2006) 2091–2102.
[18] C. Wang, J. Wei, Normal forms for NFDEs with parameters and application to the lossless transmission line, Nonlinear Dyn. 52 (2008) 199–206.
[19] C. Wang, J. Wei, Hopf bifurcation for neutral functional differential equations, Nonlinear Anal. Real World Appl. 11 (2010) 1269–1277.
[20] J. Wu, Global continua of periodic solutions to some difference–differential equations of neutral type, Tohoku Math. J. 45 (1993) 67–88.
[21] Y. Qu, M.Y. Li, J. Wei, Bifurcation analysis in a neutral differential equation, J. Math. Anal. Appl. 378 (2011) 387–402.
[22] S. Guo, Equivariant normal forms for neutral functional differential equations, Nonlinear Dyn. 61 (2010) 311–329.
[23] S. Guo, J. Lamb, Equivariant Hopf bifurcation for neutral functional differential equations, Proc. Am. Math. Soc. 136 (2008) 2031–2041.
[24] G. Liu, J. Yan, Existence of positive periodic solutions for neutral delay Gause-type predator–prey system, Appl. Math. Model. 35 (2011) 5741–5750.
[25] S. Mandal, N.C. Majee, Existence of periodic solutions for a class of Cohen–Grossberg type neural networks with neutral delays, Neurocomputing 74 (2011) 1000–1007.
[26] F. Chen, Positive periodic solutions of neutral Lotka–Volterra system with feedback control, Appl. Math. Comput. 162 (2005) 1279–1302.
[27] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer, New York, 1990.
[28] Y.A. Kuznetsov, Elements of Applied Bifurcation Theory, third ed., Springer-Verlag, New York, 2004.
[29] A.H. Nayfeh, Perturbation Methods, Wiley-Interscience, New York, 1973.
[30] A.H. Nayfeh, Introduction to Perturbation Techniques, Wiley-Interscience, New York, 1981.
[31] S.L. Das, A. Chatterjee, Multiple scales without center manifold reductions for delay differential equations near Hopf bifurcations, Nonlinear Dyn. 30 (2002) 323–335.

[32] A.H. Nayfeh, Order reduction of retarded nonlinear systems – the method of multiple scales versus center-manifold reduction, Nonlinear Dyn. 51 (2008) 483–500.
[33] T. Faria, L. Magalhães, Normal forms for retarded functional differential equation and applications to Bogdanov–Takens singularity, J. Differ. Equ. 122 (1995) 201–224.
[34] T. Faria, L. Magalhães, Normal forms for retarded functional differential equation with parameters and applications to Hopf bifurcation, J. Differ. Equ. 122 (1995) 181–200.
[35] H. Wang, W. Jiang, Hopf-pitchfork bifurcation in van der Pol's oscillator with nonlinear delayed feedback, J. Math. Anal. Appl. 368 (2010) 9–18.
[36] W. Jiang, Y. Yuan, Bogdanov–Takens singularity in van der Pol's oscillator with delayed feedback, Physica D 227 (2007) 149–161.
[37] Y. Ding, W. Jiang, P. Yu, Double Hopf bifurcation in Delayed Van Der Pol-Duffing Equation, Int. J. Bifurc. Chaos 23 (2013) (1350014, 15 pages).
[38] N.D. Kazarinoff, Y.H. Wan, P.V.D. Driessche, Hopf bifurcation and stability of periodic solutions of differential–difference and integro-differential equations, J. Inst. Math. Appl. 21 (1978) 461–477.