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DOUBLE HOPF BIFURCATION IN DELAYED VAN DER POL–DUFFING EQUATION

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In this paper, we study dynamics in delayed van der Pol–Duffing equation, with particular attention focused on nonresonant double Hopf bifurcation. Both multiple time scales and center manifold reduction methods are applied to obtain the normal forms near a double Hopf critical point. A comparison between these two methods is given to show their equivalence. Bifurcations are classified in a two-dimensional parameter space near the critical point. Numerical simulations are presented to demonstrate the applicability of the theoretical results.

Keywords: Van der Pol–Duffing equation; double Hopf bifurcation; normal form; multiple time scales; center manifold reduction.

1. Introduction

Recently, much attention has been focused on the study of high-codimensional bifurcations, since they may exhibit complex dynamical behaviors such as quasi-periodic solutions and chaos. Center manifold theory and normal form theory are usually applied to analyze stability and bifurcation of dynamical systems (e.g. see [Guckenheimer & Holmes, 1990; Kuznetsov, 2004; Wiggins, 1990]). Especially, in delayed differential equations (DDEs), due to time delay, higher-codimension singularities may occur more frequently than that in ordinary differential equations (ODEs). Even for a scalar DDE, it is possible to have bifurcation of limit cycles and even higher-codimension bifurcation phenomena, (e.g. see [Choi & LeBlanc, 2006; Hale, 1997] and references therein).

Several methods have been developed for computing the normal forms of differential equations, for example, multiple time scales (MTS) [Nayfeh, 1973, 1981] and center manifold reduction (CMR) [Guckenheimer & Holmes, 1990; Hale, 1997; Hassard *et al.*, 1981; Wiggins, 1990]. For a dynamical system described by ODEs, multiple time scales method is systematic and can be directly applied to the original nonlinear system, without the application of center manifold theory [Yu, 1998, 2002; Zheng & Wang, 2010]. In fact, this approach combines the two steps involved in using center manifold theory and normal form theory into one unified step to obtain the normal form and nonlinear transformation simultaneously. Based on multiple time scales, Yu [1998, 2001, 2002] developed Maple programs for computing the normal forms of Hopf bifurcation and other singularities. These programs can be "automatically" executed by using a computer algebra system for a given ODE system. The basic idea of the center manifold theory is employing successive coordinate transformations to systematically construct a simpler system which has

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less dimension compared to the original system, and thus greatly simplifying the dynamical analysis of the system [Guckenheimer & Holmes, 1990; Kuznetsov, 2004; Wiggins, 1990]. The multiple time scales can also be directly applied to DDEs without the application of center manifold theory [Das & Chatterjee, 2002; Hu & Wang, 2009; Nayfeh, 2008]. In contrast, by the center manifold theory, one needs to first change the retarded equations to operator differential equations, and then decompose the solution space of their linearized form into stable and center manifolds. Next, with the adjoint of the operator equations, one computes the center manifold by projecting the whole space to the center manifold, finally calculating the normal form on the center manifold [Faria & Magalhaes, 1995; Jiang & Wang, 2010; Ma *et al.*, 2008; Revel *et al.*, 2010; Wei, 2007; Xu & Yu, 2004; Yu *et al.*, 2002; Yuan & Wei, 2006].

In the research of nonlinear dynamical systems, the van der Pol–Duffing equation is one of the most intensively studied equations. This celebrated equation originally was a model for an electrical circuit with a triode valve, and was extensively studied as a host of a rich class of dynamical behaviors, including relaxation oscillations, quasi-periodicity, elementary bifurcations and chaos [Kuznetsov, 2004]. It is well known that the limit cycle oscillations with strong stability property are important in applications, hence, being able to modify their behaviors through feedback is a question of interest. On the other hand, most practical implementations have inherent or feedback delays, the presence of which leads to the appearance of complex phenomena in the autonomous van der Pol–Duffing equation, such as Hopf–pitchfork bifurcation, double Hopf bifurcation and Bogdanov–Takens singularity, etc. [Jiang & Yuan, 2007; Jiang & Wei, 2008; Wang & Jiang, 2010; Xu & Chung, 2003; Xu *et al.*, 2007].

In this paper, we consider the nonresonant double Hopf bifurcation in the following van der Pol equation with delayed feedback:

$$\begin{aligned}\ddot{x}(t) + \omega_0^2 x(t) - [b - \gamma x^2(t)]\dot{x}(t) + \beta x^3(t)\\ = A[x(t-\tau) - x(t)],\end{aligned} \tag{1}$$

where ω_0, b, γ, β are positive real constants, A is feedback strength, and τ is time delay. Xu *et al.* [2007] employed the perturbation-incremental scheme (PIS) to investigate the weakly resonant double Hopf bifurcation and dynamics of system (1). In this paper, we will study nonresonant double Hopf bifurcation of (1) by using two methods to compute the normal form of the system, namely, the multiple time scales and a combination of the method of normal forms and the center manifold theorem. A comparison between the normal forms shows that the two methods are equivalent. Furthermore, we carry out the bifurcation analysis and numerical simulations. We will show that there exist a stable fixed point, stable periodic solutions and stable quasi-periodic solutions in the neighborhood of the double Hopf critical point.

The rest of the paper is organized as follows. In Sec. 2, we consider the existence of double Hopf bifurcation in the delayed van der Pol system (1), and use two methods to derive the normal form associated with double Hopf bifurcation. Then, bifurcation analysis and numerical simulations are presented in Sec. 3. Finally, the conclusion is drawn in Sec. 4.

2. Analytical Study

In this section, we consider the van der Pol–Duffing equation with delayed feedback, described by (1), and use the MTS and CMR methods to derive the normal form of the system.

2.1. *System formulation*

The characteristic equation of the linearized equation of (1), evaluated at the trivial equilibrium $x = \dot{x} = 0$, is given by:

$$\lambda^2 - b\lambda + \omega_0^2 + A - Ae^{-\lambda\tau} = 0. \tag{2}$$

To find possible periodic solutions, which may bifurcate from a Hopf or double Hopf critical point, let $\lambda = \mathrm{i}\omega$ ($\mathrm{i}^2 = -1, \omega > 0$) be a root of (2). Substituting the root into (2) and separating the real and imaginary parts yields

$$\begin{cases}\omega_0^2 + A - \omega^2 = A\cos(\omega\tau),\\ b\omega = A\sin(\omega\tau),\end{cases} \tag{3}$$

from which we obtain

$$\omega_{1,2} = \sqrt{\frac{2A + 2\omega_0^2 - b^2 \pm \sqrt{(b^2 - 2\omega_0^2 - 2A)^2 - 4(\omega_0^4 + 2A\omega_0^2)}}{2}},$$

under the assumption:

$$\begin{cases} 2A + \omega_0^2 > 0, \\ (2A - b^2)^2 - 4\omega_0^2 b^2 > 0. \end{cases} \tag{4}$$

Further, it follows from (3) that

$$\tau_{1,2}^{(j)} = \begin{cases} \dfrac{1}{\omega_{1,2}}\left[\arccos\left(1 + \dfrac{\omega_0^2 - \omega_{1,2}^2}{A}\right) + 2j\pi\right], & \text{for } A > 0, \\ \dfrac{1}{\omega_{1,2}}\left[2\pi - \arccos\left(1 + \dfrac{\omega_0^2 - \omega_{1,2}^2}{A}\right) + 2j\pi\right], & \text{for } A < 0, \end{cases}$$

where $j = 0, 1, 2, \ldots$. Thus, a possible double Hopf bifurcation occurs when two such families of surfaces intersect, with $\tau_c = \tau_1^{(j)} = \tau_2^{(l)}$, where $j, l = 0, 1, 2, \ldots$. The equality $\tau_c = \tau_1^{(j)} = \tau_2^{(l)}$ implies that the linearized system on the trivial equilibrium has two pairs of purely imaginary eigenvalues $\pm i\omega_1$ and $\pm i\omega_2$. Assume $\omega_1 : \omega_2 = k_1 : k_2$, then a possible double Hopf bifurcation with the ratio $k_1 : k_2$ appears. When $k_1,\ k_2 \in Z_+$, it is called a $k_1 : k_2$ resonant double Hopf bifurcation; otherwise, it is called a nonresonant double Hopf bifurcation. In this paper, we only consider the nonresonant double Hopf bifurcation, for which

$$\omega_1 = \sqrt{\frac{k_1}{k_2}\omega_0\sqrt{2A + \omega_0^2}}, \qquad \omega_2 = \sqrt{\frac{k_2}{k_1}\omega_0\sqrt{2A + \omega_0^2}}. \tag{5}$$

2.2. *Multiple time scales*

We treat the feedback strength A and the delay τ as two bifurcation parameters. From $\tau_c = \tau_1^{(j)} = \tau_2^{(l)}$, $l,\ j = 0, 1, 2, \ldots$, we get the critical value A_c. Suppose system (1) undergoes a double Hopf bifurcation from the trivial equilibrium at the critical point: $A = A_c$, $\tau = \tau_c$. Further, by the MTS, the solution of (1) is assumed to take the form:

$$x(t) = \epsilon x_1(T_0, T_1, T_2, \ldots) + \epsilon^2 x_2(T_0, T_1, T_2, \ldots) + \epsilon^3 x_3(T_0, T_1, T_2, \ldots) + \cdots, \tag{6}$$

where $T_k = \epsilon^k t,\ k = 0, 1, 2, \ldots$. The derivative with respect to t is now transformed into

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2} + \cdots = \mathrm{D}_0 + \epsilon \mathrm{D}_1 + \epsilon^2 \mathrm{D}_2 + \cdots,$$

where the differential operator $\mathrm{D}_i = \frac{\partial}{\partial T_i},\ i = 0, 1, 2, \ldots$.

We take perturbations as $A = A_c + \epsilon^2 A_\epsilon$ and $\tau = \tau_c + \epsilon\tau_\epsilon$ in (1), where A_ϵ is called a detuning parameter [Nayfeh, 2008]. To deal with the delayed terms, we expand $x_j(T_0 - \tau_c - \epsilon\tau_\epsilon, T_1 - \epsilon(\tau_c + \epsilon\tau_\epsilon), T_2 - \epsilon^2(\tau_c + \epsilon\tau_\epsilon), \ldots)$ at $x_j(T_0 - \tau_c, T_1, T_2)$ for $j = 1, 2, 3, \ldots$. Then, substituting the solutions with the multiple scales into (1) and balancing the coefficients of ϵ^n $(n = 1, 2, 3, \ldots)$ yields a set of ordered linear differential equations.

First, for the ϵ^1-order terms, we have

$$\mathrm{D}_0^2 x_1 + (A_c + \omega_0^2)x_1 - b\mathrm{D}_0 x_1 - A_c x_{1\tau_c} = 0, \tag{7}$$

where $x_1 = x_1(T_0, T_1, T_2)$ and $x_{1\tau_c} = x_1(T_0 - \tau_c, T_1, T_2)$. Since $\pm i\omega_1$ and $\pm i\omega_2$ are the eigenvalues of the linear part of (1), the solution of (7) can be expressed in the form of

$$\begin{aligned} x_1(T_0, T_1, T_2) &= G_1(T_1, T_2)\sin(\omega_1 T_0) + G_2(T_1, T_2)\cos(\omega_1 T_0) \\ &\quad + G_3(T_1, T_2)\sin(\omega_2 T_0) \\ &\quad + G_4(T_1, T_2)\cos(\omega_2 T_0). \end{aligned} \tag{8}$$

Next, for the ϵ^2-order terms, we obtain

$$\begin{aligned} &\mathrm{D}_0^2 x_2 + (A_c + \omega_0^2)x_2 - b\mathrm{D}_0 x_2 - A_c x_{2\tau_c} \\ &\quad = -2\mathrm{D}_1\mathrm{D}_0 x_1 + b\mathrm{D}_1 x_1 \\ &\qquad - A_c(\tau_\epsilon \mathrm{D}_0 x_{1\tau_c} + \tau_c \mathrm{D}_1 x_{1\tau_c}). \end{aligned} \tag{9}$$

Substituting solution (8) into (9) and simplifying, we obtain the following equation:

$$\begin{aligned} &\mathrm{D}_0^2 x_2 + (A_c + \omega_0^2)x_2 - b\mathrm{D}_0 x_2 - A_c x_{2\tau_c} \\ &\quad + P_1\cos(\omega_1 T_0) + P_2\sin(\omega_1 T_0) \\ &\quad + P_3\cos(\omega_2 T_0) + P_4\sin(\omega_2 T_0) = 0, \end{aligned} \tag{10}$$

where P_i $(i = 1, 2, 3, 4)$ are given as follows:

$$\begin{aligned} P_1 &= 2\omega_1\frac{\partial G_1}{\partial T_1} - b\frac{\partial G_2}{\partial T_1} \\ &\quad + A_c\tau_\epsilon\omega_1[G_1\cos(\omega_1\tau_c) + G_2\sin(\omega_1\tau_c)] \\ &\quad + A_c\tau_c\left[\frac{\partial G_2}{\partial T_1}\cos(\omega_1\tau_c) - \frac{\partial G_1}{\partial T_1}\sin(\omega_1\tau_c)\right], \end{aligned}$$

$$
\begin{aligned}
P_2 &= A_c\tau_\epsilon\omega_1[G_1\sin(\omega_1\tau_c) - G_2\cos(\omega_1\tau_c)] \\
&\quad - 2\omega_1\frac{\partial G_2}{\partial T_1} - b\frac{\partial G_1}{\partial T_1} \\
&\quad + A_c\tau_c\left[\frac{\partial G_1}{\partial T_1}\cos(\omega_1\tau_c) + \frac{\partial G_2}{\partial T_1}\sin(\omega_1\tau_c)\right], \\
P_3 &= 2\omega_2\frac{\partial G_3}{\partial T_1} - b\frac{\partial G_4}{\partial T_1} \\
&\quad + A_c\tau_\epsilon\omega_2[G_3\cos(\omega_2\tau_c) + G_4\sin(\omega_2\tau_c)] \\
&\quad + A_c\tau_c\left[\frac{\partial G_4}{\partial T_1}\cos(\omega_2\tau_c) - \frac{\partial G_3}{\partial T_1}\sin(\omega_2\tau_c)\right], \\
P_4 &= A_c\tau_\epsilon\omega_2[G_3\sin(\omega_2\tau_c) - G_4\cos(\omega_2\tau_c)] \\
&\quad - 2\omega_2\frac{\partial G_4}{\partial T_1} - b\frac{\partial G_3}{\partial T_1} \\
&\quad + A_c\tau_c\left[\frac{\partial G_3}{\partial T_1}\cos(\omega_2\tau_c) + \frac{\partial G_4}{\partial T_1}\sin(\omega_2\tau_c)\right].
\end{aligned}
$$

To avoid occurrence of secular terms in the solution of (10), the coefficients of $\cos(\omega_i T_0)$, $\sin(\omega_i T_0)$ $(i = 1, 2)$ in (10) must be set to zero, i.e. $P_i = 0$ $(i = 1, 2, 3, 4)$. Therefore, $\frac{\partial G_1}{\partial T_1}$, $\frac{\partial G_2}{\partial T_1}$, $\frac{\partial G_3}{\partial T_1}$ and $\frac{\partial G_4}{\partial T_1}$ are solved from the four linear equations in terms of G_1, G_2, G_3 and G_4. Then, Eq. (10) is reduced to

$$
\mathrm{D}_0^2 x_2 + (A_c + \omega_0^2)x_2 - b\mathrm{D}_0 x_2 - A_c x_{2\tau_c} = 0, \tag{11}
$$

where $x_2 = x_2(T_0, T_1, T_2)$ and $x_{2\tau_c} = x_2(T_0 - \tau_c, T_1, T_2)$, and thus, the particular solution of (11) is

$$
x_2(T_0, T_1, T_2) = 0. \tag{12}
$$

Further, for the ϵ^3-order terms, we similarly obtain

$$
\begin{aligned}
&\mathrm{D}_0^2 x_3 + (A_c + \omega_0^2)x_3 - b\mathrm{D}_0 x_3 - A_c x_{3\tau_c} \\
&\quad = -2\mathrm{D}_2\mathrm{D}_0 x_1 - 2\mathrm{D}_1\mathrm{D}_0 x_2 - \mathrm{D}_1^2 x_1 - A_\epsilon x_1 \\
&\qquad - \gamma x_1^2 \mathrm{D}_0 x_1 - \beta x_1^3 + A_\epsilon x_{1\tau_c} - A_c(\tau_\epsilon \mathrm{D}_0 x_{2\tau_c} \\
&\qquad + \tau_c \mathrm{D}_1 x_{2\tau_c} + \tau_c \mathrm{D}_2 x_{1\tau_c} + \tau_\epsilon \mathrm{D}_1 x_{1\tau_c}) \\
&\qquad + b(\mathrm{D}_2 x_1 + \mathrm{D}_1 x_2) + \frac{A_c}{2}(\tau_\epsilon^2 \mathrm{D}_0^2 x_{1\tau_c} \\
&\qquad + \tau_c^2 \mathrm{D}_1^2 x_{1\tau_c} + 2\tau_\epsilon\tau_c \mathrm{D}_0\mathrm{D}_1 x_{1\tau_c}),
\end{aligned} \tag{13}
$$

where $x_3 = x_3(T_0, T_1, T_2)$ and $x_{3\tau_c} = x_3(T_0 - \tau_c, T_1, T_2)$. Substituting the solutions (8) and (12) into (13) and letting the coefficients of the terms which may generate secular terms in the solution equal to zero, yields the derivatives $\frac{\partial G_1}{\partial T_2}$, $\frac{\partial G_2}{\partial T_2}$, $\frac{\partial G_3}{\partial T_2}$ and $\frac{\partial G_4}{\partial T_2}$ expressed in terms of G_1, G_2, G_3 and G_4.

The above procedure can in principle continue indefinitely (to any high order). Finally, the equations for $\dot{G}_1$, $\dot{G}_2$, $\dot{G}_3$ and $\dot{G}_4$ are given by

$$
\begin{cases}
\dot{G}_1 = \epsilon\mathrm{D}_1 G_1 + \epsilon^2\mathrm{D}_2 G_1 + \cdots, \\
\dot{G}_2 = \epsilon\mathrm{D}_1 G_2 + \epsilon^2\mathrm{D}_2 G_2 + \cdots,
\end{cases} \tag{14}
$$

$$
\begin{cases}
\dot{G}_3 = \epsilon\mathrm{D}_1 G_3 + \epsilon^2\mathrm{D}_2 G_3 + \cdots, \\
\dot{G}_4 = \epsilon\mathrm{D}_1 G_4 + \epsilon^2\mathrm{D}_2 G_4 + \cdots.
\end{cases} \tag{15}
$$

Let $G_1 = R_1\sin(\Theta_1(t))$, $G_2 = R_1\cos(\Theta_1(t))$, $G_3 = R_2\sin(\Theta_2(t))$ and $G_4 = R_2\cos(\Theta_2(t))$. Substituting these expressions into (14) and (15), and truncating the equations at $\mathrm{O}(\epsilon^3)$ yields the following normal form in polar coordinates:

$$
\begin{cases}
\dfrac{\partial R_1}{\partial T_1} = -\dfrac{A_c\omega_1[a_1\cos(\omega_1\tau_c) + b_1\sin(\omega_1\tau_c)]}{(a_1^2 + b_1^2)\tau_c}R_1\tau_\epsilon + \dfrac{[b_1\cos(\omega_1\tau_c) - b_1 - a_1\sin(\omega_1\tau_c)]}{a_1^2 + b_1^2}R_1 A_\epsilon \\
\qquad + Q_1\tau_\epsilon^2 R_1 - \dfrac{3\beta b_1 + \omega_1\gamma a_1}{2(a_1^2 + b_1^2)}R_1 R_2^2 - \dfrac{3\beta b_1 + \omega_1\gamma a_1}{4(a_1^2 + b_1^2)}R_1^3, \\
\dfrac{\partial R_2}{\partial T_1} = -\dfrac{A_c\omega_2[a_2\cos(\omega_2\tau_c) + b_2\sin(\omega_2\tau_c)]}{(a_2^2 + b_2^2)\tau_c}R_2\tau_\epsilon + \dfrac{[b_2\cos(\omega_2\tau_c) - b_2 - a_2\sin(\omega_2\tau_c)]}{a_2^2 + b_2^2}R_2 A_\epsilon \\
\qquad + Q_2\tau_\epsilon^2 R_2 - \dfrac{3\beta b_2 + \omega_2\gamma a_2}{2(a_2^2 + b_2^2)}R_1^2 R_2 - \dfrac{3\beta b_2 + \omega_2\gamma a_2}{4(a_2^2 + b_2^2)}R_2^3, \\
\dfrac{\partial \Theta_1}{\partial T_1} = \delta_1 + H_1\tau_\epsilon^2 + \dfrac{3a_1\beta - b_1\omega_1\gamma}{4(a_1^2 + b_1^2)}R_1^2 + \dfrac{3a_1\beta - b_1\omega_1\gamma}{2(a_1^2 + b_1^2)}R_2^2, \\
\dfrac{\partial \Theta_2}{\partial T_1} = \delta_2 + H_2\tau_\epsilon^2 + \dfrac{3a_2\beta - b_2\omega_2\gamma}{2(a_2^2 + b_2^2)}R_1^2 + \dfrac{3a_2\beta - b_2\omega_2\gamma}{4(a_2^2 + b_2^2)}R_2^2,
\end{cases} \tag{16}
$$

where

$$a_i = 2\omega_i - A_c\tau_c\sin(\omega_i\tau_c), \quad b_i = A_c\tau_c\cos(\omega_i\tau_c) - b,$$

$$\begin{aligned}
Q_i &= \frac{\omega_i(a_i^2A_c^2b_i\tau_c + 2a_i^2\omega_i^2b_i^2 + A_c^2b_i^3\tau_c + \omega_i^2b_i^4 + a_i^4\omega_i^2 - 2a_iA_c^2\omega_ib_i\tau_c - a_iA_c^2\tau_c^2\omega_ib_i^2 - a_i^3A_c^2\tau_c^2\omega_i^2)\sin(\omega_i\tau_c)}{\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))}\\
&\quad + \frac{2A_c^2\tau_c(a_ib_i^2 - \omega_ib_i^2 - a_i^2\omega_i + a_i^3 + \tau_c\omega_ib_i^3 + a_i^2\tau_c\omega_ib_i)\cos(\omega_i\tau_c) - A_c\omega_i\tau_c(a_i^2+b_i^2)^2}{2\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))}\\
&\quad \times \frac{2A_c\omega_i^2(a_i^2\omega_i - a_i^3 - \omega_ib_i^2 - a_ib_i^2 - a_i^2\tau_c\omega_ib_i - \tau_c\omega_ib_i^3)\sin(2\omega_i\tau_c)}{\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))}\\
&\quad + \frac{2\omega_i^2A_c(b_i^3 - 2a_ib_i\omega_i + a_i^2b_i - a_i^3\tau_c\omega_i - a_ib_i^2\omega_i\tau_c)\cos(2\omega_i\tau_c)}{\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))},\\
\delta_i &= \frac{[a_i(\omega_i^2 + \omega_0^2 + A_c - A_c\cos(\omega_i\tau_c)) - b_iA_c\sin(\omega_i\tau_c)]\tau_\varepsilon}{(a_i^2+b_i^2)\tau_c} - \frac{[a_i(\cos(\omega_i\tau_c) - 1) + b_i\sin(\omega_i\tau_c)]A_\varepsilon}{a_i^2+b_i^2},\\
H_i &= \frac{[\omega_i^3(a_i^2+b_i^2)^2 - A_c^2\tau_c\omega_i(a_i^3\tau_c\omega_i - b_i^3 - a_i^2b_i + 2a_i\omega_ib_i + a_i\tau_c\omega_ib_i^2)]\cos(\omega_i\tau_c)}{\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))}\\
&\quad + \frac{A_c\tau_c\omega_i^2(a_i^2+b_i^2)^2 - \omega_iA_c^2\tau_c(\omega_ib_i^2 - \tau_c\omega_ib_i^3 - a_i^2\omega_i + a_i^3 + a_ib_i^2 + a_i^2\tau_c\omega_ib_i)\sin(\omega_i\tau_c)}{\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))}\\
&\quad - \frac{2A_c\omega_i(a_i^3\omega_i - a_i^2\omega_i^2 + \omega_i^2b_i^2 + \tau_c\omega_i^2b_i^3 + a_i\omega_ib_i^2 + a_i^2\tau_c\omega_i^2b_i)\cos(2\omega_i\tau_c)}{\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))}\\
&\quad - \frac{2A_c\omega_i(\omega_ib_i^3 - 2a_i\omega_i^2b_i + a_i^2\omega_ib_i - a_i^3\tau_c\omega_i^2 - a_i\tau_c\omega_i^2b_i^2)\sin(2\omega_i\tau_c)}{\tau_c(a_i^2+b_i^2)^2(A_c\tau_c - \cos(\omega_i\tau_c)b - 2\omega_i\sin(\omega_i\tau_c))}, \quad i = 1, 2.
\end{aligned}$$

2.3. *Center manifold reduction*

In this section, we compute the normal form near the double Hopf bifurcation critical point (A_c, τ_c) using the CMR method. First, let $\dot{x} = y$. Then, system (1) can be rewritten as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = y(t), \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\omega_0^2x(t) + [b - \gamma x^2(t)]y(t) - \beta x^3(t) \\ \qquad + A[x(t-\tau) - x(t)]. \end{cases} \tag{17}$$

Rescale the time by $\tilde{t} \mapsto (t/\tau)$ to normalize the delay so that system (17) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\tilde{t}} = \tau y(\tilde{t}), \\ \dfrac{\mathrm{d}y}{\mathrm{d}\tilde{t}} = b\tau y(\tilde{t}) - \omega_0^2\tau x(\tilde{t}) + A\tau[x(\tilde{t}-1) - x(\tilde{t})] \\ \qquad - \gamma\tau x^2(\tilde{t})y(\tilde{t}) - \beta\tau x^3(\tilde{t}). \end{cases} \tag{18}$$

The trivial equilibrium of (18) is $x = y = 0$. At the critical point $(A, \tau) = (A_c, \tau_c)$, we choose

$$\eta(\theta) = \begin{cases} \tau_cN_1, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -\tau_cN_2, & \theta = -1, \end{cases}$$

with

$$N_1 = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 - A_c & b \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 0 & 0 \\ A_c & 0 \end{pmatrix}.$$

Then, the linearized equation of (18) at the trivial equilibrium is

$$\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} = L_0X_{\tilde{t}},$$

where $L_0\phi = \int_{-1}^0 d\eta(\theta)\phi(\theta), \phi \in \mathrm{C} = \mathrm{C}([-1, 0], \mathrm{R}^2)$, and the bilinear form on $\mathrm{C}^* \times \mathrm{C}$ ($*$ stands for adjoint) is

$$\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \psi(\xi - \theta) d\eta(\theta) \phi(\xi) d\xi,$$

in which $\phi \in \mathrm{C}$, $\psi \in \mathrm{C}^*$. Then, the phase space C is decomposed by $\Lambda = \{\pm \mathrm{i}\omega_1, \pm \mathrm{i}\omega_2\}$ as $\mathrm{C} = P \oplus Q$, where $Q = \{\varphi \in \mathrm{C} : (\psi, \varphi) = 0, \text{for all } \psi \in P^*\}$, and the bases for P and its adjoint P^* are given respectively by

$$\Phi(\theta) = \begin{pmatrix} e^{\mathrm{i}\omega_1 \tau_c \theta} & e^{-\mathrm{i}\omega_1 \tau_c \theta} & e^{\mathrm{i}\omega_2 \tau_c \theta} & e^{-\mathrm{i}\omega_2 \tau_c \theta} \\ \mathrm{i}\omega_1 e^{\mathrm{i}\omega_1 \tau_c \theta} & -\mathrm{i}\omega_1 e^{-\mathrm{i}\omega_1 \tau_c \theta} & \mathrm{i}\omega_2 e^{\mathrm{i}\omega_2 \tau_c \theta} & -\mathrm{i}\omega_2 e^{\mathrm{i}\omega_2 \tau_c \theta} \end{pmatrix},$$

and

$$\Psi(s) = \begin{pmatrix} h_1(b - \mathrm{i}\omega_1) e^{-\mathrm{i}\omega_1 \tau_c s} & -h_1 e^{-\mathrm{i}\omega_1 \tau_c s} \\ \overline{h}_1(b + \mathrm{i}\omega_1) e^{\mathrm{i}\omega_1 \tau_c s} & -\overline{h}_1 e^{\mathrm{i}\omega_1 \tau_c s} \\ h_2(b - \mathrm{i}\omega_2) e^{-\mathrm{i}\omega_2 \tau_c s} & -h_2 e^{-\mathrm{i}\omega_2 \tau_c s} \\ \overline{h}_2(b + \mathrm{i}\omega_2) e^{\mathrm{i}\omega_2 \tau_c s} & -\overline{h}_2 e^{\mathrm{i}\omega_2 \tau_c s} \end{pmatrix},$$

where $h_j = (b - 2\mathrm{i}\omega_j - A_c \tau_c e^{-\mathrm{i}\omega_j \tau_c})^{-1}, j = 1, 2$.

We also use the same bifurcation parameters given by $A = A_c + A_\varepsilon$ and $\tau = \tau_c + \tau_\varepsilon$ in (18), where A_ε and τ_ε are perturbation parameters, and denote $\varepsilon = (A_\varepsilon, \tau_\varepsilon)$. Then (18) can be written as

$$\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} = L(\varepsilon) X_{\tilde{t}} + F(X_{\tilde{t}}, \varepsilon), \tag{19}$$

where

$$L(\varepsilon) X_{\tilde{t}} = \begin{pmatrix} (\tau_c + \tau_\varepsilon) y_{\tilde{t}}(0) \\ b(\tau_c + \tau_\varepsilon) y_{\tilde{t}}(0) - \omega_0^2 (\tau_c + \tau_\varepsilon) x_{\tilde{t}}(0) + (A_c + A_\varepsilon)(\tau_c + \tau_\varepsilon)[x_{\tilde{t}}(-1) - x_{\tilde{t}}(0)] \end{pmatrix},$$

and

$$F(X_{\tilde{t}}, \varepsilon) = \begin{pmatrix} 0 \\ -\gamma(\tau_c + \tau_\varepsilon) x_{\tilde{t}}^2(0) y_{\tilde{t}}(0) - \beta(\tau_c + \tau_\varepsilon) x_{\tilde{t}}^3(0) \end{pmatrix}.$$

We now consider the enlarged phase space BC of functions from $[-1, 0]$ to R^2, which are continuous on $[-1, 0)$ with a possible jump discontinuity at zero. This space can be identified as $\mathrm{C} \times \mathrm{R}^2$. Thus, its elements can be written in the form $\tilde{\varphi} = \varphi + X_0 c$, where $\varphi \in \mathrm{C}$, $c \in \mathrm{R}^2$ and X_0 is a 2×2 matrix-valued function, defined by $X_0(\theta) = 0$ for $\theta \in [-1, 0)$ and $X_0(0) = \mathrm{I}$. In the BC, (19) becomes an abstract ODE,

$$\frac{\mathrm{d}u}{\mathrm{d}\tilde{t}} = Au + X_0 \tilde{F}(u, \varepsilon), \tag{20}$$

where $u \in \mathrm{C}$, and A is defined by

$$A : C^1 \to BC, \quad Au = \frac{\mathrm{d}u}{\mathrm{d}\tilde{t}} + X_0 \left[L_0 u - \frac{\mathrm{d}u(0)}{\mathrm{d}\tilde{t}} \right],$$

and

$$\tilde{F}(u, \varepsilon) = [L(\varepsilon) - L_0] u + F(u, \varepsilon).$$

By the continuous projection $\pi : BC \mapsto P$, $\pi(\phi + X_0 c) = \Phi[(\Psi, \phi) + \Psi(0) c]$, we can decompose the enlarged phase space by $\Lambda = \{\pm \mathrm{i}\omega_1 \tau_c, \pm \mathrm{i}\omega_2 \tau_c\}$ as $BC = P \oplus \mathrm{Ker}^\pi$, where $\mathrm{Ker}^\pi = \{\phi + X_0 c : \pi(\phi + X_0 c) = 0\}$, denoting the Kernel under the projection π. Let $\eta = (\eta_1, \overline{\eta}_1, \eta_2, \overline{\eta}_2)^{\mathrm{T}}$, $v_{\tilde{t}} \in Q^1 := Q \cap \mathrm{C}^1 \subset \mathrm{Ker}^\pi$, and A_{Q^1} the restriction of A as an operator from Q^1 to the Banach space Ker^π. Further, denote $u_{\tilde{t}} = \Phi \eta + v_{\tilde{t}}$. Then, Eq. (20) is decomposed as

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}\tilde{t}} = B\eta + \Psi(0) \tilde{F}(\Phi\eta + v_{\tilde{t}}, \varepsilon), \\ \dfrac{\mathrm{d}v_{\tilde{t}}}{\mathrm{d}\tilde{t}} = A_{Q^1} v_{\tilde{t}} + (\mathrm{I} - \pi) X_0 \tilde{F}(\Phi\eta + v_{\tilde{t}}, \varepsilon), \end{cases} \tag{21}$$

where $B = \mathrm{diag}\{\mathrm{i}\omega_1, -\mathrm{i}\omega_1, \mathrm{i}\omega_2, -\mathrm{i}\omega_2\}$.

Next, let M_2^1 denote the operator defined in $V_2^6(\mathrm{C}^4 \times \mathrm{Ker}^\pi)$, with

$$M_2^1 : V_2^6(\mathrm{C}^4) \mapsto V_2^6(\mathrm{C}^4),$$

$$(M_2^1 p)(\eta, \varepsilon) = D_\eta p(\eta, \varepsilon) B\eta - Bp(\eta, \varepsilon),$$

where $V_2^6(\mathrm{C}^4)$ represents the linear space of the second-order homogeneous polynomials in six variables $(\eta_1, \overline{\eta}_1, \eta_2, \overline{\eta}_2, \varepsilon)$ with coefficients in C^4. Then, it is easy to verify that one may choose the decomposition $V_2^6(\mathrm{C}^4) = \mathrm{Im}(M_2^1) \oplus \mathrm{Im}(M_2^1)^c$ with complementary space $\mathrm{Im}(M_2^1)^c$ spanned by the elements $A_\varepsilon\eta_1 e_1$, $\tau_\varepsilon\eta_1 e_1$, $A_\varepsilon\overline{\eta}_1 e_2$, $\tau_\varepsilon\overline{\eta}_1 e_2$, $A_\varepsilon\eta_2 e_3$, $\tau_\varepsilon\eta_2 e_3$, $A_\varepsilon\overline{\eta}_2 e_4$, $\tau_\varepsilon\overline{\eta}_2 e_4$, where e_i $(i = 1, 2, 3, 4)$ are unit vectors.

Consequently, the normal form of (19) on the center manifold associated with the equilibrium $(0, 0)$ near $A_\varepsilon = 0$, $\tau_\varepsilon = 0$ has the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tilde{t}} = B\eta + \frac{1}{2} g_2^1(\eta, 0, \varepsilon) + \text{h.o.t.},$$

where g_2^1 is the function giving the quadratic terms in (η, ε) for $v_{\tilde{t}} = 0$, and is determined by $g_2^1(\eta, 0, \varepsilon) = \mathrm{Proj}_{(\mathrm{Im}(M_2^1))^c} \times f_2^1(\eta, 0, \varepsilon)$, where $f_2^1(\eta, 0, \varepsilon)$ is the function giving the quadratic terms in (η, ε) for $v_{\tilde{t}} = 0$ defined by the first equation of (21). Thus, the normal form, truncated at the quadratic order terms, is given by

$$\begin{cases} \dfrac{\mathrm{d}\eta_1}{\mathrm{d}\tilde{t}} = \mathrm{i}\omega_1\tau_c\eta_1 - h_1 A_\varepsilon \tau_c (e^{-\mathrm{i}\omega_1\tau_c} - 1)\eta_1 \\ \qquad\qquad + h_1[\omega_1^2 + \omega_0^2 - A_c(e^{-\mathrm{i}\omega_1\tau_c} - 1)]\tau_\varepsilon\eta_1, \\ \dfrac{\mathrm{d}\eta_2}{\mathrm{d}\tilde{t}} = \mathrm{i}\omega_2\tau_c\eta_2 - h_2 A_\varepsilon \tau_c (e^{-\mathrm{i}\omega_2\tau_c} - 1)\eta_2 \\ \qquad\qquad + h_2[\omega_2^2 + \omega_0^2 - A_c(e^{-\mathrm{i}\omega_2\tau_c} - 1)]\tau_\varepsilon\eta_2, \end{cases} \tag{22}$$

where $h_j = (b - 2\mathrm{i}\omega_j - A_c\tau_c e^{-\mathrm{i}\omega_j\tau_c})^{-1} (j = 1, 2)$.

To find the normal form up to third order, similarly, let M_3^1 denote the operator defined in $V_3^4(\mathrm{C}^4 \times \mathrm{Ker}^\pi)$, with

$$M_3^1 : V_3^4(\mathrm{C}^4) \mapsto V_3^4(\mathrm{C}^4),$$
$$(M_3^1 p)(\eta, \varepsilon) = D_\eta p(\eta, \varepsilon)B\eta - Bp(\eta, \varepsilon),$$

where $V_3^4(\mathrm{C}^4)$ denotes the linear space of the third-order homogeneous polynomials in four variables $(\eta_1, \overline{\eta}_1, \eta_2, \overline{\eta}_2)$ with coefficients in C^4. Then, one may choose the decomposition $V_3^4(\mathrm{C}^4) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$ with complementary space $\mathrm{Im}(M_3^1)^c$ spanned by the elements $\eta_1^2\overline{\eta}_1 e_1$, $\eta_1\eta_2\overline{\eta}_2 e_1$, $\eta_1\overline{\eta}_1^2 e_2$, $\overline{\eta}_1\eta_2\overline{\eta}_2 e_2$, $\eta_2^2\overline{\eta}_2 e_3$, $\eta_1\overline{\eta}_1\eta_2 e_3$, $\eta_2\overline{\eta}_2^2 e_4$, $\eta_1\overline{\eta}_1\overline{\eta}_2 e_4$, where e_i $(i = 1, 2, 3, 4)$ are unit vectors.

Therefore, the normal form up to third-order terms is given by

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tilde{t}} = B\eta + \frac{1}{2!} g_2^1(\eta, 0, \varepsilon) + \frac{1}{3!} g_3^1(\eta, 0, \varepsilon) + \text{h.o.t.}, \tag{23}$$

where

$$\frac{1}{3!} g_3^1(\eta, 0, 0) = \frac{1}{3!}(I - P_{I,3}^1) f_3^1(\eta, 0, 0),$$

and $f_3^1(\eta, 0, 0)$ is the function giving the cubic terms in $(\eta, \varepsilon, v_{\tilde{t}})$ for $\varepsilon = 0$, and $v_{\tilde{t}} = 0$ is defined by the first equation of (21). Finally, the normal form on the center manifold arising from (21) becomes

$$\begin{cases} \dfrac{\mathrm{d}\eta_1}{\mathrm{d}\tilde{t}} = \mathrm{i}\omega_1\tau_c\eta_1 - h_1 A_\varepsilon \tau_c (e^{-\mathrm{i}\omega_1\tau_c} - 1)\eta_1 \\ \qquad\qquad + h_1[\omega_1^2 + \omega_0^2 - A_c(e^{-\mathrm{i}\omega_1\tau_c} - 1)]\tau_\varepsilon\eta_1 \\ \qquad\qquad + h_1\tau_c(\mathrm{i}\omega_1\gamma + 3\beta)\eta_1^2\overline{\eta}_1 \\ \qquad\qquad + 2h_1\tau_c(\mathrm{i}\omega_1\gamma + 3\beta)\eta_1\eta_2\overline{\eta}_2, \\ \dfrac{\mathrm{d}\eta_2}{\mathrm{d}\tilde{t}} = \mathrm{i}\omega_2\tau_c\eta_2 - h_2 A_\varepsilon \tau_c (e^{-\mathrm{i}\omega_2\tau_c} - 1)\eta_2 \\ \qquad\qquad + h_2[\omega_2^2 + \omega_0^2 - A_c(e^{-\mathrm{i}\omega_2\tau_c} - 1)]\tau_\varepsilon\eta_2 \\ \qquad\qquad + h_2\tau_c(\mathrm{i}\omega_2\gamma + 3\beta)\eta_2^2\overline{\eta}_2 \\ \qquad\qquad + 2h_2\tau_c(\mathrm{i}\omega_2\gamma + 3\beta)\eta_1\overline{\eta}_1\eta_2, \end{cases} \tag{24}$$

where $h_j = (b - 2\mathrm{i}\omega_j - A_c\tau_c e^{-\mathrm{i}\omega_j\tau_c})^{-1}$, $j = 1, 2$.

With the polar coordinates: $\eta_1 = \frac{R_1}{2} e^{\mathrm{i}\Theta_1}$, $\eta_2 = \frac{R_2}{2} e^{\mathrm{i}\Theta_2}$, combining with (3), we finally obtain the amplitude and phase equations of (24) on the center manifold as

$$\begin{cases} \dfrac{\mathrm{d}R_1}{\mathrm{d}t} = \mu_1 R_1 + P_{11} R_1^3 + P_{12} R_1 R_2^2, \\ \dfrac{\mathrm{d}R_2}{\mathrm{d}t} = \mu_2 R_2 + P_{21} R_1^2 R_2 + P_{22} R_2^3, \\ \dfrac{\mathrm{d}\Theta_1}{\mathrm{d}t} = \omega_1 + \delta_1 + Q_{11} R_1^2 + 2Q_{11} R_2^2, \\ \dfrac{\mathrm{d}\Theta_2}{\mathrm{d}t} = \omega_2 + \delta_2 + 2Q_{12} R_1^2 + Q_{12} R_2^2, \end{cases} \tag{25}$$

where

$$\mu_i = -\frac{[a_i\cos(\omega_i\tau_c) + b_i\sin(\omega_i\tau_c)]A_c\omega_i\tau_\varepsilon}{(a_i^2 + b_i^2)\tau_c} + \frac{[b_i\cos(\omega_i\tau_c) - b_i - a_i\sin(\omega_i\tau_c)]A_\varepsilon}{a_i^2 + b_i^2},$$

$$\delta_i = \frac{[a_i(\omega_i^2 + \omega_0^2 + A_c - A_c\cos(\omega_i\tau_c)) - b_i A_c \sin(\omega_i\tau_c)]\tau_\varepsilon}{(a_i^2+b_i^2)\tau_c} - \frac{[a_i(\cos(\omega_i\tau_c)-1) + b_i\sin(\omega_i\tau_c)]A_\varepsilon}{a_i^2+b_i^2},$$

$$P_{ii} = -\frac{3\beta b_i + \omega_i\gamma a_i}{4(a_i^2+b_i^2)}, \quad P_{12} = 2P_{11}, \quad P_{21} = 2P_{22}, \quad Q_{1i} = \frac{3a_i\beta - b_i\omega_i\gamma}{4(a_i^2+b_i^2)},$$

with $a_i = 2\omega_i - A_c\tau_c\sin(\omega_i\tau_c)$, $b_i = A_c\tau_c\cos(\omega_i\tau_c) - b, i = 1, 2$.

Remark. In the MTS method, we assume that the solution is in a real form, while in the CMR method, we consider the solution to be in a complex form. In fact, the solution to Eq. (7) can be expressed in either a complex form or a real form, that is,

$$\begin{aligned} x_1 &= K_1 e^{i\omega_1 T_0} + \overline{K}_1 e^{-i\omega_1 T_0} \\ &\quad + K_2 e^{i\omega_2 T_0} + \overline{K}_2 e^{-i\omega_2 T_0} \\ &= G_1(T_1,T_2)\sin(\omega_1 T_0) \\ &\quad + G_2(T_1,T_2)\cos(\omega_1 T_0) \\ &\quad + G_3(T_1,T_2)\sin(\omega_2 T_0) \\ &\quad + G_4(T_1,T_2)\cos(\omega_2 T_0). \end{aligned}$$

These two different form solutions become identical under the polar coordinate transformations: $K_1 = \frac{R_1}{2}e^{i\Theta_1}$, $K_2 = \frac{R_2}{2}e^{i\Theta_2}$ and $G_1 = R_1\sin(\Theta_1)$, $G_2 = R_1\cos(\Theta_1)$, $G_3 = R_2\sin(\Theta_2)$, $G_4 = R_2\cos(\Theta_2)$. Therefore, in the CMR method, we use the polar coordinates: $\eta_1 = \frac{R_1}{2}e^{i\Theta_1}$, $\eta_2 = \frac{R_2}{2}e^{i\Theta_2}$ in order to keep consistence.

2.4. *Comparison of the MTS and CMR methods*

Equation (16) is the normal form derived by using the MTS method, and Eq. (25) is the normal form derived by using the CMR method. Comparing the two normal forms, we have found that Eq. (16) has four more terms $Q_i\tau_\epsilon^2 R_i$, $H_i\tau_\epsilon^2$ $(i = 1, 2)$ than Eq. (25) does, which come from the Taylor expansions of $x_j(T_0 - \tau_c - \epsilon\tau_\epsilon, T_1 - \epsilon(\tau_c + \epsilon\tau_\epsilon), T_2 - \epsilon^2(\tau_c + \epsilon\tau_\epsilon))$ at $x_j(T_0 - \tau_c, T_1, T_2)$ for $j = 1, 2, 3, \ldots$. In fact, in the CMR method, we treat $x(t - \tau)$ as a linear term of t, and thus no higher-order terms appear, while in the MTS method, we expand $x(t - \tau)$ in $T_1, T_2, \ldots$ as $x_j(T_0 - \tau_c - \epsilon\tau_\epsilon, T_1 - \epsilon(\tau_c + \epsilon\tau_\epsilon), T_2 - \epsilon^2(\tau_c + \epsilon\tau_\epsilon))$ at $x_j(T_0 - \tau_c, T_1, T_2)$ up to infinite orders, which results in

$$\begin{aligned} &\frac{(\epsilon\tau_\epsilon)^2}{2!} - \frac{(\epsilon\tau_\epsilon)^4}{4!} + \frac{(\epsilon\tau_\epsilon)^6}{6!} - \frac{(\epsilon\tau_\epsilon)^2}{8!} + \cdots \\ &\quad = \cos(\epsilon\tau_\epsilon) - 1 \approx 0, \end{aligned}$$

$$\begin{aligned} &-\epsilon\tau_\epsilon + \frac{(\epsilon\tau_\epsilon)^3}{3!} - \frac{(\epsilon\tau_\epsilon)^5}{5!} + \frac{(\epsilon\tau_\epsilon)^7}{7!} + \cdots \\ &\quad = -\sin(\epsilon\tau_\epsilon) \approx 0, \end{aligned}$$

but we only take terms up to ϵ^2-order in the MTS method. That is why Eq. (16) has additional $Q_i\tau_\epsilon^2 R_i$ and $H_i\tau_\epsilon^2$ $(i = 1, 2)$ terms. Or simply, in the MTS method, if we do not expand $x(t - \tau)$ in $T_1, T_2, \ldots$ (which would be the same as the CMR method), then these two normal forms are identical. Nevertheless, the small difference has very little effect on the dynamical analysis.

3. Bifurcation Analysis and Numerical Simulation

In this section, we first give a bifurcation analysis based on the normal form (25), and then present some numerical simulation results.

3.1. *Bifurcation analysis*

For the normal form (25), according to the signs of $P_{11}P_{22}$, there exist two different cases, i.e. "simple case" (with no periodic solutions) and "difficult case" (with periodic solutions) [Guckenheimer & Holmes, 1990]. Here, we are interested in the "difficult case", i.e. when $P_{11}P_{22} < 0$. Without loss of generality, we assume $P_{11} > 0$ and $P_{22} < 0$. Let $r_j = \sqrt{|P_{jj}|}R_j^2, (j = 1, 2)$. Then, we have the following planar system in terms of r_1 and r_2:

$$\begin{cases} \dot{r}_1 = r_1(\mu_1 + r_1 - \kappa r_2), \\ \dot{r}_2 = r_2(\mu_2 + \chi r_1 - r_2), \end{cases} \tag{26}$$

where $\kappa = \frac{P_{12}}{P_{22}}$, $\chi = \frac{P_{21}}{P_{11}}$.

Note that $M_0 = (0, 0)$ is always an equilibrium of (26). The two semi-trivial equilibria are given in terms of perturbation parameters as $M_1 = (-\mu_1, 0)$ and $M_2 = (0, \mu_2)$, which bifurcate from the origin on the bifurcation lines $L_1 : \mu_1 = 0$ and $L_2 : \mu_2 = 0$, respectively. There may also exist a nontrivial equilibrium $M_3 = (\frac{\mu_1 - \kappa\mu_2}{\kappa\chi - 1}, \frac{\mu_2 - \chi\mu_1}{1 - \kappa\chi})$. For this equilibrium to exist, it needs $\kappa\chi - 1 \neq 0$. The nontrivial equilibrium M_3 collides with a semi-trivial one on the bifurcation line $T_1 : \mu_1 - \kappa\mu_2 = 0$ or $T_2 : \mu_2 - \chi\mu_1 = 0$. If $(1-\chi)\mu_1 + (1-\kappa)\mu_2 < 0$, the fixed point M_3 is a sink, otherwise M_3 is a source. Therefore, we further need to consider the bifurcation line $T_3 : (1-\chi)\mu_1 + (1-\kappa)\mu_2 = 0$.

In order to give a more clear bifurcation picture, choose $A_c = 0.3519$, $\tau_c = 6.7583$, $b = 0.2774$, $\beta = 0.3$, $\gamma = 2$ and $\frac{\omega_1}{\omega_2} = \frac{\sqrt{5}}{2}$. Then, the characteristic equation (2) has two pairs of purely imaginary eigenvalues $\Lambda = \{\pm i\omega_1, \pm i\omega_2\}$, and all the other eigenvalues have negative real part. Assume that system (1) undergoes a double Hopf bifurcation from the equilibrium $(0,0)$. By a simple calculation, we obtain

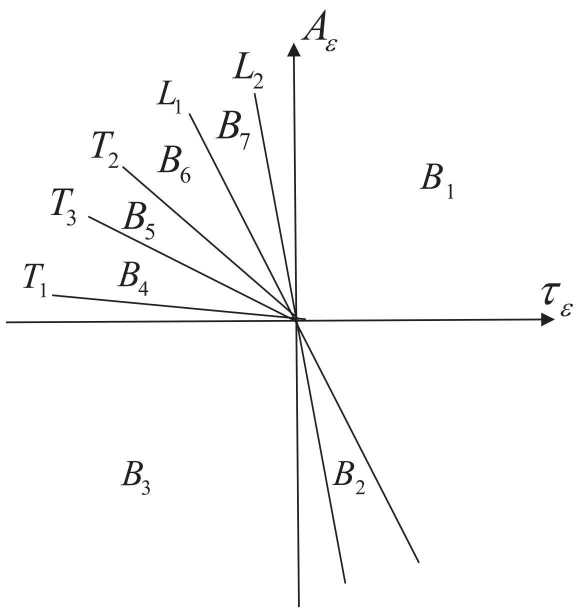
$$\begin{aligned} &\omega_1 = 1.2080, \quad \tau_1^{(0)} = 1.5572, \\ &\tau_1^{(1)} = 6.7583, \quad \mathrm{Re}'(\lambda_1) > 0, \\ &\omega_2 = 1.0805, \quad \tau_2^{(0)} = 0.9433, \\ &\tau_2^{(1)} = 6.7583, \quad \mathrm{Re}'(\lambda_2) < 0, \end{aligned} \tag{27}$$

and $\mu_1 = 0.4137\tau_\varepsilon + 1.1324A_\varepsilon$, $\mu_2 = -0.3560\tau_\varepsilon - 0.6023A_\varepsilon$, $P_{11} = 0.5224$, $P_{12} = 1.0449$, $P_{21} = -2.4338$, $P_{22} = -1.2169$, $\kappa = -0.8587$ and $\chi = -4.6585$.

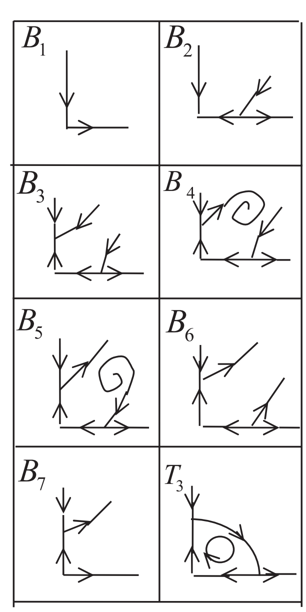
Therefore, the critical bifurcation lines become: $L_1 : A_\varepsilon = -0.3653\tau_\varepsilon$, $L_2 : A_\varepsilon = -0.5911\tau_\varepsilon$, $T_1 : A_\varepsilon = -0.1756\tau_\varepsilon$, $T_2 : A_\varepsilon = -0.3362\tau_\varepsilon$, $T_3 : A_\varepsilon = -0.3176\tau_\varepsilon$, as shown in the bifurcation diagram (see Fig. 1).

Since there does not exist unstable manifold containing the equilibrium, according to the center manifold theory, the solutions on the center manifold determine the asymptotic behavior of the solutions of the original system (1). Therefore, if Eq. (26) has one or two asymptotically stable (unstable) semi-trivial equilibria M_1 and M_2, then (1) has one or two asymptotically stable (unstable) periodic solutions in the neighborhood of the trivial equilibrium. If Eq. (26) has an asymptotically stable (unstable) equilibrium M_3, then (1) has an asymptotically stable (unstable) quasi-periodic solution in the neighborhood of $(0,0)$. So, we shall call the periodic solution the source (respectively, saddle, sink) periodic solution of (1) if the semi-trivial equilibrium of (26) is a source (respectively, saddle, sink), and call the quasi-periodic solution the source (respectively, saddle, sink) quasi-periodic solution of (1) when the nontrivial equilibrium of (26) is a source (respectively, saddle, sink).

For the bifurcation behaviors of the original system (1) in the neighborhood of the trivial equilibrium, the above critical bifurcation boundaries divide the $(A_\varepsilon, \tau_\varepsilon)$ parameter plane into seven regions (see Fig. 1). We explain the bifurcations in the clockwise direction, starting from B_1 and ending at B_1. First, in region B_1, there is only one trivial equilibrium which is a saddle. When the parameters are varied across the line L_1 from region B_1



(a)



(b)

Fig. 1. (a) Critical bifurcation lines in the $(\tau_\varepsilon, A_\varepsilon)$ parameter space near (τ_c, A_c); (b) the corresponding phase portraits in the (r_1, r_2) plane.

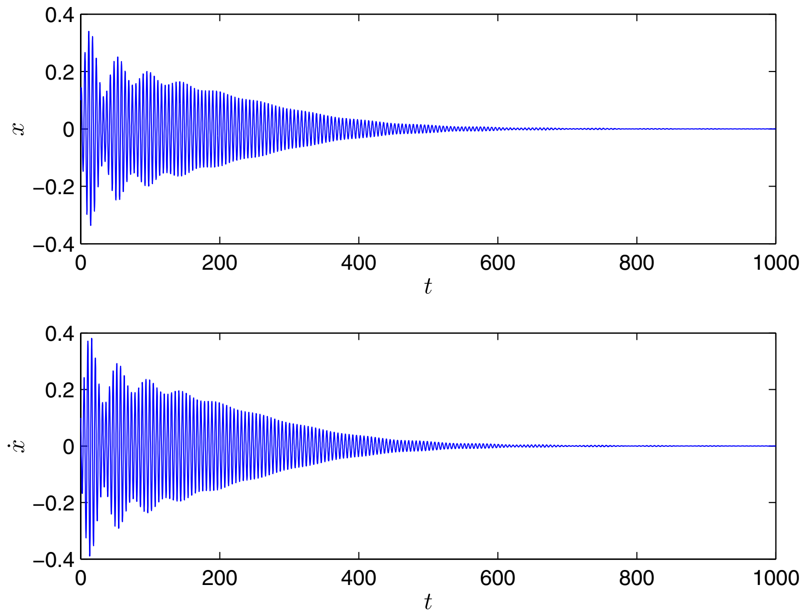


Fig. 2. Simulated solution of system (1) for $(A_\varepsilon, \tau_\varepsilon) = (-0.025, 0.05)$, showing a stable fixed point.

to B_2, the trivial equilibrium becomes a sink, and an unstable periodic solution O_1 (saddle) appears from the trivial solution due to a Hopf bifurcation. Similarly, when the parameters are changed from region B_2 to B_3, another periodic solution O_2 (sink) occurs from the trivial solution due to a Hopf bifurcation while the trivial equilibrium becomes a saddle. In region B_4, a quasi-periodic solution (sink) occurs from the periodic solution O_2 due to a Neimark–Sacker bifurcation, and O_2 is changed to a saddle from a sink. Further, the quasi-periodic solution (sink) becomes a source when the parameters are varied across line T_3 from region B_4 to B_5, and when the parameters are further changed from region B_5 to B_6 crossing the line T_2, the quasi-periodic solution collides with the periodic solution O_1 and then disappears, and O_1 becomes a source. When the parameters are further varied across line L_1 from region B_6 to B_7, the periodic solution O_1 collides with the trivial solution and then disappears, and the trivial solution becomes a source from a saddle. Finally, when the parameters are varied across line L_2 from region B_7 to B_1, the periodic solution O_2 (saddle) collides with the trivial solution and then disappears, and the trivial solution becomes a saddle from a source.

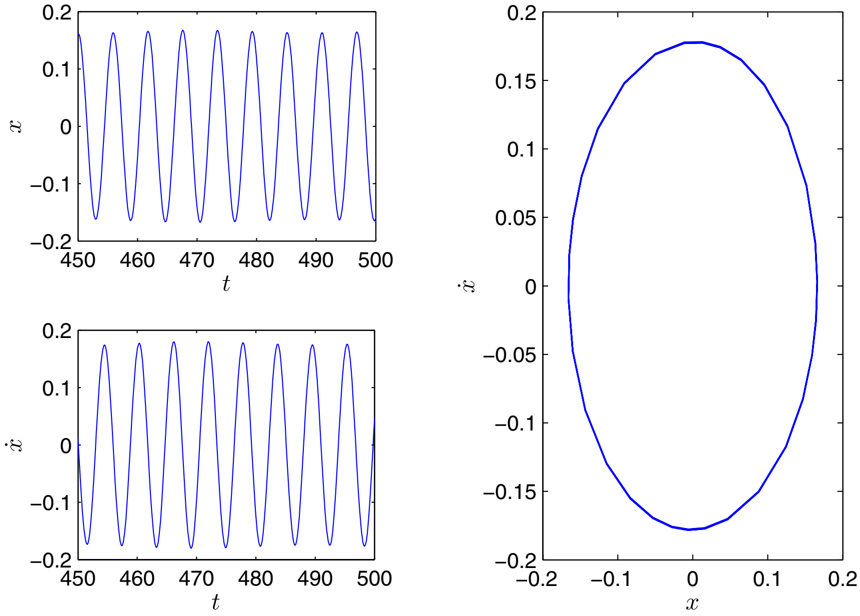
3.2. *Numerical simulation*

To demonstrate the analytic results obtained in Sec. 3.1, here we present some numerical simulation results. We choose three groups of perturbation parameter values: $(A_\varepsilon, \tau_\varepsilon) = (-0.025, 0.05)$, $(-0.01, 0.01)$ and $(0.015, -0.05)$, belonging to the regions B_2, B_3 and B_4, corresponding to a stable fixed point shown in Fig. 2, a stable periodic solution as depicted in Fig. 3, and a stable quasi-periodic solution, see Fig. 4 respectively. It is clear that the numerical simulations agree with the analytical predictions.

4. Conclusion and Discussion

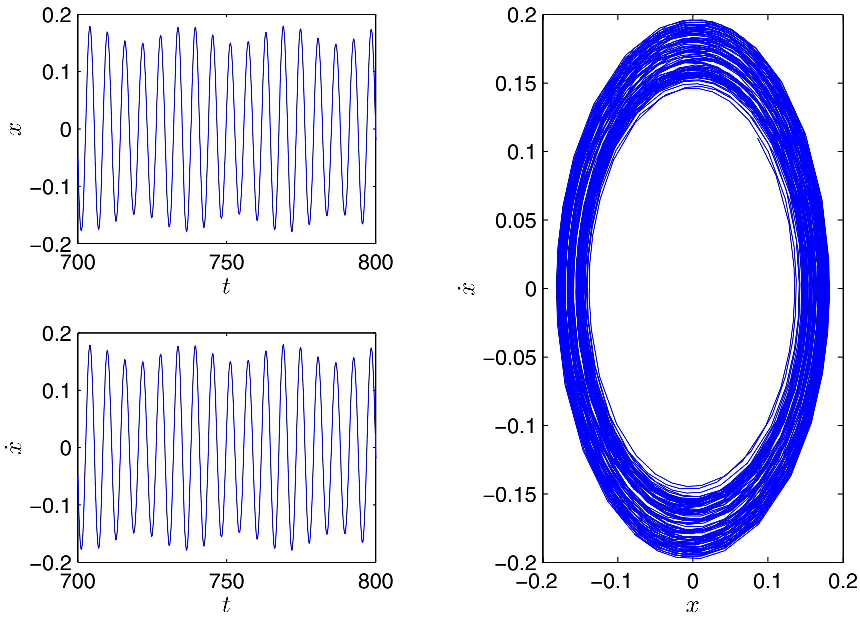
In this paper, we have discussed double Hopf bifurcation in delayed van der Pol–Duffing equation. We derived the normal form of double Hopf bifurcation by using multiple time scales and center manifold reduction methods. A comparison between the two methods shows that the two normal forms are identical if the higher-order terms obtained in the MTS method, due to the expansion in the delayed variable with respect to time scales, are ignored. Moreover, bifurcation analysis near the double Hopf critical point is given, showing that the system may exhibit a stable fixed point, periodic solutions, and quasi-periodic solutions in the neighborhood of the critical point. Numerical simulations are given to verify the analytical predictions.

The normal form method and bifurcation analysis presented in this paper are for local dynamical behaviors. But, surprisingly, we have found that even for parameter values not chosen in the



(a) (b)

Fig. 3. Simulated solution of system (1) for $(A_\varepsilon, \tau_\varepsilon) = (-0.01, -0.01)$: (a) the time history; (b) the phase portrait, showing a stable periodic solution.



(a) (b)

Fig. 4. Simulated solution of system (1) for $(A_\varepsilon, \tau_\varepsilon) = (0.015, -0.05)$: (a) the time history; (b) the phase portrait, showing a stable quasi-periodic solution.

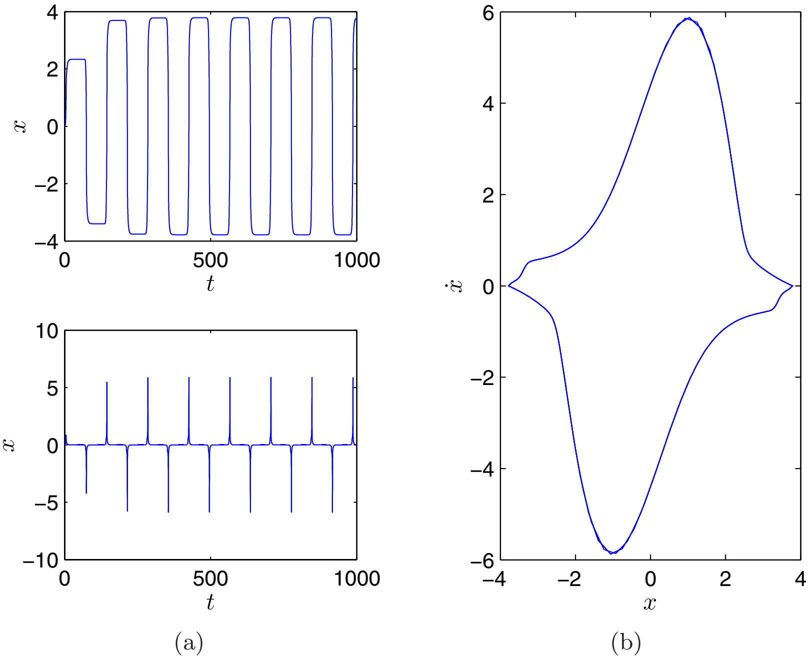


Fig. 5. Simulated solution of system (1) for $(A_\varepsilon, \tau_\varepsilon) = (-3, 60)$: (a) the time history; (b) the phase portrait, showing a stable periodic solution.

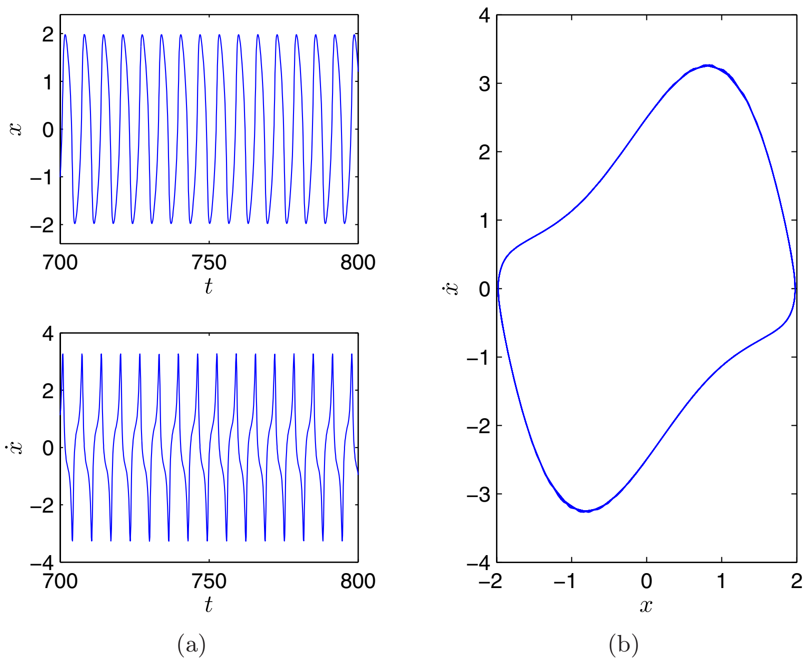


Fig. 6. Simulated solution of system (1) for $(A_\varepsilon, \tau_\varepsilon) = (-3, -6)$: (a) the time history; (b) the phase portrait, showing a stable periodic solution.

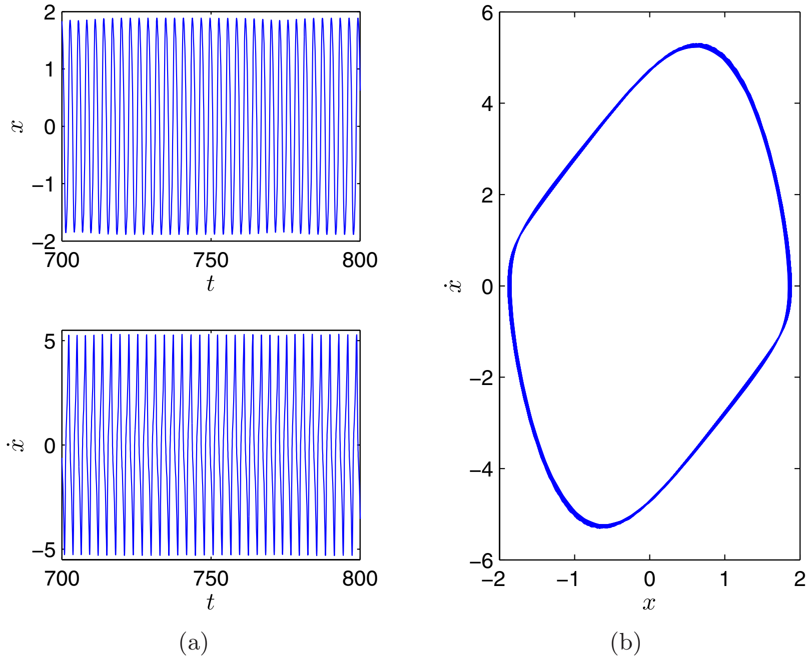


Fig. 7. Simulated solution of system (1) for $(A_\varepsilon, \tau_\varepsilon) = (3, 60)$: (a) the time history; (b) the phase portrait, showing a stable quasi-periodic solution.

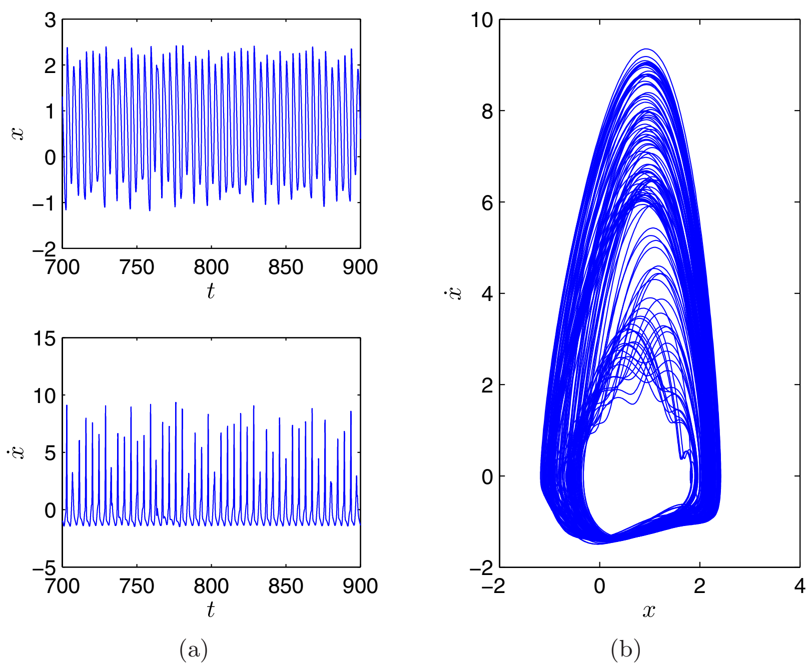


Fig. 8. Simulated solution of system (1) with a nonlinear delayed feedback: (a) the time history; (b) the phase portrait, showing a chaotic attractor.

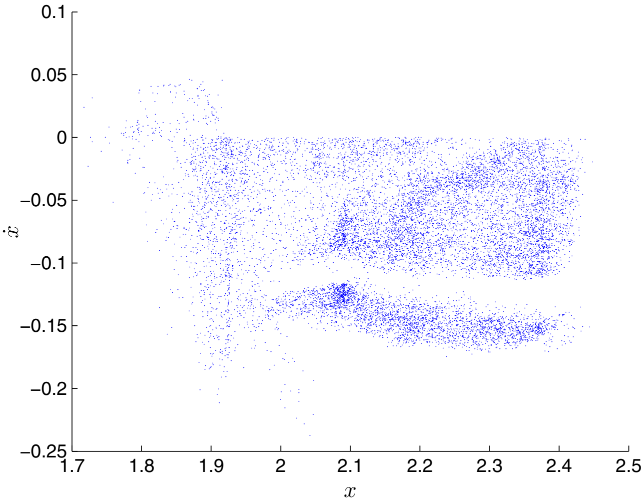


Fig. 9. The simulated Poincaré map of the chaotic attractor shown in Fig. 8(b) with Poincaré section $x(t-\tau)=0$.

neighborhood of the critical point, most simulation results are periodic or quasi-periodic solutions (see Figs. 5–7). This seemingly noncomplex dynamics is perhaps due to system (1) only containing time delay in a (feedback) linear term. If we introduce time delay in nonlinear terms, we may obtain more complicated dynamical behavior, such as chaos. Indeed, for example, when we change the linear delay feedback $A[x(t-\tau)-x(t)]$ to a nonlinear delay feedback $Ax(t-\tau)[x(t-\tau)-x(t)]$, and select $A=5, \tau=4, \omega_0=1, b=0.2774, \beta=0.3$ and $\gamma=2$, we obtain chaotic motion, as shown in Figs. 8 and 9.

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