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Hopf-zero bifurcation in a generalized Gopalsamy neural network model

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Abstract In this paper, we study Hopf-zero bifurcation in a generalized Gopalsamy neural network model. By using multiple time scales and center manifold reduction methods, we obtain the normal forms near a Hopf-zero critical point. A comparison between these two methods shows that the two normal forms are equivalent. Moreover, bifurcations are classified in two-dimensional parameter space near the critical point, and numerical simulations are presented to demonstrate the applicability of the theoretical results.

Keywords Neural network model · Hopf-zero bifurcation · Normal form · Multiple time scales · Center manifold reduction

1 Introduction

Over the past two decades, there has been an increasing interest in the study of neuron systems, for example, in mathematical modeling of neural networks and artificial representations. Neural networks have many applications, such as pattern recognition, associative memory, and combinatorial optimization [1, 2]. Such applications heavily depend on the dynamical behaviors of neural network models. Thus, analysis of the dynamical behavior is necessary for a practical neural network model. However, since the signal transmission from one neuron to another is not instantaneous, time delay should be incorporated into the neural network models in order for the analysis to be more realistic. Therefore, recent attention has been focused on the study of dynamics of neural network models with delay.

There have been many kinds of neural network models, such as Hopfield neural network model [3, 4], cellular neural network model [5, 6], bi-directional associative memory neural network model [7–9], Baldi and Atiya neural network model [10, 11], Cohen–Grossberg neural network model [12, 13], and Gopalsamy neural network model [14, 15]. For these neural network models, the main focus is on the local stability of fixed points, the existence and stability of periodic solutions (e.g. see [16–20]). On one hand, these fixed points are locally stable, and we hope to find conditions under which a fixed point can be globally stable. Several papers are devoted to study the global asymptotic stability of equilibrium solutions (e.g. see [21–25]). On the other hand, periodic solutions only exist in small neighborhoods of the critical points. We wonder whether these periodic solutions can continue to exist for a large range of parameter values. It is also an important mathematical subject to investigate

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if these non-trivial periodic solutions exist globally. A number of works are focused on global existence of periodic solutions (e.g. see [26–30]). In particular, Campbell et al. [31] and Yuan [32] analyzed the bifurcation and stability of non-trivial asynchronous oscillations from the trivial solution in neural network models using equivariant bifurcation theory and center manifold construction. Some neural network models with high co-dimensional bifurcations have also been considered. For example, Yan [33] studied Hopf-pitchfork bifurcation in a simplified tri-neuron BAM network model with multiple delays. Campbell and Yuan [34] obtained the conditions under which a general class of delay differential equations has a critical point of Bogdanov–Takens or a triple zero bifurcation. Guo et al. [35] discussed Hopf-fold and double Hopf bifurcations in a network of two neurons with two delays.

In this paper, we consider the following generalized Gopalsamy neural network model [36]:

$$\begin{cases} \dot{x_1}(t) = -x_1(t) + a_1 \tanh\bigl[x_3(t) - bx_3(t-\tau)\bigr], \\ \dot{x_2}(t) = -x_2(t) + a_2 \tanh\bigl[x_1(t) - bx_1(t-\tau)\bigr], \\ \dot{x_3}(t) = -x_3(t) + a_3 \tanh\bigl[x_2(t) - bx_2(t-\tau)\bigr], \end{cases} \quad (1)$$

where a_i $(i = 1, 2, 3)$ corresponds to the range of the continuous variable x_i, $b > 0$ is the measure of the inhibitory influence of the past history, and $\tau > 0$ is time delay. In biological fields, such a feedback is known as reverberation, while in the field of artificial neural network, it is known as an excitation or inhibition from other neurons. Gopalsamy and Leung [14] proposed a one-dimensional neuron model (called the Gopalsamy neural network model) for which Lyapunov functions are constructed to show that the trivial equilibrium is globally asymptotically stable for all the values of time delay. Liao et al. [37] studied the stability switches and bifurcation of a two-neuron Gopalsamy neural network system with a distributed time delay. Liao et al. [36] also derived sufficient delay-dependent criteria to ensure global asymptotical stability of the trivial equilibrium of (1), and obtained the stability and direction of bifurcating periodic solutions by using normal form theory and center manifold theorem.

Multiple time scales (MTS) [38, 39] and center manifold reduction (CMR) [40, 41] are two methods for computing normal forms of differential equations. The multiple time scales method is systematic and can be directly applied to the original nonlinear dynamical system, which is described by not only ordinary differential equations (ODEs) but also delayed differential equations (DDEs), without application of the center manifold theory. In fact, this approach combines the two steps involved in using the center manifold theory and normal form theory into one unified step to obtain the normal form and nonlinear transformation simultaneously [42, 43]. Based on MTS, Yu [44–46] developed Maple programs for computing the normal forms of Hopf bifurcation and other singularities for a given general n-dimensional ODE. Moreover, the MTS method only contains algebraic manipulations, which greatly facilitates computer implement in symbolic computations. For a given ODE, the basic idea of the center manifold theory is applying successive coordinate transformations to systematically construct a simpler system which has less dimension compared to the original system, and thus greatly simplifies the dynamical analysis of the system. For DDE systems, however, one needs to first change the retarded equations to operator differential equations, and then decompose the solution space of their linearized form into stable and center manifolds. Next, with adjoint operator equations, one computes the center manifold by projecting the whole space to the center manifold, and finally calculates the normal form restricted to the center manifold (e.g. see [47–49]). Nayfeh [43] used these two approaches to derive the identical normal forms of Hopf bifurcations in DDEs. Ding et al. [50] applied the two methods to obtain the normal forms near a double Hopf critical point in a delayed equation, and also showed the equivalence of the two normal forms.

In this paper, we will derive the normal form of Hopf-zero bifurcation of (1) by using the MTS and CMR methods. The multiple time scales method is the first time to be used to consider the Hopf-zero bifurcation in delay differential equations, and further shows that the multiple time scales is simpler than the center manifold reduction, though the results from the two methods are equivalent. Furthermore, we will carry out bifurcation analysis and numerical simulations, showing that there exist a stable fixed point, a pair of stable fixed points, a stable periodic solution, and co-existence of a pair of stable fixed points and a stable periodic solution in the neighborhood of the Hopf-zero critical point.

The rest of the paper is organized as follows. In Sect. 2, we consider existence of Hopf-zero bifurcation in a generalized Gopalsamy neural network model with delay, and use two methods to derive the normal forms associated with the Hopf-zero critical point. Then bifurcation analysis and numerical simulations are presented in Sect. 3, and finally, the conclusion is drawn in Sect. 4.

2 Bifurcation analysis and normal form method

In this section, we consider the generalized Gopalsamy neural network model with delay, described by (1), and use the MTS and CMR methods to derive the normal form of the system associated with Hopf-zero bifurcation. With the transformation

$$\begin{cases} y_1(t) = x_1(t) - bx_1(t-\tau), \\ y_2(t) = x_2(t) - bx_2(t-\tau), \\ y_3(t) = x_3(t) - bx_3(t-\tau), \end{cases} \tag{2}$$

system (1) can be transformed to the following equations:

$$\begin{cases} \dot{y}_1(t) = -y_1(t) + a_1 \tanh\big[y_3(t)\big] \\ \qquad\qquad - a_1 b \tanh\big[y_3(t-\tau)\big], \\ \dot{y}_2(t) = -y_2(t) + a_2 \tanh\big[y_1(t)\big] \\ \qquad\qquad - a_2 b \tanh\big[y_1(t-\tau)\big], \\ \dot{y}_3(t) = -y_3(t) + a_3 \tanh\big[y_2(t)\big] \\ \qquad\qquad - a_3 b \tanh\big[y_2(t-\tau)\big]. \end{cases} \tag{3}$$

2.1 System formulation

The characteristic equation of (3), evaluated at the trivial equilibrium $(y_1, y_2, y_3) = (0, 0, 0)$, is given by

$$\begin{aligned} &(\lambda+1)^3 - a_1 a_2 a_3 \big(1 - b\mathrm{e}^{-\lambda\tau}\big)^3 \\ &\quad = \Lambda_1(\lambda)\Lambda_2(\lambda)\Lambda_3(\lambda) = 0, \end{aligned} \tag{4}$$

where

$$\Lambda_1(\lambda) = \lambda + 1 - a\big(1 - b\mathrm{e}^{-\lambda\tau}\big),$$

$$\Lambda_2(\lambda) = \lambda + 1 + \frac{1}{2}a\big(1 - b\mathrm{e}^{-\lambda\tau}\big) - \frac{\sqrt{3}a(1 - b\mathrm{e}^{-\lambda\tau})\mathrm{i}}{2},$$

$$\Lambda_3(\lambda) = \lambda + 1 + \frac{1}{2}a\big(1 - b\mathrm{e}^{-\lambda\tau}\big) + \frac{\sqrt{3}a(1 - b\mathrm{e}^{-\lambda\tau})\mathrm{i}}{2},$$

with $a = \sqrt[3]{a_1 a_2 a_3}$.

There are several types of bifurcations.

Case 1. Fixed-point bifurcation.

Substituting $\lambda = 0$ into (4), we obtain $a(1-b) = 1$, under which the three roots of (4) with $\tau = 0$ are $\lambda_1 = 0$ and $\lambda_{2,3} = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}$, and system (3) undergoes a fixed- point bifurcation.

Case 2. Hopf bifurcation.

To find possible periodic solutions, which may bifurcate from a Hopf critical point, let $\lambda = \mathrm{i}\omega_1$ ($\mathrm{i}^2 = -1$, $\omega_1 > 0$) be a root of $\Lambda_1(\lambda) = 0$. Substituting the root into $\Lambda_1(\lambda) = 0$, and separating the real and imaginary parts yields

$$\begin{cases} a - 1 = ab\cos(\omega_1\tau), \\ \omega_1 = ab\sin(\omega_1\tau), \end{cases} \tag{5}$$

from which we obtain

$$\omega_1 = \sqrt{a^2b^2 - (a-1)^2}, \tag{6}$$

under the assumption:

$$a^2b^2 > (a-1)^2. \tag{7}$$

Further, the time delay τ can be determined from (5) as

$$\tau_1^{(j)} = \begin{cases} \frac{1}{\omega_1}\big[\arccos(\frac{a-1}{ab}) + 2j\pi\big], & \text{for } ab > 0, \\ \frac{1}{\omega_1}\big[2(j+1)\pi - \arccos(\frac{a-1}{ab})\big], & \text{for } ab < 0, \end{cases} \tag{8}$$

where $j = 0, 1, 2, \ldots$.

Let $\lambda(\tau) = \alpha(\tau) + \mathrm{i}\omega(\tau)$ be the root of $\Lambda_1(\lambda) = 0$, satisfying $\alpha(\tau_1^{(j)}) = 0$ and $\omega(\tau_1^{(j)}) = \omega_1$ ($j = 0, 1, 2, \ldots$). Then we have the transversality conditions are

$$\alpha'\big(\tau_1^{(j)}\big)^{-1} = \frac{1}{a^2b^2} > 0, \tag{9}$$

where $j = 0, 1, 2, \ldots$.

Next, let $\lambda = \mathrm{i}\omega_2$ ($\omega_2 > 0$) be a root of $\Lambda_2(\lambda) = 0$. Similarly, substituting it into $\Lambda_2(\lambda) = 0$ and separating the real and imaginary parts yields

$$\begin{cases} 1+\dfrac{a}{2}=ab\cos\left(\omega_2\tau+\dfrac{\pi}{3}\right), \\ \omega_2-\dfrac{\sqrt{3}a}{2}=-ab\sin\left(\omega_2\tau+\dfrac{\pi}{3}\right), \end{cases} \tag{10}$$

from which we obtain

$$\begin{cases} \cos(\omega_2\tau)=\dfrac{1}{b}+\dfrac{1-\sqrt{3}\omega_2}{2ab}, \\ \sin(\omega_2\tau)=-\dfrac{\omega_2+\sqrt{3}}{2ab}, \end{cases} \tag{11}$$

and

$$\omega_2^2-\sqrt{3}a\omega_2+a^2+a+1-a^2b^2=0.$$

When $a^2+a+1-a^2b^2<0$, the above quadratic polynomial equation has a positive root,

$$\omega_2^{(1)}=\frac{\sqrt{3}a+\sqrt{4a^2b^2-a^2-4a-4}}{2}. \tag{12}$$

When $a^2+a+1-a^2b^2>0$, $4a^2b^2-a^2-4a-4>0$ and $a>0$, it has two positive roots,

$$\omega_2^{(2,3)}=\frac{\sqrt{3}a\pm\sqrt{4a^2b^2-a^2-4a-4}}{2}. \tag{13}$$

Further, it follows from (11) that

$$\tau_{2,k}^{(j)}=\begin{cases} \frac{1}{\omega_2^{(k)}}[\arccos(\frac{1}{b}+\frac{1-\sqrt{3}\omega_2^{(k)}}{2ab})+2j\pi], \\ \quad \text{for } \frac{\omega_2^{(k)}+\sqrt{3}}{2ab}\le 0, \\ \frac{1}{\omega_2^{(k)}}[2(j+1)\pi-\arccos(\frac{1}{b}+\frac{1-\sqrt{3}\omega_2^{(k)}}{2ab})], \\ \quad \text{for } \frac{\omega_2^{(k)}+\sqrt{3}}{2ab}>0, \end{cases} \tag{14}$$

where $k=1,2,3$; $j=0,1,2,\ldots$.

Let $\lambda(\tau)=\alpha(\tau)+\mathrm{i}\omega(\tau)$ be the root of $\Lambda_2(\lambda)=0$, satisfying $\alpha(\tau_{2,k}^{(j)})=0$, and $\omega(\tau_{2,k}^{(j)})=\omega_2^{(k)}$ ($j=0,1,2,\ldots$; $k=1,2,3$). Then we have

$$\alpha'\left(\tau_{2,k}^{(j)}\right)^{-1}=\frac{2\omega_2^{(k)}-\sqrt{3}a}{2a^2b^2\omega_2^{(k)}},$$

where $k=1,2,3$; $j=0,1,2,\ldots$. Thus, the transversality conditions are

$$\alpha'\left(\tau_{2,1}^{(j)}\right)>0,\quad \alpha'\left(\tau_{2,2}^{(j)}\right)>0,\quad \alpha'\left(\tau_{2,3}^{(j)}\right)<0, \tag{15}$$

where $j=0,1,2,\ldots$.

Finally, let $\lambda=\mathrm{i}\omega_3$ ($\omega_3>0$) be a root of $\Lambda_3(\lambda)=0$. Substituting it into $\Lambda_3(\lambda)=0$ and separating the real and imaginary parts results in

$$\begin{cases} 1+\dfrac{a}{2}=ab\cos\left(\omega_3\tau-\dfrac{\pi}{3}\right), \\ \omega_3-\dfrac{\sqrt{3}a}{2}=-ab\sin\left(\omega_3\tau-\dfrac{\pi}{3}\right), \end{cases} \tag{16}$$

from which we obtain

$$\begin{cases} \cos(\omega_3\tau)=\dfrac{1}{b}+\dfrac{1+\sqrt{3}\omega_3}{2ab}, \\ \sin(\omega_3\tau)=-\dfrac{\sqrt{3}-\omega_3}{2ab}, \end{cases} \tag{17}$$

and

$$\omega_3^2+\sqrt{3}a\omega_3+a^2+a+1-a^2b^2=0.$$

When $a^2+a+1-a^2b^2<0$, a positive solution of the above equation is

$$\omega_3^{(1)}=\frac{-\sqrt{3}a+\sqrt{4a^2b^2-a^2-4a-4}}{2}. \tag{18}$$

When $a^2+a+1-a^2b^2>0$, $4a^2b^2-a^2-4a-4>0$ and $a<0$, the above equation has two positive roots,

$$\omega_3^{(2,3)}=\frac{-\sqrt{3}a\pm\sqrt{4a^2b^2-a^2-4a-4}}{2}. \tag{19}$$

Further, from (17), we obtain the time delay, given by

$$\tau_{3,k}^{(j)}=\begin{cases} \frac{1}{\omega_3^{(k)}}[\arccos(\frac{1}{b}+\frac{1+\sqrt{3}\omega_3^{(k)}}{2ab})+2j\pi], \\ \quad \text{for } \frac{\sqrt{3}-\omega_3^{(k)}}{2ab}\ge 0, \\ \frac{1}{\omega_3^{(k)}}[2(j+1)\pi-\arccos(\frac{1}{b}+\frac{1+\sqrt{3}\omega_3^{(k)}}{2ab})], \\ \quad \text{for } \frac{\sqrt{3}-\omega_3^{(k)}}{2ab}<0, \end{cases} \tag{20}$$

where $k=1,2,3$; $j=0,1,2,\ldots$.

Let $\lambda(\tau)=\alpha(\tau)+\mathrm{i}\omega(\tau)$ be the root of $\Lambda_3(\lambda)=0$, satisfying $\alpha(\tau_{3,k}^{(j)})=0$, and $\omega(\tau_{3,k}^{(j)})=\omega_3^{(k)}$ ($j=0,1,2,\ldots$; $k=1,2,3$). Thus,

$$\alpha'\left(\tau_{3,k}^{(j)}\right)^{-1}=\frac{2\omega_3^{(k)}+\sqrt{3}a}{2a^2b^2\omega_3^{(k)}},$$

where $k = 1, 2, 3$; $j = 0, 1, 2, \dots$. The transversality conditions are

$$\alpha'\big(\tau_{3,1}^{(j)}\big) > 0, \quad \alpha'\big(\tau_{3,2}^{(j)}\big) > 0, \quad \alpha'\big(\tau_{3,3}^{(j)}\big) < 0, \tag{21}$$

where $j = 0, 1, 2, \dots$.

Case 3. Hopf-zero bifurcation.

Combining cases 1 and 2, we have the following theorem.

Theorem 1 *Assume $a - ab - 1 = 0$. Then a Hopf-zero bifurcation occurs associated with eigenvalues*, 0 *and* $\pm i\omega$, *under each of the following three conditions*.

(1) *When $a^2 + a + 1 - a^2b^2 < 0$, $\omega = \omega_2^{(1)}$ or $\omega_3^{(1)}$, where $\omega_2^{(1)}$ and $\omega_3^{(1)}$ are given by* (12) *and* (18), *respectively, and the critical time delays and transversality conditions are given by* (14), (15) *and* (20), (21), *respectively. Moreover, when $\tau \in [0, \tau_0)$, where $\tau_0 = \min\{\tau_{2,1}^{(0)}, \tau_{3,1}^{(0)}\}$, the characteristic equation* (4) *has a zero, and all the other roots have negative real part. When $\tau > \tau_0$, the trivial equilibrium is unstable*.

(2) *When $a^2 + a + 1 - a^2b^2 > 0$, $4a^2b^2 - a^2 - 4a - 4 > 0$ and $a > 0$, $\omega = \omega_2^{(2)}$ or $\omega_2^{(3)}$, where $\omega_2^{(2,3)}$ are given by* (13), *and the critical time delays and transversality conditions are given by* (14) *and* (15), *respectively. Moreover, there exists $m \in N$, such that $\tau_{2,2}^{(0)} < \tau_{2,3}^{(0)} \dots < \tau_{2,2}^{(m)} < \tau_{2,3}^{(m)} < \tau_{2,2}^{(m+1)} < \tau_{2,2}^{(m+2)} < \dots$, thus, when $\tau \in [0, \tau_{2,2}^{(0)}) \cup \bigcup_{k=0}^{m}(\tau_{2,3}^{(k)}, \tau_{2,2}^{(k+1)})$, the characteristic equation* (4) *has a zero, and all the other roots have negative real part*; *when $\tau \in \bigcup_{k=0}^{m}(\tau_{2,2}^{(k)}, \tau_{2,3}^{(k)}) \cup (\tau_{2,2}^{(m+1)}, +\infty)$, the trivial equilibrium is unstable*.

(3) *When $a^2 + a + 1 - a^2b^2 > 0$, $4a^2b^2 - a^2 - 4a - 4 > 0$ and $a < 0$, $\omega = \omega_3^{(2)}$ or $\omega_3^{(3)}$, where $\omega_3^{(2,3)}$ are given by* (19), *and the critical time delays and transversality conditions are given by* (20) *and* (21), *respectively. Moreover, there exists $m \in N$, such that $\tau_{3,2}^{(0)} < \tau_{3,3}^{(0)} \dots < \tau_{3,2}^{(m)} < \tau_{3,3}^{(m)} < \tau_{3,2}^{(m+1)} < \tau_{3,2}^{(m+2)} < \dots$, thus, when $\tau \in [0, \tau_{3,2}^{(0)}) \cup \bigcup_{k=0}^{m}(\tau_{3,3}^{(k)}, \tau_{3,2}^{(k+1)})$, the characteristic equation* (4) *has a zero, and all the other roots have negative real part*; *when $\tau \in \bigcup_{k=0}^{m}(\tau_{3,2}^{(k)}, \tau_{3,3}^{(k)}) \cup (\tau_{3,2}^{(m+1)}, +\infty)$, the trivial equilibrium is unstable*.

2.2 Normal form of Hopf-zero bifurcation

In this subsection, we first derive the normal form of Hopf-zero bifurcation by using the multiple time scales method. We treat the measure of the inhibitory influence of the past history, b, and the time delay, τ, as two bifurcation parameters. Thus, suppose system (3) undergoes a Hopf-zero bifurcation from the trivial equilibrium at the critical point: $b = b_c$, $\tau = \tau_c$. The Taylor expansion of Eq. (3) truncated at the cubic order terms is as follows:

$$\dot{Y}(t) = M_1 Y(t) + bM_2 Y(t-\tau) + f\big(Y(t), Y(t-\tau)\big), \tag{22}$$

where

$$Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}, \qquad M_1 = \begin{pmatrix} -1 & 0 & a_1 \\ a_2 & -1 & 0 \\ 0 & a_3 & -1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & -a_1 \\ -a_2 & 0 & 0 \\ 0 & -a_3 & 0 \end{pmatrix},$$

and

$$f\big(Y(t), Y(t-\tau)\big) = \begin{pmatrix} \frac{a_1 b}{3} y_3^3(t-\tau) - \frac{a_1}{3} y_3^3(t) \\ \frac{a_2 b}{3} y_1^3(t-\tau) - \frac{a_2}{3} y_1^3(t) \\ \frac{a_3 b}{3} y_2^3(t-\tau) - \frac{a_3}{3} y_2^3(t) \end{pmatrix}.$$

Assume that linear equation

$$\dot{Y}(t) = M_1 Y(t) + bM_2 Y(t-\tau) \triangleq L_\tau\big(Y(t), Y(t-\tau)\big)$$

has a pair of purely imaginary roots $\pm i\omega$ and a zero root at critical point: $b = b_c$, $\tau = \tau_c$, and no other roots with zero real part. Let p_1 and p_2 be the two eigenvectors of the linear operator L_{τ_c} corresponding to the eigenvalues $i\omega$ and 0, respectively, and let p_1^* and p_2^* be the two normalized eigenvectors of the adjoint operator $L_{\tau_c}^*$ of the linear operator L_{τ_c} corresponding to the eigenvalues $-i\omega$ and 0, respectively, satisfying the inner product

$$\langle p_i^*, p_i \rangle = \big(\bar{p}_i^*\big)^{\mathrm{T}} p_i = 1, \quad i = 1, 2.$$

By a simple calculation, we have

$$
\begin{aligned}
p_1 &= (p_{11}, p_{12}, p_{13})^{\mathrm{T}} \\
&= \left(1, \frac{a_2(1-b_c\mathrm{e}^{-\mathrm{i}\omega\tau_c})}{1+\mathrm{i}\omega}, \frac{1+\mathrm{i}\omega}{a_1(1-b_c\mathrm{e}^{-\mathrm{i}\omega\tau_c})}\right)^{\mathrm{T}}, \\
p_2 &= (p_{21}, p_{22}, p_{23})^{\mathrm{T}} \\
&= \left(1, a_2(1-b_c), \frac{1}{a_1(1-b_c)}\right)^{\mathrm{T}}, \\
p_1^* &= (p_{11}^*, p_{12}^*, p_{13}^*)^{\mathrm{T}} \\
&= \left(\frac{1}{3}, \frac{1-\mathrm{i}\omega}{3a_2(1-b_c\mathrm{e}^{\mathrm{i}\omega\tau_c})}, \frac{a_1(1-b_c\mathrm{e}^{\mathrm{i}\omega\tau_c})}{3(1-\mathrm{i}\omega)}\right)^{\mathrm{T}}, \\
p_2^* &= (p_{21}^*, p_{22}^*, p_{23}^*)^{\mathrm{T}} \\
&= \left(\frac{1}{3}, \frac{1}{3a_2(1-b_c)}, \frac{a_1(1-b_c)}{3}\right)^{\mathrm{T}}.
\end{aligned}
\tag{23}
$$

Because the nonlinearity is cubic, we seek a uniform second-order approximate solution of system (22) in powers of $\epsilon^{1/2}$. Thus, by the MTS, the solution of (22) is assumed to take the form:

$$
Y(t) = \epsilon^{1/2}Y_1 + \epsilon^{3/2}Y_2 + \cdots, \tag{24}
$$

where $Y_j = (y_{1,j}(T_0, T_1, T_2, \ldots), y_{2,j}(T_0, T_1, T_2, \ldots), y_{3,j}(T_0, T_1, T_2, \ldots))^{\mathrm{T}}$, $j = 1, 2, 3, \ldots$; $T_k = \epsilon^k t$, $k = 0, 1, 2, \ldots$. The derivative with respect to t is now transformed into

$$
\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \cdots = \mathrm{D}_0 + \epsilon\mathrm{D}_1 + \cdots,
$$

where the differential operator $\mathrm{D}_i = \frac{\partial}{\partial T_i}$, $i = 0, 1, 2, \ldots$.

We take perturbations as $b = b_c + \epsilon b_\epsilon$ and $\tau = \tau_c + \epsilon\tau_\epsilon$ in (22). To deal with the delayed terms, we expand $y_{i,j}(T_0 - \tau_c - \epsilon\tau_\epsilon, T_1 - \epsilon(\tau_c + \epsilon\tau_\epsilon), T_2 - \epsilon(\tau_c + \epsilon\tau_\epsilon), \ldots)$ at $y_{i,j}(T_0 - \tau_c, T_1, T_2, \ldots)$ for $i = 1, 2, 3$; $j = 1, 2, 3, \ldots$. That is

$$
\begin{aligned}
& y_i\big(T_0 - \tau_c - \epsilon\tau_\epsilon, T_1 - \epsilon(\tau_c + \epsilon\tau_\epsilon), \\
&\quad T_2 - \epsilon(\tau_c + \epsilon\tau_\epsilon), \ldots\big) \\
&\quad = \epsilon^{1/2}y_{i,1}(T_0 - \tau_c, T_1, T_2, \ldots) \\
&\qquad - \epsilon^{3/2}\tau_\epsilon\mathrm{D}_0 y_{i,1}(T_0 - \tau_c, T_1, T_2, \ldots) \\
&\qquad - \epsilon^{3/2}\tau_c\mathrm{D}_1 y_{i,1}(T_0 - \tau_c, T_1, T_2, \ldots) \\
&\qquad + \epsilon^{3/2}y_{i,2}(T_0 - \tau_c, T_1, T_2, \ldots) \\
&\qquad + \cdots, \quad i = 1, 2, 3.
\end{aligned}
$$

Then, substituting the solutions with the multiple scales into (22) and balancing the coefficients of $\epsilon^{n/2}$ ($n = 1, 3, 5, \ldots$) yields a set of ordered linear differential equations.

First, for the $\epsilon^{1/2}$-order terms, we have

$$
\mathrm{D}_0Y_1 - M_1Y_1 - b_cM_2Y_{1,\tau_c} = 0, \tag{25}
$$

where $Y_{1,\tau_c} = (y_{1,1}(T_0 - \tau_c, T_1, T_2, \ldots), y_{2,1}(T_0 - \tau_c, T_1, T_2, \ldots), y_{3,1}(T_0 - \tau_c, T_1, T_2, \ldots))^{\mathrm{T}}$. Since 0 and $\pm\mathrm{i}\omega$ are the eigenvalues of the linear part of (22), then the solution of (25) can be expressed in the form of

$$
\begin{aligned}
Y_1(T_0, T_1) &= G_1(T_1)p_1\mathrm{e}^{\mathrm{i}\omega T_0} + \bar{G}_1(T_1)\bar{p}_1\mathrm{e}^{-\mathrm{i}\omega T_0} \\
&\quad + \frac{G_2(T_1)}{2}p_2,
\end{aligned}
\tag{26}
$$

where p_j $(j = 1, 2)$ is given by (23).

Next, from the $\epsilon^{3/2}$-order terms of (22), we obtain

$$
\begin{aligned}
& \mathrm{D}_0Y_2 - M_1Y_2 - b_cM_2Y_{2,\tau_c} \\
&\quad = -\mathrm{D}_1Y_1 - b_c\tau_\epsilon M_2\mathrm{D}_0Y_{1,\tau_c} - b_c\tau_cM_2\mathrm{D}_1Y_{1,\tau_c} \\
&\qquad + b_\epsilon M_2Y_{1,\tau_c} + f(Y_1, Y_{1,\tau_c}),
\end{aligned}
\tag{27}
$$

where $Y_{2,\tau_c} = (y_{1,2}(T_0 - \tau_c, T_1, T_2, \ldots), y_{2,2}(T_0 - \tau_c, T_1, T_2, \ldots), y_{3,2}(T_0 - \tau_c, T_1, T_2, \ldots))^{\mathrm{T}}$. Substituting solution (26) into (27), we obtain the following equation:

$$
\begin{aligned}
& \mathrm{D}_0Y_2 - M_1Y_2 - b_cM_2Y_{2,\tau_c} \\
&\quad = \xi_1\mathrm{e}^{\mathrm{i}\omega\mathrm{T}_0} + \bar{\xi}_1\mathrm{e}^{-\mathrm{i}\omega T_0} + \xi_2(T_1, T_2) + \mathrm{NST},
\end{aligned}
\tag{28}
$$

where NST stands for the terms that do not produce secular terms, and

$$
\begin{aligned}
\xi_1 = {} & -\frac{\partial G_1}{\partial T_1} p_1 - b_c \tau_\epsilon M_2 G_1 p_1 \mathrm{i}\omega \mathrm{e}^{-\mathrm{i}\omega\tau_c} \\
& - b_c \tau_c M_2 \frac{\partial G_1}{\partial T_1} p_1 \mathrm{e}^{-\mathrm{i}\omega\tau_c} + b_\epsilon M_2 G_1 p_1 \mathrm{e}^{-\mathrm{i}\omega\tau_c} \\
& + \left(b_c \mathrm{e}^{-\mathrm{i}\omega\tau_c} - 1\right)\xi_3 G_1^2 \bar{G}_1 \\
& + \frac{b_c \mathrm{e}^{-\mathrm{i}\omega\tau_c} - 1}{4} \xi_4 G_1 G_2^2, \\
\xi_2 = {} & -\frac{1}{2}\frac{\partial G_2}{\partial T_1} p_2 - b_c \tau_c M_2 \frac{\partial G_2}{\partial T_1} p_2 + b_\epsilon M_2 G_2 p_2 \\
& + \frac{b-1}{24}\xi_5 G_2^3 + (b-1)\xi_6 G_1 \bar{G}_1 G_2,
\end{aligned}
\tag{29}
$$

with

$$
\begin{aligned}
\xi_3 &= \left(a_1 p_{13}^2 \bar{p}_{13}, a_2, a_3 p_{12}^2 \bar{p}_{12}\right)^{\mathrm{T}}, \\
\xi_4 &= \left(a_1 p_{13} p_{23}^2, a_2, a_3 p_{12} p_{22}^2\right)^{\mathrm{T}}, \\
\xi_5 &= \left(a_1 p_{23}^3, a_2, a_3 p_{22}^3\right)^{\mathrm{T}}, \\
\xi_6 &= (a_1 p_{13} \bar{p}_{13} p_{23}, a_2, a_3 p_{12} \bar{p}_{12} p_{22})^{\mathrm{T}}.
\end{aligned}
$$

Equation (28) is a linear non-homogeneous equation for Y_2, and the non-homogeneous equation has a solution if and only if a solvability condition is satisfied. That is, the right-hand side of (28) be orthogonal to every solution of the adjoint homogeneous problem, namely, $\langle p_j^*, \xi_j \rangle = 0,\ j = 1, 2$, then $\frac{\partial G_1}{\partial T_1}$ and $\frac{\partial G_2}{\partial T_1}$ are solved to yield

$$
\begin{aligned}
\frac{\partial G_1}{\partial T_1} &= d_1 G_1 + P_1 G_1^2 \bar{G}_1 + P_2 G_1 G_2^2, \\
\frac{\partial G_2}{\partial T_1} &= d_2 G_2 + P_3 G_2^3 + P_4 G_1 \bar{G}_1 G_2,
\end{aligned}
\tag{30}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\mathrm{e}^{-\mathrm{i}\omega\tau_c}(1+\mathrm{i}\omega)(b_\epsilon - b_c\tau_\epsilon)}{(1+\mathrm{i}\omega) b_c \tau_c \mathrm{e}^{-\mathrm{i}\omega\tau_c} - 1 + b_c \mathrm{e}^{-\mathrm{i}\omega\tau_c}}, \\
d_2 &= \frac{b_\epsilon}{b_c \tau_c - 1 + b_c}, \\
P_1 &= -h\left[\frac{(1-b_c e^{-\mathrm{i}\omega\tau_c})(1-b_c e^{\mathrm{i}\omega\tau_c}) a_2^2}{1-\mathrm{i}\omega} + 1 + \mathrm{i}\omega \right. \\
& \qquad \left. + \frac{(1+\mathrm{i}\omega)^2(1-\mathrm{i}\omega)}{a_1^2(1-b_c e^{-\mathrm{i}\omega\tau_c})(1-b_c e^{\mathrm{i}\omega\tau_c})}\right], \\
P_2 &= -\frac{h}{4}\left[1 + \mathrm{i}\omega + \frac{1+\mathrm{i}\omega}{a_1^2(1-b_c)^2} \right. \\
& \qquad \left. + a_2^2(1-b)^2(1+\mathrm{i}\omega)\right],
\end{aligned}
\tag{31}
$$

$$
\begin{aligned}
P_3 &= \frac{1}{36(\tau_c a b_c - 1)} \\
& \quad \times \left[1 + \frac{1}{(1-b_c)^2 a_1^2} + a_2^2 (1-b_c)^2\right], \\
P_4 &= -\frac{2}{3(1-\tau_c a b_c)}\left[\frac{(1+\mathrm{i}\omega)(1-\mathrm{i}\omega)}{a_1^2(1-b_c e^{-\mathrm{i}\omega\tau_c})(1-b_c e^{\mathrm{i}\omega\tau_c})} \right. \\
& \quad \left. + 1 + \frac{a_2^2(1-b_c e^{-\mathrm{i}\omega\tau_c})(1-b_c e^{\mathrm{i}\omega\tau_c})}{(1+\mathrm{i}\omega)(1-\mathrm{i}\omega)}\right],
\end{aligned}
$$

with $h = (3 + \frac{3 b_c \tau_c (1+\mathrm{i}\omega) e^{-\mathrm{i}\omega\tau_c}}{b_c e^{-\mathrm{i}\omega\tau_c} - 1})^{-1}$.

The derivation of the normal form near the Hopf-zero critical point (b_c, τ_c) using the CMR method is given in the Appendix. Equation (30) is the normal form truncated the third order derived by using the MTS method, and Eq. (40) is the normal form derived by using the CMR method (see the Appendix). Comparing the two normal forms, these two normal forms are identical.

Now, let $G_1 = r\mathrm{e}^{\mathrm{i}\Theta}$ and $G_2 = z$. Substituting these expressions into (30), we obtain the following normal form in cylindrical coordinates:

$$
\begin{cases}
\dfrac{\mathrm{d}r}{\mathrm{d}t} = \alpha_1 r + Q_1 r^3 + Q_2 r z^2, \\
\dfrac{\mathrm{d}z}{\mathrm{d}t} = \alpha_2 z + Q_3 z^3 + Q_4 z r^2, \\
\dfrac{\mathrm{d}\Theta}{\mathrm{d}t} = \alpha_3 + Q_5 z^2 + Q_6 r^2,
\end{cases}
\tag{32}
$$

where $\alpha_1 = \mathrm{Re}(d_1)$, $\alpha_2 = d_2$, $\alpha_3 = \mathrm{Im}(d_1)$, $Q_1 = \mathrm{Re}(P_1)$, $Q_2 = \mathrm{Re}(P_2)$, $Q_3 = P_3$, $Q_4 = P_4$, $Q_5 = \mathrm{Im}(P_2)$ and $Q_6 = \mathrm{Im}(P_1)$.

3 Hopf-zero bifurcation analysis and numerical simulation

In this section, we first give a bifurcation analysis based on the first two equations of the normal form (32), and then present some numerical simulation results.

$$
\begin{cases}
\dfrac{\mathrm{d}r}{\mathrm{d}t} = \alpha_1 r + Q_1 r^3 + Q_2 r z^2, \\
\dfrac{\mathrm{d}z}{\mathrm{d}t} = \alpha_2 z + Q_3 z^3 + Q_4 z r^2,
\end{cases}
\tag{33}
$$

Equilibrium solutions of (33) are obtained by simply setting $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} = 0$. Note that $F_0 = (r, z) = (0, 0)$

corresponds to the original trivial equilibrium, and the other ones are

$$F_1 = \left(\sqrt{-\frac{\alpha_1}{Q_1}}, 0\right) \quad \text{for } \frac{\alpha_1}{Q_1} < 0,$$

$$F_2^{\pm} = \left(0, \pm\sqrt{-\frac{\alpha_2}{Q_3}}\right) \quad \text{for } \frac{\alpha_2}{Q_3} < 0,$$

$$F_3^{\pm} = \left(\sqrt{\frac{\alpha_2 Q_2 - \alpha_1 Q_3}{Q_1 Q_3 - Q_2 Q_4}}, \pm\sqrt{\frac{\alpha_2 Q_1 - \alpha_1 Q_4}{Q_2 Q_4 - Q_1 Q_3}}\right)$$

$$\text{for } \frac{\alpha_2 Q_2 - \alpha_1 Q_3}{Q_1 Q_3 - Q_2 Q_4} > 0, \ \frac{\alpha_2 Q_1 - \alpha_1 Q_4}{Q_2 Q_4 - Q_1 Q_3} > 0.$$

The semi-trivial equilibria F_1 and $F_2^{\pm}$ are bifurcating from the origin on the critical lines L_1: $\alpha_1 = 0$ and L_2: $\alpha_2 = 0$, respectively. The pair of non-trivial equilibria $F_3^{\pm}$ collide with the semi-trivial equilibria F_1 and $F_2^{\pm}$, respectively on the critical lines L_3: $\alpha_2 Q_1 - \alpha_1 Q_4 = 0$ and L_4: $\alpha_2 Q_2 - \alpha_1 Q_3 = 0$.

When $b = b_c$ and $\tau = \tau_c$, the solutions on the center manifold determine the local asymptotic behavior of solutions of the original system (1). So, for (33), equilibria on the z-axis remain equilibria, while equilibria away from the z-axis become periodic orbits (with period $\approx 2\pi/\omega$).

In order to give a more clear bifurcation picture, we choose $a_1 = 1$, $a_2 = 2$, $a_3 = -4$, $b = 1.5$, which satisfy the assumption $a - ab = 1$ ($a = \sqrt[3]{a_1 a_2 a_3}$) of Theorem 1. Under these parameter values, the characteristic equation (4) with $\tau = 0$ has a zero root and a pair of complex conjugate roots with negative real part. By Theorem 1, we obtain

$$\omega_2^{(1)} = 3 - \sqrt{3},$$

$$\tau_{2,1}^{(0)} = \frac{1}{\omega_2^{(1)}} \arccos\left(\frac{1}{b} + \frac{1 - \sqrt{3}\omega_2^{(1)}}{2ab}\right) \approx 0.4129493351,$$

$$\mathrm{Re}\left(\lambda'\left(\tau_{2,1}^{(0)}\right)\right) > 0,$$

$$\omega_3^{(1)} = 3 + \sqrt{3},$$

$$\tau_{3,1}^{(0)} = \frac{1}{\omega_3^{(1)}} \arccos\left(\frac{1}{b} + \frac{1 + \sqrt{3}\omega_3^{(1)}}{2ab}\right) \approx 0.5532472039,$$

$$\mathrm{Re}\left(\lambda'\left(\tau_{3,1}^{(0)}\right)\right) > 0.$$

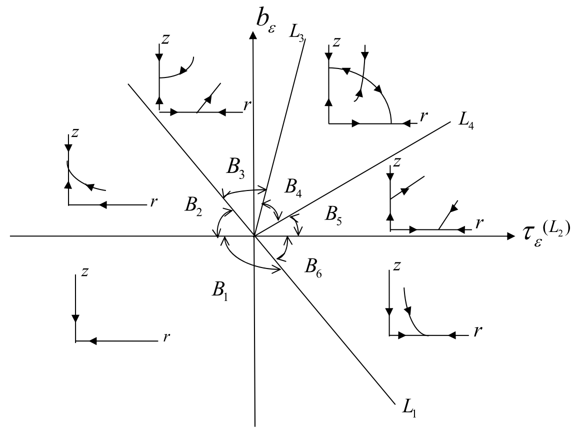

Fig. 1 Critical bifurcation lines in the $(\tau_\varepsilon, b_\varepsilon)$ parameter plane near (τ_c, b_c) and the corresponding phase portraits in the (r, z) plane

We take $\tau_c = \tau_{2,1}^{(0)} = 0.4129493351$. Thus, the characteristic equation (4) has a zero root and a pair of purely imaginary eigenvalues $\pm i\omega_2^{(1)} = \pm(3 - \sqrt{3})i$, and all the other eigenvalues have negative real part. Assume that system (1) undergoes a Hopf-zero bifurcation from the equilibrium (0, 0, 0). By a simple calculation, we obtain

$$\alpha_1 = 1.500682911\tau_\varepsilon + 0.9774934427 b_\varepsilon,$$

$$\alpha_2 = 0.8933165608 b_\varepsilon,$$

$$Q_1 = -2.028446218, \qquad Q_2 = -0.5071115547,$$

$$Q_3 = -0.07444304675, \qquad Q_4 = -1.786633120.$$

For the above chosen parameter values, the critical bifurcation lines become:

$$L_1: \quad b_\varepsilon = -1.535235783\tau_\varepsilon,$$

$$L_2: \quad b_\varepsilon = 0,$$

$$L_3: \quad b_\varepsilon = 40.85751446\tau_\varepsilon,$$

$$L_4: \quad b_\varepsilon = 0.2937996958\tau_\varepsilon,$$

as shown in the bifurcation diagram (see Fig. 1).

Figure 1 shows the critical bifurcation lines in the $(\tau_\varepsilon, b_\varepsilon)$ parameter plane near the critical point (τ_c, b_c) and the corresponding phase portraits in the (r, z) plane, whose origin is the Hopf-zero critical point. The bifurcation behaviors of the original system (1) in the neighborhood of (0, 0, 0) can be observed from Fig. 1. Note that the bifurcation boundaries divide the $(\tau_\varepsilon, b_\varepsilon)$ parameter plane into six regions. Also, it is seen from the phase portraits that the orbits are symmetric with respect to the r coordinate, therefore, we only draw the orbits in the first quadrant.

For system (1), we use Fig. 1 to describe the bifurcations in the clockwise direction, starting from B_1 and ending at B_1. First, in region B_1, there is only one trivial equilibrium which is a sink. When the parameters are varied across line L_2 (the τ_ε axis) from region B_1 to B_2, the trivial equilibrium becomes a saddle, and a pair of stable fixed points (for convenience, we call them $E_1^{\pm}$) appear from the trivial solution due to a pitchfork bifurcation. When the parameters are changed from region B_2 to B_3, an unstable periodic solution (called O_1) occurs from the trivial solution due to a Hopf bifurcation, and the trivial solution becomes a source from a saddle. In region B_4, a pair of periodic solutions (called $O_2^{\pm}$), which are sources, occur from O_1 due to a pitchfork bifurcation, and O_1 becomes a sink from a saddle. When the parameters are further changed from region B_4 to B_5 crossing line L_4, the pair of periodic solutions $O_2^{\pm}$ collide with the pair of semi-trivial fixed points $E_1^{\pm}$, respectively, and then disappear, and $E_1^{\pm}$ become saddles. When the parameters are further varied across line L_2 from region B_5 to B_6, the pair of semi-trivial fixed points $E_1^{\pm}$ collide with the trivial solution and then disappear, and the trivial solution becomes a saddle from a source. Finally, when the parameters are varied across line L_1 from region B_6 to B_1, the stable periodic solution O_1 collides with the trivial solution and then disappears, and the trivial solution becomes a sink from a saddle.

For simulations, we choose four groups of perturbation parameter values: $(\tau_\varepsilon, b_\varepsilon) = (-0.1, -0.1)$, $(-0.1, 0.1)$, $(0.1, 0.1)$, and $(0.1, -0.1)$, which belong to the regions B_1, B_2, B_4, and B_6, respectively, resulting in a stable fixed point shown in Fig. 2, a pair of stable fixed points as depicted in Fig. 3, co-existence of a pair of stable fixed points and a stable periodic solution shown in Fig. 4, and a stable periodic solution as depicted in Fig. 5. It is clear that the numerical simulations agree with the analytical predictions.

Remark It should be noted that if we take $\tau_c = \tau_{3,1}^{(0)} = 0.5532472039$, then the characteristic equation (4) has a zero root and a pair of purely imaginary eigenvalues $\pm i\omega_3^{(1)} = \pm(3 + \sqrt{3})i$, and in addition it has a pair of complex conjugate eigenvalues with positive real part. Thus, the system contains a two-dimensional unstable manifold, which has very little interest in practical application and thus is not investigated in this paper.

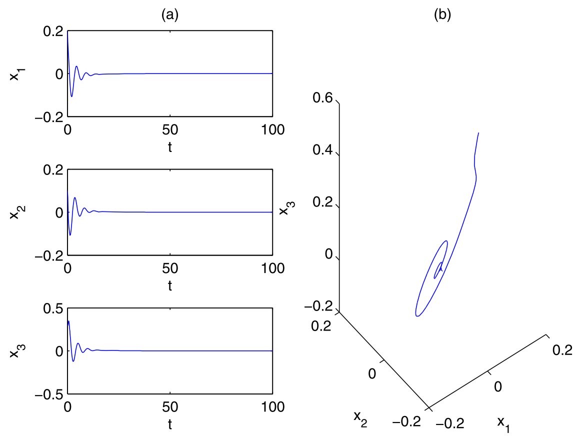

Fig. 2 Simulated solution of system (1) for $\tau_c = 0.4129493351$, $b_c = 1.5$ and $(\tau_\varepsilon, b_\varepsilon) = (-0.1, -0.1)$: (**a**) the time history; and (**b**) the phase portrait, showing a stable fixed point

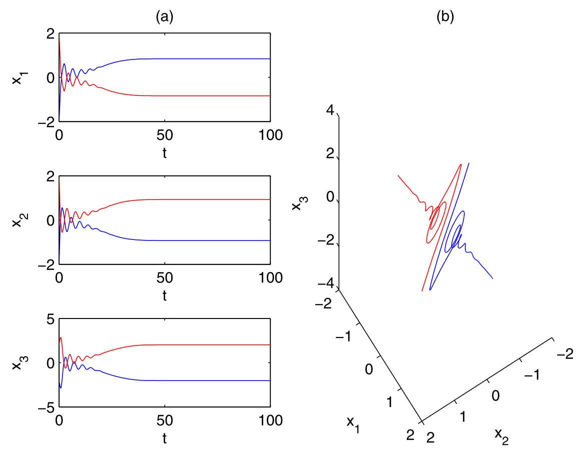

Fig. 3 Simulated solution of system (1) for $\tau_c = 0.4129493351$, $b_c = 1.5$ and $(\tau_\varepsilon, b_\varepsilon) = (-0.1, 0.1)$: (**a**) the time history; and (**b**) the phase portrait, showing coexistence of a pair of stable fixed points with the initial values $(x_1(0), x_2(0), x_3(0)) = (1, 1, 1)$ (*red lines*) and $(x_1(0), x_2(0), x_3(0)) = (-1, -1, -1)$ (*blue lines*), respectively (Color figure online)

4 Conclusion

In this paper, we have studied Hopf-zero bifurcation in a generalized Gopalsamy neural network model. We

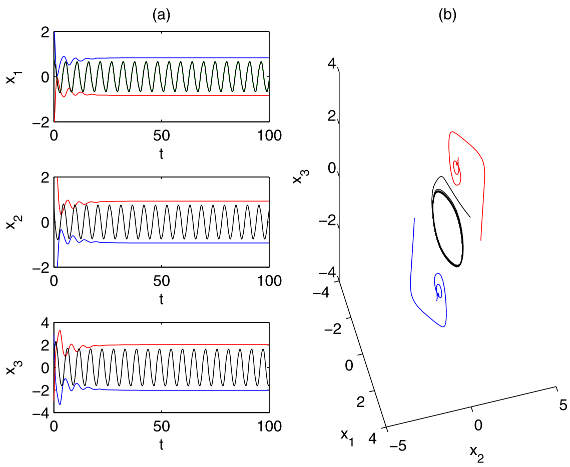

Fig. 4 Simulated solution of system (1) for $\tau_c = 0.4129493351$, $b_c = 1.5$ and $(\tau_\varepsilon, b_\varepsilon) = (0.1, 0.1)$: (**a**) the time history; and (**b**) the phase portrait, showing co-existence of a pair of stable fixed points with the initial values $(x_1(0), x_2(0), x_3(0)) = (3, -3, 3)$ (*blue lines*) and $(x_1(0), x_2(0), x_3(0)) = (-3, 3, -3)$ (*red lines*) and a stable periodic solution with the initial value $(x_1(0), x_2(0), x_3(0)) = (1, 1, 1)$ (*black lines*), respectively (Color figure online)

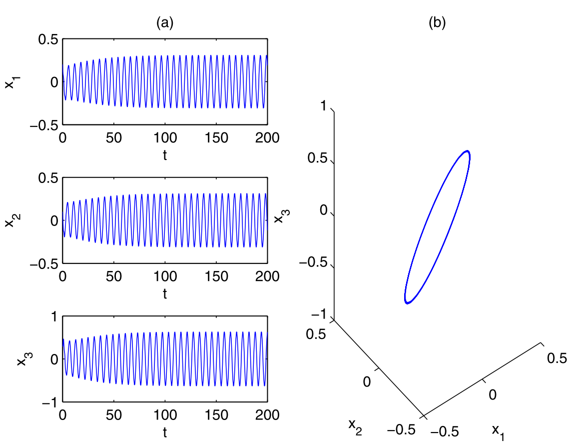

Fig. 5 Simulated solution of system (1) for $\tau_c = 0.4129493351$, $b_c = 1.5$ and $(\tau_\varepsilon, b_\varepsilon) = (0.1, -0.1)$: (**a**) the time history; and (**b**) the phase portrait, showing a stable periodic solution

derived the normal form of Hopf-zero bifurcation by using multiple time scales and center manifold reduction methods. A comparison between the two methods shows that the two normal forms are identical. The multiple time scales method is the first time to be used to consider the Hopf-zero bifurcation in delay differential equations, and further shown that the multiple time scales is simpler than the center manifold reduction, though the results from the two methods are equivalent. Moreover, bifurcation analysis near the Hopf-zero critical point is given, showing that the system may exhibit a stable fixed point, a pair of stable fixed points, a stable periodic solution, and co-existence of a pair of stable fixed points and a stable periodic solution in the neighborhood of the critical point. Numerical simulations are presented to verify the analytical predictions.

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Appendix

In this Appendix, we compute the normal form of (22) on center manifold near the Hopf-zero bifurcation critical point (b_c, τ_c) using the center manifold reduction (CMR) method. To achieve this, first re-scale time by $\tilde{t} \mapsto (t/\tau)$ to normalize the delay so that system (22) becomes

$$\begin{cases} \dot{y_1}(\tilde{t}) = -\tau y_1(\tilde{t}) + \tau a_1 y_3(\tilde{t}) - \tau a_1 b y_3(\tilde{t}-1) \\ \qquad - \dfrac{\tau a_1}{3} y_3^3(\tilde{t}) + \dfrac{\tau a_1 b}{3} y_3^3(\tilde{t}-1), \\ \dot{y_2}(\tilde{t}) = -\tau y_2(\tilde{t}) + \tau a_2 y_1(\tilde{t}) - \tau a_2 b y_1(\tilde{t}-1) \\ \qquad - \dfrac{\tau a_2}{3} y_1^3(\tilde{t}) + \dfrac{\tau a_2 b}{3} y_1^3(\tilde{t}-1), \\ \dot{y_3}(\tilde{t}) = -\tau y_3(\tilde{t}) + \tau a_3 y_2(\tilde{t}) - \tau a_3 b y_2(\tilde{t}-1) \\ \qquad - \dfrac{\tau a_3}{3} y_2^3(\tilde{t}) + \dfrac{\tau a_3 b}{3} y_2^3(\tilde{t}-1). \end{cases} \tag{34}$$

The trivial equilibrium of (34) is $y_1 = y_2 = y_3 = 0$. At the critical point $(b, \tau) = (b_c, \tau_c)$, we choose

$$\eta(\theta) = \begin{cases} \tau_c N_1, & \theta = 0, \\ 0, & \theta \in (-1, 0), \\ -\tau_c N_2, & \theta = -1, \end{cases}$$

with

$$N_1 = \begin{pmatrix} -1 & 0 & a_1 \\ a_2 & -1 & 0 \\ 0 & a_3 & -1 \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 0 & 0 & -a_1 b_c \\ -a_2 b_c & 0 & 0 \\ 0 & -a_3 b_c & 0 \end{pmatrix}.$$

Then the linearized equation of (34) at the trivial equilibrium can be written as

$$\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} = L_0 X_{\tilde{t}},$$

where $L_0\phi = \int_{-1}^{0} \mathrm{d}\eta(\theta)\phi(\theta)$, $\phi \in \mathrm{C} = \mathrm{C}([-1,0], \mathrm{R}^3)$, and the bilinear form [51] on $\mathrm{C}^* \times \mathrm{C}$ ($*$ stands for adjoint) is

$$\langle \psi(s), \phi(\theta) \rangle = \psi(0)\phi(0) - \int_{-1}^{0} \int_{\xi=0}^{\theta} \psi(\xi - \theta)\,\mathrm{d}\eta(\theta)\phi(\xi)\,\mathrm{d}\xi,$$

in which $\phi \in \mathrm{C}$, $\psi \in \mathrm{C}^*$. Then the phase space C is decomposed by $\Lambda = \{\pm \mathrm{i}\omega\tau_c, 0\}$ as $\mathrm{C} = P \oplus Q$, where $Q = \{\varphi \in \mathrm{C}\colon\ (\psi, \varphi) = 0,\ \text{for all } \psi \in P^*\}$, and the bases for P and its adjoint P^* are given respectively by

$$\Phi(\theta) = \begin{pmatrix} e^{\mathrm{i}\omega\tau_c\theta} & e^{-\mathrm{i}\omega\tau_c\theta} & \frac{1}{2} \\ \frac{a_2(1-b_c e^{-\mathrm{i}\omega\tau_c})e^{\mathrm{i}\omega\tau_c\theta}}{1+\mathrm{i}\omega} & \frac{a_2(1-b_c e^{\mathrm{i}\omega\tau_c})e^{-\mathrm{i}\omega\tau_c\theta}}{1-\mathrm{i}\omega} & \frac{a_2(1-b_c)}{2} \\ \frac{(1+\mathrm{i}\omega)e^{\mathrm{i}\omega\tau_c\theta}}{a_1(1-b_c e^{-\mathrm{i}\omega\tau_c})} & \frac{(1-\mathrm{i}\omega)e^{-\mathrm{i}\omega\tau_c\theta}}{a_1(1-b_c e^{\mathrm{i}\omega\tau_c})} & \frac{1}{2a_1(1-b_c)} \end{pmatrix}$$

and

$$\Psi(s) = \begin{pmatrix} h e^{-\mathrm{i}\omega\tau_c s} & \frac{h(1+\mathrm{i}\omega)e^{-\mathrm{i}\omega\tau_c s}}{a_2(1-b_c e^{-\mathrm{i}\omega\tau_c})} & \frac{a_1 h(1-b_c e^{-\mathrm{i}\omega\tau_c})e^{-\mathrm{i}\omega\tau_c s}}{1+\mathrm{i}\omega} \\ \bar{h} e^{\mathrm{i}\omega\tau_c s} & \frac{\bar{h}(1-\mathrm{i}\omega)e^{\mathrm{i}\omega\tau_c s}}{a_2(1-b_c e^{\mathrm{i}\omega\tau_c})} & \frac{a_1 \bar{h}(1-b_c e^{\mathrm{i}\omega\tau_c})e^{\mathrm{i}\omega\tau_c s}}{1-\mathrm{i}\omega} \\ \frac{2}{3(1-\tau_c a b_c)} & \frac{2a_1 a_3(1-b_c)^2}{3(1-\tau_c a b_c)} & \frac{2a_1(1-b_c)}{3(1-\tau_c a b_c)} \end{pmatrix},$$

where $h = (3 + \frac{3b_c\tau_c(1+\mathrm{i}\omega)e^{-\mathrm{i}\omega\tau_c}}{b_c e^{-\mathrm{i}\omega\tau_c} - 1})^{-1}$.

We also use the same bifurcation parameters given by $b = b_c + b_\varepsilon$ and $\tau = \tau_c + \tau_\varepsilon$ in (34), where b_ε and τ_ε are perturbation parameters, and denote $\varepsilon = (b_\varepsilon, \tau_\varepsilon)$. Then (34) can be written as

$$\frac{\mathrm{d}X(\tilde{t})}{\mathrm{d}\tilde{t}} = L(\varepsilon)X_{\tilde{t}} + F(X_{\tilde{t}}, \varepsilon), \tag{35}$$

where

$$L(\varepsilon)X_{\tilde{t}} = \begin{pmatrix} -(\tau_c+\tau_\varepsilon)y_{1,\tilde{t}}(0) + a_1(\tau_c+\tau_\varepsilon)y_{3,\tilde{t}}(0) \\ -(\tau_c+\tau_\varepsilon)y_{2,\tilde{t}}(0) + a_2(\tau_c+\tau_\varepsilon)y_{1,\tilde{t}}(0) \\ -(\tau_c+\tau_\varepsilon)y_{3,\tilde{t}}(0) + a_3(\tau_c+\tau_\varepsilon)y_{2,\tilde{t}}(0) \end{pmatrix} - \begin{pmatrix} a_1(b_c+b_\varepsilon)(\tau_c+\tau_\varepsilon)y_{3,\tilde{t}}(-1) \\ a_2(b_c+b_\varepsilon)(\tau_c+\tau_\varepsilon)y_{1,\tilde{t}}(-1) \\ a_3(b_c+b_\varepsilon)(\tau_c+\tau_\varepsilon)y_{2,\tilde{t}}(-1) \end{pmatrix},$$

and

$$F(X_{\tilde{t}}, \varepsilon) = \begin{pmatrix} \frac{a_1(b_c+b_\varepsilon)(\tau_c+\tau_\varepsilon)}{3} y_{3,\tilde{t}}^3(-1) - \frac{a_1(\tau_c+\tau_\varepsilon)}{3} y_{3,\tilde{t}}^3(0) \\ \frac{a_2(b_c+b_\varepsilon)(\tau_c+\tau_\varepsilon)}{3} y_{1,\tilde{t}}^3(-1) - \frac{a_2(\tau_c+\tau_\varepsilon)}{3} y_{1,\tilde{t}}^3(0) \\ \frac{a_3(b_c+b_\varepsilon)(\tau_c+\tau_\varepsilon)}{3} y_{2,\tilde{t}}^3(-1) - \frac{a_3(\tau_c+\tau_\varepsilon)}{3} y_{2,\tilde{t}}^3(0) \end{pmatrix}.$$

We now consider the enlarged phase space BC of functions from $[-1,0]$ to R^3, which are continuous on $[-1,0)$ with a possible jump discontinuity at zero. This space can be identified as $\mathrm{C} \times \mathrm{R}^3$. Thus, its elements can be written in the form $\tilde{\varphi} = \varphi + X_0 c$, where $\varphi \in \mathrm{C}$, $c \in \mathrm{R}^3$ and X_0 is a 3×3 matrix-valued function, defined by $X_0(\theta) = 0$ for $\theta \in [-1,0)$ and $X_0(0) = \mathrm{I}$. In the BC, (35) becomes an abstract ODE,

$$\frac{\mathrm{d}u}{\mathrm{d}\tilde{t}} = Au + X_0\tilde{F}(u, \varepsilon), \tag{36}$$

where $u \in \mathrm{C}$, and A is defined by

$$A:\ C^1 \to \mathrm{BC},\ Au = \frac{\mathrm{d}u}{\mathrm{d}\tilde{t}} + X_0\left[L_0 u - \frac{\mathrm{d}u(0)}{\mathrm{d}\tilde{t}}\right],$$

and

$$\tilde{F}(u,\varepsilon) = \left[L(\varepsilon) - L_0\right]u + F(u,\varepsilon).$$

By using the continuous projection $\pi:\ \mathrm{BC} \mapsto P$, $\pi(\phi + X_0 c) = \Phi[(\Psi,\phi) + \Psi(0)c]$, we can decompose the enlarged phase space by $\Lambda = \{\pm \mathrm{i}\omega\tau_c, 0\}$ as $\mathrm{BC} = P \oplus \mathrm{Ker}^{\pi}$, where $\mathrm{Ker}^{\pi} = \{\phi + X_0 c:\ \pi(\phi + X_0 c) = 0\}$, denoting the Kernel under the projection π. Let $\eta = (\eta_1, \bar{\eta}_1, \eta_2)^{\mathrm{T}}$, $v_{\tilde{t}} \in Q^1 := Q \cap \mathrm{C}^1 \subset \mathrm{Ker}^{\pi}$, and A_{Q^1} the restriction of A as an operator from Q^1 to the Banach space Ker^{π}. Further, denote $u_{\tilde{t}} = \Phi\eta + v_{\tilde{t}}$. Then Eq. (36) is decomposed to the form:

$$\begin{cases} \dfrac{\mathrm{d}\eta}{\mathrm{d}\tilde{t}} = B\eta + \Psi(0)\tilde{F}(\Phi\eta + v_{\tilde{t}}, \varepsilon), \\ \dfrac{\mathrm{d}v_{\tilde{t}}}{\mathrm{d}\tilde{t}} = A_{Q^1}v_{\tilde{t}} + (\mathrm{I} - \pi)X_0\tilde{F}(\Phi\eta + v_{\tilde{t}}, \varepsilon), \end{cases} \tag{37}$$

where $B = \mathrm{diag}\{\mathrm{i}\omega\tau_c, -\mathrm{i}\omega\tau_c, 0\}$.

Next, let M_2^1 denote the operator defined in $V_2^5(\mathrm{C}^3 \times \mathrm{Ker}^{\pi})$, with

$$M_2^1: V_2^5(\mathrm{C}^3) \mapsto V_2^5(\mathrm{C}^3),$$
$$\left(M_2^1 p\right)(\eta,\varepsilon) = D_\eta p(\eta,\varepsilon)B\eta - Bp(\eta,\varepsilon),$$

where $V_2^5(\mathrm{C}^3)$ represents the linear space of the second-order homogeneous polynomials in five variables $(\eta_1, \bar{\eta}_1, \eta_2, b_\varepsilon, \tau_\varepsilon)$ with coefficients in C^3. Then we may choose the decomposition $V_2^5(\mathrm{C}^3) = \mathrm{Im}(M_2^1) \oplus \mathrm{Im}(M_2^1)^c$ with complementary space $\mathrm{Im}(M_2^1)^c$ spanned by the elements $b_\varepsilon\eta_1 e_1$, $\tau_\varepsilon\eta_1 e_1$, $\eta_1\eta_2 e_1$, $b_\varepsilon\bar{\eta}_1 e_2$, $\tau_\varepsilon\bar{\eta}_1 e_2$, $\bar{\eta}_1\eta_2 e_2$, $\tau_\varepsilon\eta_2 e_3$, $b_\varepsilon\eta_2 e_3$, $\eta_2^2 e_3$, $\eta_1\bar{\eta}_1 e_3$, where e_i $(i = 1,2,3)$ are unit vectors.

Consequently, the normal form of (35) on the center manifold near the equilibrium $(0,0,0)$ associated with the critical point $(b_\varepsilon, \tau_\varepsilon) = (0,0)$ has the form

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tilde{t}} = B\eta + \frac{1}{2}g_2^1(\eta, 0, \varepsilon) + \mathrm{h.o.t.},$$

where g_2^1 is the function giving the quadratic terms in (η,ε) for $v_{\tilde{t}} = 0$, and is determined by $g_2^1(\eta,0,\varepsilon) = Proj_{(\mathrm{Im}(M_2^1))^c} \times f_2^1(\eta,0,\varepsilon)$, where $f_2^1(\eta,0,\varepsilon)$ is the function giving the quadratic terms in (η,ε) for $v_{\tilde{t}} = 0$ defined by the first equation of (37). Thus, the normal form, truncated at the quadratic order terms, is given by

$$\begin{cases} \dfrac{\mathrm{d}\eta_1}{\mathrm{d}\tilde{t}} = \mathrm{i}\omega\tau_c\eta_1 + 3\mathrm{i}\omega h\tau_\varepsilon\eta_1 \\ \qquad - \dfrac{3h\tau_c e^{-\mathrm{i}\omega\tau_c}(1+\mathrm{i}\omega)}{1 - b_c e^{-\mathrm{i}\omega\tau_c}} b_\varepsilon\eta_1, \\ \dfrac{\mathrm{d}\bar{\eta}_1}{\mathrm{d}\tilde{t}} = -\mathrm{i}\omega\tau_c\bar{\eta}_1 - 3\mathrm{i}\omega\bar{h}\tau_\varepsilon\bar{\eta}_1 \\ \qquad - \dfrac{3\bar{h}\tau_c e^{\mathrm{i}\omega\tau_c}(1-\mathrm{i}\omega)}{1 - b_c e^{\mathrm{i}\omega\tau_c}} b_\varepsilon\bar{\eta}_1, \\ \dfrac{\mathrm{d}\eta_2}{\mathrm{d}\tilde{t}} = \dfrac{a\tau_c}{\tau_c a b_c - 1} b_\varepsilon\eta_2, \end{cases} \tag{38}$$

where $h = (3 + \frac{3b_c\tau_c(1+\mathrm{i}\omega)e^{-\mathrm{i}\omega\tau_c}}{b_c e^{-\mathrm{i}\omega\tau_c} - 1})^{-1}$.

To find the normal form up to third order, similarly, let M_3^1 denote the operator defined in $V_3^3(\mathrm{C}^3 \times \mathrm{Ker}^{\pi})$, with

$$M_3^1: V_3^3(\mathrm{C}^3) \mapsto V_3^5(\mathrm{C}^3),$$
$$\left(M_3^1 p\right)(\eta,\varepsilon) = D_\eta p(\eta,\varepsilon)B\eta - Bp(\eta,\varepsilon),$$

where $V_3^3(\mathrm{C}^3)$ denotes the linear space of the third-order homogeneous polynomials in three variables: η_1, $\bar{\eta}_1$, and η_2 with coefficients in C^3. Then one may choose the decomposition $V_3^3(\mathrm{C}^3) = \mathrm{Im}(M_3^1) \oplus \mathrm{Im}(M_3^1)^c$ with complementary space $\mathrm{Im}(M_3^1)^c$ spanned by the elements $\eta_1\eta_2^2 e_1$, $\eta_1^2\bar{\eta}_1 e_2$, $\bar{\eta}_1\eta_2^2 e_2$, $\eta_1\bar{\eta}_1^2 e_3$, $\eta_2^3 e_1$, $\eta_1\bar{\eta}_1\eta_2 e_3$, where e_i $(i = 1,2,3)$ are unit vectors.

Therefore, the normal form up to third-order terms is given by

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tilde{t}} = B\eta + \frac{1}{2!}g_2^1(\eta,0,\varepsilon) + \frac{1}{3!}g_3^1(\eta,0,\varepsilon) + \mathrm{h.o.t.}, \tag{39}$$

where

$$\frac{1}{3!}g_3^1(\eta,0,0) = \frac{1}{3!}\left(I - P_{I,3}^1\right)f_3^1(\eta,0,0),$$

and $f_3^1(\eta,0,0)$, is the function giving the cubic terms in $(\eta,\varepsilon,v_{\tilde{t}})$ for $\varepsilon = 0$, and $v_{\tilde{t}} = 0$, is defined by the first equation of (37). Finally, the normal form on the center manifold arising from (37) becomes

$$\begin{cases} \frac{d\eta_1}{dt} = i\omega\eta_1 + d_1\eta_1 + P_1\eta_1^2\bar{\eta}_1 + P_2\eta_1\eta_2^2, \\ \frac{d\bar{\eta}_1}{dt} = -i\omega\bar{\eta}_1 + \bar{d}_1\bar{\eta}_1 + \bar{P}_1\eta_1\bar{\eta}_1^2 + \bar{P}_2\bar{\eta}_1\eta_2^2, \\ \frac{d\eta_2}{dt} = d_2\eta_2 + P_3\eta_2^3 + P_4\eta_1\bar{\eta}_1\eta_2, \end{cases} \tag{40}$$

where d_i $(i = 1, 2)$ and P_i $(i = 1, 2, 3, 4)$ are given in (31).

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