Applied Mathematics Letters 100 (2020) 106019

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Identifying weak focus and center in a convection model ☆

Laigang Guo [a,b,c], Pei Yu [c,*], Yufu Chen [b]

[a] NCMIS, KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China
[b] School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, PR China
[c] Department of Applied Mathematics, Western University, London, Ontario N6A5B7, Canada

ARTICLE INFO

Article history:
Received 24 May 2019
Received in revised form 20 August 2019
Accepted 21 August 2019
Available online 5 September 2019

Keywords:
Convection model
3-dimensional quadratic system
Normal form
Limit cycle

ABSTRACT

In this paper, we consider a 3-dimensional convection model and identify its weak focus and center. This model is described by quadratic polynomial differential equations, having three equilibria, two of them may be center-focus type. Center manifold theory and normal form theory are applied to prove that at least three limit cycles can bifurcate from a Hopf critical point around one of the two equilibria.

1. Introduction

Identifying whether a singular point in a planar differential system is a center or focus has received great attention for more than one century and is still a hot topic. The order of the degeneracy of a singular point, related to Lyapunov constants in the polynomial ring of coefficients over the real field, determines the number of limit cycles bifurcating from the singular point. Poincaré [1] initiated the research of limit cycles, and later, the well-known Hilbert's 16th problem [2] promoted the development of this study. The second part of Hilbert's 16th problem is to study the upper bound on the number of limit cycles in planar polynomial systems of degree n, and the upper bound is called Hilbert number, denoted by $H(n)$. For general quadratic polynomial systems, the best result is four limit cycles with $(3,1)$ distribution, which was obtained 40 years ago [3], showing that $H(2) \geq 4$. For cubic polynomial systems, many results have been obtained on the lower bound of the Hilbert number [4,5]. So far, the best result obtained for cubic systems is $H(3) \geq 13$ [6,7]. Recently, many researchers paid attention to the center-focus problem for 3-dimensional (3-d) dynamical systems, for example, see [8,9]. In general, to study bifurcation in higher-dimensional

☆ This work was partially supported by the Natural Sciences and Engineering Research Council of Canada (No. R2686A2), the National Natural Science Foundation of China (11271363), and China Postdoctoral Science Foundation (No. 2019TQ0343).
* Corresponding author.
E-mail address: pyu@uwo.ca (P. Yu).

https://doi.org/10.1016/j.aml.2019.106019
0893-9659/

systems, the system is first reduced to a simpler one on its center manifold (e.g., see [8,10]), and then to study the limit cycle bifurcation restricted to the center manifold. It is difficult to investigate Hopf bifurcation in general n-dimensional systems, while many results have been obtained for 3-d dynamical systems [11–13]. Very recently, Guo et al. found 12 limit cycles in a class of 3-d quadratic systems with Z_3 symmetry [14].

The convection model we will consider in this paper is described by the following ordinary differential equations [15]:

$$\begin{aligned} \dot{P} &= r_1 q_1 - \chi P - \eta_1 N - \eta_2 N^2 + \eta_3 PN - \varphi_1 F - \varphi_2 F^2 + \varphi_3 PF, \\ \dot{N} &= N(\gamma P - \beta_1 - \beta_2 N - \alpha_1 F), \\ \dot{F} &= F(\alpha_2 N - \mu), \end{aligned} \tag{1}$$

where P is the average potential energy related to the pressure profile, N is the average fluctuation energy and F is the zonal flow energy; $r_1 q_1$ and χ represent the source and diffusion, respectively. Simulation has shown that the fluctuation energy N can cause decrease in P, expressed by $-(\eta_1 N + \eta_2 N^2)$ taking linear and quadratic effects. However, as N initially begins to increase, P also increases, yielding a small jump, which is measured by $\eta_3 PN$. When the pressure gradient becomes sufficiently large, the equilibrium solution of the system (in S-shape) becomes unstable and a fluctuating flow is generated, which is described by γPN. The two terms $-(\beta_1 N + \beta_2 N^2)$ describe the effect of the dissipation that causes the fluctuation energy N to be self-damped. α_1 and α_2 measure the rates of mutual effects of N and F, causing N to decrease and F to increase, respectively. The value of α_1 is usually estimated by a linear fit from plotting $\gamma P_H - \beta_1 - \beta_2 N_H$ as a function in F_H, and the value of α_2 is chosen to approximately reproduce the frequency of oscillations. μ is a debugging parameter, and the coefficients φ_1, φ_2 and φ_3 are chosen to obtain the best approximation of F_H. Note that in system (1) the state variables P, N and F are nonnegative, and all the parameters take positive real values.

In [15], the authors carried out a bifurcation analysis on system (1) and studied the stability of equilibrium solutions and numerically demonstrated the existence of limit cycles. However, the number of bifurcating limit cycles and the center-focus conditions were not considered in [15]. It has been noted that identifying weak foci and centers in 3-d systems can cause great complexity in computation (e.g. see [16]). Moreover, the criteria for determining whether a singular point in planar dynamical systems is a center or focus, such as time-reversibility and integrability [17], are no longer applicable for 3-d systems, because application of center manifold theory to higher-dimensional dynamical systems merely yields approximation on invariant manifolds. To determine the number and stability of limit cycles bifurcating from a singular point, it needs to compute Lyapunov constants, which are usually obtained by applying one of the three methods: the method of normal forms [8,18,19], Poincaré return map [20], and Lyapunov function method [21,22]. For 3-d dynamical systems associated with Hopf bifurcation, we can combine center manifold theory and normal form theory to compute Lyapunov constants L_k with the aid of a computer algebra system such as Maple or Mathematica (e.g., see [8,19,23]).

2. Simplification of model (1)

To simplify the analysis for model (1), we first introduce the following scalings,

$$\begin{aligned} &P \to x_1, \quad N \to x_2, \quad F \to x_3, \quad r_1 \to r^2\mu, \quad q_1 \to q^2, \quad \chi \to a_1^2\mu, \quad \eta_1 \to n_1^2\mu, \\ &\eta_2 \to n_2^2\mu, \quad \eta_3 \to n_3^2\mu, \quad \varphi_1 \to c_1^2\mu, \quad \varphi_2 \to c_2^2\mu, \quad \varphi_3 \to c_3^2\mu, \quad \beta_1 \to b_1^2\mu, \\ &\beta_2 \to b_2^2\mu, \quad \gamma \to a_2^2\mu, \quad \alpha_1 \to a_3^2\mu, \quad \alpha_2 \to a_4^2\mu, \quad t \to \tfrac{1}{\mu}t, \end{aligned} \tag{2}$$

under which system (1) becomes

$$\begin{aligned} \dot{x}_1 &= q^2 r^2 - a_1^2 x_1 - n_1^2 x_2 - c_1^2 x_3 - n_2^2 x_2^2 - c_2^2 x_3^2 + c_3^2 x_1 x_3 + n_3^2 x_1 x_2, \\ \dot{x}_2 &= x_2 (a_2^2 x_1 - b_1^2 - b_2^2 x_2 - a_3^2 x_3), \\ \dot{x}_3 &= x_3 (a_4^2 x_2 - 1), \end{aligned} \tag{3}$$

where the state variables x_i, $i = 1, 2, 3$ are nonnegative, and all the parameters q, r, n_1, $n_2, a_i\,(i = 1, 2, 3, 4)$, b_1, b_2 and $c_i\,(i = 1, 2, 3)$ take nonzero real values. Note that we have transformed the original positive parameters to nonzero parameters for convenience in computation.

It is easy to show that system (3) has three equilibrium solutions:

$$\mathrm{E_s} = \left(\frac{r^2q^2}{a_1^2}, 0, 0\right), \quad \mathrm{E_1} = \left(\frac{b_2^2\mathrm{I_1} + b_1^2}{a_2^2}, \mathrm{I_1}, 0\right), \quad \mathrm{E_2} = \left(\frac{a_3^2\mathrm{I_2} + b_1^2a_4^2 + b_2^2}{a_2^2a_4^2}, \frac{1}{a_4^2}, \frac{\mathrm{I_2}}{a_4^2}\right),$$

where $\mathrm{I_1}$ and $\mathrm{I_2}$ are determined from the following quadratic polynomial equations:

$$\begin{aligned}
&(a_2^2n_2^2 - b_2^2n_3^2){\mathrm{I_1}}^2 + (a_1^2b_2^2 + a_2^2n_1^2 - b_1^2n_3^2)\mathrm{I_1} - r^2q^2a_2^2 + a_1^2b_1^2 = 0,\\
&(a_2^2c_2^2 - a_3^2c_3^2){\mathrm{I_2}}^2 + [(a_1^2a_3^2 + a_2^2c_1^2 - b_1^2c_3^2)a_4^2 - a_3^2n_3^2 - b_2^2c_3^2]\mathrm{I_2}\\
&(a_1^2b_1^2 - a_2^2q^2r^2)a_4^4 + (a_1^2b_2^2 + a_2^2n_1^2 - b_1^2n_3^2)a_4^2 - b_2^2n_3^2 + n_2^2a_2^2 = 0.
\end{aligned} \tag{4}$$

The solutions of $\mathrm{I_1}$ and $\mathrm{I_2}$ obtained from the above two polynomial equations can be classified into two categories under the conditions: (A) $a_2^2n_2^2 - b_2^2n_3^2 = a_2^2c_2^2 - a_3^2c_3^2 = 0$; and (B) $a_2^2n_2^2 \neq b_2^2n_3^2$ or $a_2^2c_2^2 \neq a_3^2c_3^2$. For Case (A) the model can have three equilibria; while for Case (B), there may exist at most five equilibria. In this paper, we focus on Case (A) for which the three equilibria may be stable for certain parameter values. Therefore, solving the equation $a_2^2n_2^2 - b_2^2n_3^2 = 0$ for a_2 and $a_2^2c_2^2 - a_3^2c_3^2 = 0$ for c_2 we obtain

$$a_2^2 = \frac{b_2^2n_3^2}{n_2^2}, \quad c_2^2 = \frac{a_3^2c_3^2n_2^2}{b_2^2n_3^2},$$

which are substituted into system (3) to yield the following 3-d quadratic system,

$$\begin{aligned}
\dot{x}_1 &= q^2r^2 - a_1^2\,x_1 - n_1^2\,x_2 - c_1^2\,x_3 - n_2^2\,x_2^2 - \frac{a_3^2c_3^2n_2^2}{b_2^2n_3^2}\,x_3^2 + n_3^2\,x_1x_2 + c_3^2\,x_1x_3,\\
\dot{x}_2 &= x_2\Big(\frac{b_2^2n_3^2}{n_2^2}\,x_1 - b_1^2 - b_2^2\,x_2 - a_3^2\,x_3\Big),\\
\dot{x}_3 &= x_3\,(a_4^2\,x_2 - 1).
\end{aligned} \tag{5}$$

Then, the equilibrium solutions of system (5) can be rewritten (with new names $\mathrm{E_L}$ and $\mathrm{E_H}$) as

$$\mathrm{E_s} = \left(\frac{r^2q^2}{a_1^2},\ 0,\ 0\right), \quad \mathrm{E_L} = \left(\frac{S_1\,n_2^2}{S_3\,b_2^2},\ \frac{S_2}{S_3},\ 0\right), \quad \mathrm{E_H} = \left(\frac{n_2^2F_1}{F_2a_4^2b_2^2n_3^2},\ \frac{1}{a_4^2},\ \frac{a_4^2S_2 - S_3}{F_2}\right),$$

where

$$\begin{aligned}
S_1 = {}& b_2^4q^2r^2 + b_1^2(b_2^2n_1^2 - b_1^2n_2^2),\\
S_2 = {}& b_2^2n_3^2q^2r^2 - a_1^2b_1^2n_2^2,\\
S_3 = {}& a_1^2b_2^2n_2^2 + n_3^2(b_2^2n_1^2 - b_1^2n_2^2),\\
F_1 = {}& (a_4^2c_1^2n_3^2 - c_3^2n_2^2)b_2^4 + [(a_3^2q^2r^2 + b_1^2c_1^2)n_3^2a_4^4 - (a_3^2n_1^2n_3^2 + 2b_1^2c_3^2n_2^2)a_4^2 - a_3^2n_2^2n_3^2]b_2^2 - a_4^4b_1^4c_3^2n_2^2,\\
F_2 = {}& [(a_1^2a_3^2 - b_1^2c_3^2)a_4^2 - a_3^2n_3^2 - b_2^2c_3^2]n_2^2 + a_4^2b_2^2c_1^2n_3^2.
\end{aligned}$$

It is easy to see that the equilibrium $\mathrm{E_s}$ exists for any positive parameter values; the equilibrium $\mathrm{E_L}$ exists if $S_1/S_3 > 0$ and $S_2/S_3 > 0$; and the equilibrium $\mathrm{E_H}$ exists if $F_1/F_2 > 0$ and $(a_4^2S_2 - S_3)/F_2 > 0$. Further, note that both $\mathrm{E_L}$ and $\mathrm{E_H}$ are possible to be of center-focus type, while $\mathrm{E_s}$ can be only node or saddle. We apply the method of normal forms to prove that system (5) can have three small-amplitude limit cycles bifurcating from $\mathrm{E_H}$. Moreover, it can be shown that the equilibrium $\mathrm{E_L}$ is a center, restricted to a global center manifold,

3. Weak focus at the equilibrium $\mathrm{E_H}$

In [15], the authors present some qualitative properties for the equilibria of system (1). The reduced system (5) with less number of parameters helps us in simplifying the computation of normal forms and focus values. In the following, we first consider the qualitative properties of $\mathrm{E_s}$. The Jacobian matrix of system (5) evaluated at $\mathrm{E_s}$ is given by

$$J(\mathrm{E_s}) = \begin{bmatrix} -a_1^2 & \dfrac{n_3^2 q^2 r^2 - a_1^2 n_1^2}{a_1^2} & c_3^2 q^2 r^2 - a_1^2 c_1^2 a_1^2 \\ 0 & \dfrac{S_2}{a_1^2 n_2^2} & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{6}$$

which has eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = -a_1^2, \quad \lambda_3 = \frac{S_2}{a_1^2 n_2^2}.$$

All the three eigenvalues are real, and λ_1 and λ_2 are negative constants. Thus, $\mathrm{E_s}$ is a stable node if $S_2 < 0$ and a saddle if $S_2 > 0$.

Next, consider the equilibrium $\mathrm{E_H}$. In [15], the stability of equilibria was discussed, and numerical bifurcation software was used to plot bifurcation diagrams where Hopf bifurcation was identified, which is based on a linear analysis. In this section, we will further investigate the Hopf bifurcation in system (5), and prove the existence of three limit cycles. In order for system (5) to have a Hopf critical point on the equilibrium solution $\mathrm{E_H}$ and make the computation of focus values manageable, we further let

$$\begin{aligned} & r = c_1 = a_3 = a_1 = 1, \quad q = \frac{\sqrt{2}}{c_3}, \quad b_1 = \frac{b_2 n_3}{c_3 n_2}, \\ & S_4 = a_4^2 b_2^2 n_3^2 - b_2^2 c_3^2 n_2^2 + a_4^2 n_2^2 - n_2^2 n_3^2, \\ & n_1 = \frac{\sqrt{S_4[a_4^2 b_2^2 n_3^4 (a_4^2 + b_2^2 - n_3^2) + (2n_3^4 - b_2^2 n_3^2 - c_3^2 n_2^2) S_4]}}{S_4 n_3 c_3}. \end{aligned} \tag{7}$$

Thus, under the above conditions, the linearized system of (5) evaluated at the $\mathrm{E_H}$ contains a non-zero real eigenvalue and two purely imaginary eigenvalues. Next, introducing the transformation,

$$\begin{aligned} x_1 &= \frac{a_4^2 n_2^2 n_3^2 (a_4^2 + b_2^2 - n_3^2) + (a_4^2 n_3^2 + c_3^2 n_2^2) S_4}{S_4 a_4^2 c_3^2 n_3^2} + \frac{n_2^2}{n_3^2} z_1 + \frac{a_4^2 b_2 n_2 (a_4^2 n_3^2 - c_3^2 n_2^2 - n_2^2)}{S_4 n_3 c_3} z_2 \\ &\quad + \frac{n_2^2 F_3}{S_4 a_4^4 b_2^4 n_3^4} z_3, \\ x_2 &= \frac{1}{a_4^2} + z_1 - \frac{n_2^2 c_3^2 (a_4^2 - b_2^2 c_3^2 - n_3^2)}{a_4^4 b_2^2 n_3^2} z_3, \\ x_3 &= \frac{b_2^2 n_3^2 (a_4^2 + b_2^2 - n_3^2)}{S_4 c_3^2} - \frac{b_2 a_4^2 n_2 n_3 (a_4^2 + b_2^2 - n_3^2)}{S_4 c_3} z_2 + z_3, \\ t &\to \frac{c_3 n_2}{b_2 n_3} t, \end{aligned} \tag{8}$$

where

$$\begin{aligned} F_3 = \; & \{ \, a_4^4 b_2^2 n_3^2 - b_2^2 n_2^2 (a_4 - n_3)(a_4 + n_3) c_3^4 \\ & + [(-b_2^2 n_3^2 + n_2^2) a_4^4 - n_3^2 (b_2^4 - b_2^2 n_3^2 + 2n_2^2) a_4^2 + n_2^2 n_3^4] c_3^2 \, \} \, S_4 + a_4^4 b_2^4 c_3^2 n_3^4 (a_4^2 + b_2^2 - n_3^2), \end{aligned}$$

into (5), we obtain

$$\begin{aligned}\dot{z_1} &= z_2 + A_{002}z_3^2 + A_{011}z_2z_3 + A_{020}z_2^2 + A_{101}z_1z_3 + A_{110}z_1z_2,\\ \dot{z_2} &= -z_1 + B_{002}z_3^2 + B_{011}z_2z_3 + B_{020}z_2^2 + B_{101}z_1z_3 + B_{110}z_1z_2,\\ \dot{z_3} &= C_{001}z_3 + C_{002}z_3^2 + C_{011}z_2z_3 + C_{020}z_2^2 + C_{101}z_1z_3 + C_{110}z_1z_2,\end{aligned} \tag{9}$$

where the coefficients A_{ijk}, B_{ijk} and C_{ijk} are given in the website: http://www.apmaths.uwo.ca/~pyu.

Clearly, under the conditions given in (7), the singular point $\mathrm{E_H}$ of system (5) corresponds to the origin of (9), which is a Hopf-type critical point. We have the following result.

Theorem 3.1. *Under the conditions given in (7), there exist parameter values for system (1) such that at least three limit cycles can bifurcate from* $\mathrm{E_H}$ *within some, hence every, two-dimensional invariant manifold through* $\mathrm{E_H}$.

Proof. In order to make the computation of focus values manageable and find the parameter values satisfying the conditions, we further set $a_4 = 10$ and $n_2 = n_3 = 1$ in system (9). Then, we apply center manifold theory and the method of normal forms, as well as the Maple program in [23] to system (9) to obtain the focus values. In particular, v_1 and v_2 are given by

$$\begin{aligned}v_1 &= -\frac{125000(b_2^2c_3^2 - 100b_2^2 - 99)b_2(b_2^2 + 99)c_3^3F_4}{(b_2^2c_3^2 - 99)F_6F_7},\\ v_2 &= -\frac{312500000(b_2^2c_3^2 - 100b_2^2 - 99)(b_2^2 + 99)c_3^3F_5}{9b_2(b_2^2c_3^2 - 99)^3F_6^3F_7^3F_8},\end{aligned}$$

where the lengthy expressions of the polynomials $F_i, i = 4, 5, \ldots, 8$ are given in terms of b_2 and c_3, and can be found in the website: http://www.apmaths.uwo.ca/~pyu.

In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin of system (9), we use the coefficients b_2 and c_3 to solve the two polynomial equations, $F_4 = F_5 = 0$. If a solution of $F_4 = F_5 = 0$ yields $v_3 \neq 0$, we may obtain two small-amplitude limit cycles by properly perturbing b_2 and c_3. To achieve this, we first compute the resultant of F_4 and F_5, yielding

$$\begin{aligned}F_{45} = \;& 4.45990583806\cdots \times 10^{940}c_3^{124}(c_3^2 - 99)^{56}(c_3^4 - 200c_3^2 + 299)^4\\ &\times (275c_3^{10} - 109175c_3^8 - 2128419c_3^6 + 3004576243c_3^4 - 137557848532c_3^2 + 71109368756)^4\\ &\times (33275c_3^{14} - 13105026c_3^{12} + 464987348c_3^{10} + 6341100342c_3^8 - 7908729138339c_3^6\\ &\quad + 296555246006244c_3^4 - 265954686172548c_3^2 + 982858205776)^4\, F_{45a}^2,\end{aligned}$$

where F_{45a} is a polynomial in c_3^2, given in the website: http://www.apmaths.uwo.ca/~pyu.

Now, to find the solutions of $F_4 = F_5 = 0$, we must solve $F_{45} = 0$. It is seen from F_{45} that besides the factor F_{45a}, there are five polynomial factors which may also yield solutions c_3^2 for $F_{45} = 0$. However, a careful verification shows that the solutions solved from these five factors do not satisfy the equations $F_4 = F_5 = 0$. Thus, possible solutions for $F_{45} = 0$ only come from the positive roots of F_{45a} (since $c_3^2 > 0$). It can be shown that the polynomial F_{45a} has 7 positive roots for c_3^2, which in turn yield corresponding 14 solutions for b_2, among them only eight sets of solutions c_3 and b_2 satisfy the equations, $F_4 = F_5 = 0$. Then, under the conditions, $C_{001} < 0$ and $N_1 \geq 0$, we choose one of the solutions as follows:

$$b_2 = -0.53868839285\cdots, \quad c_3 = -0.61448088908\cdots,$$

under which

$$v_1 = v_2 = 0, \quad v_3 = -5.92361180765\cdots \times 10^8 \neq 0.$$

Moreover, a direct calculation shows that the Jacobian evaluated at the critical values is

$$\det\left[\frac{\partial(v_1,v_2)}{\partial(b_2,c_3)}\right]=-6.44212213245\cdots\times 10^8\neq 0,$$

implying that system (9) can indeed have two small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Further, a linear perturbation on n_1 generates a third small-amplitude limit cycle. Thus, system (1) can have at least three limit cycles around the equilibrium $\mathrm{E_H}$.

The proof of Theorem 3.1 is complete. □

Finally, we point out that for the singular point $\mathrm{E_L}$, one can identify the Hopf bifurcation condition and then apply normal form theory and global center manifold theory [24] to prove that when $\mathrm{E_L}$ is a Hopf singular point, it must be a center on a global center manifold. The proof is straightforward and is thus omitted.

4. Conclusion

In this paper, we have studied a convection model and paid particular attention on bifurcation of limit cycles. We have shown that the model has three distinct equilibria, and that there exist parameter values for the model to have three small-amplitude limit cycles around one of the equilibria.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.aml.2019.106019.

References

[1] H. Poincaré, Mémoire sur les courbes définies par une équation différentielle (II), J. Math. Pures Appl. 8 (1882) 251–296.
[2] D. Hilbert, Mathematical problems, Bull. Am. Math. 8 (1902) 437–479.
[3] S. Shi, A concrete example of the existence of four limit cycles for plane quadratic systems, Sci. Sin. 23 (1980) 153–158.
[4] H. Zołądek, Eleven small limit cycles in a cubic vector field, Nonlinearity 8 (5) (1995) 843.
[5] P. Yu, M. Han, A study on Żołądek's example, J. Appl. Anal. Comput. 1 (2011) 143–153.
[6] J. Li, Y. Liu, New results on the study of Z_q-equivariant planar polynomial vector fields, Qual. Theory Dyn. Syst. 9 (1) (2010) 167–219.
[7] C. Li, C. Liu, J. Yang, A cubic system with thirteen limit cycles, J. Differential Equations 246 (9) (2009) 3609–3619.
[8] M. Han, P. Yu, Normal Forms, Melnikov Functions and Bifurcations of Limit Cycles, Springer-Verlag, London, 2012.
[9] W. Zhang, L.M. Wahl, P. Yu, Viral blips may not need a trigger: How transient viremia can arise in deterministic in-host models, SIAM Rev. 56 (1) (2014) 127–155.
[10] J. Guckenheimer, P.J. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, first ed., Springer-Verlag, New York, 1983.
[11] V.G. Romanovski, D.S. Shafer, Centers and limit cycles in polynomial systems of ordinary differential equations, Adv. Stud. Pure Math. 68 (2016) 267–373.
[12] Q. Wang, Y. Liu, H. Chen, Hopf bifurcation for a class of three-dimensional nonlinear dynamic systems, Bull. Sci. Math. 134 (7) (2010) 786–798.
[13] Y. Tian, P. Yu, Seven limit cycles around a focus point in a simple three-dimensional quadratic vector field, Int. J. Bifurcation Chaos 24 (6) (2014) 1450083.
[14] L.G. Guo, P. Yu, Y.F. Chen, Twelve limit cycles in 3-D quadratic vector fields with Z_3 symmetry, Int. J. Bifurcation Chaos 28 (11) (2018) 1850139.
[15] M. Dam, M. Brøns, J. Juul Rasmussen, V. Naulin, Jan S. Hesthaven, Sparse identification of a predator–prey system from simulation data of a convection model, Phys. Plasmas 24 (2) (2017) 022310.
[16] P. Yu, M. Han, Ten limit cycles around a center-type singular point in a 3-d quadratic system with quadratic perturbation, Appl. Math. Lett. 44 (2015) 17–20.
[17] V.G. Romanovski, D.S. Shafer, The Center and Cyclicity Problems: A Computational Algebra Approach, Birkhäuser, Boston-Basel-Berlin, 2009.
[18] W.W. Farr, C. Li, I.S. Labouriau, W. Langford, Degenerate hopf bifurcation formulas and Hilbert's 16th problem, SIAM J. Math. Anal. 20 (1) (1989) 13–30.

[19] P. Yu, Computation of normal forms via a perturbation technique, J. Sound Vib. 211 (1) (1998) 19–38.
[20] A.A. Andronov, Theory of Bifurcations of Dynamic Systems on a Plane, Wiley, New York, 1973.
[21] S. Shi, On the structure of Poincaré–Lyapunov constants for the weak focus of polynomial vector fields, J. Differential Equations 52 (1) (1984) 52–57.
[22] A. Gasull, J. Torregrosa, A new approach to the computation of the Lyapunov constants, J. Comput. Appl. Math. 20 (2001) 149–177.
[23] Y. Tian, P. Yu, An explicit recursive formula for computing the normal form and center manifold of general n-dimensional differential systems associated with Hopf bifurcation, Int. J. Bifurcation Chaos 23 (6) (2013) 1350104.
[24] V.G. Romanovski, D.S. Shafer, The Center and Cyclicity Problems: A Computational Algebra Approach, Birkhäuser, Boston-Basel-Berlin, 2009.