Identifying weak focus and center in a convection model

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\section{Introduction}

Identifying whether a singular point in a planar differential system is a center or focus has received great attention for more than one century and is still a hot topic. The order of the degeneracy of a singular point, related to Lyapunov constants in the polynomial ring of coefficients over the real field, determines the number of limit cycles bifurcating from the singular point. Poincaré \cite{1} initiated the research of limit cycles, and later, the well-known Hilbert’s 16th problem \cite{2} promoted the development of this study. The second part of Hilbert’s 16th problem is to study the upper bound on the number of limit cycles in planar polynomial systems of degree \( n \), and the upper bound is called Hilbert number, denoted by \( H(n) \). For general quadratic polynomial systems, the best result is four limit cycles with (3,1) distribution, which was obtained 40 years ago \cite{3}, showing that \( H(2) \geq 4 \). For cubic polynomial systems, many results have been obtained on the lower bound of the Hilbert number \cite{4,5}. So far, the best result obtained for cubic systems is \( H(3) \geq 13 \) \cite{6,7}. Recently, many researchers paid attention to the center-focus problem for 3-dimensional (3-d) dynamical systems, for example, see \cite{8,9}. In general, to study bifurcation in higher-dimensional systems, the focus and center problems are of great importance.

\section{Abstract}

In this paper, we consider a 3-dimensional convection model and identify its weak focus and center. This model is described by quadratic polynomial differential equations, having three equilibria, two of them may be center-focus type. Center manifold theory and normal form theory are applied to prove that at least three limit cycles can bifurcate from a Hopf critical point around one of the two equilibria.
systems, the system is first reduced to a simpler one on its center manifold (e.g., see [8,10]), and then to study the limit cycle bifurcation restricted to the center manifold. It is difficult to investigate Hopf bifurcation in general $n$-dimensional systems, while many results have been obtained for 3-d dynamical systems [11–13]. Very recently, Guo et al. found 12 limit cycles in a class of 3-d quadratic systems with $Z_3$ symmetry [14].

The convection model we will consider in this paper is described by the following ordinary differential equations [15]:

$$\begin{align*}
P &= r_1 q_1 - \chi P - \eta_1 N - \eta_2 N^2 + \eta_3 PN - \varphi_1 F - \varphi_2 F^2 + \varphi_3 PF, \\
N &= N(\gamma P - \beta_1 - \beta_2 N - \alpha_1 F), \\
F &= F(\alpha_2 N - \mu),
\end{align*}$$

(1)

where $P$ is the average potential energy related to the pressure profile, $N$ is the average fluctuation energy and $F$ is the zonal flow energy; $r_1 q_1$ and $\chi$ represent the source and diffusion, respectively. Simulation has shown that the fluctuation energy $N$ can cause decrease in $P$, expressed by $-(\eta_1 N + \eta_2 N^2)$ taking linear and quadratic effects. However, as $N$ initially begins to increase, $P$ also increases, yielding a small jump, which is measured by $\eta_3 PN$. When the pressure gradient becomes sufficiently large, the equilibrium solution of the system (in S-shape) becomes unstable and a fluctuating flow is generated, which is described by $\gamma PN$. The two terms $-(\beta_1 N + \beta_2 N^2)$ describe the effect of the dissipation that causes the fluctuation energy $N$ to be self-damped. $\alpha_1$ and $\alpha_2$ measure the rates of mutual effects of $N$ and $F$, causing $N$ to decrease and $F$ to increase, respectively. The value of $\alpha_1$ is usually estimated by a linear fit from plotting $\gamma PN - \beta_1 - \beta_2 N$, as a function in $F_H$, and the value of $\alpha_2$ is chosen to approximately reproduce the frequency of oscillations. $\mu$ is a debugging parameter, and the coefficients $\varphi_1$, $\varphi_2$ and $\varphi_3$ are chosen to obtain the best approximation of $F_H$. Note that in system (1) the state variables $P$, $N$ and $F$ are nonnegative, and all the parameters take positive real values.

In [15], the authors carried out a bifurcation analysis on system (1) and studied the stability of equilibrium solutions and numerically demonstrated the existence of limit cycles. However, the number of bifurcating limit cycles and the center-focus conditions were not considered in [15]. It has been noted that identifying weak foci and centers in 3-d systems can cause great complexity in computation (e.g. see [16]). Moreover, the criteria for determining whether a singular point in planar dynamical systems is a center or focus, such as time-reversibility and integrability [17], are no longer applicable for 3-d systems, because application of center manifold theory to higher-dimensional dynamical systems merely yields approximation on invariant manifolds. To determine the number and stability of limit cycles bifurcating from a singular point, it needs to compute Lyapunov constants, which are usually obtained by applying one of the three methods: the method of normal forms [8,18,19], Poincaré return map [20], and Lyapunov function method [21,22]. For 3-d dynamical systems associated with Hopf bifurcation, we can combine center manifold theory and normal form theory to compute Lyapunov constants $L_k$ with the aid of a computer algebra system such as Maple or Mathematica (e.g., see [8,19,23]).

2. Simplification of model (1)

To simplify the analysis for model (1), we first introduce the following scalings,

$$\begin{align*}
P &\to x_1, \quad N \to x_2, \quad F \to x_3, \quad r_1 \to r^2 \mu, \quad q_1 \to q^2, \quad \chi \to a_1^2 \mu, \quad \eta_1 \to n_1^2 \mu, \\
\eta_2 &\to n_2^2 \mu, \quad \eta_3 \to n_3^2 \mu, \quad \varphi_1 \to c_1^2 \mu, \quad \varphi_2 \to c_2^2 \mu, \quad \varphi_3 \to c_3^2 \mu, \quad \beta_1 \to b_1^2 \mu, \\
\beta_2 &\to b_2^2 \mu, \quad \gamma \to a_2^2 \mu, \quad \alpha_1 \to a_3^2 \mu, \quad \alpha_2 \to a_4^2 \mu, \quad t \to \frac{1}{\alpha_1} t,
\end{align*}$$

(2)

under which system (1) becomes

$$\begin{align*}
\dot{x}_1 &= q^2 r^2 - a_1^2 x_1 - n_1^2 x_2 + c_2^2 x_3 - n_2^2 x_2^2 - c_2^2 x_3^2 + c_3^2 x_1 x_3 + n_3^2 x_1 x_2, \\
\dot{x}_2 &= x_2 (a_2^2 x_1 - b_2^2 x_2 - a_3^2 x_3), \\
\dot{x}_3 &= x_3 (a_4^2 x_2 - 1),
\end{align*}$$

(3)
where the state variables $x_i, i = 1, 2, 3$ are nonnegative, and all the parameters $q, r, n_1, n_2, a_i (i = 1, 2, 3, 4), b_1, b_2$ and $c_i (i = 1, 2, 3)$ take nonzero real values. Note that we have transformed the original positive parameters to nonzero parameters for convenience in computation.

It is easy to show that system (3) has three equilibrium solutions:

$$
E_a = \left(\frac{r^2 q^2}{a_1^2}, 0, 0\right), \quad E_1 = \left(\frac{b_1^2 I_1 + b_2^2}{a_2^2}, I_1, 0\right), \quad E_2 = \left(\frac{a_1^2 I_2 + b_2^2 a_4^2 + b_2^2}{a_4^2 a_4^2}, 1, \frac{I_2}{a_4^2}\right),
$$

where $I_1$ and $I_2$ are determined from the above two polynomial equations:

\begin{align*}
(a_2^2 n_2^2 - b_2^2 n_3^2) I_1^2 + (a_2^2 b_2^2 + a_2^2 n_1^2 - b_2^2 n_3^2) I_1 - r^2 q^2 a_2^2 + a_3^2 b_1^2 &= 0, \\
(a_2^2 c_2 - a_3^2 c_3) I_2^2 + [(a_2^2 a_3^2 + a_2^2 c_1^2 - b_1^2 c_3^2) a_4^2 - a_3^2 n_3^2 - b_2^2 c_3^2] I_2 &= 0, \\
(a_2^2 b_1^2 - a_2^2 q^2 r^2) a_4^2 + (a_2^2 b_2^2 + a_3^2 n_1^2 - b_2^2 n_3^2) a_4^2 - b_2^2 n_3^2 + n_2^2 a_4^2 &= 0.
\end{align*}

The solutions of $I_1$ and $I_2$ obtained from the above two polynomial equations can be classified into two categories under the conditions: (A) $a_2^2 n_2^2 - b_2^2 n_3^2 = a_2^2 c_2 - a_3^2 c_3 = 0$; and (B) $a_2^2 n_2^2 \neq b_2^2 n_3^2$ or $a_2^2 c_2 \neq a_3^2 c_3$. For Case (A) the model can have three equilibria; while for Case (B), there may exist at most five equilibria. In this paper, we focus on Case (A) for which the three equilibria may be stable for certain parameter values. Therefore, solving the equation $a_2^2 n_2^2 - b_2^2 n_3^2 = 0$ for $a_2$ and $a_2^2 c_2 - a_3^2 c_3 = 0$ for $c_2$ we obtain

$$a_2^2 = \frac{b_2^2 n_3^2}{n_2^2}, \quad c_2 = \frac{a_2^2 c_2 n_3^2}{b_2^2 n_3^2},$$

which are substituted into system (3) to yield the following 3-d quadratic system,

\begin{align*}
&x_1 = q^2 r^2 - a_1^2 x_1 - n_1^2 x_2 - c_1^2 x_3 - n_2^2 x_2^2 - \frac{a_3^2 c_2^2 n_3^2}{b_2^2 n_3^2} x_3^2 + n_3^2 x_1 x_2 + c_3^2 x_1 x_3, \\
&x_2 = x_2 \left(\frac{b_2^2 n_3^2}{n_2^2} x_1 - b_1^2 - b_2^2 x_2 - a_3^2 x_3\right), \\
&x_3 = x_3 (a_4^2 x_2 - 1).
\end{align*}

Then, the equilibrium solutions of system (5) can be rewritten (with new names $E_L$ and $E_H$) as

$$E_a = \left(\frac{r^2 q^2}{a_1^2}, 0, 0\right), \quad E_L = \left(S_1 \frac{n_2}{S_3}, S_2 \frac{n_2}{S_3}, 0\right), \quad E_H = \left(\frac{n_2^2 F_1}{F_2 a_4^2 b_2^2 n_3^2}, \frac{1}{a_4^2}, \frac{a_2^2 S_2 - S_3}{F_2}\right),$$

where

\begin{align*}
S_1 &= b_1^2 q^2 r^2 + b_1^2 (b_2^2 n_1^2 - b_1^2 n_2^2), \\
S_2 &= b_2^2 q^2 r^2 - a_1^2 b_1^2 n_2^2, \\
S_3 &= a_1^2 b_1^2 n_2^2 + n_2^2 (b_2^2 n_1^2 - b_1^2 n_2^2), \\
F_1 &= (a_2^2 c_1^2 n_3^2 - c_1^2 n_3^2) b_2^2 + [(a_2^2 q^2 r^2 + b_2^2 c_1^2) n_3^2 a_4^2 - (a_2^2 n_1^2 n_3^2 + 2 b_1^2 c_3^2 n_2^2) a_4^2 - a_3^2 n_2^2 n_3^2] b_2^2 - a_4^2 b_1^4 c_3^2 n_2^2, \\
F_2 &= [(a_2^2 c_1^2 n_3^2 - b_1^2 c_1^2) a_4^2 - a_3^2 n_2^2 - b_2^2 c_3^2 n_2^2] + a_2^2 b_1^2 c_1^2 n_3^2.
\end{align*}

It is easy to see that the equilibrium $E_a$ exists for any positive parameter values; the equilibrium $E_L$ exists if $S_1/S_3 > 0$ and $S_2/S_3 > 0$; and the equilibrium $E_H$ exists if $F_1/F_2 > 0$ and $(a_4^2 S_2 - S_3)/F_2 > 0$. Further, note that both $E_L$ and $E_H$ are possible to be of center-focus type, while $E_a$ can be only node or saddle. We apply the method of normal forms to prove that system (5) can have three small-amplitude limit cycles bifurcating from $E_H$. Moreover, it can be shown that the equilibrium $E_L$ is a center, restricted to a global center manifold,
3. Weak focus at the equilibrium $E_H$

In [15], the authors present some qualitative properties for the equilibria of system (1). The reduced system (5) with less number of parameters helps us in simplifying the computation of normal forms and focus values. In the following, we first consider the qualitative properties of $E_s$. The Jacobian matrix of system (5) evaluated at $E_s$ is given by

$$J(E_s) = \begin{bmatrix}
-a_1^2 & n_3 q^2 r^2 - a_1^2 n_1^2 & c_3 q^2 r^2 - a_1^2 c_1^2 a_1^2 \\
0 & \frac{S_2}{a_1^2 n_2^2} & 0 \\
0 & 0 & -1
\end{bmatrix},$$

where

$$\lambda_1 = -1, \quad \lambda_2 = -a_1^2, \quad \lambda_3 = \frac{S_2}{a_1^2 n_2^2}.$$ 

All the three eigenvalues are real, and $\lambda_1$ and $\lambda_2$ are negative constants. Thus, $E_s$ is a stable node if $S_2 < 0$ and a saddle if $S_2 > 0$.

Next, consider the equilibrium $E_H$. In [15], the stability of equilibria was discussed, and numerical bifurcation software was used to plot bifurcation diagrams where Hopf bifurcation was identified, which is based on a linear analysis. In this section, we will further investigate the Hopf bifurcation in system (5), and prove the existence of three limit cycles. In order for system (5) to have a Hopf critical point on the equilibrium solution $E_H$ and make the computation of focus values manageable, we further let

$$r = c_1 = a_3 = a_1 = 1, \quad q = \frac{\sqrt{2}}{c_3}, \quad b_1 = \frac{b_2 n_3}{c_3 n_2},$$

$$S_4 = a_4^2 b_2^2 n_3^2 - b_2^2 c_3^2 n_2^2 + a_4^2 n_2^2 - n_2^2 n_3^2,$$

$$n_1 = \frac{\sqrt{S_4[a_4^2 b_2^2 n_3^2(a_4^2 + b_2^2 - n_3^2) + (2n_3^2 - b_2^2 n_3^2 - c_3^2 n_2^2)]}}{S_4 n_3 c_3}.$$ 

Thus, under the above conditions, the linearized system of (5) evaluated at the $E_H$ contains a non-zero real eigenvalue and two purely imaginary eigenvalues. Next, introducing the transformation,

$$x_1 = \frac{n_3^2 F_3}{S_4 a_1^2 b_2^2 n_3^2} z_3,$$

$$x_2 = \frac{1}{a_1^2} + z_1 - \frac{n_3^2 c_3^2 F_3(a_4^2 - b_2^2 c_3^2 - n_3^2)}{a_1^2 b_2^2 n_3^2} z_3,$$

$$x_3 = \frac{a_4^2 b_2^2 n_3^2(a_4^2 + b_2^2 - n_3^2)}{S_4 c_3} - \frac{b_2 a_1^2 n_3^2(a_4^2 + b_2^2 - n_3^2)}{S_4 c_3} z_2 + z_3,$$

$$t \rightarrow \frac{c_3 n_2}{b_2 n_3} t,$$

where

$$F_3 = \left\{ a_4^4 b_2^2 n_3^2 - b_2^2 n_3^2(a_4 - n_3)(a_4 + n_3)c_3^4 + \left[ (-b_2^2 n_3^2 + n_3^2)a_4^2 - n_3^2[b_2^2 - b_2^2 n_3^2 + 2n_3^2]a_4^2 + n_2^2 n_3^2]c_3^2 \right] S_4 + a_4^4 b_2^2 c_3^2 n_4^2(a_4^2 + b_2^2 - n_3^2), \right\}$$

and make the computation of focus values manageable, we further let
into (5), we obtain
\begin{align*}
\dot{z}_1 &= z_2 + A_{002}z_2^2 + A_{011}z_2z_3 + A_{020}z_3^2 + A_{101}z_1z_3 + A_{110}z_1z_2, \\
\dot{z}_2 &= -z_1 + B_{002}z_2^2 + B_{011}z_2z_3 + B_{020}z_3^2 + B_{101}z_1z_3 + B_{110}z_1z_2, \\
\dot{z}_3 &= C_{001}z_3 + C_{002}z_3^2 + C_{011}z_2z_3 + C_{020}z_3^2 + C_{101}z_1z_3 + C_{110}z_1z_2,
\end{align*}

(9)
where the coefficients $A_{ijk}$, $B_{ijk}$ and $C_{ijk}$ are given in the website: http://www.apmaths.uwo.ca/~pyu.

Clearly, under the conditions given in (7), the singular point $E_H$ of system (5) corresponds to the origin of (9), which is a Hopf-type critical point. We have the following result.

**Theorem 3.1.** Under the conditions given in (7), there exist parameter values for system (1) such that at least three limit cycles can bifurcate from $E_H$ within some, hence every, two-dimensional invariant manifold through $E_H$.

**Proof.** In order to make the computation of focus values manageable and find the parameter values satisfying the conditions, we further set $a_4 = 10$ and $n_2 = n_3 = 1$ in system (9). Then, we apply center manifold theory and the method of normal forms, as well as the Maple program in [23] to system (9) to obtain the focus values. In particular, $v_1$ and $v_2$ are given by
\begin{align*}
v_1 &= -\frac{125000(b_2^2c_3^2 - 100b_2^2 - 99)b_2(b_2^2 + 99)c_3^3F_4}{(b_2^2c_3^2 - 99)F_6F_7}, \\
v_2 &= -\frac{312500000(b_2^2c_3^2 - 100b_2^2 - 99)(b_2^2 + 99)c_3^3F_5}{9b_2(b_2^2c_3^2 - 99)^3F_6^3F_7^2F_8},
\end{align*}
where the lengthy expressions of the polynomials $F_i, i = 4, 5, \ldots, 8$ are given in terms of $b_2$ and $c_3$, and can be found in the website: http://www.apmaths.uwo.ca/~pyu.

In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin of system (9), we use the coefficients $b_2$ and $c_3$ to solve the two polynomial equations, $F_4 = F_5 = 0$. If a solution of $F_4 = F_5 = 0$ yields $v_3 \neq 0$, we may obtain two small-amplitude limit cycles by properly perturbing $b_2$ and $c_3$. To achieve this, we first compute the resultant of $F_4$ and $F_5$, yielding
\begin{align*}
F_{45} &= 4.459990583806 \times 10^{940}c_3^{124}(c_3^3 - 99)^{56}(c_3^4 - 200c_3^2 + 299)^4 \\
&\times (275c_3^{10} - 109175c_3^6 + 2128419c_3^6 + 3004576243c_3^4 - 137557848532c_3^2 + 71109368756)^4 \\
&\times (33275c_3^{14} - 13105026c_3^{12} + 464987348c_3^{10} + 6341100342c_3^8 - 7908729138339c_3^6 \\
&+ 296555246006244c_3^4 - 265954686172548c_3^2 + 982858205776)^4 F_{45a},
\end{align*}
where $F_{45a}$ is a polynomial in $c_3^2$, given in the website: http://www.apmaths.uwo.ca/~pyu.

Now, to find the solutions of $F_4 = F_5 = 0$, we must solve $F_{45} = 0$. It is seen from $F_{45}$ that besides the factor $F_{45a}$, there are five polynomial factors which may also yield solutions $c_3^2$ for $F_{45} = 0$. However, a careful verification shows that the solutions solved from these five factors do not satisfy the equations $F_4 = F_5 = 0$. Thus, possible solutions for $F_{45} = 0$ only come from the positive roots of $F_{45a}$ (since $c_3^2 > 0$). It can be shown that the polynomial $F_{45a}$ has 7 positive roots for $c_3^2$, which in turn yield corresponding 14 solutions for $b_2$, among them only eight sets of solutions $c_3$ and $b_2$ satisfy the equations, $F_4 = F_5 = 0$. Then, under the conditions, $C_{001} < 0$ and $N_1 \geq 0$, we choose one of the solutions as follows:
\begin{align*}
b_2 &= -0.53868839285 \cdots, \\
c_3 &= -0.61448088908 \cdots,
\end{align*}
under which
\begin{align*}
v_1 &= v_2 = 0, \\
v_3 &= -5.92361180765 \cdots \times 10^8 \neq 0.
\end{align*}
Moreover, a direct calculation shows that the Jacobian evaluated at the critical values is

\[
\det \left[ \frac{\partial (v_1,v_2)}{\partial (b_2,c_3)} \right] = -6.44212213245 \cdots \times 10^8 \neq 0,
\]

implying that system (9) can indeed have two small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Further, a linear perturbation on \( n_1 \) generates a third small-amplitude limit cycle. Thus, system (1) can have at least three limit cycles around the equilibrium \( E_H \).

The proof of Theorem 3.1 is complete. \( \Box \)

Finally, we point out that for the singular point \( E_L \), one can identify the Hopf bifurcation condition and then apply normal form theory and global center manifold theory [24] to prove that when \( E_L \) is a Hopf singular point, it must be a center on a global center manifold. The proof is straightforward and is thus omitted.

4. Conclusion

In this paper, we have studied a convection model and paid particular attention on bifurcation of limit cycles. We have shown that the model has three distinct equilibria, and that there exist parameter values for the model to have three small-amplitude limit cycles around one of the equilibria.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.aml.2019.106019.

References