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Twelve Limit Cycles in 3D Quadratic Vector Fields with Z_3 Symmetry

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This paper is concerned with the number of limit cycles bifurcating in three-dimensional quadratic vector fields with Z_3 symmetry. The system under consideration has three fine focus points which are symmetric about the z-axis. Center manifold theory and normal form theory are applied to prove the existence of 12 limit cycles with 4–4–4 distribution in the neighborhood of three singular points. This is a new lower bound on the number of limit cycles in three-dimensional quadratic systems.

Keywords: Three-dimensional quadratic system; Z_3-equivariant vector field; normal form; limit cycle.

1. Introduction

Dynamical systems can exhibit the well-known self-sustained oscillation, leading to limit cycles, which arise in almost all fields of science and engineering including physics, mechanics, electronics, ecology, economy, biology, finance, etc. Limit cycle theory plays an important role not only in the theoretical study of dynamical systems, but also in practical applications. The phenomenon of limit cycle bifurcation was first discovered by Poincaré [1882] in late 19th century, he developed a breakthrough qualitative theory of differential equations, which can be used to study the general behavior of a dynamical system without solving the differential equations. The later development was most motivated by the well-known Hilbert's 16th problem, one of the 23 mathematical problems proposed by Hilbert in 1900 [Hilbert, 1902]. Recently, a modern version of the second part of Hilbert's 16th problem was formulated by Smale [1998], which was chosen as one of his 18 most challenging mathematical problems for the 21st century. The second part of the Hilbert's 16th problem is to find the upper bound, called the Hilbert number denoted by $H(n)$, on the number of limit cycles that planar polynomial systems of degree n can have. In early 1990's, Il'yashenko and Yakovenko [1991], and Écalle [1992] independently proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, four limit cycles with $(3, 1)$ distribution were obtained almost 40 years ago [Shi, 1980; Sun & Shu, 1979], showing

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that $H(2) \geq 4$. This result was also proved recently for near-integrable quadratic systems [Yu & Han, 2012]. However, this problem is not even completely solved for general quadratic systems, i.e. $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the lower bound of the Hilbert number. So far, the best result for cubic polynomial systems is $H(3) \geq 13$ [Li & Liu, 2010; Li *et al.*, 2009]. Note that in these studies, the 13 limit cycles are distributed around several singular points. If the problem is restricted to the vicinity of isolated singular points, it is equivalent to investigating generalized Hopf bifurcations, and the main tasks will be computing the focus values and determining the center conditions. For quadratic systems, it is well known that the maximum number of small-amplitude limit cycles around an isolated singular point of such a system is three [Bautin, 1952], and center conditions have been obtained and classified explicitly. For cubic systems, many results have been obtained, divided into two categories. For systems with an elementary focus, the best result obtained so far is nine limit cycles [Yu & Corless, 2009; Chen *et al.*, 2013; Lloyd & Pearson, 2012]. On the other hand, for systems with a center, there are also a few results obtained in the past two decades [Żołądek, 1995; Yu & Han, 2011; Tian & Yu, 2016; Bondar & Sadovskii, 2008]. Recently, the existence of 12 small-amplitude limit cycles around a single singular point was proved by Yu and Tian [2014]. However, the center problem has not been solved for cubic polynomial systems. A comprehensive review on the study of Hilbert's 16th problem may be found in a survey article [Li, 2003].

Bifurcation of limit cycles due to Hopf bifurcation is common in real applications, but real systems often have dimension higher than two [Han & Yu, 2012; Zhang *et al.*, 2014]. When the dimension of a dynamical system associated with Hopf bifurcation is more than two, center manifold theory is usually applied together with normal form theory. In the 1990's, computation of center manifold and normal forms was extensively studied and some efficient computational methods were developed (e.g. see [Han & Yu, 2012; Guckenheimer & Holmes, 1983]). For three-dimensional dynamical systems, a lot of results have been obtained on bifurcation of limit cycles. Surprisingly, it has been shown that unlike two-dimensional systems, a simple quadratic three-dimensional system can have infinitely many small-amplitude limit cycles, which appear on an infinite family of algebraic invariant surfaces [Romanovski & Shafer, 2016; Bulgakov & Grin, 1996]. However, finding an upper bound for the cyclicity of singular points in quadratic three-dimensional systems, restricted to finite center manifolds, is still a very challenging task. Recently, some results have been obtained on Hopf bifurcation of three-dimensional polynomial differential systems. Wang *et al.* [2010] studied Hopf bifurcation in a class of three-dimensional nonlinear dynamical systems and obtained five small-amplitude limit cycles around a singular point. Tian and Yu [2014] constructed a simple three-dimensional quadratic system and, with the help of an explicit recursive formula for computing the normal form of general n-dimensional differential systems associated with Hopf bifurcation [Tian & Yu, 2013], proved the existence of seven small-amplitude limit cycles in the vicinity of a singular point. Further, Yu and Han [2015] showed that ten small-amplitude limit cycles can bifurcate from an isolated center-type singular point in a three-dimensional quadratic system with quadratic perturbation. Recently, Du *et al.* [2016] investigated Hopf bifurcation in a class of three-dimensional quadratic systems and proved the existence of ten small-amplitude limit cycles around two symmetric singular points.

To determine the number and stability of bifurcating limit cycles associated with a singular point, we need to compute Lyapunov constants. There mainly exist three methodologies for computing Lyapunov constants: the method of normal forms [Han & Yu, 2012; Farr *et al.*, 1989; Yu, 1998], the method of Poincaré return map [Andronov, 1973; Liu *et al.*, 2008], and the Lyapunov function method [Shi, 1984; Gasull & Torregrosa, 2001]. Without loss of generality, assume that the system under consideration has a singularity at the origin, and that the Jacobian of the system evaluated at the origin has a purely imaginary pair: $\pm i\omega_c$. With the aid of a computer algebra system such as Maple or Mathematica (e.g. see [Han & Yu, 2012; Tian & Yu, 2013; Yu, 1998]), we can compute the normal form of the system at the origin to obtain the Lyapunov constants L_k.

We start from the following general three-dimensional quadratic system,

$$\frac{dx_1}{dt} = \sum_{i+j+k=0}^{2} A_{ijk} x_1^i x_2^j x_3^k,$$

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$$\frac{dx_2}{dt} = \sum_{i+j+k=0}^{2} B_{ijk} x_1^i x_2^j x_3^k,$$
$$\frac{dx_3}{dt} = \sum_{i+j+k=0}^{2} C_{ijk} x_1^i x_2^j x_3^k. \tag{1}$$

In order to let the system (1) have an equilibrium at $(1, 0, 0)$, and make it invariant under the following transformation of rotation,

$$\begin{aligned} x_1 &= \tilde{x}_1 \cos\left(\frac{2\pi}{3}\right) - \tilde{x}_2 \sin\left(\frac{2\pi}{3}\right), \\ x_2 &= \tilde{x}_1 \sin\left(\frac{2\pi}{3}\right) + \tilde{x}_2 \cos\left(\frac{2\pi}{3}\right), \\ x_3 &= \tilde{x}_3, \end{aligned} \tag{2}$$

we set

$$\begin{aligned} A_{000} &= B_{000} = A_{002} = A_{001} = B_{001} \\ &= B_{002} = C_{100} = C_{101} = C_{010} \\ &= C_{011} = C_{110} = 0, \\ A_{010} &= \frac{1}{2} A_{110} = -B_{020} = -B_{100} = B_{200}, \\ A_{101} &= B_{011}, \quad A_{011} = -B_{101}, \\ A_{020} &= A_{100} = B_{010} = -A_{200} = \frac{1}{2} B_{110}, \\ C_{020} &= C_{200}. \end{aligned}$$

In addition, we set $C_{000} = -C_{200}$ for which the system has an equilibrium at $(1, 0, 0)$. Further, let $(1, 0, 0)$ be a Hopf singular point, we let $B_{110} + C_{001} < 0$, $B_{200} > 0$, $C_{002} B_{110} B_{011} \neq 0$, and

$$\begin{aligned} C_{200} &= -\frac{1}{8B_{011}} (7B_{110}^2 + 4B_{110} C_{001} + 12B_{200}^2 + 4C_{001}^2), \\ B_{101} &= \frac{(13B_{110}^3 - 18B_{110}^2 C_{001} + 36B_{110} B_{200}^2 - 12B_{110} C_{001}^2 - 24B_{200}^2 C_{001} - 8C_{001}^3) B_{011}}{6B_{200}(7B_{110}^2 + 4B_{110} C_{001} + 12B_{200}^2 + 4C_{001}^2)}. \end{aligned} \tag{3}$$

In order to impose perturbation on the Hopf bifurcation, we introduce $A_{011} \to A_{011} - \delta$ and $B_{101} \to B_{101} + \delta$ which still satisfy $A_{011} = -B_{101}$. Under the above conditions, system (1) becomes

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{1}{2} B_{110} x_1 + B_{200} x_2 - \frac{1}{2} B_{110} x_1^2 + \frac{1}{2} B_{110} x_2^2 + 2B_{200} x_1 x_2 + B_{011} x_1 x_3 - (B_{101} + \delta) x_2 x_3, \\ \frac{dx_2}{dt} &= -B_{200} x_1 + \frac{1}{2} B_{110} x_2 + B_{200} x_1^2 - B_{200} x_2^2 + B_{110} x_1 x_2 + (B_{101} + \delta) x_1 x_3 + B_{011} x_2 x_3, \\ \frac{dx_3}{dt} &= -C_{200} + C_{001} x_3 + C_{200} x_1^2 + C_{200} x_2^2 + C_{002} x_3^2, \end{aligned} \tag{4}$$

where δ, B_{ijk} and C_{ijk} are real parameters, and $|\delta| \ll 1$. Note that system (4) is Z_3-equivariant (see [Li, 2003]) and has three Hopf critical points at $(1, 0, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$. In this paper, we study bifurcation of limit cycles in the three-dimensional system (4).

This paper is organized as follows. In the next section, for completeness and convenience, some basic formulations and preliminary results are briefly described, which will be used for proving the existence of small-amplitude limit cycles. In Sec. 3, by means of proper scaling and transformation, we compute the first four focus values associated with the singular point $(1, 0, 0)$ of system (4), and then show that this point can be a fine focus of order four, implying that at most four limit cycles can be found around this point. In Sec. 4, we prove that proper perturbations can be applied to generate four limit cycles around $(1, 0, 0)$, and thus a total of 12 limit cycles exist around the three singular points. Conclusion is drawn in Sec. 5.

2. Some Preliminary Results

In this section, we present some basic methods and preliminary results which will be used in the following sections. For computing the focus values of planar vector fields, there mainly exist three computational methods [Han & Yu, 2012]: Poincaré–Takens method [Guckenheimer & Holmes, 1983],

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the perturbation method [Yu, 1998], and the singular point value method [Liu & Li, 1990]. However, it is more computationally demanding for higher-dimensional dynamical systems. In the following, the normal form theory is briefly described for computing the focus values of general n-dimension dynamical systems. The general normal form theory can be found in [Guckenheimer & Holmes, 1983; Chow *et al.*, 1994] and computations using computer algebra systems can be found in [Han & Yu, 2012; Tian & Yu, 2013].

Consider the following general n-dimensional differential system:

$$\dot{z} = Az + f(z), \quad z \in R^n, \quad f : R^n \to R^n, \tag{5}$$

where Az and $f(z)$ represent the linear and nonlinear terms of the system, respectively. We suppose that $f(0) = Df(0) = 0$. Further, it is assumed that $f(z)$ is analytic and can be expanded in Taylor series about z. Moreover, from the viewpoint of real applications, we assume that system (5) only contains stable and center manifolds. In normal form computation, the first step is usually to introduce a linear transformation into (5) such that its linear part becomes the Jordan canonical form. Suppose under the linear transformation $z = T(x, y)^t$, system (5) becomes

$$\begin{aligned} \dot{x} &= J_1 x + f_1(x, y), \quad x \in R^k, \quad f_1 : R^n \to R^k, \\ \dot{y} &= J_2 y + f_2(x, y), \quad y \in R^{n-k}, \quad f_2 : R^n \to R^{n-k}, \end{aligned} \tag{6}$$

where $J_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$, and $J_2 = \mathrm{diag}(\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n)$, with $\mathrm{Re}(\lambda_j) = 0$, $j = 1, 2, \ldots, k$, and $\mathrm{Re}(\lambda_j) < 0$, $j = k+1, \ldots, n$. The second step is to apply center manifold theory [Carr, 2012] to system (6) so that y can be expressed as $y = H(x)$, satisfying $H(0) = DH(0) = 0$. Therefore, the first equation of (6) can be rewritten as

$$\begin{aligned} \dot{x} &= J_1 x + f_1(x, H(x)) \\ &= J_1 x + f_1^2(x) + f_1^3(x) + \cdots + f_1^s(x) + \cdots, \end{aligned} \tag{7}$$

where $f_1^j \in M_j$, $j = 2, 3, \ldots$, M_j, defining a linear space of vector fields whose elements are homogeneous polynomials of degree j. Equation (7) describes the dynamics on the center manifold of system (6), and $H(x)$ can be determined from the following equation:

$$\begin{aligned} &DH(x)[J_1 x + f_1(x, H(x))] \\ &\quad - J_2(H(x)) - f_2(x, H(x)) = 0. \end{aligned} \tag{8}$$

Next, using normal form theory, we introduce the near-identity transformation:

$$\begin{aligned} x &= u + Q(u) \\ &= u + q_2(u) + q_3(u) + \cdots + q_s(u) + \cdots, \end{aligned} \tag{9}$$

where $q_j \in M_j$, $j = 2, 3, \ldots$ into (7) to obtain the normal form,

$$\begin{aligned} \dot{u} &= J_1 u + C(u) = J_1 u + N_2(u) + N_3(u) \\ &\quad + \cdots + N_s(u) + \cdots, \end{aligned} \tag{10}$$

where $N_j \in M_j$, $j = 2, 3, \ldots$.

Theoretically, it seems computing center manifold and normal form is straightforward. However, practically it is not an easy task to design an efficient algorithm. Recently, an explicit recursive formula has been developed for computing the normal form together with center manifold. We omit the detailed formulas and algorithms, as well as the Maple program here, which can be found in [Tian & Yu, 2013].

Suppose that we have obtained the normal form of system (5), given in the polar coordinates up to the $(2k+1)$th-order term:

$$\begin{aligned} \dot{r} &= r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k}), \\ \dot{\theta} &= \omega_c + t_1 r^2 + t_2 r^4 + \cdots + t_k r^{2k}, \end{aligned} \tag{11}$$

where r and θ denote the amplitude and phase of motion, respectively. v_k and t_k are expressed in terms of the original system's coefficients. v_k is called the kth-order focus value of the origin. The zero-order focus value v_0 is obtained from linear analysis.

To find k small-amplitude limit cycles of system (5) around the origin, we first find the conditions based on the original system's coefficients such that $v_0 = v_1 = v_2 = \cdots = v_{k-1} = 0$, but $v_k \neq 0$. Then appropriate small perturbations are performed to prove the existence of k limit cycles. Note that any focus value v_k is a polynomial in terms of the coefficients of system (5). Thus, we need to use the Maple built-in command "resultant" to solve a system of multivariate polynomial equations. Denote by $\mathrm{R}[x_1, x_2, \ldots, x_r]$ the polynomial

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ring of multivariate polynomials in $x_1, x_2, \ldots, x_r$ with coefficients in R. Let

$$p(x_1, x_2, \ldots, x_r) = \sum_{i=0}^{m} p_i(x_1, x_2, \ldots, x_{r-1}) x_r^i, \quad q(x_1, x_2, \ldots, x_r) = \sum_{i=0}^{n} q_i(x_1, x_2, \ldots, x_{r-1}) x_r^i \tag{12}$$

be two polynomials in $\mathrm{R}[x_1, x_2, \ldots, x_r]$ respectively with positive degrees m and n in x_r. The following matrix is called the Sylvester matrix of p and q with respect to x_r,

$$\mathrm{Syl}(p, q, x_r) = \begin{pmatrix} p_m & p_{m-1} & \cdots & p_0 & & & \\ & p_m & p_{m-1} & \cdots & p_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & p_m & p_{m-1} & \cdots & p_0 \\ q_n & q_{n-1} & \cdots & q_0 & & & \\ & q_n & q_{n-1} & \cdots & q_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \\ & & & q_n & q_{n-1} & \cdots & q_0 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} m \end{matrix} \tag{13}$$

whose determinant is called the resultant of p and q with respect to x_r, denoted by $\mathrm{Res}(p, q, x_r)$. We have the following lemma.

Lemma 1 [Mishra, 1993]. *Consider two multivariate polynomials $p(x_1, x_2, \ldots, x_r)$ and $q(x_1, x_2, \ldots, x_r)$ in $\mathrm{R}[x_1, x_2, \ldots, x_r]$ given by (12). Let $\mathrm{Res}(p, q, x_r) = h(x_1, \ldots, x_{r-1})$. Then, the following holds.*

(1) *If the real vector $\langle \alpha_1, \alpha_2, \ldots, \alpha_r \rangle \in R^r$ is a common zero of the two equations $p(x_1, x_2, \ldots, x_r)$ and $q(x_1, x_2, \ldots, x_r)$, then $h(\alpha_1, \ldots, \alpha_{r-1}) = 0$.*

(2) *Conversely, if $h(\alpha_1, \ldots, \alpha_{r-1}) = 0$, then at least one of the following four conditions is true*:

(a) $p_m(\alpha_1, \ldots, \alpha_{r-1}) = \cdots = p_0(\alpha_1, \ldots, \alpha_{r-1}) = 0$,
(b) $q_n(\alpha_1, \ldots, \alpha_{r-1}) = \cdots = q_0(\alpha_1, \ldots, \alpha_{r-1}) = 0$,
(c) $p_m(\alpha_1, \ldots, \alpha_{r-1}) = q_n(\alpha_1, \ldots, \alpha_{r-1}) = 0$,
(d) *for some $\alpha_r \in R$, $\langle \alpha_1, \ldots, \alpha_r \rangle$ is a common zero of both $p(x_1, \ldots, x_r)$ and $q(x_1, \ldots, x_r)$.*

The following lemma gives sufficient conditions for the existence of small-amplitude limit cycles. (The proof can be found in [Han & Yu, 2012].)

Lemma 2. *Suppose that the focus values depend on k parameters, expressed as*

$$v_j = v_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_k), \quad j = 0, 1, \ldots, k, \tag{14}$$

satisfying

$$v_j(0, \ldots, 0) = 0, \quad j = 0, 1, \ldots, k-1,$$

$$v_k(0, \ldots, 0) \neq 0 \quad \text{and}$$

$$\det\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)}(0, \ldots, 0)\right] \neq 0. \tag{15}$$

Then, for any given $\epsilon_0 > 0$, there exist $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ and $\delta > 0$ with $|\epsilon_j| < \epsilon_0$, $j = 1, 2, \ldots, k$ such that the equation $\dot{r} = 0$ has exactly k real positive roots [i.e. system (5) has exactly k limit cycles] in a δ-ball with its center at the origin.

3. Focus Values of System (4)

In this section, we compute the focus values of system (4). Consider the three-dimensional quadratic polynomial system (4). Due to the symmetry of the system, the focus values associated with the singular points, $(1, 0, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ are same. Hence, we only need to consider the Hopf bifurcation at the singular point $(1, 0, 0)$.

To simplify the analysis, introducing the scalings,

$$\begin{aligned} &x_1 \to x_1, \quad x_2 \to x_2, \\ &x_3 \to \frac{B_{200}}{C_{002}} x_3, \quad t \to \frac{1}{B_{200}} t, \\ &B_{110} \to b_{110} B_{200}, \quad B_{011} \to C_{002} b_{011}, \\ &C_{001} \to B_{200} c_{001}, \quad \delta \to C_{002} \delta, \end{aligned} \tag{16}$$

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into system (4), we obtain

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{1}{2}b_{110}x_1 - \frac{1}{2}b_{110}x_1^2 + b_{011}x_3x_1 - b_{101}x_3x_2 + 2x_1x_2 + x_2 + \frac{1}{2}b_{110}x_2^2 - \delta x_3x_2,\\ \frac{dx_2}{dt} &= b_{011}x_2x_3 + b_{101}x_3x_1 + b_{110}x_1x_2 + x_1^2 - x_2^2 + \frac{1}{2}b_{110}x_2 - x_1 + \delta x_3x_1,\\ \frac{dx_3}{dt} &= x_3^2 + c_{200}x_1^2 + c_{200}x_2^2 + c_{001}x_3 - c_{200},\end{aligned} \tag{17}$$

where

$$c_{200} = -\frac{S_2}{8b_{011}}, \quad b_{101} = \frac{(13b_{110}^3 - 18b_{110}^2c_{001} + 36b_{110} - 12b_{110}c_{001}^2 - 24c_{001} - 8c_{001}^3)b_{011}}{6S_2}, \tag{18}$$

with

$$\begin{aligned}S_1 &= b_{110} + c_{001},\\ S_2 &= 7b_{110}^2 + 4b_{110}c_{001} + 4c_{001}^2 + 12,\end{aligned} \tag{19}$$

satisfying $S_1 < 0$ and $b_{110}b_{011} \neq 0$, and $|\delta| \ll 1$.

In order to study the limit cycle bifurcation around the Hopf critical point $(1, 0, 0)$, we need to compute the focus values. To further simplify the system, we introduce the following transformation,

$$\begin{aligned}x_1 &= \frac{4b_{011}c_{001}}{S_2}z_1 + \frac{4S_1b_{011}}{S_2}z_2 + z_3 + 1,\\ x_2 &= -\frac{(3b_{110}^2 - 6b_{110}c_{001} + 12)b_{011}}{3S_2}z_1\\ &\quad + \frac{2S_1(b_{110} - 2c_{001})b_{011}}{3S_2}z_2\\ &\quad + \frac{S_2 + 2b_{110}(2S_1 + b_{110})}{12b_{110}}z_3,\\ x_3 &= z_1 - \frac{S_2}{4b_{011}b_{110}}z_3, \quad t \to \frac{-1}{b_{110} + c_{001}}t,\end{aligned} \tag{20}$$

under which (17) becomes

$$\begin{aligned}\frac{dz_1}{dt} &= A_{10}z_1 + z_2 + A_{30}z_3 + A_{11}z_1^2 + A_{12}z_1z_2\\ &\quad + A_{13}z_1z_3 + A_{22}z_2^2 + A_{23}z_2z_3 + A_{33}z_3^2,\\ \frac{dz_2}{dt} &= B_{10}z_1 + B_{30}z_3 + B_{11}z_1^2 + B_{12}z_1z_2\\ &\quad + B_{13}z_1z_3 + B_{22}z_2^2 + B_{23}z_2z_3 + B_{33}z_3^2,\\ \frac{dz_3}{dt} &= C_{10}z_1 + C_{30}z_3 + C_{11}z_1^2 + C_{12}z_1z_2\\ &\quad + C_{13}z_1z_3 + C_{22}z_2^2 + C_{23}z_2z_3 + C_{33}z_3^2,\end{aligned} \tag{21}$$

where the coefficients A_{ij}, B_{ij} and C_{ij} are expressed in terms of δ, b_{110}, b_{011} and c_{001}, as listed in Appendix.

Clearly, the singular point $(1, 0, 0)$ of system (4) corresponds to the origin of system (21), which is a Hopf-type critical point. Now we use the method of normal forms and the Maple program in [Tian & Yu, 2013] to compute the focus values of system (21) associated with the Hopf critical point at the origin. We have the following result.

Theorem 1. *The first four focus values at the origin of system* (*21*) *are given as follows*:

$$\begin{aligned}v_0 &= -\frac{3S_2\delta}{16b_{011}S_1^3},\\ v_1 &= -\frac{1}{10240S_1^6S_2^2}F_1,\\ v_2 &= -\frac{1}{2359296000S_1^{12}S_2^4}F_2,\\ v_3 &= -\frac{1}{532275296993280000S_1^{18}S_2^6}F_3,\\ v_4 &= -\frac{1}{813078194472678850560000000S_1^{24}S_2^8}F_4,\end{aligned} \tag{22}$$

where the lengthy polynomials F_1, F_2, F_3 and F_4 are expressed in terms of the coefficients b_{110}, b_{011} and c_{001}.

4. Existence of 12 Limit Cycles in System (4)

In this section, we present our main result of this paper, we prove the existence of four small-amplitude limit cycles around the origin of system (21), and thus the original system (4) has 12

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limit cycles due to Z_3 symmetry. Since the three focus values, v_i $(i = 1, 2, 3)$ given in (22) involve three free coefficients, b_{110}, b_{011}, and c_{001}, it may be possible to choose appropriate values of these parameters such that $v_i = 0$, $i = 1, 2, 3$, but $v_4 \neq 0$, implying that four small-amplitude limit cycles may bifurcate from the origin of system (21). Indeed we have the following result.

Theorem 2. *System (4) can have 12 limit cycles with four around each of the three singular points* $(1, 0, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ *and* $(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$.

Proof. First, we set $\delta = 0$ to make $v_0 = 0$. In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin of (21), we use the coefficients b_{110}, b_{011}, c_{001} to solve the three polynomial equations: $F_1 = F_2 = F_3 = 0$. If a solution of these equations yields $v_4 \neq 0$, then we may obtain four small-amplitude limit cycles by properly perturbing the coefficients b_{110}, b_{011} and c_{001}. To achieve this, we eliminate b_{011} from the equations $F_1 = F_2 = F_3 = 0$ to obtain a solution $b_{011} = g(b_{110}, c_{001})$, and two resultants:

$$F_{12} = S_3 F_{12a}(b_{110}, c_{001}),$$
$$F_{13} = S_3 F_{13a}(b_{110}, c_{001}),$$

where

$$\begin{aligned} S_3 = S_1 S_2 (b_{110}^2 - 4b_{110}c_{001} + 4c_{001}^2 + 4) \\ \times (13b_{110}^4 - 44b_{110}^3 c_{001} + 24b_{110}^2 c_{001}^2 \\ + 16b_{110}c_{001}^3 + 16c_{001}^4 + 72b_{110}^2 \\ - 144b_{110}c_{001} + 144). \end{aligned}$$

Here, the lengthy expressions of the polynomials $g(b_{110}, c_{001})$, $F_{12a}(b_{110}, c_{001})$ and $F_{13a}(b_{110}, c_{001})$ are omitted here for brevity.

Finally, we need to solve the two polynomial equations: $F_{12a}(b_{110}, c_{001}) = F_{13a}(b_{110}, c_{001}) = 0$ to find the solutions of b_{110} and c_{001}. It can be shown that these two equations have 21 sets of real solutions, but only three of them satisfy the equations $F_1 = F_2 = F_3 = 0$ with $b_{011} = g(b_{110}, c_{001})$ and $b_{110} + c_{001} < 0$. We choose one of them as follows:

$$b_{011} = 2.029425183\ldots,$$
$$b_{110} = -1.797263080\ldots,$$
$$c_{001} = -2.745841812\ldots,$$

under which

$$v_1 = v_2 = v_3 = 0, \quad v_4 = -0.000329518\ldots \neq 0.$$

Moreover, a direct calculation shows that the Jacobian evaluated at the critical values is equal to

$$\det\left[\frac{\partial(v_1, v_2, v_3)}{\partial(b_{110}, b_{011}, c_{001})}\right] = -0.0000030880\ldots \neq 0,$$

implying, by Lemma 2, that system (21) can indeed have four small-amplitude limit cycles bifurcating from the center-type singular point (the origin), and thus, system (4) can have 12 limit cycles.

The proof of Theorem 2 is complete. ∎

5. Simulation of 12 Limit Cycles

In this section, we present simulations to illustrate the existence of the 12 limit cycles arising from Hopf bifurcation, with four around each of the three equilibria: $(1, 0, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$ of system (17).

As we all know, simulating single limit cycle is straightforward and is still easy for two limit cycles. However, it is somewhat challenging for simulating three limit cycles [Kuznetsov *et al.*, 2013]. Simulating four limit cycles around a singular point is more difficult. The main difficulty comes from how to appropriately choose perturbations of the parameters from the critical point so that the truncated normal form can have four real positive roots. If the perturbations can be performed step by step and perturbing one parameter at each step, then the process is still straightforward. However, if the polynomial equations from the normal form are coupled, then finding the perturbation is much challenging. In the following, we present a method for finding the 12 limit cycles.

Assume that the associated normal form of the system is given in polar coordinates as

$$\begin{aligned} \frac{dr}{dt} &= r(v_0 + v_1 r^2 + v_2 r^4 + v_3 r^6 + v_4 r^8 + \cdots), \\ \frac{d\theta}{dt} &= 1 + \tau_0 + \tau_1 r^2 + \tau_2 r^4 + \tau_3 r^6 + \cdots, \end{aligned} \tag{23}$$

where v_k is the kth-order focus value. The first equation of (23) can be rewritten as

$$\frac{dR}{dt} = 2R(v_0 + v_1 R + v_2 R^2 + v_3 R^3 + v_4 R^4 + \cdots), \tag{24}$$

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where $R = r^2$. Then, solving for r^2 from $\frac{dr}{dt} = 0$ of (23) is equivalent to solving for positive R from $\frac{dR}{dt} = 0$ of (24). Since we are interested in small-amplitude limit cycles, we introduce the scaling $R \to \epsilon R$ $(0 < \epsilon \ll 1)$ into (24) to obtain

$$\frac{d(\epsilon R)}{dt} = 2\epsilon R(v_0 + v_1\epsilon R + v_2\epsilon^2 R^2 + v_3\epsilon^3 R^3 + v_4\epsilon^4 R^4 + \cdots). \tag{25}$$

Next, we suppose the perturbed focus values are given in the form of

$$\begin{aligned} &v_0 = K_0\epsilon^4 + o(\epsilon^4), \quad v_1 = K_1\epsilon^3 + o(\epsilon^3),\\ &v_2 = K_2\epsilon^2 + o(\epsilon^2), \quad v_3 = K_3\epsilon + o(\epsilon),\\ &v_4 = K_4 + o(1), \quad v_5 = K_5 + o(1), \ldots \end{aligned} \tag{26}$$

under which Eq. (25) becomes

$$\frac{d(\epsilon R)}{dt} = 2\epsilon^5 R[(K_0 + K_1 R + K_2 R^2 + K_3 R^3 + K_4 R^4) + \epsilon R G(\epsilon, R)], \tag{27}$$

where $G(\epsilon, R)$ is analytic at $(0, 0)$.

By implicit function theorem, when ϵ is small enough, it follows from (27) that if the equation

$$K_0 + K_1 x + K_2 x^2 + K_3 x^3 + K_4 x^4 = 0 \tag{28}$$

has four positive roots R_1, R_2, R_3 and R_4, then the equation $\frac{d(\epsilon R)}{dt} = 0$ also has four positive roots, which are sufficiently close to R_1, R_2, R_3 and R_4. This implies that system (17) has 12 limit cycles, with four on each of three center manifolds, near the circles which are the intersection of the center manifolds and the balls $(x-1)^2 + y^2 + z^2 = \epsilon R_i$, $(x+\frac{1}{2})^2 + (y - \frac{\sqrt{3}}{2})^2 + z^2 = \epsilon R_i$ and $(x+\frac{1}{2})^2 + (y + \frac{\sqrt{3}}{2})^2 + z^2 = \epsilon R_i$, $i = 1, 2, 3, 4$. Thus, we have the following result.

Theorem 3. *For system (17), with the following perturbed parameter values:*

$$\begin{aligned} \delta &= (-0.0949761581\ldots)\epsilon^4,\\ b_{011} &= -(0.29494257879\ldots)\epsilon^3 - (0.8659673227\ldots)\epsilon^2\\ &\quad - (0.1474737377\ldots)\epsilon + (2.0294251839\ldots),\\ b_{110} &= (2.3969638306\ldots)\epsilon^2 - (0.2712863236\ldots)\epsilon\\ &\quad - (1.7972630809\ldots),\\ c_{001} &= -(0.1867194953\ldots)\epsilon - (2.7458418128\ldots), \end{aligned} \tag{29}$$

the equation $\frac{d(\epsilon R)}{dt} = 0$ in (27) has four real positive roots, which are sufficiently close to 1, 2, 3 and 4, and thus correspondingly, system (17) has 12 limit cycles, with four on each of three center manifolds, near the circles which are the intersection of the center manifolds and the balls $(x-1)^2 + y^2 + z^2 = j\epsilon$, $(x + \frac{1}{2})^2 + (y - \frac{\sqrt{3}}{2})^2 + z^2 = j\epsilon$ and $(x + \frac{1}{2})^2 + (y + \frac{\sqrt{3}}{2})^2 + z^2 = j\epsilon$, $j = 1, 2, 3, 4$.

Proof. Suppose that the four positive roots of Eq. (28) are $x = 1, 2, 3, 4$. Then, we can use (28) to find K_i as

$$\begin{aligned} K_0 &= -0.0079084422\ldots,\\ K_1 &= 0.0164759212\ldots,\\ K_2 &= -0.0115331449\ldots,\\ K_3 &= 0.0032951842\ldots,\\ K_4 &= -0.0003295184\ldots. \end{aligned} \tag{30}$$

Since δ can be linearly solved from the first focus value v_0, we first consider the equations $v_1 = v_2 = v_3 = 0$ to determine the perturbed parameters b_{011}, b_{110} and c_{001}, and then use these values to find a proper perturbation on δ for v_0. So, without loss of generality, we assume that

$$\begin{aligned} b_{011} &= b_{011c} + k_{11}\epsilon + k_{12}\epsilon^2 + k_{13}\epsilon^3,\\ b_{110} &= b_{110c} + k_{21}\epsilon + k_{22}\epsilon^2 + k_{23}\epsilon^3,\\ c_{001} &= c_{001c} + k_{31}\epsilon + k_{32}\epsilon^2 + k_{33}\epsilon^3, \end{aligned} \tag{31}$$

where b_{011c}, b_{110c} and c_{001c} are critical values such that $v_1 = v_2 = v_3 = 0$. Substituting (31) into the expressions of v_1, v_2 and v_3, and expanding them in Taylor series up to ϵ^3 order, we obtain

$$\begin{aligned} v_1 &= E_{10} + E_{11}\epsilon + E_{12}\epsilon^2 + E_{13}\epsilon^3 + o(\epsilon^3),\\ v_2 &= E_{20} + E_{21}\epsilon + E_{22}\epsilon^2 + o(\epsilon^2),\\ v_3 &= E_{30} + E_{31}\epsilon + o(\epsilon), \end{aligned} \tag{32}$$

where E_{ij} are functions of b_{011}, b_{110} and c_{001}, and so the functions of k_{ij}, $i, j = 1, 2, 3$.

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Combining (26) and (30), and balancing the coefficients of like powers of ϵ give the following equations,

$$\begin{aligned} E_{10}(b_{011}, b_{110}, c_{001}) &= E_{11}(b_{011}, b_{110}, c_{001}) \\ &= E_{12}(b_{011}, b_{110}, c_{001}) = 0, \\ E_{20}(b_{011}, b_{110}, c_{001}) &= E_{21}(b_{011}, b_{110}, c_{001}) \\ &= E_{30}(b_{011}, b_{110}, c_{001}) = 0, \\ E_{13}(b_{011}, b_{110}, c_{001}) &= K_1 = 0.0164759212\ldots, \\ E_{22}(b_{011}, b_{110}, c_{001}) &= K_2 = -0.0115331449\ldots, \\ E_{31}(b_{011}, b_{110}, c_{001}) &= K_3 = 0.0032951842\ldots. \end{aligned} \tag{33}$$

Then, solving the equations in (33) yields the solutions:

$$\begin{aligned} k_{31} &= -0.1867194953\ldots, \\ k_{22} &= 2.3969638306\ldots, \\ k_{13} &= -0.2949425787\ldots, \\ k_{11} &= -0.1474737377\ldots, \\ k_{21} &= -0.2712863236\ldots, \\ k_{12} &= -0.8659673227\ldots, \\ k_{23} &= k_{33} = k_{32} = 0. \end{aligned} \tag{34}$$

Thus, the perturbed values of the parameters b_{011}, b_{110}, and c_{001} are obtained from (31) as

$$\begin{aligned} b_{011} &= -(0.29494257879\ldots)\epsilon^3 \\ &\quad - (0.8659673227\ldots)\epsilon^2 - (0.1474737377\ldots)\epsilon \\ &\quad + (2.0294251839\ldots), \\ b_{110} &= (2.3969638306\ldots)\epsilon^2 - (0.2712863236\ldots)\epsilon \\ &\quad - (1.7972630809\ldots), \\ c_{001} &= -(0.1867194953\ldots)\epsilon - (2.7458418128\ldots). \end{aligned} \tag{35}$$

Next, we assume that

$$\delta = k_{41}\epsilon + k_{42}\epsilon^2 + k_{43}\epsilon^3 + k_{44}\epsilon^4. \tag{36}$$

Substituting (35) and (36) into v_0 and expanding it in Taylor series up to ϵ^4 order results in

$$\begin{aligned} v_0 &= E_{00} + E_{01}\epsilon + E_{02}\epsilon^2 + E_{03}\epsilon^3 \\ &\quad + E_{04}\epsilon^4 + o(\epsilon^4), \end{aligned} \tag{37}$$

where E_{ij} are functions of δ and so functions of k_{4i}, $i = 1, 2, 3, 4$. Again, combining (26) and (30), and balancing the coefficients of like powers of ϵ yield the following solutions,

$$\begin{aligned} E_{00} &= E_{01} = E_{02} = E_{03} = 0, \\ E_{04} &= K_0 = -0.0079084422\ldots. \end{aligned} \tag{38}$$

Then, solving the equations in (38), we obtain the solutions,

$$\begin{aligned} k_{41} &= k_{42} = k_{43} = 0, \\ k_{44} &= -0.0949761581\ldots, \end{aligned} \tag{39}$$

for which δ is given by

$$\delta = -(0.0949761581\ldots)\epsilon^4. \tag{40}$$

Now, having obtained all the perturbed parameter values given in (29), we obtain the perturbed focus values:

$$\begin{aligned} v_0 &= (-0.0079084422\ldots)\epsilon^4 + o(\epsilon^4), \\ v_1 &= (0.0164759212\ldots)\epsilon^3 + o(\epsilon^3), \\ v_2 &= (-0.0115331449\ldots)\epsilon^2 + o(\epsilon^2), \\ v_3 &= (0.0032951842\ldots)\epsilon + o(\epsilon), \\ v_4 &= (-0.0003295184\ldots) + o(1). \end{aligned} \tag{41}$$

With the above perturbed focus values, the equation $\frac{d(\epsilon R)}{dt} = 0$ in (27) has four real positive roots which are sufficiently close to $1, 2, 3$ and 4, and thus system (17) has 12 limit cycles near the circles which are the intersections of three center manifolds and the balls $(x-1)^2 + y^2 + z^2 = j\epsilon$, $(x + \frac{1}{2})^2 + (y - \frac{\sqrt{3}}{2})^2 + z^2 = j\epsilon$ and $(x + \frac{1}{2})^2 + (y + \frac{\sqrt{3}}{2})^2 + z^2 = j\epsilon$, $j = 1, 2, 3, 4$.

The proof is complete. ■

For simulation, we choose $\epsilon = 7 \times 10^{-8}$, for which the four positive roots are obtained from $\frac{dr}{dt} = 0$ as follows:

$$r_1 = 0.0002645751\ldots, \quad r_2 = 0.0003741657\ldots,$$
$$r_3 = 0.0004582575\ldots, \quad r_4 = 0.0005291502\ldots,$$

which are approximations of the amplitudes of the four limit cycles bifurcating from $(1, 0, 0)$, and due to symmetry, other eight limit cycles are around the equilibriums $(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0)$, respectively. By Theorem 3, using $\epsilon = 7 \times 10^{-8}$, we can get the approximate values of the parameters δ, b_{011}, b_{110} and c_{001}. Then, with the program "ode45" in

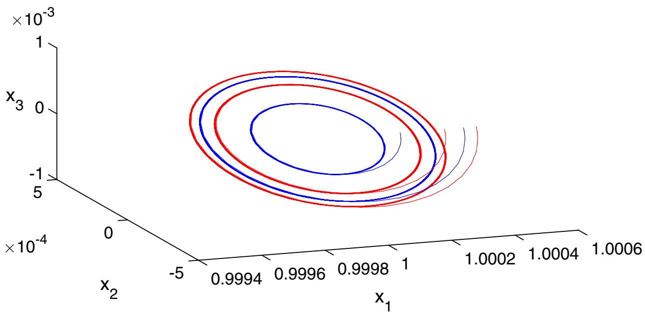

Fig. 1. Simulated four limit cycles around the equilibrium $(1, 0, 0)$ of system (17) for $\epsilon = 7 \times 10^{-8}$.

MATLAB, we obtain the simulation, as shown in Fig. 1, clearly indicating that the simulated four limit cycles agree very well with the analytical predictions.

6. Conclusion

In this paper, normal form theory and Maple software program for computing normal forms have been applied to compute the focus values of dynamical systems associated with Hopf singular point, and then to determine the number of bifurcating limit cycles near the critical points. We have shown that three-dimensional quadratic polynomial vector fields can have at least 12 small-amplitude limit cycles around three critical points. This is a new lower bound obtained for three-dimensional quadratic systems.

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Appendix

The coefficients A_{ij}, B_{ij} and C_{ij} in system (21) are listed below, where S_1 and S_2 are given in (19).

$$
\begin{aligned}
A_{10} &= \frac{3S_2\delta}{8b_{011}S_1^3},\\
A_{30} &= \frac{3S_2^2\delta}{b_{110}b_{011}^2S_1^3},\\
A_{11} &= \frac{1}{192S_1^3S_2}\{S_1[S_1(S_1(S_1(S_1b_{011}(65S_1-662c_{001})+2631b_{011}c_{001}^2+1100b_{011}-1848)\\
&\quad -36c_{001}(145b_{011}c_{001}^2+188b_{011}-120))-4752c_{001}^2+4656b_{011}+5535b_{011}c_{001}^4\\
&\quad +14856b_{011}c_{001}^2-5184)-6c_{001}(513b_{011}c_{001}^4-400c_{001}^2+2312b_{011}c_{001}^2+2064b_{011}-960)]\\
&\quad +9(c_{001}^2+4)(81b_{011}c_{001}^4-56c_{001}^2+232b_{011}c_{001}^2+144b_{011}-96)\\
&\quad +\delta[12S_2(S_1(S_1(b_{110}-6c_{001})+15c_{001}^2+4)-9c_{001}(c_{001}^2+4))]\},\\
A_{12} &= \frac{-1}{72S_1^2S_2}\{b_{011}[S_1(S_1(S_1(S_1(26S_1-231c_{001})+12(63c_{001}^2+26))-18c_{001}(63c_{001}^2+104))\\
&\quad +18(45c_{001}^4+132c_{001}^2+32))-81c_{001}(3c_{001}^4+16c_{001}^2+16)]+3S_2\delta[S_1(b_{110}-5c_{001})+9(c_{001}^2+4)]\},
\end{aligned}
$$

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$$A_{23} = \frac{1}{288b_{110}b_{011}S_1^2}\{b_{011}[S_1(S_1(S_1(S_1(91b_{110} - 464c_{001}) + 1242c_{001}^2 + 528) - 18c_{001}(81c_{001}^2 + 128)) + 891c_{001}^4 + 2592c_{001}^2 + 144) - 81c_{001}(3c_{001}^4 + 16c_{001}^2 + 16)] + 3S_2\delta[S_1(b_{110} - 5c_{001}) + 9(c_{001}^2 + 4)]]\},$$

$$A_{22} = \frac{1}{144S_1S_2}\{S_1[S_1(S_1(13b_{110} - 83c_{001}) + 6(39c_{001}^2 + 44)) - 216c_{001}^3] + 81c_{001}^4 + 216c_{001}^2 - 432\},$$

$$A_{13} = \frac{-1}{1152b_{011}b_{110}S_1^3}\{S_1[S_1(S_1(S_1(S_1b_{011}(715S_1 - 5358c_{001}) + 8964b_{011} - 5544 + 15669b_{011}c_{001}^2) - 36c_{001}(669b_{011}c_{001}^2 - 360 + 1012b_{011})) + 23184b_{011} - 14256c_{001}^2 - 15552 + 60984c_{001}^2b_{011} + 21141b_{011}c_{001}^4) - 18c_{001}(567b_{011}c_{001}^4 - 400c_{001}^2 + 2600c_{001}^2b_{011} - 960 + 2352b_{011})] + 27(c_{001}^2 + 4)(81b_{011}c_{001}^4 - 56c_{001}^2 + 232b_{011}c_{001}^2 - 96 + 144b_{011}) + 12S_2\delta[S_1(S_1(8b_{110} - 31c_{001}) + 6(9c_{001}^2 + 8)) - 27c_{001}(c_{001}^2 + 4)]\},$$

$$A_{33} = \frac{S_2}{3072b_{011}^2b_{110}^2S_1}\{S_1[S_1(S_1(S_1(S_1(845S_1b_{011} - 4030b_{011}c_{001}) - 1848 + 8991b_{011}c_{001}^2 + 5868b_{011}) - 12c_{001}(951b_{011}c_{001}^2 + 1556b_{011} - 360)) + 8667b_{011}c_{001}^4 + 26088c_{001}^2b_{011} + 10224b_{011} - 4752c_{001}^2 - 5184) - 6c_{001}(621b_{011}c_{001}^4 + 2888c_{001}^2b_{011} - 400c_{001}^2 - 960 + 2640b_{011})] + 9(c_{001}^2 + 4)(81b_{011}c_{001}^4 + 232b_{011}c_{001}^2 - 56c_{001}^2 + 144b_{011} - 96) + S_2\delta[S_1(S_1(52b_{110} - 176c_{001}) + 84(3c_{001}^2 + 4)) - 108c_{001}(c_{001}^2 + 4)]\},$$

$$B_{10} = -\frac{1}{8b_{011}S_1^3}[3\delta(10b_{110}c_{001} + 3c_{001}^2 - 12) + (8b_{011}S_1 - 21\delta)S_1^2],$$

$$B_{30} = -\frac{3S_2^2\delta}{32b_{110}b_{011}^2S_1^3},$$

$$B_{11} = \frac{-1}{192S_1^3S_2}\{S_1[S_1(S_1(S_1(S_1b_{011}(169S_1 - 1534c_{001}) + 1804b_{011} - 504 + 5271b_{011}c_{001}^2) - 12c_{001}(735b_{011}c_{001}^2 - 88 + 772b_{011})) - 1488c_{001}^2 + 5808b_{011} + 7911b_{011}c_{001}^4 + 18504c_{001}^2b_{011} - 2880) - 6c_{001}(621b_{011}c_{001}^4 - 176c_{001}^2 + 2536c_{001}^2b_{011} + 2256b_{011} - 576)] + 9(c_{001}^2 + 4)(81b_{011}c_{001}^4 - 56c_{001}^2 + 232b_{011}c_{001}^2 + 144b_{011} - 96) + S_2\delta[S_1(S_1(60b_{110} - 216c_{001}) + 12(27c_{001}^2 + 20)) - 108c_{001}(c_{001}^2 + 4)]\},$$

$$B_{12} = \frac{1}{72S_1^2S_2}\{b_{011}[S_1(S_1(S_1(S_1(52b_{110} - 371c_{001}) + 24(51c_{001}^2 + 1)) - 18c_{001}(87c_{001}^2 + 88)) + 2376c_{001}^2 + 972c_{001}^4 + 288) - 81c_{001}(c_{001}^2 + 4)(3c_{001}^2 + 4)] + 3S_2\delta[S_1(7b_{110} - 3c_{001}) + 7c_{001}^2 + 12]\},$$

$$B_{22} = -\frac{1}{144S_1S_2}[b_{011}(b_{110}^2 - 4b_{110}c_{001} + 4c_{001}^2 - 12)(13b_{110}^2 + 8b_{110}c_{001} + 4c_{001}^2 + 36)],$$

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$$
\begin{aligned}
B_{13} = {} & \frac{-S_2}{9216 b_{110}^2 b_{011}^2 S_1^3}\{S_1[S_1(S_1(S_1(S_1 b_{011}(1547 S_1 - 10566 c_{001}) - 1512 + 11076 b_{011} + 27477 b_{011} c_{001}^2) \\
& - 36 c_{001}(1041 b_{011} c_{001}^2 + 1284 b_{011} - 88)) - 4464 c_{001}^2 - 8640 + 28917 b_{011} c_{001}^4 + 71928 c_{001}^2 b_{011} \\
& + 25488 b_{011}) - 18 c_{001}(675 b_{011} c_{001}^4 - 176 c_{001}^2 + 2824 c_{001}^2 b_{011} - 576 + 2544 b_{011})] \\
& + 27(c_{001}^2 + 4)(81 b_{011} c_{001}^4 + 232 b_{011} c_{001}^2 - 56 c_{001}^2 + 144 b_{011} - 96) \\
& + S_2 \delta[S_1(S_1(480 b_{110} - 708 c_{001}) + 72(15 c_{001}^2 + 16)) - 324 c_{001}(c_{001}^2 + 4)]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{23} = {} & -\frac{-1}{96 b_{110} b_{011} S_1^2}\{b_{011}[S_1(S_1(S_1(S_1(39 S_1 - 249 c_{001}) + 10(8 + 57 c_{001}^2)) \\
& - 42 c_{001}(15 c_{001}^2 + 16)) + 3(-16 + 288 c_{001}^2 + 117 c_{001}^4)) - 27 c_{001}(16 + 16 c_{001}^2 + 3 c_{001}^4)] \\
& + S_2 \delta[S_1(5 b_{110} - 13 c_{001}) + 9(c_{001}^2 + 4)]\},
\end{aligned}
$$

$$
\begin{aligned}
B_{33} = {} & \frac{-S_2}{9216 b_{110}^2 b_{011}^2 S_1^3}\{S_1[S_1(S_1(S_1(13 S_1 b_{011}(299 S_1 - 1530 c_{001}) - 1512 + 42669 b_{011} c_{001}^2 + 21252 b_{011}) \\
& - 36 c_{001}(1395 b_{011} c_{001}^2 - 88 + 1892 b_{011})) + 34425 b_{011} c_{001}^4 + 89208 c_{001}^2 b_{011} - 4464 c_{001}^2 \\
& + 31824 b_{011} - 8640) - 18 c_{001}(729 b_{011} c_{001}^4 + 3112 c_{001}^2 b_{011} - 176 c_{001}^2 - 576 + 2832 b_{011})] \\
& + 27(c_{001}^2 + 4)(81 b_{011} c_{001}^4 + 232 b_{011} c_{001}^2 - 56 c_{001}^2 + 144 b_{011} - 96) \\
& + S_2 \delta[S_1(S_1(780 b_{110} - 768 c_{001}) + 396(3 c_{001}^2 + 4)) - 324 c_{001}(c_{001}^2 + 4)]\},
\end{aligned}
$$

$$
\begin{aligned}
C_{11} = {} & \frac{b_{110} b_{011}}{48 S_1^3 S_2^2}\{S_1[S_1(S_1(S_1(S_1 b_{011}(41 S_1 - 470 c_{001}) + 908 b_{011} - 504 + 2103 b_{011} c_{001}^2) \\
& - 12 c_{001}(387 b_{011} c_{001}^2 - 200 + 500 b_{011})) + 4272 b_{011} - 3408 c_{001}^2 + 5319 b_{011} c_{001}^4 - 2880 \\
& + 13896 c_{001}^2 b_{011}) - 6 c_{001}(513 b_{011} c_{001}^4 - 400 c_{001}^2 + 2312 c_{001}^2 b_{011} + 2064 b_{011} - 960)] \\
& + 9(c_{001}^2 + 4)(81 b_{011} c_{001}^4 + 232 b_{011} c_{001}^2 - 56 c_{001}^2 + 144 b_{011} - 96) \\
& + S_2 \delta[S_1(S_1(12 b_{110} - 72 c_{001}) + 12(15 c_{001}^2 + 4)) - 108 c_{001}(c_{001}^2 + 4)]\},
\end{aligned}
$$

$$
C_{10} = -\frac{3 b_{110} \delta}{2 S_1^3}, \quad C_{30} = -\frac{S_1^2(8 b_{011} b_{110} + 8 b_{011} c_{001} - 21\delta) + 3\delta(10 b_{110} c_{001} + 3 c_{001}^2 - 12)}{8 S_1^3 b_{011}},
$$

$$
\begin{aligned}
C_{12} = {} & \frac{-b_{110} b_{011}}{18 S_1^2 S_2^2}\{b_{011}[S_1(S_1(S_1(S_1(14 b_{110} - 133 c_{001}) + 24(24 c_{001}^2 + 11)) - 18 c_{001}(57 c_{001}^2 + 80)) \\
& + 18(3 c_{001}^2 + 8)(15 c_{001}^2 + 4)) - 81 c_{001}(c_{001}^2 + 4)(3 c_{001}^2 + 4)] \\
& + 3 S_2 \delta[S_1(7 b_{110} - 3 c_{001}) + 7 c_{001}^2 + 12]\},
\end{aligned}
$$

$$
C_{22} = \frac{b_{110} b_{011}^2(b_{110}^2 - 4 b_{110} c_{001} + 4 c_{001}^2 - 12)(5 b_{110}^2 - 8 b_{110} c_{001} - 4 c_{001}^2 + 36)}{36 S_1 S_2^2},
$$

$$
\begin{aligned}
C_{13} = {} & \frac{-1}{288 S_1^3 S_2}\{S_1[S_1(S_1(S_1(S_1 b_{011}(403 S_1 - 3678 c_{001}) + 7428 b_{011} - 1512 + 12789 b_{011} c_{001}^2) \\
& - 36 c_{001}(609 b_{011} c_{001}^2 - 200 + 900 b_{011})) + 20493 b_{011} c_{001}^4 - 10224 c_{001}^2 + 22032 b_{011} - 8640 \\
& + 58104 c_{001}^2 b_{011}) - 18 c_{001}(567 b_{011} c_{001}^4 - 400 c_{001}^2 + 2600 c_{001}^2 b_{011} - 960 + 2352 b_{011})]
\end{aligned}
$$

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$$+ 27(c_{001}^2 + 4)(81b_{011}c_{001}^4 - 56c_{001}^2 + 232b_{011}c_{001}^2 - 96 + 144b_{011})$$
$$+ S_2\delta[S_1(S_1(96b_{110} - 372c_{001}) + 72(9c_{001}^2 + 8)) - 324c_{001}(c_{001}^2 + 4)]\},$$

$$C_{23} = \frac{1}{24S_1^2S_2}\{b_{011}(4c_{001}^2 + 12 + 8b_{110}c_{001} + 13b_{110}^2)(b_{110}^3 - 4b_{110}^2c_{001} + 4b_{110} + 4b_{110}c_{001}^2 - 32c_{001})$$
$$+ S_2\delta[S_1(7b_{110} - 3c_{001}) + 7c_{001}^2 + 12]\},$$

$$C_{33} = \frac{1}{2304b_{110}b_{011}S_1^3}\{S_1[S_1(S_1(S_1(S_1b_{011}(1183S_1 - 8346c_{001}) + 13956b_{011} + 22509b_{011}c_{001}^2 - 1512)$$
$$- 36c_{001}(879b_{011}c_{001}^2 - 200 + 1396b_{011})) - 10224c_{001}^2 + 25353b_{011}c_{001}^4 - 8640 + 29520b_{011}$$
$$+ 75384c_{001}^2b_{011}) - 18c_{001}(621b_{011}c_{001}^4 + 2888c_{001}^2b_{011} - 400c_{001}^2 - 960 + 2640b_{011})]$$
$$+ 27(c_{001}^2 + 4)(81b_{011}c_{001}^4 + 232b_{011}c_{001}^2 - 56c_{001}^2 + 144b_{011} - 96)$$
$$+ 12S_2\delta[S_1(S_1(13b_{110} - 44c_{001}) + 7(9c_{001}^2 + 12)) - 27c_{001}(c_{001}^2 + 4)]\}.$$

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