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Bifurcation analysis on a class of Z_2-equivariant cubic switching systems showing eighteen limit cycles ☆

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Abstract

In this paper, the method developed for computing the Lyapunov constants of planar switching systems associated with an elementary singular point is applied to study bifurcation of limit cycles in a cubic switching system. A complete classification on the center conditions and 16 limit cycles of this system are obtained around the two foci $(1, 0)$ and $(-1, 0)$. Further, with the method, an example of cubic switching systems is constructed to show the existence of 18 small-amplitude limit cycles bifurcating from centers. This is a new lower bound on the maximal number of small-amplitude limit cycles obtained in such cubic switching systems. Finally, a method is present to show the realization of the 18 limit cycles.

MSC: 34C07; 34C23

Keywords: Switching system; Z_2-equivariant system; Lyapunov constant; Center; Limit cycle

1. Introduction

It is well known that Hopf bifurcation plays an important role in the study of nonlinear dynamical systems. There have been plenty of studies on Hopf bifurcation for smooth systems. Developing limit cycle theory is not only theoretically important, but also practically significant.

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The best result for quadratic systems, obtained 40 years ago, is four limit cycles, see [1,2]. For cubic systems, many results have been obtained on the low bound of the Hilbert number. In 1991, James and Lloyd [3] constructed a special class of cubic systems to obtain 8 limit cycles around an elemental focus. In 2009, Yu and Corless [4] used symbolic and numerical computations to obtain 9 limit cycles in a cubic system, which was reconsidered using purely symbolic computation to find all real solutions [5]. Later, Lloyd and Pearson [6] constructed another cubic system and also used purely symbolic computation to find 9 limit cycles. So far, the best result for cubic systems is 13 limit cycles. In 2009, Li et al. [7] proved that 13 limit cycles can bifurcate in a cubic Hamiltonian system under small perturbation. Around the same time, Liu and Li [8] investigated the cyclicity problem for a Z_2-equivariant cubic system, and showed the existence of 13 limit cycles. Note that the 13 limit cycles obtained in [7,8] are distributed around several singular points. Very recently, Yu and Tian [9] have shown the existence of 12 limit cycles around an elementary center in a planar cubic-degree polynomial system. This is the best result obtained so far for cubic polynomial systems with all limit cycles bifurcating from a single singular point.

However, since many problems arising from mechanics, electrical engineering and automatic control are described by non-smooth dynamical systems (see, e.g., [10,11]), increasing interest has been attracted to the qualitative analysis of those systems, which can display rich complex dynamical phenomena. Non-smooth systems can exhibit not only the classical bifurcations, but also more complicated bifurcations that only non-smooth systems can have, such as border-collision bifurcation [12,13]. During the past few decades, many contributions have been made to generalize the classical bifurcation methods for smooth systems to study non-smooth systems. For example, Kukučka [14] investigated the occurrence of homoclinic solutions in non-smooth systems and showed the existence of a homoclinic solution in a perturbed system. Li and Huang [15] studied the concurrent homoclinic bifurcation and Hopf bifurcation for a class of planar perturbed non-smooth Filippov systems.

A class of non-smooth systems is called switching system, if such a system has different definitions for the continuous vector fields in two or more different regions divided by a line or a curve. In this paper, we focus on switching planar systems, described by

$$(\dot{x}, \dot{y}) = \begin{cases} (\delta x - y + f^{+}(x, y),\ x + \delta y + g^{+}(x, y)), & \text{if } y > 0, \\ (\delta x - y + f^{-}(x, y),\ x + \delta y + g^{-}(x, y)), & \text{if } y < 0, \end{cases} \tag{1}$$

where $f^{\pm}(x, y)$ and $g^{\pm}(x, y)$ are analytic functions in x and y, starting from at least second-order terms. Thus, the origin of system (1) is an equilibrium. Actually, (1) includes two systems: one is called upper system, defined for $y > 0$, and the other is called lower system, defined for $y < 0$.

The investigation of switching systems started a half century ago [16–18]. Filippov [17] established some basic qualitative theory on switching equations and defined three types of pseudo-focus singular points: focus-focus (FF), parabolic-focus (PF) and parabolic-parabolic (PP). Coll and Gasull [19] derived the formulas for computing the first three Lyapunov quantities associated with the three types of singularities. In particular, they proved that at least 4 limit cycles can bifurcate from the weak focus in the FF-type case. Gasull and Torregrosa [20] obtained 5 limit cycles in a quadratic switching system, two more than that of smooth quadratic systems. Han and Zhang [21] proved that 2 limit cycles can bifurcate from a focus of either FF, FP or PP type in piecewise linear systems. Center conditions have also been obtained for switching Kukles system [20] and switching Liénard system [22]. Chen and Du [23] constructed a switching Bautin system and proved the existence of 9 limit cycles around a center of the system. Recently, Chen and Romanovski [24] constructed a class of discontinuous quadratic Bautain system, which has at least 5 and 8 limit cycles bifurcating from weak foci and centers, respectively. Tian and Yu [25] provided a complete classification on the conditions of a singular point being a center in the switching Bautin system, and constructed an example to show the existence of 10 limit cycles bifurcating from the center. However, very fewer results have been obtained for cubic switching systems. Llibre et al. [26] obtained 12 limit cycles that bifurcate from the periodic orbits in a family of isochronous cubic polynomial systems. Li *el at.* [27] constructed a switching cubic system which exhibits 15 limit cycles.

To study bifurcation of limit cycles associated with a singular point in a switching system, we need Lyapunov constants to determine the number and stability of bifurcating limit cycles. We will introduce a recursive procedure based on the method of Poincaré return map [28,29] to compute the Lyapunov constants near the origin of the general system (1). Then we apply this method to study the center conditions and bifurcation of limit cycles in a class of Z_2-equivariant cubic switching system. Without loss of generality, Z_2-equivariant cubic switching systems can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{00} - \frac{1}{2}\delta(x - x^3) + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ \quad +a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ b_{00} + b_{10}x + (\delta + b_{01})y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 \\ \quad +b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} -a_{00} - \frac{1}{2}\delta(x - x^3) + a_{10}x + a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 \\ \quad +a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ -b_{00} + b_{10}x + (\delta + b_{01})y - b_{20}x^2 - b_{11}xy - b_{02}y^2 + b_{30}x^3 \\ \quad +b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{if } y < 0, \end{cases} \tag{2}$$

where δ, a_{ij}'s and b_{ij}'s are real parameters, satisfying $|\delta| \ll 1$. Without loss of generality, suppose $b_{10} \neq 0$ and $a_{02} \neq 0$. Then, we apply scaling on the state variables and parameters in (2) so that $b_{10} = a_{02} = 1$. Moreover, to make $(\pm 1, 0)$ be Hopf-type singular points, we let

$$a_{20} + a_{00} = b_{20} + b_{00} = a_{30} + a_{10} = b_{30} + b_{10} = b_{11} - 2a_{00} = b_{21} - 2a_{10} + b_{01} = 0.$$

Further, we set $a_{00} = b_{00} = a_{10} = a_{11} = 0$ and $a_{21} = \frac{1}{2}(1 - 2a_{01})$ so that higher Lyapunov constants can be obtained from the resulting Z_2-equivariant cubic switching system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{01}y - \frac{1}{2}\delta(x - x^3) + y^2 + (\frac{1}{2} - a_{01})x^2y + a_{12}xy^2 + a_{03}y^3 \\ x + (\delta + b_{01})y + b_{02}y^2 - x^3 - b_{01}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} a_{01}y - \frac{1}{2}\delta(x - x^3) - y^2 + (\frac{1}{2} - a_{01})x^2y + a_{12}xy^2 + a_{03}y^3 \\ x + (b_{01} + \delta)y - b_{02}y^2 - x^3 - b_{01}x^2y + b_{12}xy^2 + b_{03}y^3 \end{pmatrix}, & \text{if } y < 0. \end{cases} \tag{3}$$

The main goal of this paper is to consider the center conditions and bifurcation of limit cycles in the switching Z_2-equivariant cubic system (3). We will apply a recursive procedure to obtain a new lower bound on the number of limit cycles. We first compute the first nine Lyapunov constants for the singular point $(1, 0)$ of system (3) to obtain the center conditions and prove the existence of 16 limit cycles. Then, we choose one of the center conditions with proper perturbations to construct a perturbed system, and then compute the Lyapunov constants associated with the singular point of the perturbed system to prove the existence of 9 limit cycles around $(1, 0)$, yielding a total of 18 limit cycles around the two symmetric singular points. In the next section, we present some basic formulas and preliminary results which are needed in proving our main results in Sections 3 and 4. Realization of 18 limit cycles is presented in section 5. Conclusion is drawn in section 6.

2. Methodology

In this section, we present some basic methods and preliminary results which will be used in the following sections. First, we introduce the classical method for solving center problems of discontinuous systems based on the computation of Lyapunov constants. The details of the method can be found in [31]. Consider the general switching differential system,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y + \sum\limits_{k=2}^{n} X_k^{+}(x, y) \\ x + \delta y + \sum\limits_{k=2}^{n} Y_k^{+}(x, y) \end{pmatrix}, & \text{if } \ y > 0, \\ \begin{pmatrix} \delta x - y + \sum\limits_{k=2}^{n} X_k^{-}(x, y) \\ x + \delta y + \sum\limits_{k=2}^{n} Y_k^{-}(x, y) \end{pmatrix}, & \text{if } \ y < 0, \end{cases} \tag{4}$$

where $X_k^{\pm}(x, y)$ and $Y_k^{\pm}(x, y)$ are homogeneous polynomials in x and y. Under the polar coordinates transformation, $x = r\cos\theta$ and $y = r\sin\theta$, (4) can be rewritten as

$$\frac{dr}{d\theta} = \begin{cases} \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_k^{+}(\theta) r^k}{1 + \sum_{k=2}^{n} \Theta_k^{+}(\theta) r^{k-1}}, & \text{for } \theta \in (0, \pi), \\ \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_k^{-}(\theta) r^k}{1 + \sum_{k=2}^{n} \Theta_k^{-}(\theta) r^{k-1}}, & \text{for } \theta \in (-\pi, 0), \end{cases} \tag{5}$$

where $\Upsilon_k^{\pm}(\theta)$ and $\Theta_k^{\pm}(\theta)$ are polynomials in $\sin\theta$ and $\cos\theta$ of degrees $k + 1$. By the method of small parameters of Poincaré, the solutions of the upper and lower systems of (5) are given by

$$r^{+}(h, \theta) = \sum_{k \geq 1} u_k(\theta) h^k, \qquad r^{-}(h, \theta) = \sum_{k \geq 1} v_k(\theta) h^k, \tag{6}$$

where $u_1(0) = v_1(0) = 1$, $u_k(0) = v_k(0) = 0$, $\forall k \geq 2$. Substituting the above solutions into (5) yields

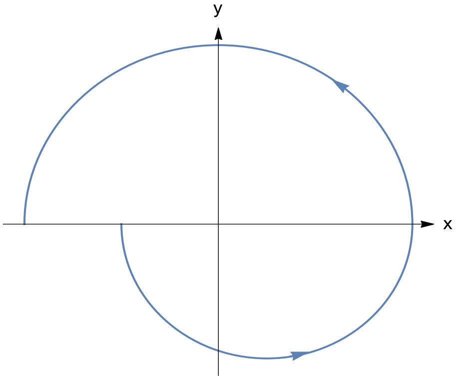

Fig. 1. Poincaré map for system (1).

$$
\begin{aligned}
u_1'(\theta) &= \delta u_1(\theta),\\
u_2'(\theta) &= \delta u_2(\theta) + P_1(\theta)u_1^2(\theta),\\
&\vdots\\
u_m'(\theta) &= \delta u_m(\theta) + P_1(\theta)\Omega_{2,m}(\theta) + \cdots + P_{m-1}(\theta)\Omega_{m,m}(\theta),
\end{aligned}
$$

and

$$
\begin{aligned}
v_1'(\theta) &= \delta v_1(\theta),\\
v_2'(\theta) &= \delta v_2(\theta) + R_1(\theta)v_1^2(\theta),\\
&\vdots\\
v_m'(\theta) &= \delta v_m(\theta) + R_1(\theta)\bar{\Omega}_{2,m}(\theta) + \cdots + R_{m-1}(\theta)\bar{\Omega}_{m,m}(\theta),
\end{aligned}
$$

where $P_i(\theta)$, $R_i(\theta)$, $\Omega_{ij}(\theta)$ and $\bar{\Omega}_{ij}(\theta)$ are polynomials functions of $\sin\theta$ and $\cos\theta$. Thus, we may solve $u_k(\theta)$ and $v_k(\theta)$ one by one. Consequently, we can define the following successive functions,

$$\triangle^+(h) = r^+(h,\pi) - h, \quad \triangle^-(h) = h - r^-(h,-\pi),$$

for the upper and lower systems of (5), respectively. Then, the successive function for the switching system (4) can be defined as

$$\triangle(h) = \triangle^+(h) + \triangle^-(h) = r^+(h,\pi) - r^-(h,-\pi), \tag{7}$$

as illustrated in Fig. 1.

It has been shown in [31] that the displacement function $\triangle(h)$ can be expanded as

$$\triangle(h) = \sum_{k=1}^{n}(u_k(\pi) - v_k(-\pi))h^k = \sum_{k=0}^{n-1} V_k h^{k+1}, \tag{8}$$

where V_k is called the kth-order Lyapunov constant of the switching system (4). Obviously, the origin is a center of system (4) if and only if $\triangle(h) \equiv 0$ for $0 < h \ll 1$, which means that all the Lyapunov constants in (8) vanish. The isolated zeros of $\triangle(h) = 0$ near $h = 0$ correspond to the number of limit cycles around the origin. It is easy to get that $V_0 = \frac{1}{e^{\pi\delta}}(e^{2\pi\delta} - 1)$, since $u_1(\theta) = v_1(\theta) = e^{\delta\theta}$. Thus, $V_0 = 0$ if and only if $\delta = 0$. It is well known that k must be odd for the V_k of smooth systems [30]. However, for the switching system (4), in general, $V_k \neq 0$ with k being any positive integer.

Now we turn to discuss how to determine the maximal number of limit cycles which may bifurcate from a Hopf critical point. To find k small-amplitude limit cycles of system (4) around the origin, we first find the conditions based on the original system's coefficients, a_{ij} and b_{ij} such that $V_0 = V_1 = V_2 = \cdots = V_{k-1} = 0$, but $V_k \neq 0$. For convenience, we call these coefficients as $c_1, c_2, \ldots, c_k$ and use the corresponding critical values to define a critical point C. Then appropriate small perturbations from the critical values are performed to prove the existence of k limit cycles. More generally, the following theorem gives sufficient conditions for the existence of small-amplitude limit cycles in the switching system (4). (The proof can be found in [25].)

Theorem 2.1. ([25]) *Suppose that there exists a sequence of Lyapunov constants of system* (4)*, $V_{i_0}, V_{i_1}, \ldots, V_{i_k}$, with $1 = i_0 < i_1 < \cdots < i_k$, such that $V_j = O(|V_{i_0}, \ldots, V_{i_l}|)$ for any $i_l < j < i_{l+1}$. Further, if at the critical point* C*, $V_{i_0} = V_{i_1} = \cdots = V_{i_{k-1}} = 0$, $\;V_{i_k} \neq 0$, and*

$$\det\left[\frac{\partial(V_{i_0}, V_{i_1}, \cdots, V_{i_{k-1}})}{\partial(c_1, c_2, \cdots, c_k)}\right]_{\mathrm{C}} \neq 0, \tag{9}$$

then system (4) *has exactly k limit cycles in a δ-ball with its center at the origin.*

The center problem in switching systems is more complicated than that of smooth systems. The origin of system (4) can be a center even if it is not a center of either the upper system or the lower system. On the other hand, the origin of system (4) may not be a center, even if both the upper and lower systems have the centers at the origin. In order to prove the center conditions for system (4), we have the following lemmas.

Lemma 2.1. ([32]) *If the upper and lower systems of* (4) *have the first integrals $H^+(x, y)$ and $H^-(x, y)$ near the origin, respectively, and either both $H^+(x, y)$ and $H^-(x, y)$ are even functions in x or $H^+(x, 0) \equiv H^-(x, 0)$, then the origin of system* (4) *is a center.*

Lemma 2.2. ([27]) *Assuming that $\delta = 0$, if system* (4) *is symmetric with respect to the x-axis, i.e. the functions on the right-hand side of system* (4) *satisfy*

$$X_k^+(x, y) = -X_k^-(x, -y), \quad Y_k^+(x, y) = Y_k^-(x, -y),$$

or if system (4) *is symmetric with respect to the y-axis, i.e. the functions on the right-hand side of system* (4) *satisfy*

$$\begin{aligned} X_k^+(x, y) &= X_k^+(-x, y), \qquad & X_k^-(x, y) &= X_k^-(-x, y), \\ Y_k^+(x, y) &= -Y_k^+(-x, y), \qquad & Y_k^-(x, y) &= -Y_k^-(-x, y), \end{aligned}$$

then the origin of system (4) *is a center.*

3. Bi-center conditions and Hopf bifurcation for system (3)

In this section, we consider the center conditions and bifurcation of limit cycles for the switching Z_2-equivariant cubic system (3). Due to the symmetry of system (3), the Lyapunov constants associated with the singular points $(1,0)$ and $(-1,0)$ are same, hence we only need to consider the center conditions and Hopf bifurcation at the singular point $(1,0)$.

In order to study the center conditions and limit cycle bifurcation around the Hopf critical point $(1,0)$, we need to compute its Lyapunov constants associated with the Hopf critical point. To achieve this, we introduce the following transformation,

$$x = 1 - \frac{1}{2}x_1, \quad y = y_1,$$

into system (3) to obtain

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x_1 - y_1 - \frac{3}{4}\delta x_1^2 + (1-2a_{01})x_1y_1 - (2+2a_{12})y_1^2 + \frac{1}{8}\delta x_1^3 \\ \quad +a_{12}x_1y_1^2 + (\frac{1}{2}a_{01} - \frac{1}{4})x_1^2y_1 - 2a_{03}y_1^3 \\ x_1 + \delta y_1 - \frac{3}{4}x_1^2 + b_{01}x_1y_1 + (b_{12}+b_{02})y_1^2 + \frac{1}{8}x_1^3 - \frac{1}{2}b_{12}x_1y_1^2 \\ \quad -\frac{1}{4}b_{01}x_1^2y_1 + b_{03}y_1^3 \end{pmatrix}, & \text{if } y_1 > 0, \\ \begin{pmatrix} \delta x_1 - y_1 - \frac{3}{4}\delta x_1^2 + (1-2a_{01})x_1y_1 + (2-2a_{12})y_1^2 + \frac{1}{8}\delta x_1^3 \\ \quad +a_{12}x_1y_1^2 + (\frac{1}{2}a_{01} - \frac{1}{4})x_1^2y_1 - 2a_{03}y_1^3 \\ x_1 + \delta y_1 - \frac{3}{4}x_1^2 + b_{01}x_1y_1 + (b_{12}-b_{02})y_1^2 + \frac{1}{8}x_1^3 - \frac{1}{2}b_{12}x_1y_1^2 \\ \quad -\frac{1}{4}b_{01}x_1^2y_1 + b_{03}y_1^3 \end{pmatrix}, & \text{if } y_1 < 0. \end{cases} \tag{10}$$

Clearly, the singular point $(1,0)$ of system (10) corresponds to the origin of (10), which is a Hopf-type critical point. In the following, we will use the method presented in the previous section to compute the Lyapunov constants for the origin of system (10), and use them to derive the center conditions and to consider limit cycle bifurcation.

3.1. Center conditions for system (10)

With the aid of a computer algebra system, we have obtained the Lyapunov constants associated with the origin of system (10) (i.e. associated with the two symmetric singular points $(1,0)$ of system (3)), as given in the following theorem.

Theorem 3.1. *The first four Lyapunov constants at the origin of system* (10) *are obtained as*

$$\begin{aligned} V_0 &= \frac{1}{e^{\pi\delta}}(e^{2\pi\delta} - 1), \\ V_1 &= \frac{8}{3}b_{02}, \\ V_2 &= \frac{\pi}{8}(8a_{01}a_{12} - 8a_{12}b_{12} - 2b_{01}b_{12} - 2a_{12} + b_{01} + 6b_{03}), \end{aligned}$$

$$V_3 = \frac{32}{45}(8a_{12} - b_{01})\,b_{12} - \frac{128}{45}(2a_{01} - 1)\,a_{12} + \frac{8}{45}(4a_{01} + 1)\,b_{01},$$

where $V_{k-1} = 0$ has been used in computing V_k, $k = 1, 2, 3$. Higher Lyapunov constants are given as follows.

(I) *If $8a_{12} - b_{01} = 0$, then*

$$\begin{aligned}
V_3 &= \frac{64}{15}a_{12},\\
V_4 &= \frac{1}{60}\,a_{12}\,\big\{128\big[1 + 8(b_{12} - a_{01})\big] + 5\pi\big[48a_{12}^2(3 - 6b_{12} - 2a_{01})\\
&\quad + 6a_{03}(1 + 2a_{01} - 2b_{12}) - 4\,a_{01}b_{12}(3 + 3b_{12} - a_{01}) + 8a_{01}^3 - 10a_{01}^2 + 9a_{01} - 40\big]\big\},\\
V_5 &= \frac{1}{75600}\,a_{12}\,(\cdots),\\
V_6 &= \frac{1}{3628800}\,a_{12}\,(\cdots),\\
&\ \ \vdots
\end{aligned}$$

which gives a necessary condition for the origin of system (10) to be a center: $8a_{12} - b_{01} = a_{12} = 0$.

(II) *If $8a_{12} - b_{01} \neq 0$, then*

$$\begin{aligned}
V_4 = \frac{-\pi\, b_{01}}{288(8a_{12} - b_{01})^2}\big\{&36(8a_{12} - b_{01})(2a_{12} - b_{01})a_{03} - 36(8a_{12} - b_{01})(2a_{12} + b_{01})a_{01}^2\\
&+ \big[8b_{01}(8a_{12} - b_{01})^2(10a_{12} + b_{01}) - 9(2a_{12} - b_{01})(32a_{12} - b_{01})\big]a_{01}\\
&+ 2(8a_{12} - b_{01})(2a_{12} - b_{01})(10a_{12} + b_{01})(32a_{12} - b_{01}) - 90(8a_{12} - b_{01})^2\big\}
\end{aligned}$$

Further,

($\mathrm{II_a}$) *if $2a_{12} - b_{01} \neq 0$, then*

$$\begin{aligned}
V_5 &= \frac{16b_{01}F_1}{4725(8a_{12} - b_{01})(2a_{12} - b_{01})},\\
V_6 &= \frac{b_{01}F_2}{7257600(8a_{12} - b_{01})^2(2a_{12} - b_{01})^2},\\
V_7 &= \frac{-\,b_{01}F_3}{5486745600(8a_{12} - b_{01})^3(2a_{12} - b_{01})^2},\\
V_8 &= \frac{b_{01}F_4}{21946982400(8a_{12} - b_{01})^4(2a_{12} - b_{01})^3};
\end{aligned}$$

($\mathrm{II_b}$) *or if $2a_{12} - b_{01} = 0$, then*

$$V_4 = \frac{\pi\, a_{12}}{24}(4a_{01}^2 - 32a_{01}a_{12}^2 + 15),$$

$$
\begin{aligned}
V_5 =& \frac{-a_{12}}{50400}\big[(1474560a_{12}^2 - 197120)a_{01} - 245760a_{12}^2 - 165888a_{03} - 622080\big],\\
V_6 =& \frac{-5\pi\, a_{12}}{1679616}\big[18579456a_{01}a_{12}^6 - (1124352a_{01} + 8409600)a_{12}^4\\
&\quad + (576288 - 1000576a_{01})a_{12}^2 - 4374a_{01} - 165975\big],\\
V_7 =& \frac{a_{12}}{44442639360}\big\{92897280a_{01}(99225\pi + 524288)a_{12}^6 - \big[(557819136000\pi\\
& + 3143110754304)a_{01} + 4172212800000\pi + 21708920586240\big]a_{12}^4\\
& + \big[285910884000\pi + 1753787400192 - (496410768000\pi + 2785429946368)a_{01}\big]a_{12}^2\\
& - (2170050750\pi + 11466178560)a_{01} - 82344346875\pi - 414100045824\big\}.
\end{aligned}
$$

The polynomials F_1 and F_2 in the above expressions are given by

$$
\begin{aligned}
F_1 =& 972b_{01}(8a_{12} - b_{01})a_{01}^2 - 9\big[24b_{01}(8a_{12} - b_{01})(40a_{12}^2 - b_{01}^2)\\
& - 5(172a_{12} - 7b_{01})(2a_{12} - b_{01})\big]a_{01} - 10(2a_{12} - b_{01})(6784a_{12}^3 - 384a_{12}^2b_{01}\\
& - 168a_{12}b_{01}^2 + 7b_{01}^3) + 1215(8a_{12} - b_{01})^2,
\end{aligned}
$$

$$
\begin{aligned}
F_2 =& 6350400\pi b_{01}^2(8a_{12} - b_{01})^2a_{01}^4 - 25200\pi b_{01}(8a_{12} - b_{01})\big[56b_{01}(8a_{12} - b_{01})(82a_{12}^2\\
& - a_{12}b_{01} - 2b_{01}^2) - 9(2a_{12} - b_{01})(444a_{12} - 17b_{01})\big]a_{01}^3 - 8\big\{5971968b_{01}(64a_{12}^2\\
& - b_{01}^2)(2a_{12} - b_{01}) - 175\pi\big[224b_{01}^2(8a_{12} - b_{01})^3(10a_{12} + b_{01})(5a_{12} - b_{01})(4a_{12} + b_{01})\\
& - 36b_{01}(8a_{12} - b_{01})(2a_{12} - b_{01})(35768a_{12}^3 - 1918a_{12}^2b_{01} - 851a_{12}b_{01}^2 + 34b_{01}^3)\\
& + 81(14080a_{12}^4 + 8136a_{12}^3b_{01} + 14028a_{12}^2b_{01}^2 - 3734a_{12}b_{01}^3 + 243b_{01}^4)\big]\big\}a_{01}^2\\
& + 2\big\{(221184(8a_{12} + b_{01})(2a_{12} - b_{01})\big[24b_{01}(8a_{12} - b_{01})(40a_{12}^2 - b_{01}^2)\\
& - 5(2a_{12} - b_{01})(172a_{12} - 7b_{01})\big] + 175\pi\big[32b_{01}(8a_{12} - b_{01})^2(2a_{12} - b_{01})(4a_{12}\\
& + b_{01})(21560a_{12}^3 - 3150a_{12}^2b_{01} - 339a_{12}b_{01}^2 + 17b_{01}^3) - 216(371200a_{12}^6\\
& - 96080a_{12}^5b_{01} + 257680a_{12}^4b_{01}^2 - 64312a_{12}^3b_{01}^3 + 124a_{12}^2b_{01}^4 + 1033a_{12}b_{01}^5 - 71b_{01}^6)\\
& + 135(8a_{12} - b_{01})(2a_{12} - b_{01})(6880a_{12}^2 + 4322a_{12}b_{01} - 191b_{01}^2)\big]\big\}a_{01} + 245760(2a_{12}\\
& - b_{01})(8a_{12} + b_{01})\big[2(2a_{12} - b_{01})(6784a_{12}^3 - 384a_{12}^2b_{01} - 168a_{12}b_{01}^2 + 7b_{01}^3)\\
& - 243(8a_{12} - b_{01})^2\big] + 875\pi(8a_{12} - b_{01})\big[32(2a_{12} - b_{01})^2(4a_{12} + b_{01})(32a_{12}\\
& - b_{01})(1036a_{12}^3 - 168a_{12}^2b_{01} - 15a_{12}b_{01}^2 + b_{01}^3) - 108(2a_{12} - b_{01})(29120a_{12}^4\\
& + 20328a_{12}^3b_{01} - 2796a_{12}^2b_{01}^2 - 334a_{12}b_{01}^3 + 15b_{01}^4) - 81(24448a_{12}^3 - 45036a_{12}^2b_{01}\\
& + 10068a_{12}b_{01}^2 - 611b_{01}^3)\big],
\end{aligned}
$$

and the lengthy polynomials F_3 and F_4 in a_{01}, b_{01} and b_{12} can be found from the supplement posted on the journal website.

Now, we turn to discuss the center conditions of system (10). From Theorem 3.1 we have the following result.

Theorem 3.2. *System (10) has a center at the origin if and only if one of the following conditions is satisfied:*

(a) $\delta = b_{02} = b_{01} = b_{03} = a_{12} = 0$.
(b) $\delta = b_{02} = b_{01} = 3b_{03} + a_{12} = 2a_{01} - 2b_{12} - 1 = 0$.

That is, the above conditions are the necessary and sufficient conditions for the symmetric singular points $(\pm 1, 0)$ *of system (3) to be bi-centers.*

Proof. To show that the two conditions in the theorem give the complete classification on the center conditions, we first prove these two conditions are only necessary conditions for the origin of system (10) to be center. Assume that the origin of system (10) is a center. Then, all the Lyapunov constants at the origin of the system should vanish. From the Lyapunov constants given in Theorem 3.1, it is easy to see that $V_0 = 0$ yields $\delta = 0$. Next, we solve $V_1 = 0$ to obtain $b_{02} = 0$. Then, solving $V_2 = 0$ yields

$$b_{03} = \frac{1}{6}\big[(8b_{12} - 8a_{01} + 2)a_{12} + b_{01}(2b_{12} - 1)\big]. \tag{11}$$

Now it can be seen from V_3 that there are two cases. First, consider the case $8a_{12} - b_{01} = 0$, with the Lyapunov constants given in (I) of Theorem 3.1. For this case, $V_3 = \frac{64}{15}a_{12}$, and all higher Lyapunov constants have a factor a_{12}. Thus, letting $a_{12} = 0$ yields $V_3 = V_4 = \cdots = 0$, from which we obtain the necessary condition for the origin of system (10) to be a center: $a_{12} = b_{01} = b_{03} = 0$, leading to the condition (a).

Next, consider the case $8a_{12} - b_{01} \neq 0$, under which we use b_{12} to linearly solve $V_3 = 0$ to obtain

$$b_{12} = \frac{16a_{12}(2a_{01} - 1) - (4a_{01} + 1)\,b_{01}}{4(8a_{12} - b_{01})}. \tag{12}$$

Then it is obvious to see from (II_a) that under the condition $2a_{12} - b_{01} \neq 0$, setting $b_{01} = 0$ results in $V_4 = V_5 = \cdots = 0$. This gives another necessary condition for the origin of system (10) to be a center: $b_{01} = 2a_{01} - 2b_{12} - 1 = 3b_{03} + a_{12} = 0$, leading to the condition (b).

On the other hand, if $2a_{12} - b_{01} = 0$, it can be seen from (II_b) that setting $a_{12} = 0$ yields the condition (a). Suppose $a_{12} \neq 0$. Then, solving $V_4 = 0$ gives $a_{12}^2 = \frac{4a_{01}^2 + 15}{32a_{01}}$, and then solving $V_5 = 0$ yields $a_{30} = \frac{5}{324a_{01}}(72a_{01}^3 - 89a_{01}^2 + 27a_{01} - 45)$. Then, V_6 and V_7 become two rational functions of a_{01}, which cannot be simultaneously vanished. Thus, (II_b) does not yield center conditions.

Thus, the only other possibility for the origin of system (10) to be a center comes from the conditions: $V_4 = F_1 = F_2 = F_3 = F_4 = 0$ ($b_{01} \neq 0$). This will be discussed in the next subsection.

Now we turn to prove the sufficiency of the conditions (a) and (b). When the condition (a) holds, system (10) can be rewritten as

$$
\begin{cases}
\dfrac{dx_1}{dt} = -y_1 + (1-2a_{01})x_1y_1 - 2y_1^2 - \dfrac{1}{4}(1-2a_{01})x_1^2y_1 - 2a_{03}y_1^3, \\
\dfrac{dy_1}{dt} = x_1 - \dfrac{3}{4}x_1^2 + b_{12}y_1^2 + \dfrac{1}{8}x_1^3 - \dfrac{1}{2}b_{12}x_1y_1^2,
\end{cases} \quad (y_1 > 0);
$$

$$
\begin{cases}
\dfrac{dx_1}{dt} = -y_1 + (1-2a_{01})x_1y_1 + 2y_1^2 - \tfrac{1}{4}(1-2a_{01})x_1^2y_1 - 2a_{03}y_1^3, \\
\dfrac{dy_1}{dt} = x_1 - \dfrac{3}{4}x_1^2 + b_{12}y_1^2 + \dfrac{1}{8}x_1^3 - \dfrac{1}{2}b_{12}x_1y_1^2,
\end{cases} \quad (y_1 < 0), \tag{13}
$$

showing that the system is symmetric with the x_1-axis, and thus by Lemma 2.2, the origin of system (10) is a center.

When the condition (b) is satisfied, system (10) becomes

$$
\begin{cases}
\dfrac{dx_1}{dt} = -y_1 + (1-2a_{01})x_1y_1 - 2(1+a_{12})y_1^2 \\
\qquad - \dfrac{1}{4}(1-2a_{01})x_1^2y_1 + a_{12}x_1y_1^2 - 2a_{03}y_1^3, \\
\dfrac{dy_1}{dt} = x_1 - \dfrac{3}{4}x_1^2 - \dfrac{1}{2}(1-2a_{01})y_1^2 + \dfrac{1}{8}x_1^3 + \dfrac{1}{4}(1-2a_{01})x_1y_1^2 - \dfrac{1}{3}a_{12}y_1^3,
\end{cases} \quad (y_1 > 0);
$$

$$
\begin{cases}
\dfrac{dx_1}{dt} = -y_1 + (1-2a_{01})x_1y_1 + 2(1-a_{12})y_1^2 \\
\qquad - \dfrac{1}{4}(1-2a_{01})x_1^2y_1 + a_{12}x_1y_1^2 - 2a_{03}y_1^3, \\
\dfrac{dy_1}{dt} = x_1 - \dfrac{3}{4}x_1^2 - \dfrac{1}{2}(1-2a_{01})y_1^2 + \dfrac{1}{8}x_1^3 + \dfrac{1}{4}(1-2a_{01})x_1y_1^2 - \dfrac{1}{3}a_{12}y_1^3,
\end{cases} \quad (y_1 < 0). \tag{14}
$$

The upper and lower systems have analytic first integrals,

$$
\begin{aligned}
H_1(x_1, y_1) = & -\frac{1}{2}(x_1^2 + y_1^2) + \frac{1}{4}x_1^3 + \frac{1}{2}(1-2a_{01})x_1y_1^2 - \frac{2}{3}(1+a_{12})y_1^3 - \frac{1}{32}x_1^4 \\
& + \frac{1}{3}a_{12}x_1y_1^3 - \frac{1}{8}(1-2a_{01})x_1^2y_1^2 - \frac{1}{2}a_{03}y_1^4,
\end{aligned}
$$

and

$$
\begin{aligned}
H_2(x_1, y_1) = & -\frac{1}{2}(x_1^2 + y_1^2) + \frac{1}{4}x_1^3 + \frac{1}{2}(1-2a_{01})x_1y_1^2 + \frac{2}{3}(1-a_{12})y_1^3 - \frac{1}{32}x_1^4 \\
& + \frac{1}{3}a_{12}x_1y_1^3 - \frac{1}{8}(1-2a_{01})x_1^2y_1^2 - \frac{1}{2}a_{03}y_1^4,
\end{aligned}
$$

respectively, indicating that $H_1(x_1, 0) = H_2(x_1, 0)$. So by Lemma 2.1, the origin of system (10) is a center.

The proof of Theorem 3.2 is complete. □

3.2. Bifurcation of limit cycles in system (3)

In this section, we consider the limit cycles bifurcating from the symmetric singular points $(\pm 1, 0)$ of (3). We prove the following theorem using the Lyapunov constants given in Theorem 3.1.

Theorem 3.3. *System (10) can have 8 small-amplitude limit cycles bifurcating from the origin, that is, system (3) can have 16 limit cycles with the $8 \cup 8$ distribution around the singular points $(1, 0)$ and $(-1, 0)$.*

Proof. It has been shown in proving Theorem 3.2 that setting V_0, V_1 and V_2 zero respectively yields $\delta = 0$, $b_{02} = 0$ and the solution b_{03} given in (11). In order to obtain maximal number of small-amplitude limit cycles bifurcating from the origin of (10), we have to assume that

$$(8a_{12} - b_{01})(2a_{12} - b_{01})b_{01} \neq 0. \tag{15}$$

Otherwise, if $8a_{12} - b_{01} = 0$ we can get maximal 3 limit cycles around the origin; and if $2a_{12} - b_{01} = 0$, we can obtain maximal 7 limit cycles, as shown in the proof of Theorem 3.2.

So suppose the condition (15) holds. Then, solving $V_3 = 0$ yields the solution b_{12} given in (12). Further, solving $V_4 = 0$ for a_{03} we obtain

$$\begin{aligned} a_{03} = {} & \frac{1}{36(8a_{12} - b_{01})(2a_{12} - b_{01})} \big\{ 36(8a_{12} - b_{01})(2a_{12} + b_{01})a_{01}^2 \\ & - \big[8b_{01}(8a_{12} - b_{01})^2(10a_{12} + b_{01}) - 9(2a_{12} - b_{01})(32a_{12} - b_{01}) \big] a_{01} \\ & - 2(8a_{12} - b_{01})(2a_{12} - b_{01})(10a_{12} + b_{01})(32a_{12} - b_{01}) + 90(8a_{12} - b_{01})^2 \big\}. \end{aligned}$$

Next, to solve $V_5 = V_6 = V_7 = 0$, i.e. $F_1 = F_2 = F_3 = 0$, we use the Maple built-in command "eliminate", eliminate($\{F_1, F_2, F_3\}, a_{01}$), to obtain the expression of $a_{01} = \frac{a_{01N}}{a_{01D}}$, where

$$\begin{aligned} a_{01N} = {} & -1215(8a_{12} - b_{01})^2(64a_{12}^2 + 23476a_{12}b_{01} - 689b_{01}^2) + 4(2170880a_{12}^6 \\ & - 2314804736a_{12}^5 b_{01} + 3406137856a_{12}^4 b_{01}^2 - 1252652096a_{12}^3 b_{01}^3 + 195375952a_{12}^2 b_{01}^4 \\ & - 13917038a_{12}b_{01}^5 + 373277b_{01}^6) + 192b_{01}(8a_{12} - b_{01})(14819328a_{12}^6 \\ & - 71382720a_{12}^5 b_{01} + 26937688a_{12}^4 b_{01}^2 - 3022718a_{12}^3 b_{01}^3 + 112215a_{12}^2 b_{01}^4 + 232a_{12}b_{01}^5 \\ & + 10b_{01}^6) + 161280a_{12}^2 b_{01}^2(8a_{12} - b_{01})^2(2a_{12} - b_{01})(6784a_{12}^3 - 384a_{12}^2 b_{01} \\ & - 168a_{12}b_{01}^2 + 7b_{01}^3) \end{aligned}$$

and

$$\begin{aligned} a_{01D} = {} & 18(55040a_{12}^4 + 3456752a_{12}^3 b_{01} - 1293012a_{12}^2 b_{01}^2 + 141454a_{12}b_{01}^3 - 5861b_{01}^4) \\ & + 432b_{01}(8a_{12} - b_{01})(10752a_{12}^4 + 2631716a_{12}^3 b_{01} - 979246a_{12}^2 b_{01}^2 + 122008a_{12}b_{01}^3 \\ & - 5123b_{01}^4) - 20736a_{12}^2 b_{01}^2(8a_{12} - b_{01})^2(8312a_{12}^2 - 7010a_{12}b_{01} + 329b_{01}^2) \\ & - 3483648a_{12}^2 b_{01}^3(8a_{12} - b_{01})^3(40a_{12}^2 - b_{01}^2), \end{aligned}$$

and two resultants F_{12} and F_{13}:

$$F_{12} = \pi b_{01}(8a_{12} - b_{01})(2a_{12} - b_{01})F_{12a}, \quad F_{13} = b_{01}(8a_{12} - b_{01})(2a_{12} - b_{01})F_{13a},$$

in which F_{12a}, F_{13a} are lengthy polynomials in a_{12} and b_{01}, which can be found from the supplement posted on the journal website. To solve the two equations $F_{12a} = F_{13a} = 0$, we have

$$F_{1213a} = \mathrm{Res}(F_{12a}, F_{13a}, a_{12}) = C_1 b_{01}^{108} G_1 G_2,$$

where C_1 is a constant, G_1 and G_2 are respectively 84th- and 172th-degree polynomials in b_{01}, which can be found from the supplement posted on the journal website.

To this end, we need to solve b_{01} from the equation $F_{1213a} = 0$. It can be shown that G_1 has 20 real solutions, and G_2 has 60 real solutions, which in turn yield corresponding 80 solutions for a_{12}. By verifying the original equations $F_{12a} = F_{13a} = 0$, we take one of them:

$$b_{01} = 9.845873500 \cdots, \quad a_{12} = -0.0831832490 \cdots.$$

Then the other five perturbation parameters are equal to

$$\begin{aligned} &a_{01} = -21.54010342 \cdots, \quad a_{03} = -0.075427946 \cdots, \quad b_{12} = -21.33758543 \cdots, \\ &b_{03} = -71.72022380 \cdots, \quad b_{02} = 0. \end{aligned}$$

At the above critical values, the Lyapunov constants become

$$V_i = 0, \ \ i = 1, 2, \ldots, 7, \quad V_8 = -22.67645322 \cdots \neq 0.$$

Moreover, at the above critical values, a direct calculation shows that

$$\det\left[\frac{\partial(V_1, V_2, V_3, V_4, V_5, V_6, V_7)}{\partial(b_{02}, b_{03}, b_{12}, a_{03}, a_{01}, b_{01}, a_{12})}\right] = 1.316056443 \cdots \times 10^{10} \neq 0,$$

implying, by Theorem 2.1, that system (10) can have 7 small-amplitude limit cycles bifurcating from the origin. Further, a linear perturbation on δ yields additional one limit cycle, giving a total of 8 limit cycles around the origin of system (10). Thus, system (3) can have 16 limit cycles around the two symmetric singular points $(\pm 1, 0)$.

From the above discussions, we do not find any solutions such that $V_5 = V_6 = V_7 = V_8 = 0$ under the restriction (15). Therefore, except for the two center conditions given in Theorem 3.2, no more bi-center conditions are found.

Finally, we check if it is possible to have limit cycles bifurcating from the origin of system (3) under the above critical parameter values. A simple computation shows that the zero-order Lyapunov constant at the origin of system (3) is equal to $e^{2\pi(\delta + b_{01})} = e^{2\pi b_{01}} > 0$ for $\delta = 0$, $b_{01} > 0$, implying that under the above critical parameter values, the origin of system (3) is an unstable focus, and so no more limit cycles can bifurcate from the origin of system (3).

Summarizing the above results we have shown that system (3) has at least 16 limit cycles with $8 \cup 8$ distribution around the singular points $(1, 0)$ and $(-1, 0)$. $\square$

4. 18 limit cycles generated by perturbing system (3) under the center condition (a)

In this section, we present our main result of this paper. We want to perturb the system (3) to generate small-amplitude limit cycles around the two centers $(\pm 1, 0)$. We choose the center condition (a) and add cubic perturbations to system (3) to obtain the following perturbed system:

$$
\begin{cases}
\dfrac{dx_2}{dt} = a_{01}y_2 + y_2^2 + (\frac{1}{2} - a_{01})x_2^2 y_2 + a_{03}y_2^3 + \epsilon \big[p_{00} + \frac{1}{2}\delta(x_2^3 - x_2) \\
\qquad + p_{10}x_2 + p_{01}y_2 + p_{20}x_2^2 + p_{11}x_2y_2 + p_{02}y_2^2 + p_{30}x_2^3 + p_{21}x_2^2y_2 \\
\qquad + p_{12}x_2y_2^2 + p_{03}y_2^3\big], & (y_2 > 0); \\
\dfrac{dy_2}{dt} = x_2 - x_2^3 + b_{12}x_2y_2^2 + \epsilon \big[q_{00} + q_{10}x_2 + (\delta + q_{01})y_2 + q_{20}x_2^2 + q_{11}x_2y_2 \\
\qquad + q_{02}y_2^2 + q_{30}x_2^3 + q_{21}x_2^2y_2 + q_{12}x_2y_2^2 + q_{03}y_2^3\big],
\end{cases}
$$

$$
\begin{cases}
\dfrac{dx_2}{dt} = a_{01}y_2 - y_2^2 + (\frac{1}{2} - a_{01})x_2^2 y_2 + a_{03}y_2^3 + \epsilon \big[-p_{00} + \frac{1}{2}\delta(x_2^3 - x_2) \\
\qquad + p_{10}x_2 + p_{01}y_2 - p_{20}x_2^2 - p_{11}x_2y_2 - p_{02}y_2^2 + p_{30}x_2^3 + p_{21}x_2^2y_2 \\
\qquad + p_{12}x_2y_2^2 + p_{03}y_2^3\big], & (y_2 < 0), \\
\dfrac{dy_2}{dt} = x_2 - x_2^3 + b_{12}x_2y_2^2 + \epsilon \big[-q_{00} + q_{10}x_2 + (\delta + q_{01})y_2 - q_{20}x_2^2 \\
\qquad - q_{11}x_2y_2 - q_{02}y_2^2 + q_{30}x_2^3 + q_{21}x_2^2y_2 + q_{12}x_2y_2^2 + q_{03}y_2^3\big],
\end{cases}
\tag{16}
$$

where δ, p_{ij}'s and q_{ij}'s are real parameters, satisfying $|\delta| \ll 1$ and $0 < \epsilon \ll 1$.

Without of loss of generality, suppose system (16) has Hopf singular points at $(\pm 1, 0)$, which requires that

$$p_{20} = -p_{00}, \quad q_{20} = -q_{00}, \quad p_{30} = -p_{10}, \quad q_{30} = -q_{10}, \quad q_{11} = 2p_{00}, \quad q_{21} = 2p_{10} - q_{01}.$$

In order to avoid complicated transformation in computing the Lyapunov constants, we set $p_{00} = p_{10} = 0$. Further, solving higher Lyapunov constant equations shows that we may set the superfluous parameters (which are not used to solve the equations) to equal zero:

$$p_{01} = p_{02} = p_{21} = p_{03} = q_{10} = q_{12} = 0,$$

and choose q_{00} as a free parameter, which can be used to adjust the order of Lyapunov constants.

Then, for the perturbed system (16) we have the following theorem.

Theorem 4.1. *The perturbed system* (16) *can have at least* 18 *small-amplitude limit cycles with* 9 *each around the singular points* $(1, 0)$ *and* $(-1, 0)$.

Proof. It is easy to verify that system (16) is a switching Z_2-equivariant cubic system. Due to the symmetry of the system, the Lyapunov constants associated with the singular points $(1, 0)$ and $(-1, 0)$ are same, hence we only need to consider the Hopf bifurcation at the singular point $(1, 0)$. In order to study the limit cycles bifurcating from the Hopf critical point $(1, 0)$, we need to compute its Lyapunov constants. To achieve this, we introduce the transformation,

$$x_2 = 1 - X, \quad y_2 = \frac{2\sqrt{(1 + q_{00}\epsilon)(1 + 2\epsilon p_{11})}}{1 + 2\epsilon p_{11}} Y, \quad t \to \frac{1}{\sqrt{(1 + q_{00}\epsilon)(1 + 2\epsilon p_{11})}} t,$$

into the upper system of (16), and the transformation

$$x_2 = 1 - X, \quad y_2 = \frac{2\sqrt{(1-q_{00}\epsilon)(1-2\epsilon p_{11})}}{1-2\epsilon p_{11}}\, Y, \quad t \to \frac{1}{\sqrt{(1-q_{00}\epsilon)(1-2\epsilon p_{11})}}\, t,$$

into the lower system of (16), so that the singular point $(1, 0)$ of (16) becomes the origin of the following system which have been expanded up to ϵ order,

$$\begin{cases} \dfrac{dX}{dt} = -Y + 2(1-2a_{01})XY - 4Y^2 - (1-2a_{01})X^2Y - 8a_{03}Y^3 \\ \quad + \epsilon\big[\delta\big(X - \frac{3}{2}X^2 + \frac{1}{2}X^3\big) - 2p_{11}(1-4a_{01})XY - 2(q_{00}+2p_{12}-6p_{11})Y^2 \\ \quad + 2p_{11}(1-2a_{01})X^2Y + 4p_{12}XY^2 - 8a_{03}(q_{00}-4p_{11}))Y^3\big], \\ \dfrac{dY}{dt} = X - \frac{3}{2}X^2 + 2b_{12}Y^2 + \frac{1}{2}X^3 - 2b_{12}XY^2 \\ \quad + \epsilon\big[\delta Y + q_{00}X^2 + 2q_{01}XY + 2(q_{02} - 2b_{12}p_{11})Y^2 \\ \quad - \frac{1}{2}q_{00}X^3 - q_{01}X^2Y + 4b_{12}p_{11}XY^2 + 4q_{03}Y^3\big], \end{cases} \quad (Y > 0);$$

$$\begin{cases} \dfrac{dX}{dt} = -Y + 2(1-2a_{01})XY + 4Y^2 - (1-2a_{01})X^2Y - 8a_{03}Y^3 \\ \quad + \epsilon\big[\delta\big(X - \frac{3}{2}X^2 + \frac{1}{2}X^3\big) + 2p_{11}(1-4a_{01})XY - 2(q_{00}+2p_{12}-6p_{11})Y^2 \\ \quad - 2p_{11}(1-2a_{01})X^2Y + 4p_{12}XY^2 + 8a_{03}(q_{00}-4p_{11})Y^3\big], \\ \dfrac{dY}{dt} = X - \frac{3}{2}X^2 + 2b_{12}Y^2 + \frac{1}{2}X^3 - 2b_{12}XY^2 \\ \quad + \epsilon\big[\delta Y - q_{00}X^2 + 2q_{01}XY - 2(q_{02} - 2b_{12}p_{11})Y^2 \\ \quad + \frac{1}{2}q_{00}X^3 - q_{01}X^2Y - 4b_{12}p_{11}XY^2 + 4q_{03}Y^3\big], \end{cases} \quad (Y < 0). \tag{17}$$

To prove the existence of 18 small-amplitude limit cycles, we need to find the ϵ-order Lyapunov constants ϵV_{1i}, $i = 0, 1, 2 \cdots$. First, we have $V_{10} = 2\pi\delta$, thus letting $\delta = 0$ yields $V_{10} = 0$. Then, we obtain

$$V_{11} = \frac{4}{3}\big[4q_{02} + 8(a_{01} - b_{12})p_{11} + q_{00} - 2p_{11}\big].$$

Solving $V_{11} = 0$ for q_{02} yields

$$q_{02} = -\frac{1}{4}\big[8(a_{01} - b_{12})p_{11} + q_{00} - 2p_{11}\big]. \tag{18}$$

Next, we solve $V_{12} = 0$ for q_{03} to get

$$q_{03} = -\frac{1}{6}\big[4(a_{01} - b_{12})(q_{00} - 6p_{11} + 2p_{12}) - (2b_{12} - 1)q_{01} + 2(6p_{11} - p_{12})\big] \tag{19}$$

Now, in order to solve higher Lyapunov constant equations using the remaining perturbations parameters, we assume that $F_0 = F_{01}F_{02}F_{03} \neq 0$, where

$$
\begin{aligned}
F_{01} =& (2a_{01} - 2b_{12} - 1),\\
F_{02} =& 6(4a_{01} - 4b_{12} - 3)a_{03} + 2a_{01}(4a_{01} - 12b_{12} - 3)b_{12} + 16a_{01}^3 - 8a_{01}^2 + 9a_{01} + 60,\\
F_{03} =& 720a_{01}^3 b_{12}^3 - 3240a_{03}^3 + 10a_{01}(356a_{01} + 616b_{12} - 409)a_{03}^2 - \big\{4b_{12}\big[2a_{01}^2(578a_{01}\\
&+ 475b_{12}) - 1265a_{01}^2 + 450\big] + 9(144a_{01}^4 - 352a_{01}^3 + 185a_{01}^2 + 80a_{01} - 120)\big\}a_{03}\\
&+ 6a_{01}(252a_{01}^3 - 255a_{01}^2 + 280)b_{12}^2 + a_{01}(880a_{01}^4 - 2008a_{01}^3 + 1035a_{01}^2 + 864a_{01}\\
&- 1890)b_{12} + 128a_{01}^6 - 600a_{01}^5 + 658a_{01}^4 - 249a_{01}^3 - 498a_{01}^2 + 525a_{01} - 360.
\end{aligned}
$$

Then, solving $V_{13} = 0$ for p_{12} we have

$$
\begin{aligned}
p_{12} =& \frac{1}{8}q_{01} + \frac{1}{32}(3a_{01} + 12a_{03} - 16)q_{00} + \frac{3}{16}\big[a_{01}(4a_{01} + 4b_{12} - 3) - 8a_{03} + 16\big]p_{11}\\
&+ \frac{3}{64F_{01}}\big\{4q_{01} + \big[4a_{03} - a_{01}(4a_{01} - 5)\big]q_{00}\big\},
\end{aligned}
\tag{20}
$$

and solving $V_{14} = 0$ for q_{01} yields

$$
\begin{aligned}
q_{01} =& -\frac{1}{4F_{02}}\big\{12F_{01}\big[2a_{03}\big(40a_{03} - a_{01}(36a_{01} + 44b_{12} - 27)\big) + 2b_{12}(4a_{01}^2(5a_{01} + 3b_{12})\\
&- 15a_{01}^2 + 24) + 16a_{01}^4 - 24a_{01}^3 + 9a_{01}^2 + 12a_{01} - 24\big]p_{11} - \big[6a_{03}\big(20(4a_{01} - 4b_{12}\\
&- 1)a_{03} - 16a_{01}b_{12}(a_{01} - 3b_{12} + 2) - 32a_{01}^3 + 32a_{01}^2 + 9a_{01} - 40\big) + 2b_{12}(12(3a_{01}^2\\
&- 8)b_{12} + 24a_{01}^3 - 45a_{01}^2 + 176a_{01} + 48) - a_{01}(36a_{01}^2 - 107a_{01} + 460)\big]q_{00}\big\}
\end{aligned}
\tag{21}
$$

There are two remaining perturbation parameters p_{11} and q_{00}, but only one is independent. We use p_{11} to solve $V_{15} = 0$ to obtain

$$
\begin{aligned}
p_{11} =& -\frac{q_{00}}{32F_{01}F_{03}}\big\{5760(10a_{01} - 10b_{12} - 3)a_{03}^3 - 80\big[(4a_{01}(142a_{01} - 236b_{12} + 141)b_{12}\\
&+ 376a_{01}^3 - 792a_{01}^2 + 83a_{01} + 144\big]a_{03}^2 - 2\big[9600a_{01}^2 b_{12}^3 - 176(8a_{01}^3 + 115a_{01}^2 - 40)b_{12}^2\\
&- 16b_{12}(424a_{01}^4 - 434a_{01}^3 + 380 - 935a_{01}^2 + 620a_{01}) - 1408a_{01}^5 + 5616a_{01}^4 - 3776a_{01}^3\\
&- 12375a_{01}^2 + 22800a_{01} + 1200\big]a_{03} - 3840a_{01}^3 b_{12}^4 - 160a_{01}(4a_{01}^3 - 3a_{01}^2 - 144)b_{12}^3\\
&+ 64a_{01}(50a_{01}^4 - 134a_{01}^3 + 195a_{01}^2 - 426a_{01} - 345)b_{12}^2 + 2(640a_{01}^6 - 3632a_{01}^5 + 6432a_{01}^4\\
&- 11593a_{01}^3 + 20904a_{01}^2 + 1440a_{01} - 8320)b_{12} - 960a_{01}^6 + 3408a_{01}^5 - 364a_{01}^4\\
&+ 14045a_{01}^3 - 19380a_{01}^2 + 23600a_{01} + 8000\big\}.
\end{aligned}
\tag{22}
$$

It is noted that p_{11} has a factor q_{00}, and so does q_{01}, and so on, showing that q_{00} can indeed treated as a free parameter.

Now, for the above solutions, higher Lyapunov constants are obtained as follows:

$$V_{16} = \frac{5\pi\, q_{00}}{64 F_{03}} F_{16}, \qquad V_{17} = -\frac{q_{00}}{846720 F_{03}} F_{17},$$
$$V_{18} = \frac{q_{00}}{508032 F_{03}} F_{18}, \qquad V_{19} = -\frac{q_{00}}{167650560 F_{03}} F_{19},$$

where

$$\begin{aligned}
F_{16} = {} & 322560a_{03}^6 - 7680a_{01}(74a_{01} + 178b_{12} - 145)a_{03}^5 + 64(37680a_{01}^2 b_{12}^2 + 8(4083a_{01}^3 \\
& - 7680a_{01}^2 + 1450)b_{12} + 5256a_{01}^4 - 24948a_{01}^3 + 23715a_{01}^2 - 13700a_{01} - 6920)a_{03}^4 \\
& - 8(285440a_{01}^3 b_{12}^3 + 16a_{01}(23776a_{01}^3 - 43305a_{01}^2 + 16500)b_{12}^2 + 16a_{01}(8176a_{01}^4 \\
& - 36602a_{01}^3 + 33330a_{01}^2 - 10818a_{01} - 22695)b_{12} + 9728a_{01}^6 - 93904a_{01}^5 + 215200a_{01}^4 \\
& - 289487a_{01}^3 + 139716a_{01}^2 + 116400a_{01} + 60480)a_{03}^3 + 4(308480a_{01}^4 b_{12}^4 \\
& + 160a_{01}^2(3448a_{01}^3 - 6135a_{01}^2 + 3612)b_{12}^3 + 16(18640a_{01}^6 - 79636a_{01}^5 + 70055a_{01}^4 \\
& - 6132a_{01}^3 - 75795a_{01}^2 + 8960)b_{12}^2 + 2(24832a_{01}^7 - 217968a_{01}^6 + 470288a_{01}^5 \\
& - 533773a_{01}^4 + 60540a_{01}^3 + 411750a_{01}^2 - 5280a_{01} - 83360)b_{12} + 1536a_{01}^8 \\
& - 33536a_{01}^7 + 153040a_{01}^6 - 386240a_{01}^5 - 65832a_{01}^3 + 514735a_{01}^4 - 31160a_{01}^2 \\
& - 7200a_{01} + 46400)a_{03}^2 - 2(181760a_{01}^5 b_{12}^5 + 320a_{01}^3(1740 - 2199a_{01}^2 + 1256a_{01}^3)b_{12}^4 \\
& + 64a_{01}(4624a_{01}^6 - 19158a_{01}^5 + 16415a_{01}^4 + 3210a_{01}^3 - 24865a_{01}^2 + 6700)b_{12}^3 \\
& + 4a_{01}(20480a_{01}^7 - 164720a_{01}^6 + 338432a_{01}^5 - 334789a_{01}^4 - 116164a_{01}^3 + 412740a_{01}^2 \\
& - 63264a_{01} - 193200)b_{12}^2 + 4(1536a_{01}^9 - 28288a_{01}^8 + 117456a_{01}^7 - 257720a_{01}^6 \\
& + 277423a_{01}^5 + 77576a_{01}^4 - 191589a_{01}^3 + 54508a_{01}^2 + 114800a_{01} + 32960)b_{12} \\
& - 3840a_{01}^9 + 37632a_{01}^8 - 174528a_{01}^7 + 423472a_{01}^6 - 366805a_{01}^5 + 192628a_{01}^4 \\
& + 106974a_{01}^3 - 45240a_{01}^2 - 20240a_{01} - 83840)a_{03} + 46080a_{01}^6 b_{12}^6 + 384a_{01}^4(308a_{01}^3 \\
& - 535a_{01}^2 + 500)b_{12}^5 + 192a_{01}^2(568a_{01}^6 - 2312a_{01}^5 + 1940a_{01}^4 + 1080a_{01}^3 - 3755a_{01}^2 \\
& + 1560)b_{12}^4 + 24(1792a_{01}^9 - 13552a_{01}^8 + 26928a_{01}^7 - 22765a_{01}^6 - 23636a_{01}^5 \\
& + 42890a_{01}^4 - 6528a_{01}^3 - 33480a_{01}^2 + 6400)b_{12}^3 + 12(512a_{01}^1 0 - 7680a_{01}^9 + 29296a_{01}^8 \\
& - 59408a_{01}^7 + 48749a_{01}^6 + 47240a_{01}^5 - 77626a_{01}^4 + 20040a_{01}^3 + 64740a_{01}^2 - 4480a_{01} \\
& - 20800)b_{12}^2 - 2(3840a_{01}^1 0 - 31488a_{01}^9 + 119488a_{01}^8 - 265264a_{01}^7 + 170709a_{01}^6 \\
& + 58284a_{01}^5 - 231042a_{01}^4 + 57096a_{01}^3 + 53328a_{01}^2 - 13440a_{01} - 67200)b_{12} \\
& + 2304a_{01}^1 0 - 24064a_{01}^9 + 90528a_{01}^8 - 162848a_{01}^7 + 192489a_{01}^6 - 66576a_{01}^5 \\
& - 90174a_{01}^4 + 102576a_{01}^3 - 48708a_{01}^2 - 77760,
\end{aligned}$$

while F_{17}, F_{18}, F_{19} are lengthy polynomials in a_{01}, a_{03}, and b_{12}, which can be found from the supplement posted on the journal website. Therefore, the best choice for obtaining maximal number of limit cycles is to find the solutions of a_{03}, a_{01} and b_{12} such that $F_{16} = F_{17} = F_{18} = 0$, but $F_0 F_{19} \neq 0$, which results in at most 9 small-amplitude limit cycles from the origin of system (17).

To find the solutions of $F_{16} = F_{17} = F_{18} = 0$, we again apply the Maple built-in command "eliminate" to compute eliminate($\{F_{16}, F_{17}, F_{18}\}, a_{03}$), yielding $a_{03} = \dfrac{a_{03N}}{2a_{03D}}$, and two resultants F_{67} and F_{68},

$$F_{67} = G_{678} F_{67a}, \quad F_{68} = G_{678} F_{68a},$$

where a_{03N}, a_{03D}, F_{67a}, F_{68a} and G_{678} are polynomials in a_{01}, b_{12}, which can be found from the supplement of the paper (as posted on the journal website). We need to find solutions of a_{01} and b_{12} such that $F_{67a} = F_{68a} = 0$, but $a_{03D} F_0 F_{19} \neq 0$. For $F_{67a} = F_{68a} = 0$, we have

$$F_{6768a} = \mathrm{Res}(F_{67a}, F_{68a}, b_{12}),$$

and F_{6768a} is a lengthy polynomial in a_{01}. Now we solve the polynomial equation $F_{6768a} = 0$ to find the solutions of a_{01}. It can be shown that this polynomial has 23 real roots, which in turn yield 23 corresponding solutions for b_{12}. By verifying that $F_{67a} = F_{68a} = 0$ and $a_{03D} F_0 F_{19} \neq 0$, we take one of the solutions:

$$a_{01} = 0.771804218\cdots, \quad b_{12} = 1.884124074\cdots \quad \Longrightarrow \quad a_{03} = 0.859592555\cdots, \tag{23}$$

for which the other perturbation parameters are equal to

$$\begin{aligned} &p_{11} = 0.160897025\cdots q_{00}, \quad p_{12} = 0.285394345\cdots q_{00}, \quad q_{01} = 0.140458877\cdots q_{00}, \\ &q_{02} = 0.188386424\cdots q_{00}, \quad q_{03} = 0.287078710\cdots q_{00}. \end{aligned}$$

The above critical values can be used to define a critical point, called p_c, for which the ϵ-order Lyapunov constants become

$$V_{1i} = 0, \ i = 1, 2, \ldots, 8, \quad V_{19} = 82.19417558\cdots q_{00} \neq 0, \quad (q_{00} \neq 0).$$

Moreover, a direct calculation shows that

$$\det\left[\frac{\partial(V_{11}, V12, V13, V_{14}, V_{15}, V_{16}, V_{17}, V_{18})}{\partial(p_{11}, p_{12}, q_{01}, q_{02}, q_{03}, a_{01}, b_{12}, a_{03})}\right] = 0.78482376\cdots \times 10^{13} q_{00}^3 \neq 0, \quad (q_{00} \neq 0),$$

implying, by Theorem 2.1, that system (17) can indeed have 8 small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Finally, a linear perturbation is applied to the parameter δ to yield one more small limit cycles, and so system (17) has a total 9 limit cycles around the origin. Thus, system (16) can have 18 limit cycles.

The proof of Theorem 4.1 is complete. □

5. Realization of 18 limit cycles

In this section, we present a method to show how to realize the 18 limit cycles arising from Hopf bifurcation by perturbing the singular points $(1, 0)$ and $(-1, 0)$ of system (16).

In general, determining the location of single limit cycle is straightforward, and is still easy for two limit cycles. However, it is somewhat challenging for finding the locations of three limit cycles [33]. Determining the locations of 9 limit cycles around a singular point is extremely hard.

The main difficulty comes from how to appropriately choosing perturbations of the parameters from the critical point so that the truncated Poincaré succession function can have 9 positive real roots. If the perturbations can be performed step by step and choose one parameter at each step, then the process is still straightforward. However, if the polynomial equations from the Poincaré succession function are coupled, then finding the perturbation is much more difficult. In the following, motivated by Liu's idea [34], we develop a method to realize the 18 limit cycles.

Suppose in general we want to prove the existence of N small-amplitude limit cycles around an isolated singular point (the origin), and thus assume that the associated displacement function of the system is given by

$$\triangle(h) = h\left(\sum_{i=0}^{N} v_i\, h^i + \cdots\right), \tag{24}$$

where v_i is the ith-order Lyapunov constant. As discussed in Section 4, $v_i = \epsilon V_{1i} + o(\epsilon)$, $i = 0, 1, 2\cdots$. Thus, equation (24) can be rewritten as

$$\triangle(h) = \epsilon h\left(\sum_{i=0}^{N} V_{1i}\, h^i + \cdots\right) + o(\epsilon) = \epsilon h\bar{\triangle}(h) + o(\epsilon). \tag{25}$$

Since we are interested in finding small-amplitude limit cycle solutions from the truncated equation $\sum_{i=0}^{N} V_{1i}\, h^i$, we introduce the scaling $h \to \varepsilon h$ $(0 < \varepsilon \ll 1)$ into (25) to obtain

$$\bar{\triangle}(\varepsilon h) = \sum_{i=0}^{N} V_{1i}\, \varepsilon^i h^i + \cdots \tag{26}$$

When ε is small enough, solving for positive real solution of h from $\triangle(\varepsilon h) = 0$ is equivalent to solving for positive real solution of h from $\bar{\triangle}(\varepsilon h) = 0$.

Next, we suppose the perturbed Lyapunov constants are given in the form of

$$V_{1i} = K_i \varepsilon^{N-i} + o(\varepsilon^{N-i}), \;\; i = 0, 1, 2, \cdots, N-1, \quad \text{and} \quad V_{1j} = K_j + o(1), \;\; j \geq N, \tag{27}$$

under which equation $\bar{\triangle}(\varepsilon h) = 0$ becomes

$$\varepsilon^N \left[\sum_{i=0}^{N} K_i\, h^i + \varepsilon h G(\varepsilon, h)\right] = 0, \tag{28}$$

where $G(\varepsilon, h)$ is analytic at $(0, 0)$. By implicit function theorem, when ε is small enough, it follows from (28) that if the equation

$$\sum_{i=0}^{N} K_i\, x^i = 0 \tag{29}$$

has N positive real roots h_i, $i = 1, 2, \ldots, N$, then the equation $\bar{\triangle}(\varepsilon h) = 0$ has N positive real roots, and so the system has N limit cycles near the origin.

Now, for (16), we have the following theorem.

Theorem 5.1. *With the following perturbed parameter values:*

$$\begin{array}{ll} \delta = 0.1 \times 10^{-77}, & q_{00} = -0.001 \\ q_{02} = -0.0001883900\cdots, & q_{03} = -0.0002870751\cdots, \\ p_{12} = -0.0002853968\cdots, & q_{01} = -0.0001404514\cdots, \\ p_{11} = -0.0001608979\cdots, & a_{01} = 0.7718034681\cdots, \\ b_{12} = 1.8841266284\cdots, & a_{03} = 0.8595955621\cdots, \end{array}$$

the equation $\bar{\Delta}(\varepsilon h) = 0$ *has* 9 *positive real roots, implying that system* (16) *has* 9 *limit cycles around the origin, and thus correspondingly, system* (16) *has* 18 *limit cycles around the two systematic singular points* $(\pm 1, 0)$.

Proof. Since in the previous section we have shown that the first 6 Lyapunov constant equations: $V_{1i} = 0$, $i = 0, 1, \ldots, 5$ can be solved one by one using the 6 parameters: δ, q_{02}, q_{03}, p_{12}, q_{01} and p_{11}, respectively, we may use the above described method to first determine the perturbed values of the coefficients a_{01}, b_{12} and a_{03} such that the truncated equation $\bar{\Delta}(\varepsilon h)$ has 3 positive roots. Once, the 3 roots are found, then we perturb the above mentioned 6 parameters to obtain the other 6 positive roots.

Therefore, we use the values of a_{01}, b_{12} and a_{03} given in (23) to define a critical point $(a_{01c}, b_{12c}, a_{03c})$, and suppose that the 3 positive roots satisfies the equation:

$$f(x) = x^6(K_6 + K_7 x + K_8 x^2 + K_9 x^3),$$

where $K_9 = -0.0821941755\cdots$ with $x = 1, 2, 3$. Then, solving the equations: $f(i) = 0$, $i = 1, 2, 3$ yields

$$K_6 = K_8 = 0.4931650534\cdots, \quad K_7 = -0.9041359314\cdots. \tag{30}$$

Next, without loss of generality, we assume that

$$\begin{array}{l} a_{01} = a_{01c} + k_{11}\varepsilon + k_{12}\varepsilon^2 + k_{13}\varepsilon^3, \\ b_{12} = b_{12c} + k_{21}\varepsilon + k_{22}\varepsilon^2 + k_{23}\varepsilon^3, \\ a_{03} = a_{03c} + k_{31}\varepsilon + k_{32}\varepsilon^2 + k_{33}\varepsilon^3. \end{array} \tag{31}$$

Substituting (31) into V_{16}, V_{17} and V_{18}, and expanding them in Taylor series, we obtain

$$\begin{array}{l} V_{16} = E_{60} + E_{61}\varepsilon + E_{62}\varepsilon^2 + E_{63}\varepsilon^3 + o(\varepsilon^3), \\ V_{17} = E_{70} + E_{71}\varepsilon + E_{72}\varepsilon^2 + o(\varepsilon^2), \\ V_{18} = E_{80} + E_{81}\varepsilon + o(\varepsilon), \end{array} \tag{32}$$

where E_{ij} are functions of a_{01}, b_{12} and a_{03}, and so the functions of k_{ij}, $i, j = 1, 2, 3$.

Combining (27), (30) and (32), and balancing the coefficients of like powers of ε give the following equations,

$$\begin{aligned}
&E_{60}=E_{61}=E_{62}=0,\\
&E_{70}=E_{71}=E_{80}=0,\\
&E_{63}=K_6=0.4931650534\cdots,\\
&E_{72}=K_7=-0.9041359314\cdots,\\
&E_{81}=K_8=0.4931650534\cdots.
\end{aligned} \tag{33}$$

Then, solving the equations in (33) yields the solutions:

$$\begin{aligned}
&k_{11}=-1.5005559300\cdots, \quad && k_{12}=77.4164584360\cdots, \quad && k_{13}=4308.6840042932\cdots\\
&k_{21}=5.1087234435\cdots, && k_{22}=-52.3053218513\cdots, && k_{31}=6.0142433205\cdots,\\
&k_{23}=k_{32}=k_{33}=0.
\end{aligned}$$

Thus, the perturbed values of the parameters a_{01}, b_{12}, and a_{03} are obtained from (31) as

$$\begin{aligned}
a_{01}&=0.7718042183\cdots-1.5005559300\cdots\varepsilon+77.4164584360\cdots\varepsilon^2\\
&\quad+4308.6840042932\cdots\varepsilon^3,\\
b_{12}&=1.8841240740\cdots+5.1087234435\cdots\varepsilon-52.3053218513\cdots\varepsilon^2,\\
a_{03}&=0.8595925550\cdots+6.0142433205\cdots\varepsilon.
\end{aligned}$$

Thus, the perturbed Lyapunov constants V_{16}, V_{17} and V_{18} becomes

$$V_{16}\approx 0.4931650534\cdots\varepsilon^3, \quad V_{17}\approx -0.9041359314\cdots\varepsilon^2, \quad V_{18}\approx 0.4931650534\cdots\varepsilon.$$

In order for the truncated equation $h^6(V_{16}+V_{17}h+V_{18}h^2+V_{19}h^3)$ to have 3 positive roots, we choose $\varepsilon=0.0000005$ and obtain the parameter values:

$$a_{01}=0.7718034681\cdots, \quad b_{12}=1.8841266284\cdots, \quad a_{03}=0.8595955621\cdots,$$

and 3 positive roots:

$$h=0.15035837\cdots\times 10^{-5}, \ 0.99512380\cdots\times 10^{-6}, \ 0.50125398\cdots\times 10^{-6}.$$

Now we perturb p_{11} given in (22) as

$$p_{11}=p_{11}(a_{01},b_{12},a_{03})-10^{-28},$$

which yields $V_{15}=-0.1847974495\cdots\times 10^{-26}$, and then the truncated equation $h^5(V_{15}+V_{16}h+V_{17}h^2+V_{18}h^3+V_{19}h^4$ gives four positive roots:

$$\begin{aligned}
h=\ &0.14703859\cdots\times 10^{-5}, \ 0.10800418\cdots\times 10^{-5}, \ 0.41545695\cdots\times 10^{-6},\\
&0.34076758\cdots\times 10^{-7},
\end{aligned}$$

the first three of which are very close to that obtained above. Similarly, we can perturb q_{01} given in (21), p_{12} in (20), q_{03} in (19), q_{02} in (18) and δ as

$$
\begin{aligned}
&q_{01} = q_{01}(a_{01}, b_{12}, a_{03}, p_{11}) - 10^{-36},\\
&p_{12} = p_{12}(a_{01}, b_{12}, a_{03}, p_{11}, q_{01}) - 10^{-46},\\
&q_{03} = q_{03}(a_{01}, b_{12}, a_{03}, p_{11}, q_{01}, p_{12}) + 10^{-55},\\
&q_{02} = q_{02}(a_{01}, b_{12}, a_{03}, p_{11}, q_{01}, p_{12}, q_{03}) - 10^{-66},\\
&\delta = 10^{-78},
\end{aligned}
$$

so that the perturbed Laypunov constants are

$$
\begin{aligned}
&V_{10} = 0.628318530\cdots \times 10^{-77}, && V_{11} = -0.533333333\cdots \times 10^{-65},\\
&V_{12} = 0.942477796\cdots \times 10^{-54}, && V_{13} = -0.733786184\cdots \times 10^{-44},\\
&V_{14} = 0.900230757\cdots \times 10^{-35}, && V_{15} = -0.184797448\cdots \times 10^{-26},\\
&V_{16} = 0.652939927\cdots \times 10^{-19}, && V_{17} = -0.225929658\cdots \times 10^{-12},\\
&V_{18} = 0.246578621\cdots \times 10^{-6}, && V_{19} = -0.082193928\cdots,
\end{aligned}
$$

and then the truncated equation $\sum_{i=0}^{9} V_{1i} h^i = 0$ yields 9 positive roots:

$$
\begin{aligned}
&h_1 = 0.1657109744\cdots \times 10^{-11}, && h_2 = 0.4214638433\cdots \times 10^{-11},\\
&h_3 = 0.1506345222\cdots \times 10^{-9}, && h_4 = 0.8244924206\cdots \times 10^{-9},\\
&h_5 = 0.4932305243\cdots \times 10^{-8}, && h_6 = 0.2511209732\cdots \times 10^{-7},\\
&h_7 = 0.5011772531\cdots \times 10^{-6}, && h_8 = 0.9133019498\cdots \times 10^{-6},\\
&h_9 = 0.1554457849\cdots \times 10^{-5},
\end{aligned}
$$

which are approximated amplitudes of the 9 limit cycles.

The proof is complete. □

6. Conclusion

In this paper, we have studied planar switching systems, and applied a computationally efficient algorithm to compute the Lyapunov constants for planar switching systems. With the help of this algorithm and Maple build-in commands, we have obtained the center conditions and proved the existence of 16 limit cycles for a class of cubic switching systems. Moreover, we have used one of the center conditions to construct a special integrable system and then perturbed this system to obtain 18 small-amplitude limit cycles, which is a new lower bound on the maximal number of small-amplitude limit cycles obtained in such cubic switching systems.

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Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.jde.2018.07.071.

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