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INFINITE ORDER PARAMETRIC NORMAL FORM OF HOPF SINGULARITY

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In this paper, we introduce a suitable algebraic structure for efficient computation of the parametric normal form of Hopf singularity based on a notion of formal decompositions. Our parametric state and time spaces are respectively graded parametric Lie algebra and graded ring. As a consequence, the parametric state space is also a graded module. Parameter space is observed as an integral domain as well as a vector space, while the near-identity parameter map acts on the parametric state space. The method of multiple Lie bracket is used to obtain an infinite order parametric normal form of codimension-one Hopf singularity. Filtration topology is revisited and proved that state, parameter and time (near-identity) maps are continuous. Furthermore, parametric normal form is a convergent process with respect to filtration topology. All the results presented in this paper are verified by using Maple.

Keywords: Unique normal form; bifurcation parameter; graded Lie algebra; graded module; formal basis.

1. Introduction

The history of normal form theory goes back to more than one hundred years ago, when Poincaré approached the problem of integrating nonlinear differential equations and developed normal form theory (see [Poincaré, 1879]). Since then normal form theory has played a fundamental role in the study of qualitative behavior of dynamical systems, e.g. see [Chow & Hale, 1982; Chow *et al.*, 1994; Dumortier *et al.*, 1991; Kuznetsov, 2004; Liao *et al.*, 2007]. Classical normal forms are useful, yet neither unique nor sufficient to fulfil all of its possible applications. Takens [1973] noticed that classical normal forms could be further simplified. This finding, with the invention of computer algebra systems and its vast applications, has attracted the attention of many researchers to work in this field. Various methods have been developed for computing unique normal forms. A unique normal form is the simplest normal form (SNF) in its own style.

The results in the existing literature widely vary from methods and techniques in practical and efficient computations to abstract concepts. The abstract concepts bridges the normal form theory to other areas of mathematics. Interestingly, some of these results sometimes naively seem unrelated but they end up being quite helpful. For example, Sanders [2003, 2005] has recently paid attention to applications of cohomology theory and spectral sequences in the computation of normal forms. Kokubu *et al.* [1996] raised the notion of multiple Lie bracket method and quasi homogeneous normal form theory, see also [Algaba *et al.*, 2003; Ashkenazi & Chow,

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1988; Chen & Della Dora, 2000; Chen *et al.*, 2000; Peng & Wang, 2004; Wang *et al.*, 2000]. Baider, Churchill and Sanders [Baider & Churchill, 1988; Baider, 1989; Baider & Sanders, 1991, 1992] brought up the concept of filtered Lie algebra and its associated topology in normal form theory for which the normal forms are convergent. Yu *et al.* [Yu, 1999; Yu & Yuan, 2000, 2001; Yu & Leung, 2002; Yu & Yuan, 2003a, 2003b] developed the theory for efficient computation of normal forms, based on a refinement of conventional normal forms. Kuznetsov [2005] considered the normal theory in an application-oriented way for computation of high dimensional systems and applied to some practical problems. Time rescaling was used by Stróżyna and Zoladek [2002, 2003], and Sadovskiĭ [1985] to obtain the results on orbital equivalence of vector fields, while Dumortier *et al.* [1991] paid attention to C^∞-conjugates, see also [Algaba *et al.*, 2003]. There are many other important contributions made by the aforementioned researchers, their colleagues and other well-known mathematicians in this field.

Most results obtained, and approaches developed so far have focused on the systems without perturbation parameters (unfolding). It is evident that the parametric normal form theory is much more complicated than, and different from, normal form without unfolding. This is the main reason that most results and theorems have been obtained by using simplified systems without unfolding. However, in reality all systems contain some parameters, and thus parametric normal forms are the only useful tool in the direct analysis of engineering and practical problems. Recently, several researchers [Yu, 2002; Yu & Leung, 2003; Liao *et al.*, 2007; Yu & Chen, 2007] have paid particular attention to this problem. They have extended the efficient computing normal form theory with their novel formulas in which the computation of the simplest normal form (SNF) with unfolding is only involved with some successive algebraic equations at each degree. They have also developed efficient Maple programs to compute the SNF of the systems with parameters for several different singularities. This problem, however, has not been touched by using other well-known approaches which have been widely used to consider systems without unfolding.

In this paper, we introduce an algebraic structure for computation of infinite order parametric normal form of Hopf singularity. Our structures generalize multiple Lie bracket method to parametric normal form theory which can be modified to consider other singularities. We prove that the simplest normal form obtained in [Yu & Leung, 2003] is an infinite order parametric normal form and is unique. While the efficient computing method represents the simplest normal form, it is instructive to reconsider it by other well-known methods such as multiple Lie bracket method. We compare and unify the two different approaches. The implementation of the formulas and results obtained here generates simpler and more systematic Maple programs. It has been noticed that the parametric simplest normal form may not be obtained without time rescaling and reparametrization [Yu, 2002; Yu & Leung, 2003; Liao *et al.*, 2007; Yu & Chen, 2007]. Thus, our algebraic structure develops the necessary tools for time rescaling and reparametrization. For a good background and some original ideas used in this paper, we refer the reader to [Jacobson, 1966; Baider & Churchill, 1988; Chua & Kokubu, 1988; Baider, 1989; Baider & Sanders, 1991, 1992; Wang, 1993; Kokubu *et al.*, 1996; Chen & Dora, 2000; Wang *et al.*, 2000; Yu, 2002; Algaba *et al.*, 2003; Yu & Leung, 2003; Peng & Wang, 2004; Liao *et al.*, 2007; Yu & Chen, 2007].

The structure of the near-identity change of state variable, reparametrization, and time rescaling are totally different from each other and therefore, they have been studied separately. Unlike the normal form of a system with no parameters, the computation of lower order terms depend on higher order terms. Thus, we have to predict "all" possible changes and outcomes that may get involved and then take the best choices. To this end, we have to determine a solid strategy with regard to the parametric "time" space and the "complementary" subspaces, based on the specific singularity and its conditions. We present this strategy by means of formal decomposition (and basis) of parametric time and state spaces. Thus, all the graded structures and our algebraic setting are presented in terms of formal basis and decomposition. This concept is one of the main tools used in this paper. In order to introduce this concept (analogous concept of formal power series) we revisit filtration topology which is among the best approaches of representing this concept (since, roughly speaking, direct sum is dense in direct product).

The rest of this paper is organized as follows. Section 2 introduces formal bases and its associated decompositions via filtration topology, followed by a presentation of a Lie algebraic structure for parametric state space. Parameter space and parametric time space are discussed in Sec. 3. Section 4 presents a method in obtaining an infinite order parametric normal form. The general theory and methodology are then applied to consider parametric normal form of codimension-1 Hopf singularity in Sec. 5, and finally conclusions are given in Sec. 6.

2. Formal Basis and Parametric Lie Algebra $\mathscr{L}^2$

In this section, we present concepts of formal bases, formal decompositions and parametric state space. The reader should note that the concepts presented in the following are known but they are necessary in defining formal basis and formal decomposition, which are needed for setting our time rescaling scheme. We denote $\mathbb{F}$ for a field of characteristic zero throughout this paper.

In this paper, a graded vector space over $\mathbb{F}$ is a vector space V which can be written as a direct product of the form $V = \prod_{i=1}^{\infty} V_i$, where V_i is a finite dimensional vector space. For a given n, any element $v \in V_n$ is called a homogeneous element of grade n. This graded structure is associated with a filtration $\mathscr{F}_{\{V_i\}} = \{\mathscr{F}_j = \prod_{i=j}^{\infty} V_i\}$. Obviously, $\{\mathscr{F}_j\}$ constitutes an open local base for zero whose induced topology is called filtration topology denoted by $\tau = \tau_{\mathscr{F}_{\{V_i\}}}$. It is easy to see that τ is a metric topology. Any sequence of $\{v_n\}_{n=1}^{\infty} \subset V$ is convergent to $v \in V$ if and only if for any $N \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $v - v_n \in \mathscr{F}_N$ for all $n \geq k_0$. It is well known that V is the τ-closure of $\oplus_{n=1}^{\infty} V_n$ (direct sum).

Now consider $\mathscr{B}_n = \{e_n^i | 1 \leq i \leq N_n\}$ $(n \in \mathbb{N})$ as a vector basis of V_n. Then, any $v \in V$ can be uniquely represented by $v = \sum_{n=1}^{\infty} \sum_{i=1}^{N_n} a_n^i e_n^i \quad (a_n^i \in \mathbb{F})$. Since we are interested in the order of $\mathscr{B} = \cup_{n=1}^{\infty} \mathscr{B}_n$, and $\mathscr{B}$ is a countable set, we fix an order on $\mathscr{B}$, i.e. $\mathscr{B} = \{e_k\}_{k=1}^{\infty}$. Thus, $\mathscr{B}$ decomposes the vector space V to a product of one-dimensional vector spaces, i.e. $V = \prod_{n=1}^{\infty} (\prod_{i=1}^{N_n} \mathrm{span}_{\mathbb{F}}\{e_n^i\})$ is isomorphic to $\prod_{k=1}^{\infty} \mathrm{span}_{\mathbb{F}}\{e_k\}$, while the induced topology remains unchanged $(\tau_{\mathscr{B}} = \tau_{\mathscr{F}_{\mathbb{F}e_k}} = \tau_{\mathscr{F}_{\{V_i\}}})$. Thus, $V = \{\sum_{k=1}^{\infty} a_k e_k | a_k \in \mathbb{F}\}$. For a more detailed discussion see [Baider & Churchill, 1988; Baider, 1989; Baider & Sanders, 1991, 1992] and the references therein. We call $\mathscr{B}$ a formal basis (FB) for V.

Example 2.1. Consider the set of all formal power series over the field $\mathbb{F}$, i.e. $\mathbb{F}[[x]]$. Then, $\mathbb{F}[[x]]$ is a graded vector space where the homogeneous elements of grade n are exactly the monomials of grade n. Then $\{1, x, x^2, \ldots\}$ with its natural order is a formal basis.

It is useful to consider decomposition of V where the components are subspaces of finite dimensions of larger than one. A sequence $\{W_k | k \in \mathbb{N}\}$ of vector subspaces is called a formal decomposition (FD) of a graded vector space $V = \prod_{k=1}^{\infty} V_k$ (where $\mathscr{B}$ is the fixed FB), denoted by $V = \prod_{k=1}^{\infty} W_k$, if

1. W_k is finite dimensional and every $v \in V$ has a unique representation of the form $v = \sum_{k=1}^{\infty} w_k$ ($w_k \in W_k$, for every k);
2. $\mathscr{B} \cap W_k$ is an ordered basis for W_k and $\mathscr{B} = \cup_{k=1}^{\infty} (\mathscr{B} \cap W_k)$.

Note that the order of basis (FB) is important in our setting. Obviously, $\tau_{\mathscr{B}} = \tau_{\mathscr{F}_{\{W_i\}}} = \tau_{\mathscr{F}_{\{V_i\}}}$. Since all graded structures defined in this paper are also graded vector space over $\mathbb{F}$, to avoid unnecessary repetition, the filtration topologies considered throughout this paper is based on their graded vector space structure.

Notation 2.2. Let W be a vector subspace of V. Then, we use π_W to denote its projection on W, i.e. $\pi_W : V \to W$ is a surjective linear map and $\pi_W \circ \pi_W = \pi_W^2 = \pi_W$.

Proposition 2.3. *Let V and W be vector spaces over $\mathbb{F}$ and $L : W \to V$ be a linear map, where V has an* FB *$\mathscr{B} = \{e_n\}_{n=1}^{\infty}$ or a finite ordered basis $\mathscr{B} = \{e_n\}_{n=1}^{\dim_{\mathbb{F}} V}$. Then, there exists a "unique" vector subspace $\mathcal{N} \subseteq V$ such that*

1. *$V = L(W) \oplus \mathcal{N}$. In other words, for any $v \in V$ there exist unique vectors $\hat{w} \in L(W)$ and $v_{\mathcal{N}} \in \mathcal{N}$ such that $v = \hat{w} + v_{\mathcal{N}}$.*
2. *$\mathscr{B} \cap \mathcal{N} = \{e_{n_k}\}$ is either an ordered basis or an* FB *for $\mathcal{N}$.*
3. *For any $e_m \in \mathscr{B}$ there exist a unique vector $\hat{w} = L(w)$ (for some $w \in W$) and unique scalars $a_{n_1}, a_{n_2}, \ldots, a_{n_N} (n_N \leq m)$ satisfying $e_m = \hat{w} + \sum_{k=1}^{N} a_{n_k} e_{n_k}$.*

In particular, when $W \subseteq V$ and L is the identity map, $\mathcal{N}$ represents a unique complementary space for W in V.

Proof. $\mathcal{N}$ is a complement space of $L(W)$ in V. It is unique since its basis (FB) is considered to be unique (conditions 2.3.2 and 2.3.3). To construct such complementary space, we just need to introduce its basis (FB). Based on condition 2.3, we choose inductively the least natural number n_k in which e_{n_k} is not an element of $L(W) \oplus \mathrm{span}_{\mathbb{F}}\{e_{n_i} | i < k\}$. If this procedure is terminated in a finite process, say m, then $\mathscr{B} \cap \mathscr{N} = \{e_{n_k} | 1 \leq k \leq m\}$ and $\mathscr{N} = \mathrm{span}_{\mathbb{F}}(\mathscr{B} \cap \mathscr{N})$. Otherwise, $\mathscr{B} \cap \mathscr{N} = \{e_{n_k} | k \in \mathbb{N}\}$ and $\mathscr{N} = \{\sum_{k=1}^{\infty} a_{n_k} e_{n_k} | a_{n_k} \in \mathbb{F}\}$. ■

We now introduce an algebraic structure which represents parametric state space $\mathscr{L}^2$. While any $V \in \mathscr{L}^2$ can alternatively be represented by a formal two-vector power series, working with this algebraic structure is much easier and more convenient. Since any vector $V \in \mathscr{L}^2$ can be associated with a formal two-vector field and any vector field $f(x, \mu)$ is associated with a system of differential equation $\dot{x} = f(x, \mu)$, we may call $V (V \in \mathscr{L}^2)$ a vector field or a system whenever it is appropriate. Let $\mathscr{L}^2_{S,k}$ denote the free vector space over $\mathbb{F}$ generated by the set

$$\mathscr{B}_{k,S} = \{X_{ij}, Y_{ij} | i, j \in \mathbb{N}_0, i + j = k + 1\}, \qquad (1)$$

where X_{ij} and Y_{ij} are mutually distinct objects.

Definition 2.4. Let $\mathscr{L}^2_S = \prod_{k=0}^{\infty} \mathscr{L}^2_{S,k}$ denote the state space without parameters. We define a map (denoted by square bracket) from $\mathscr{B}_S \times \mathscr{B}_S$ ($\mathscr{B}_S = \cup_{k=0}^{\infty} \mathscr{B}_{k,S}$) into $\mathscr{L}^2_S$ by

1. $[X_{ij}, X_{mn}] = (i - m)X_{(i+m-1)(j+n)} + jX_{(i+n)(j+m-1)} - nX_{(m+j)(n+i-1)}$,
2. $[X_{mn}, Y_{pq}] = (m - p)Y_{(p+m-1)(q+n)} - nY_{(m+q)(p+n-1)} - qY_{(n+p)(q+m-1)}$,
3. $[Y_{pq}, Y_{mn}] = (m - p)X_{(p+m-1)(q+n)} - nX_{(m+q)(p+n-1)} + qX_{(n+p)(q+m-1)}$,

for any $i, j, m, n, p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, see also [Peng & Wang, 2004]. Note that X_{00} and Y_{00} are not elements of $\mathscr{L}^2_S$. We call

$$\mathscr{L}^2 = \mathscr{L}^2_S[[\mu]] = \left\{ \sum_{k=0}^{\infty} V_k \mu^n | n \in \mathbb{N}_0, V_k \in \mathscr{L}^2_{S,k} \right\}$$

the parametric state space.

Note that the graded structures defined in this paper are based on direct product of vector spaces rather than direct sum which are usually applied in the literature for defining graded structures, e.g. see [Jacobson, 1966] and also Remark 1.2 in [Baider, 1989].

Theorem 2.5. *The bracket defined in Definition 2.4 can be uniquely extended on $\mathscr{L}^2$ such that $(\mathscr{L}^2, [\cdot, \cdot])$ is a graded, of type $\mathbb{Z}$, parametric Lie algebra over $\mathbb{F}$ (where $[V_{k_1}\mu^{n_1}, V_{k_2}\mu^{n_2}] = [V_{k_1}, V_{k_2}]\mu^{n_1+n_2}$), and the Lie bracket is continuous with respect to filtration topology $\tau_{\mathscr{B}}$. Besides $(\mathscr{L}^2_S, [\cdot, \cdot])$ is a $\tau_{\mathscr{B}}$-closed graded Lie subalgebra of $(\mathscr{L}^2, [\cdot, \cdot])$. More precisely,*

1. *There exist an* FB *for $\mathscr{L}^2$, namely $\mathscr{B}$, and a grading function δ defined on $\mathscr{B}$ such that $\mathscr{L}^2_{S,n} = \mathrm{span}_{\mathbb{F}}(\delta|^{-1}_{\mathscr{B}_S}(n))$, $\mathscr{L}^2_n = \mathrm{span}_{\mathbb{F}}(\delta^{-1}(n))$, $\mathscr{L}^2 = \prod_{k=0}^{\infty} \mathscr{L}^2_k$, $[\mathscr{L}^2_m, \mathscr{L}^2_n] \subseteq \mathscr{L}^2_{m+n}$, and $[\mathscr{L}^2_{S,m}, \mathscr{L}^2_{S,n}] \subseteq \mathscr{L}^2_{S,m+n}$, for any $m, n \in \mathbb{N}_0$.*
2. *If $\{v_n\}$ and $\{w_n\} \subset \mathscr{L}^2$ converge to v and w, respectively, with respect to $\tau_{\mathscr{B}}$. Then, $[v_n, v_m]$ converges to $[v, w]$. Furthermore, $v \in \mathscr{L}^2_S$, when $\{v_n\} \subset \mathscr{L}^2_S$ and v_n converges to v.*

Proof. Let $\mathscr{B} = \{X_{ij}\mu^n, Y_{ij}\mu^n | n \in \mathbb{N}_0\}$. We consider a fixed order for $\mathscr{B} = \{e_n\}_{n=1}^{\infty}$ ($\mathscr{B}_S = \{e_{n_k}\}_{k=1}^{\infty} \subset \mathscr{B}$) in the following manner:

1. Lower order elements are designated by lower degree terms based on a grading function δ. (Our grading function δ is given below.)
2. Y_{ij} is before X_{mn}, when both of them have the same degree.
3. The terms without parameters are before the terms with parameters where they have the same degree.

Thus, $\mathscr{B}(\mathscr{B}_S)$ is an FB for $\mathscr{L}^2(\mathscr{L}^2_S)$. Now define the grading function $\delta : \mathscr{B} \to \mathbb{N}_0$ as:

$$\delta(X_{ij}\mu^r) = \delta(Y_{ij}\mu^r) = i + j - 1 + r,$$

where $r \in \mathbb{N}_0$. Then, we have $\mathscr{L}^2_{S,N} = \mathrm{span}\{X_{ij}, Y_{ij} | i + j - 1 = N\}$,

$$\mathscr{L}^2_N = \mathscr{L}^2_{S,N} \oplus \sum_{i+r=N} \mathscr{L}^2_{S,i}\mu^r \quad (\forall\, N \geq 0),$$

and $\mathscr{L}^2_N = \mathscr{L}^2_{S,N} = \{\mathbf{0}\}$, when $N < 0$. Obviously, $\mathscr{L}^2_{S,n} = \mathrm{span}_{\mathbb{F}}(\delta|^{-1}_{\mathscr{B}_S}(n))$, and $\mathscr{L}^2 = \prod_{k=0}^{\infty} \mathscr{L}^2_k = \{\sum_{n=0}^{\infty} a_n e_n | a_n \in \mathbb{F}\}$. On the other hand, the bracket $[\cdot, \cdot]$ is defined on the formal basis $\mathscr{B}$ of $\mathscr{L}^2$. Thus, it can be uniquely extended on $\oplus_{n=0}^{N} \mathscr{L}^2_n$ ($\forall\, N \in \mathbb{N}_0$) as a bilinear map. Therefore, $[\mathscr{L}^2_m, \mathscr{L}^2_n] \subseteq \mathscr{L}^2_{m+n}$, and $[\mathscr{L}^2_{S,m}, \mathscr{L}^2_{S,n}] \subseteq \mathscr{L}^2_{S,m+n}$ for any $m, n \in \mathbb{N}_0$. Then, the bracket $[\cdot, \cdot]$ can be uniquely extended as a continuous bilinear operation on $\mathscr{L}^2$, i.e.

$$\left[\prod_{k=n}^{\infty} \mathscr{L}^2_k, \prod_{k=m}^{\infty} \mathscr{L}^2_k \right] = [\mathscr{F}_m, \mathscr{F}_n] \subseteq \prod_{k=m+n}^{\infty} \mathscr{L}^2_k = \mathscr{F}_{m+n}$$

and $[\mathscr{F}_{m,S}, \mathscr{F}_{n,S}] \subseteq \mathscr{F}_{m+n,S}$, where $\mathscr{F}_{n,S} = \prod_{k=n}^{\infty} \mathscr{L}^2_{S,k}$. Since it also satisfies Jacobian identity, the bracket is a Lie bracket and $(\mathscr{L}^2_S, [\cdot,\cdot])$ is a closed Lie subalgebra of $(\mathscr{L}^2, [\cdot,\cdot])$. ■

It should be noted that $\mathscr{B}(\mathscr{B}_S)$ is not a basis for $\mathscr{L}^2(\mathscr{L}^2_S)$. In fact, any basis for $\mathscr{L}^2(\mathscr{L}^2_S)$ has an uncountable cardinal number, while any formal basis is a sequence.

Proposition 2.6. *Let* $\mathscr{L}^2_H = \mathrm{span}_{\mathbb{F}}\{X_{(k+1)k}\mu^r, Y_{(k+1)k}\mu^r \,|\, k \in \mathbb{N}_0,\, r \in \mathbb{N}_0\}$,

$$\mathscr{L}^2_X = \{X_{ij}\mu^r | i,j,r \in \mathbb{N}_0, i+j \geq 1\}, \quad \text{and}$$
$$\mathscr{L}^2_Y = \{Y_{ij}\mu^r | i,j,r \in \mathbb{N}_0, i+j \geq 1\}.$$

Then, $\mathscr{L}^2_H$ *and* $\pi_{\mathscr{L}^2_X}(\mathscr{L}^2_H)$ *are* $\tau_{\mathscr{B}}$*-closed graded Lie subalgebras of* $\mathscr{L}^2$*, while* $\pi_{\mathscr{L}^2_Y}(\mathscr{L}^2_H)$ *is a* $\tau_{\mathscr{B}}$*-closed commutative Lie subalgebra. Furthermore, for any vector* $V \in \mathscr{L}^2 \backslash \mathscr{L}^2_H$*, we have* $\pi_{\mathscr{L}^2_H}([V, \mathscr{L}^2_H]) = \{\mathbf{0}\}$.

Proof. The claim can be easily shown based on the following formulas:

$$\begin{aligned}
[X_{ij}, X_{(n+1)n}] &= (i+j-n-1)X_{(i+n)(j+n)} \\
&\quad - nX_{(n+1+j)(n+i-1)}, \\
[X_{mn}, Y_{(q+1)q}] &= (m-n-q-1)Y_{(m+q)(q+n)} \\
&\quad - qY_{(n+q+1)(q+m-1)}, \\
[Y_{pq}, Y_{(n+1)n}] &= (n+q-p+1)X_{(p+n)(q+n)} \\
&\quad - nX_{(n+1+q)(p+n-1)}, \\
[X_{(n+1)n}, Y_{pq}] &= (n-p-q+1)Y_{(p+n)(q+n)} \\
&\quad - nY_{(n+1+q)(p+n-1)},
\end{aligned}$$

and

$$\begin{aligned}
[X_{(j+1)j}, X_{(n+1)n}] &= 2(j-n)X_{(j+n+1)(j+n)}, \\
[X_{(n+1)n}, Y_{(q+1)q}] &= -2qY_{(n+q+1)(q+n)}, \\
[Y_{(q+1)q}, Y_{(n+1)n}] &= 0.
\end{aligned}$$

Note that the latter structure has been studied in [Peng & Wang, 2004]. ■

Consider the time one-mapping ϕ^{μ}_Y ($\mu \in \mathbb{R}$), given by the flow ϕ^t_Y generated from $\dot{\mathbf{x}} = Y(\mathbf{x}, \mu)$ ($\mathbf{x} \in \mathbb{R}^n$ and $Y(\mathbf{x}, \mu)$ is a vector field with no linear terms). Then, the transformation $\phi^{\mu}_Y(\mathbf{x})$, a near-identity change of the state variable, sends the system $\dot{\mathbf{x}} = f(\mathbf{x}, \mu)$ to

$$\begin{aligned}
(\phi^{\mu}_Y)_*(f(\mathbf{x}, \mu)) &= \exp(\mathrm{ad}_Y)(f(\mathbf{x}, \mu)) \\
&= f(\mathbf{x}, \mu) + \mathrm{ad}_Y f(\mathbf{x}, \mu) + \cdots \\
&\quad + \frac{1}{n!}\mathrm{ad}^n_Y f(\mathbf{x}, \mu) + \cdots,
\end{aligned}$$

where $\mathrm{ad}_Y(f(\mathbf{x}, \mu)) = [Y, f(\mathbf{x}, \mu)]$ and $\mathrm{ad}^n_Y = \mathrm{ad}_Y \circ \mathrm{ad}^{n-1}_Y$ ($\forall\, n \in \mathbb{N}$), e.g. see [Chua & Kokubu, 1988] and Sec. 3 in [Kokubu *et al.*, 1996]. Thus, $\phi^S_{Y*} : \mathscr{L}^2 \to \mathscr{L}^2$ defined by

$$\begin{aligned}
\phi^S_{Y*}(V) &= \exp(\mathrm{ad}_Y)(V) \\
&= \lim_n \sum_{k=0}^{n} \frac{1}{k!}\mathrm{ad}^k_Y V \quad (\forall\, V \in \mathscr{L}^2) \qquad (2)
\end{aligned}$$

(the limit is with respect to $\tau_{\mathscr{B}}$) may represent the state map associated with $Y \in \mathscr{L}^2$. It is not hard to prove that $\phi^S_{Y*}(V)$ (Eq. 2) is $\tau_{\mathscr{B}}$-convergent and well defined for any $Y, V \in \mathscr{L}^2$.

For any $V^{(0)}_{k_0} \in \mathscr{L}^2_{k_0}$ we define the first-order linear state operator as:

$$L^{S,N}_{(1)}(V^{(0)}_{k_0}) : \begin{array}{l} \mathscr{L}^2_{N-k_0} \to \mathscr{L}^2_N \\ Y_{N-k_0} \mapsto [Y_{N-k_0}, V^{(0)}_{k_0}], \quad \forall\, N > k_0. \end{array}$$

One of the key ideas in computing the SNF is to use all possible vectors to eliminate terms as many as possible. Therefore, we use multiple Lie bracket method (e.g. see [Kokubu *et al.*, 1996]) to define an nth-order state operator associated with a sequence

$$\begin{aligned}
V^{(0)}_{k_0} = V^{(k_0)}_{k_0}, V^{(k_0+1)}_{k_0+1}, \ldots, V^{(k_0+n-1)}_{k_0+n-1}, \ldots \\
(n \in \mathbb{N},\ V^{(k_0+n-1)}_{k_0+n-1} \in \mathscr{L}^2_{k_0+n-1}). \qquad (3)
\end{aligned}$$

Note that the sequence 3 is computed in the normal form process, e.g. see Theorem 4.3 given in Sec. 4.

Definition 2.7. Define $L^{S,N}_{(n)} = L^{S,N}_{(n)}(V^{(k_0)}_{k_0}, V^{(k_0+1)}_{k_0+1}, \ldots, V^{(k_0+n-1)}_{k_0+n-1})$ by

$$L^{S,N}_{(n)} : \begin{array}{l} \ker(L^{S,N-1}_{(n-1)}) \times (\mathscr{L}^2_{N-k_0}) \to \mathscr{L}^2_N \\ (Y_{N-k_0-n+1}, \ldots, Y_{N-k_0}) \mapsto \displaystyle\sum_{i=0}^{n-1}[Y_{N-k_0-i}, V^{(k_0+i)}_{k_0+i}] \end{array} \quad (\text{for } n \leq N-k_0),$$

and $L^{S,N}_{(n)} = L^{S,N}_{(n-1)}$ (for $n > N-k_0$). We call the linear map $L^{S,N}_{(n)}$ the nth-order state operator at degree N.

Lemma 2.8. *For any* $m, n \in \mathbb{N}$ *we have* $\mathrm{Im}(L^{S,N}_{(n)}) \subseteq \mathrm{Im}(L^{S,N}_{(n+1)})$.

Proof. It is straightforward to prove the lemma by definition and a similar argument used in Lemma 3.6 of [Kokubu *et al.*, 1996]. ■

3. Parameter Space $\mathscr{P}^m$ and Parametric Time Space $\mathscr{R}$

We call the space of all formal power series in terms of parameter μ, i.e. $\mathscr{P}^m = \mathbb{F}[[\mu]]$, as the parameter space. Let $\mathscr{P}_0^m = \mathbb{F}$ and $\delta_{\mathscr{P}^m}(\mu^r) = r$. Then $\mathscr{P}^m = \prod_{n=0}^{\infty} \mathscr{P}_n^m$, where $\mathscr{P}_n^m = \delta_{\mathscr{P}^m}^{-1}(n)$, for all $n \in \mathbb{N}$. We consider $\mathscr{B}_{\mathscr{P}}^m = \{\mu^r\}_{r=0}^{\infty}$ as the formal basis for $\mathscr{P}^m$. Let $\mathscr{P}_I^m = \{\mu + Y^P | Y^P \in \prod_{n=2}^{\infty} \mathscr{P}_n^m\}$ denote the affine space of the near-identity reparametrization. Obviously, $(\mathscr{P}_I^m, \tau_{\mathscr{B}_{\mathscr{P}}^m}|_{\mathscr{P}_I^m})$ is homeomorphic to $(\prod_{n=2}^{\infty} \mathscr{P}_n^m, \tau_{\mathscr{F}_{\{\mathscr{P}_n^m\}}})$ where $\tau_{\mathscr{B}_{\mathscr{P}}^m}|_{\mathscr{P}_I^m}$ denotes the relative topology.

Since the elements of $\mu + \mathscr{P}_k^m$ and $\mathscr{L}_k^2$ can be respectively formulated as

$$\phi_{Y_k^P}^P(\cdot) : \mathbb{F} \to \mathbb{F}$$
$$\phi_{Y_k^P}^P(\mu) = \mu + Y_k^P(\mu),$$

where $Y_k^P \in \mathscr{P}_k^m$; and

$$V_k(\cdot) : \mathbb{F} \to \mathbb{F},$$

in which $V_k = V_k(\mu)$ for any $\mu \in \mathbb{F}$ and $V_k \in \mathscr{L}_k^2$, $\mathscr{P}_I^m$ may act on parametric Lie algebra $\mathscr{L}^2$ (via the parametric map $\phi_{Y^P *}^P$) by

$$\phi_*^P(\cdot) : \mathscr{P}_I^m \times \mathscr{L}^2 \to \mathscr{L}^2$$
$$\phi_*^P(Y^P, V(\mu)) = \phi_{Y^P *}^P(V(\mu)) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{D}_\mu^n\big(V(\mu)\big)\big(Y^P\big)^n, \quad (4)$$

where $\mathcal{D}_\mu^n$ denotes the nth-order formal Frechet derivative of $V(\mu)$ with respect to μ, see [Kuznetsov, 2005; Liao *et al.*, 2007]. It is easy to see that the formal sum (4) is $\tau_{\mathscr{B}}$-convergent and well defined, i.e. $\phi_{Y^P *}^P(V) \in \mathscr{L}^2$ for any $V \in \mathscr{L}^2$ and $Y^P \in \mathscr{P}_I^m$. $\phi_*^P(\cdot)$ is a continuous map from $(\mathscr{P}_I^m \times \mathscr{L}^2, \tau)$ to $(\mathscr{L}^2, \tau_{\mathscr{B}})$, where τ denotes the Tikhonov topology with respect to $\tau_{\mathscr{B}_{\mathscr{P}}^m}|_{P_I}$ and $\tau_{\mathscr{B}}$.

Definition 3.1. Let k be the least number satisfying $\mathcal{D}_\mu(V_k^{(k)}) \neq \mathbf{0}$, $k_0 \leq k < N$. Consider the linear map

$$L_{(k)}^{P,N} : \mathscr{P}_{N-k+1}^m \to \mathscr{L}_N^2$$
$$Y_{N-k+1}^P \mapsto D_\mu(V_k^{(k)}) Y_{N-k+1}^P$$

and $L_{(n)}^{P,N} = L_{(n-1)}^{P,N}$, for $n > k$. When $n \leq k$, let $L_{(k)}^{P,n}$ and $L_{(n-1)}^{P,N}$ be a zero operator. Then, we call $L_{(n)}^{P,N}$ the parameter operator at degree N. So, we have $\mathrm{Im}(L_{(n)}^{P,N}) \subseteq \mathrm{Im}(L_{(n+1)}^{P,N})$ $\forall\, n, N \in \mathbb{N}$.

Note that $\mathcal{D}_\mu^n(V_k(\mu))(Y^P)^n = \mathbf{0}$ $(V_k(\mu) \in \mathscr{L}_k^2)$ implies $\mathcal{D}_\mu^m(V_k(\mu))(Y^P)^m = \mathbf{0}$ $(\forall\, m, m \geq n)$. When the bifurcation parameter is multidimensional, however, this statement is not true in general and then, the structure of the parameter operators is more complicated. This requires developing a new approach, which will be discussed in a separate work.

Next, we turn to construct an appropriate ring structure for parametric time space.

Theorem 3.2. *Let $\mathscr{R}_0 = \prod_{i=0}^{\infty} \mathscr{R}^i$, where $\mathscr{R}^i = \mathbb{F}Z_i$ and $\{Z_i\}_i$ is an infinite sequence of distinct objects. Then, $\mathscr{R}_0$ can be associated with a ring structure for which the parametric time space $\mathscr{R} = \mathscr{R}_0[[\mu]]$ is an integral domain. There exists a grading function $\delta_{\mathscr{R}}$ defined on $\mathscr{B}_{\mathscr{R}} = \{Z_i\mu^r\}$ satisfying $\{0\} = \mathrm{span}_{\mathbb{F}}\delta_{\mathscr{R}}{}^{-1}(k) = \mathrm{span}_{\mathbb{F}}\emptyset$ $(\forall\, k < 0)$, $\mathbb{F}Z_0 = \mathrm{span}_{\mathbb{F}}\delta_{\mathscr{R}}{}^{-1}(0)$, and $\mathscr{R}_k = \mathrm{span}_{\mathbb{F}}\delta_{\mathscr{R}}{}^{-1}(k)$ such that $\mathscr{R} = \prod_{k=0}^{\infty} \mathscr{R}_k$, $\mathscr{R}_m\mathscr{R}_n \subseteq \mathscr{R}_{m+n}$, and $\mathscr{R}^m\mathscr{R}^n \subseteq \mathscr{R}^{m+n}$, for all $m, n \in \mathbb{N}_0$. In other words, $\mathscr{R}$ is a graded ring and $\mathscr{R}_0$ is a graded subring of $\mathscr{R}$.*

Proof. Consider the natural vector space structure on $\mathscr{R}_0$ over $\mathbb{F}$ as $(\mathscr{R}_0, +, \cdot)$. Define

$$\delta_{\mathscr{R}} : \bigcup_{i,r} \mathbb{F}Z_i\mu^r \to \mathbb{Z} \quad (5)$$
$$\delta_{\mathscr{R}}(aZ_i\mu^r) = 2i + r, \quad \forall\, a \in \mathbb{F}.$$

Then,

$$\mathscr{R}_N = \sum_{2i+r=N} \mathbb{F}Z_i\mu^r, \quad \forall\, N \in \mathbb{N}_0,$$

and $\mathscr{R}_N = \{0\}$ for any $N < 0$. Let $\mathscr{B}_{\mathscr{R}} = \{Z_i\mu^r\}$ $(\mathscr{B}_{\mathscr{R}_0} \subset \mathscr{B}_{\mathscr{R}})$ be the fixed FB for $\mathscr{R}\,(\mathscr{R}_0)$ whose order must obey the following rules:

1. The terms of lower degrees are in lower orders.
2. The terms without parameters are before the terms with parameters, provided they have the same degree.

Obviously, $\mathscr{B}_{\mathscr{R}_0} = \{Z_i\}_{i=0}^{\infty}$. Thus, $\mathscr{R}_0 = \{\sum_{i=0}^{\infty} a_iZ_i | a_i \in \mathbb{F}\}$. Now for any $\mathbf{z}_1, \mathbf{z}_2 \in \mathscr{R}_0$ define the operation $*$ on $\mathscr{R}_0$ by

$$\begin{aligned}\pi_{\mathscr{R}_0^n}(\mathbf{z}) &= \pi_{\mathscr{R}_0^n}(\mathbf{z}_1 * \mathbf{z}_2)\\ &= \sum_{k=0}^{n} \pi_{\mathscr{R}_0^{n-k}}(\mathbf{z}_1) * \pi_{\mathscr{R}_0^k}(\mathbf{z}_2)\\ &= \sum_{k=0}^{n} a_{1(n-k)} a_{2k} Z_n, \end{aligned} \tag{6}$$

where $\mathbf{z}_i = \sum_{n=0}^{\infty} a_{in} Z_n$, for $i = 1, 2$. It is not very difficult to verify that Eq. (6) is $\tau_{\mathscr{B}_{\mathscr{R}_0}}$-convergent, $*$ is a well-defined continuous operation, and $(\mathscr{R}_0, +, *)$ is an integral domain. Thus, $*$ can be uniquely extended on $\mathscr{R}$ as a continuous operation such that $Z_{i_1}\mu^{r_1} * Z_{i_2}\mu^{r_2} = Z_{i_1+i_2}\mu^{r_1+r_2}$ $(\forall\, i_1, i_2, r_1, r_2 \in \mathbb{N}_0)$. As a result, $\mathscr{R} = \mathscr{R}_0[[\mu]]$ is an integral domain and $Z_0 = 1_{\mathscr{R}}$. Furthermore, $\mathscr{R}_{N_1}\mathscr{R}_{N_2} \subseteq \mathscr{R}_{N_1+N_2}$, where the notation $*$ is omitted for simplicity. ■

Theorem 3.3. *There exists a continuous scalar product from $\mathscr{R} \times \mathscr{L}^2$ to $\mathscr{L}^2$ such that $\mathscr{L}^2$ is a torsion-free graded left $\mathscr{R}$-module of type $\mathbb{Z}$ over the graded ring $\mathscr{R}$. More precisely, $\mathscr{R}_{N_1}\mathscr{L}^2_{N_2} \subseteq \mathscr{L}^2_{N_1+N_2}$ $(\forall\, N_1, N_2 \in \mathbb{N}_0)$. In particular, $\mathscr{L}^2_H$ is a graded $\mathscr{R}$-submodule of $\mathscr{L}^2$, while $\mathscr{L}^2_S$ is just an $\mathscr{R}_0$-submodule of $\mathscr{L}^2$.*

Proof. We define the scalar product as follows:

$$\begin{aligned} &\star : \mathscr{B}_{\mathscr{R}} \times \mathscr{B}_{\mathscr{L}^2} \to \mathscr{B}_{\mathscr{L}^2}\\ &Z_i\mu^{r_1} \star X_{mn}\mu^{r_2} = X_{(i+m)(i+n)}\mu^{r_1+r_2}\\ &Z_i\mu^{r_1} \star Y_{mn}\mu^{r_2} = Y_{(i+m)(i+n)}\mu^{r_1+r_1}. \end{aligned}$$

Then, it can be uniquely extended as a continuous operator $\star : \mathscr{R} \times \mathscr{L}^2 \to \mathscr{L}^2$. Since $\mathscr{R}$ is an integral domain, $\mathscr{L}^2_H$ is a torsion-free $\mathscr{R}$-module. It is easy to see that $\mathscr{R}_{N_1}\mathscr{L}^2_{N_2} \subset \mathscr{L}^2_{N_1+N_2}$ and $\mathscr{R}^{N_1}\mathscr{L}^2_{N_2} \subset \mathscr{L}^2_{N_1+N_2}$, where $\star$ is dropped for convenience. Because $\mathscr{R}\mathscr{L}^2_H = \mathscr{L}^2_H$ and $\mathscr{R}_0\mathscr{L}^2_S = \mathscr{L}^2_S$, $\mathscr{L}^2_H$ is a graded $\mathscr{R}$-submodule of $\mathscr{L}^2$ and $\mathscr{L}^2_S$ is a graded $\mathscr{R}_0$-submodule. ■

In practice, the near-identity time rescaling is a function of real numbers, given as

$$\begin{aligned} &\phi^T_{Y^T}(\cdot) : \mathbb{R} \to \mathbb{R}\\ &\phi^T_{Y^T}(t) = (1 + Y^T(T))t, \end{aligned}$$

and time map $\phi^T_{Y^T{}_*}(\cdot)$ is defined by $\phi^T_{Y^T{}_*}(V) = V + Y^TV$, see [Sadovskiĭ, 1985; Dumortier, 1991; Stróżyna & Zoladek, 2002; Algaba *et al.*, 2003; Yu & Leung, 2003]. So, the affine space $\mathscr{R}_I = 1 + \prod_{k=1}^{\infty}\mathscr{R}_k$ may represent the space of near-identity time rescalings, while time rescaling map is defined by

$$\begin{aligned} &\phi^T_* : \mathscr{R}_I \times \mathscr{L}^2 \to \mathscr{L}^2,\\ &\phi^T_*(Y^T, V) = \phi^T_{Y^T*}(V) = V + Y^TV. \end{aligned}$$

Obviously, ϕ^T_* is well defined and is continuous with respect to the filtration topologies.

Remark 3.4. One may want to define the first-order time linear operator as

$$\begin{aligned} &L^{T,N} : \mathscr{R}_{N-k_0} \to \mathscr{L}^2_N\\ &Y^T_{N-k_0} \mapsto Y^T_{N-k_0}V^{(k_0)}_{k_0}. \end{aligned}$$

However, the parametric time space is a torsion-free ring. Therefore, the above linear operator is an injective transformation and thus its kernel does not have nonzero terms in order to define higher-order time operators. It, however, is evident that some terms in its domain may not be useful via this linear operator while we need to use them to eliminate some higher-order terms. This, in fact, has led us to the innovation of FD.

Based on the sequence (3) and a parametric time space decomposition:

$$\mathscr{R}_I = 1 + \prod_{n=1}^{\infty}\mathscr{R}_n \quad \left(\text{where } \mathscr{R}_n = \mathscr{R}^U_n \bigoplus_{i=0}^{n} \mathscr{R}^{k_0+i}_n\right), \tag{7}$$

we shall define the time operators, namely $L^{T,N}_{(n)}$. $L^{T,N}_{(n)}$ is used to determine the time solution $Y^{T,N}$ at step N, which is specifically designed to eliminate terms not only at degree N but also some terms with degrees higher than N. To achieve this, we use the notation $\mathscr{R}^{k_0+i}_{N-k_0}$ to indicate that the homogeneous time terms of degree $N - k_0$ in this subspace of $\mathscr{R}$ are used to eliminate any remaining terms in the system which belong to $\mathbb{F}\mathscr{B}_{\mathscr{L}^2} \cap \mathscr{R}^{k_0+i}_{N-k_0}V^{(k_0+i)}_{k_0+i}$. Therefore, $L^{T,N}_{(n)}$ is defined by

$$\begin{aligned} &L^{T,N}_{(n)} : \bigoplus_{i=0}^{n} \mathscr{R}^{k_0+i}_{N-k_0} \to \sum_{i=0}^{N-k_0} \mathscr{L}^2_{N+i}\\ &(Y^{T,k_0+n}_{N-k_0}, \ldots, Y^{T,k_0+1}_{N-k_0}, Y^{T,k_0}_{N-k_0}) \mapsto \sum_{i=0}^{n} Y^{T,k_0+i}_{N-k_0}V^{(k_0+i)}_{k_0+i}, \end{aligned}$$

where $Y^{T,k_0+i}_{N-k_0} \in \mathscr{R}^{k_0+i}_{N-k_0}$ $(n \le N - k_0)$. Let $L^{T,N}_{(n)} = L^{T,N}_{(n-1)}$ $(\forall\, n > N - k_0)$. Note that $\mathscr{R}^U_n$ denotes the unused time subspace. Our normal form largely depends on parametric time space decomposition (7). Thus, this decomposition is very important in order to achieve an infinite order normal form, see Eqs. (15) and (16). We have the following result, similar to Lemma 2.8.

Lemma 3.5. $\mathrm{Im}(L^{T,N}_{(n)}) \subseteq \mathrm{Im}(L^{T,N}_{(n+1)})\ \forall\, n, N \in \mathbb{N}$.

4. Infinite Order Parametric Normal Forms

The notion of finite order parametric normal forms is different from that of systems with no parameters. Although we have enough tools to discuss the notion here, it is beyond the scope of this paper. Therefore, we directly describe infinite order parametric normal forms.

Remark 4.1. Note that the terms in the space

$$\sum_{i=1}^{[\frac{N-k_0}{2}]} \pi_{\mathscr{L}^2_N} \circ L^{T,N-i}_{(N-i)} = \bigoplus_{i=1}^{[\frac{N-k_0}{2}]} \mathscr{R}^{k_0+i}_{N-k_0-i} V^{(k_0+i)}_{k_0+i} \tag{8}$$

are associated with time operators (solutions) of degrees less than N, which have actual designed effect on terms of degree N. Thus, these terms appear in $Y^{T,N-i}$ ($1 \le i \le [\frac{N-k_0}{2}]$) as unknowns which are to be predicted (computed in advance until step N). For instance, let

$$\mathscr{L}^2_N = \bigoplus_{i=1}^{[\frac{N-k_0}{2}]} \mathscr{R}^{k_0+i}_{N-k_0-i} V^{(k_0+i)}_{k_0+i} \oplus \mathcal{N}_N, \tag{9}$$

where $\mathcal{N}_N$ is the unique subspace obtained (with a simple argument) via Proposition 2.3. Then, $\mathcal{N}_N$ indeed denotes the space terms (degree N) which are not eliminated by time solutions $Y^{T,k}$ ($k < N$).

Notation 4.2. Let

$$\pi_{S,P,T}(\mathscr{L}^2_n) = \mathrm{Im}(L^{S,n}_{(n)}) + \mathrm{Im}(L^{P,n}_{(n)}) + \sum_{i=0}^{[\frac{n-k_0}{2}]} \mathscr{R}^{k_0+i}_{n-k_0-i} V^{(k_0+i)}_{k_0+i},$$

and $\mathcal{N}^{S,P}_{n,T}$ denote the unique space obtained from Proposition 2.3 by $\pi_{S,P,T}(\mathscr{L}^2_n) = \big(\mathrm{Im}(L^{S,n}_{(n)}) + \mathrm{Im}(L^{P,n}_{(n)})\big) \oplus \mathcal{N}^{S,P}_{n,T}$.

Let $V^{(0)} = V^{(k_0)} = \sum_{k=k_0}^{\infty} V^{(k_0)}_k$, where $V^{(k_0)}_k \in \mathscr{L}^2_k$ and $V^{(k_0)}_{k_0} \neq \mathbf{0}$. Assume the parametric time space decomposition (7) is given. Obviously, the operators $L^{S,k_0+1}_{(1)} = L^{S,k_0+1}_{(k_0+1)}$, and $L^{P,k_0+1}_{(1)} = L^{P,k_0+1}_{(k_0+1)}$ can be defined. Thus, via a simple argument, Proposition 2.3 provides a unique complementary space $\mathcal{N}^{S,P,T}_N$ which satisfies

$$\mathscr{L}^2_N = \pi_{S,P,T}(\mathscr{L}^2_N) \oplus \mathcal{N}^{S,P,T}_N, \tag{10}$$

when $N = k_0 + 1$. Then, there exist state solution Y^{S,k_0+1}, parameter solution Y^{P,k_0+1} and time solution $\hat{Y}^{T,k_0+1}$ such that

$$\begin{aligned} V^{(k_0+1)} &= \phi^T_{\hat{Y}^{T,k_0+1}_*} \circ \phi^P_{Y^{P,k_0+1}_*} \circ \phi^S_{Y^{S,k_0+1}_*}(V^{(k_0)}) \\ &= V^{(k_0)}_{k_0} + V^{(k_0+1)}_{k_0+1} + \text{h.o.t.}, \end{aligned}$$

where $V^{(k_0+1)}_{k_0+1} = V^{(k_0),S,P,T}_{k_0+1} \in \mathcal{N}^{S,P,T}_{k_0+1}$, and $\phi^S_{Y^{S,k_0+1}_*}, \phi^P_{Y^{P,k_0+1}_*}$ and $\phi^T_{\hat{Y}^{T,k_0+1}_*}$ are the state, parameter, and time maps associated with Y^{S,k_0+1}, Y^{P,k_0+1}, and $\hat{Y}^{T,k_0+1}$, respectively. Note that the time terms associated with $\mathscr{R}^{k_0+1}_1$ in $\hat{Y}^{T,k_0+1}$ are unknown. In fact, the sign of $\hat{Y}^{T,\cdot}$ is used to distinguish a completely known time solution ($Y^{T,\cdot}$) from a partially known time solution ($\hat{Y}^{T,\cdot}$) at each step. Obviously, although $V^{(k_0+1)}_{k_0+1}$ (and h.o.t) depend on these unknowns, it still can be used to determine the operators $L^{S,k_0+2}_{(k_0+2)}$ and $L^{P,k_0+2}_{(k_0+2)}$. By taking into account the decomposition (10) when $N = k_0 + 2$, there exist state solution Y^{S,k_0+2}, parameter solution Y^{P,k_0+2}, and time solution $\hat{Y}^{T,k_0+2}$ such that

$$\begin{aligned} V^{(k_0+2)} &= \phi^T_{\hat{Y}^{T,k_0+2}_*} \circ \phi^P_{Y^{P,k_0+2}_*} \circ \phi^S_{Y^{S,k_0+2}_*}(V^{(k_0)}) \\ &= \sum_{i \le k_0} V^{(i)}_i + V^{(k_0+2)}_{k_0+2} + \text{h.o.t.}, \end{aligned}$$

where $V^{(k_0+2)}_{k_0+2} = V^{(k_0+1),S,P,T}_{k_0+2} \in \mathcal{N}^{S,P,T}_{k_0+2}$, and the time solution $Y^{T,k_0+2} = \hat{Y}^{T,k_0+2}$ is completely determined. However, some terms associated with $\mathscr{R}^{k_0+2}_2$ and $\mathscr{R}^{k_0+1}_2$ in $\hat{Y}^{T,k_0+2}$ are the new unknowns in the system.

Now consider the decomposition (10) ($N = k + 1$), where $L^{S,k+1}_{(k+1)}$ and $L^{P,k+1}_{(k+1)}$ are determined by the mathematical induction hypothesis. There exist state solution $Y^{S,k+1}$, parameter solution $Y^{P,k+1}$, time solution $\hat{Y}^{T,k+1}$, and unique vector $V^{(k+1)}_{k+1}$ such that

$$\begin{aligned} V^{(k+1)} &= \phi^T_{\hat{Y}^{T,k+1}_*} \circ \phi^P_{Y^{P,k+1}_*} \circ \phi^S_{Y^{S,k+1}_*}(V^{(k_0)}) \\ &= \sum_{i \le k} V^{(i)}_i + V^{(k+1)}_{k+1} + \text{h.o.t.}, \end{aligned}$$

where $V_{k+1}^{(k+1)} \in \mathcal{N}_{k+1}^{S,P,T}$ and the unknowns are the time terms associated with

$$\mathscr{R}_{m-k_0}^{k_0+i} \quad \text{in} \quad \hat{Y}^{T,m} \quad \text{where} \quad k_0 < m \leq k+1 \ \& \ k+1-m < i \leq m-k_0.$$

Clearly, for a natural number $m \in \mathbb{N}$ at the step $k+1$ with $k > 2m-k_0$, the time solution $Y^{T,m} = \hat{Y}^{T,m}$ is completely determined. Thus, in general, to obtain the time solution $Y^{T,N}$, we may need to predict the process for all steps less than and equal to $2N - k_0 + 1$. Therefore, for any natural number N there exist $Y^{S,N} \in \mathscr{L}^2$, $Y^{P,N} \in \mathscr{P}^m$, $Y^{T,N} \in \mathscr{R}$, and unique vector $V_N^{(N)} \in \mathcal{N}_N^{S,P,T} \subseteq \mathscr{L}_N^2$ such that $V^{(N-1)}$ is transformed to

$$\begin{aligned}\Phi_N(V^{(N-1)}) &= \phi^T_{Y^{t,N}*} \circ \phi^P_{Y^{P,N}*} \circ \phi^S_{Y^{S,N}*}(V^{(N-1)}) \\ &= \phi^T_{Y^{T,N}*} \circ \phi^P_{Y^{P,N}*}(V^{(N-1),S}) \\ &= \phi^T_{Y^{T,N}*}(V^{(N-1),S,P}) \\ &= \sum_{i=0}^{N-1} V_{k_0+i}^{(k_0+i)} + V_N^{(N)} + \text{h.o.t.} \\ &= V^{(N)},\end{aligned} \tag{11}$$

implying that

$$V^{(\infty)} = \sum_{N=k_0}^{\infty} \pi_{\mathscr{L}_N^2} \circ \Phi_N \circ \Phi_{N-1} \cdots \circ \Phi_{k_0}(V^{(k_0)}),$$

where $\Phi_{k_0}(V^{(k_0)}) = V^{(k_0)}$. Since $\pi_{\mathscr{L}_N^2} \circ \Phi_N \circ \Phi_{N-1} \cdots \circ \Phi_{k_0}(V^{(k_0)}) \in \mathcal{N}_N^{S,P,T}$ and $\mathcal{N}_N^{S,P,T}$ is spanned by $\mathscr{B} \cap \mathcal{N}_N^{S,P,T}$ (where $\mathscr{B}$ is the FB), $V^{(\infty)} \in \mathscr{L}^2$ is well defined. Besides, $V^{(\infty)} - V^{(N)} \in \mathscr{F}_{N+1}$ and thus, $V^{(\infty)}$ is convergent. In other words, the computation of normal form is a convergent process with respect to filtration topology $\tau_{\mathscr{B}}$. Summarizing above discussions leads to the following theorem.

Theorem 4.3. *Let $\mathscr{B}$ be the formal basis in Theorem 2.5 and*

$$\mathscr{R}_I = 1 + \prod_{n=1}^{\infty} \mathscr{R}_n \quad \left(\text{where } \mathscr{R}_n = \mathscr{R}_n^U \bigoplus_{i=0}^{n} \mathscr{R}_n^{k_0+i}\right), \tag{12}$$

be a parametric time space decomposition. Then, for any $V \in \mathscr{L}^2$,

$$V = V^{(0)} = \sum_{k=k_0}^{\infty} V_k^{(0)} \quad (V_k^{(0)} \in \mathscr{L}_k^2, V_{k_0}^{(0)} \neq \mathbf{0}), \tag{13}$$

there exists a sequence of vectors $\{V_{k_0+i}^{(k_0+i)}\}_{i=0}^{\infty} \subset \mathscr{L}^2$ such that by a sequence of near-identity change of state variable, time rescaling and reparametrization, V can be sent to a normal form

$$V^{(\infty)} = \sum_{i=0}^{\infty} V_{k_0+i}^{(k_0+i)} \quad (V_{k_0+i}^{(k_0+i)} \in \mathcal{N}_{k_0+i}^{S,P,T}), \tag{14}$$

where

$$\mathscr{L}_n^2 = \left(\text{Im}(L_{(n)}^{S,n}) + \text{Im}(L_{(n)}^{P,n}) + \sum_{i=0}^{[\frac{n-k_0}{2}]} \mathscr{R}_{n-k_0-i}^{k_0+i} V_{k_0+i}^{(k_0+i)}\right) \oplus \mathcal{N}_n^{S,P,T},$$

and $\mathcal{N}_n^{S,P,T}$ follows Proposition 2.3. In addition, $V^{(\infty)}$ is convergent with respect to filtration topology $\tau_{\mathscr{B}}$.

We call the vector field $V^{(\infty)}$ defined in Theorem 4.3 as an infinite order parametric normal form associated with parametric time space decomposition (12). It is evident that an ideal approach is to properly use time terms as much as possible to simplify the system. Therefore, an admissible approach may follow the conditions:

$$\mathscr{L}_n^2 = \text{Im}(L_{(n)}^{S,n}) \oplus \text{Im}(L_{(n)}^{P,n}) \oplus \mathcal{N}_{n,T}^{S,P} \oplus \mathcal{N}_n^{S,P,T}, \tag{15}$$

$$\dim_{\mathbb{F}}(\mathcal{N}_{n,T}^{S,P}) = \sum_{i=0}^{[\frac{n-k_0}{2}]} \dim_{\mathbb{F}}(\mathscr{R}_{n-k_0-i}^{k_0+i}), \quad \text{and} \quad \mathscr{R}_{n-k}^U V_k^{(k)} \subseteq \pi_{S,P,T}(\mathscr{L}_n^2) \quad (\forall\, n, k \in \mathbb{N}_0). \tag{16}$$

Throughout this paper we assume the conditions (15) and (16) are satisfied for infinite order normal forms. Note that the conditions (15) and (16) lead to the computation of normal form process (at each step N) to be freely split into four phases which can be briefly expressed by the following decompositions (uniquely obtained via Proposition 2.3): decomposition (9), $\mathcal{N}_N = \text{Im}(L_{(N)}^{S,N}) \oplus \mathcal{N}_N^S, \mathcal{N}_N^S = \text{Im}(L_{(N)}^{P,N}) \oplus \mathcal{N}_N^{S,P}$, and $\mathcal{N}_N^{S,P} = \mathscr{R}_{N-k_0}^{k_0} V_{k_0}^{(k_0)} \oplus \mathcal{N}_N^{S,P,t}$. One may notice that finding the solutions $(Y^{S,N}, Y^{P,N}, Y^{T,N}$, and $V_N^{(N)})$ via a four-phase step is much easier and more efficient than finding them in a one-phase step (when the conditions (15) and (16) do not hold).

5. Parametric Normal Form of Hopf Singularity

In this section, we apply the results obtained in the previous sections to derive the parametric normal form of a planar formal vector field with codimension-1 Hopf singularity. Let $\mathbb{H}^2 = Y_{10} + \prod_{k=2}^{\infty} \mathscr{L}_k^2$ denote the parametric Hopf singularity space.

Theorem 5.1. *Assume* $a_{101}^{(0)}(a_{210}^{(0)} - a_{200}^{(0)}b_{110}^{(0)} - a_{110}^{(0)}b_{200}^{(0)}) \neq 0$. *Consider* $V^{(0)} \in \mathbb{H}^2$, *given by*

$$V^{(0)} = Y_{10} + \sum_{i+j+k=2, i+j\geq 1}^{\infty} a_{ijk}^{(0)} X_{ij}\mu^k + \sum_{i+j+k=2, i+j\geq 1}^{\infty} b_{ijk}^{(0)} Y_{ij}\mu^k, \tag{17}$$

and a parametric time space decomposition

$$\mathscr{R}_I = 1 + \prod_{n=1}^{\infty} \mathscr{R}_n \quad \left(\mathscr{R}_n = \mathscr{R}_n^U \oplus_{i=0}^{n-1} \mathscr{R}_n^{k_0+i}\right), \tag{18}$$

where

$$\begin{gathered} \mathscr{R}_{2k+1}^0 = \mathbb{F}\mu^{2k+1}, \quad \mathscr{R}_{2k+1}^2 = 0, \\ \mathscr{R}_{2k}^0 = \mathbb{F}\mu^{2k}, \quad \mathscr{R}_{2k}^2 = \mathbb{F}Z_k, \quad \text{and} \\ \mathscr{R}_n^i = 0 \quad (\forall\, i,\ 0 \neq i \neq 2), \end{gathered} \tag{19}$$

while $\mathscr{R}_n^U = \oplus_{k=1}^{[\frac{n-1}{2}]} \mathbb{F} Z_k \mu^{n-2k}$ $(\forall\, n \in \mathbb{N})$. *Then, by a sequence of near-identity change of state variables, time rescaling maps and reparametrization maps,* $V^{(0)}$ *given in (17) can be transformed to an infinite order parametric normal form*:

$$V^{(\infty)} = Y_{10} + a_{101}^{(0)} X_{10}\mu + a_{210}^{(2)} X_{21} + \sum_{i=1}^{\infty} b_{(i+1)i}^{(2i)} Y_{(i+1)i}, \tag{20}$$

where $a_{210}^{(2)} = a_{210}^{(0)} - (a_{200}^{(0)}b_{110}^{(0)} + a_{110}^{(0)}b_{200}^{(0)})$.

Proof. The FB of $\mathscr{L}^2$ is $\mathscr{B} = \bigcup_{N=0}^{\infty} \mathscr{B}_N = \{X_{ij}\mu^r, Y_{ij}\mu^r | N = i+j+r-1\}$, where $\mathscr{B}_N$ is the basis of $\mathscr{L}_N^2$. Note that the same order described in Theorem 2.5 is assumed for $\mathscr{B}$. Consider (for $N \geq 1$)

$$\begin{gathered} L_{(1)}^{S,N} : \mathscr{L}_N^2 \to \mathscr{L}_N^2 \\ Y_N \mapsto [Y_N, Y_{10}]. \end{gathered}$$

Then,

$$\begin{aligned} \mathrm{Im}(L_{(1)}^{S,N}) = \mathrm{span}_{\mathbb{F}}\{&X_{ij}\mu^r, Y_{ij}\mu^r | N = i+j+r-1, \\ &i \neq j+1\} \subset \mathscr{L}_N^2 \backslash \mathscr{L}_H^2, \end{aligned}$$

with $\ker(L_{(1)}^{S,N}) = \mathscr{L}_H^2$. Let

$$Y_{(1)}^{S,N} = \sum_{m+n+r=N+1, n+1\neq m} \frac{a_{mnr}^{(N-1)}}{m-n-1} Y_{mn}\mu^r + \frac{b_{mnr}^{(N-1)}}{n+1-m} X_{mn}\mu^r,$$

and $\mathscr{B}_{H,N}$ denote the ordered basis of $\mathscr{L}_{H,N}^2 = \mathscr{L}_{H_N}^2$. Therefore, we have

$$\mathscr{B}_{H,2k} = \{X_{(k-r+1)(k-r)}\mu^{2r}, Y_{(k-r+1)(k-r)}\mu^{2r}\}_{r=0}^k,$$

and

$$\begin{aligned} \mathscr{B}_{H,2k+1} = \{&X_{(k-r+1)(k-r)}\mu^{2r+1}, \\ &Y_{(k-r+1)(k-r)}\mu^{2r+1}\}_{r=0}^k, \end{aligned}$$

where their orders follow the order of $\mathscr{B}$. Thus, following Proposition 2.3 we obtain

$$\mathscr{N}_{(1)}^{S,2k} = \mathrm{span}_{\mathbb{F}}\mathscr{B}_{H,2k} \quad \text{and} \quad \mathscr{N}_{(1)}^{S,2k+1} = \mathrm{span}_{\mathbb{F}}\mathscr{B}_{H,2k+1}.$$

As a consequence, we have $\mathscr{N}_{(1)}^S = \mathrm{span}_{\mathbb{F}}\{X_{10}\mu, Y_{10}\mu\}$. Since $D_\mu(Y_{10}) = \mathbf{0}$, we have $\mathrm{Im}(L_{(1)}^{P,1}) = \mathbf{0}$. Obviously, $\mathrm{Im}(L_{(1)}^{T,1}) = \mathrm{span}_{\mathbb{F}}\{Y_{10}\mu\}$. So, we choose $Y^{T,1} = -b_{101}^{(0)}\mu \in \mathscr{R}_1 = \mathscr{R}_1^0$ as the time solution. Thus, by Proposition 2.3 we have

$$\mathscr{L}_1^2 = \left(\mathrm{Im}(L_{(1)}^{S,1}) + \mathrm{Im}(L_{(1)}^{P,1}) + \mathrm{Im}(L_{(1)}^{T,1})\right) \oplus \mathscr{N}_1^{S,P,T},$$

where $\mathscr{N}_1^{S,P,T} = \mathrm{span}_{\mathbb{F}}\{X_{10}\mu\}$. On the other hand, $\pi_{\pi_{S,P,T}(\mathscr{L}_1^2)}(\pi_{S,P,T}(\mathscr{L}_1^2)) = \pi_{S,P,T}(\mathscr{L}_1^2)$. Hence,

$$V_1^{(1)} = a_{101}^{(0)} X_{10}\mu,$$

and

$$\begin{gathered} L_{(2)}^{S,N} : \ker(L_{(1)}^{S,N-1}) \times \mathscr{L}_N^2 \to \mathscr{L}_N^2 \\ (Y_{(2)}^N, Y_{(1)}^N) \mapsto [Y_{(2)}^N, V_1^{(1)}] + [Y_{(1)}^N, Y_{10}]. \end{gathered}$$

Now based on the formulas,

$$[X_{(n+1)n}, V_1^{(1)}] = 2\, n\, a_{101}^{(0)} X_{(n+1)n}\mu,$$

and

$$[Y_{(n+1)n}, V_1^{(1)}] = 2\, n\, a_{101}^{(0)} Y_{(n+1)n}\mu,$$

we choose (for $k \geq 1$)

$$Y_{(2)}^{2k} = -\sum_{r=1}^{k-1} \frac{a_{(k-r+1)(k-r)2r}^{(2k-1)}}{2(k-r)a_{101}^{(0)}} X_{(k-r+1)(k-r)}\mu^{2r-1} - \sum_{r=1}^{k-1} \frac{b_{(k-r+1)(k-r)2r}^{(2k-1)}}{2(k-r)a_{101}^{(0)}} Y_{(k-r+1)(k-r)}\mu^{2r-1},$$

$$Y_{(2)}^{2k+1} = -\sum_{r=0}^{k-1} \frac{a_{(k-r+1)(k-r)(2r+1)}^{(2k)}}{2(k-r)a_{101}^{(0)}} X_{(k-r+1)(k-r)}\mu^{2r} - \sum_{r=0}^{k-1} \frac{b_{(k-r+1)(k-r)(2r+1)}^{(2k)}}{2(k-r)a_{101}^{(0)}} Y_{(k-r+1)(k-r)}\mu^{2r},$$

and $Y^{S,N} = Y_{(1)}^{S,N} + Y_{(2)}^{S,N}$. This yields

$$\mathcal{N}_{(2)}^{S,2N} = \mathrm{span}_{\mathbb{F}}\{X_{10}\mu^{2N}, Y_{10}\mu^{2N}, X_{(N+1)N}, Y_{(N+1)N}\}, \tag{21}$$

and

$$\mathcal{N}_{(2)}^{S,2N+1} = \mathrm{span}_{\mathbb{F}}\{X_{10}\mu^{2N+1}, Y_{10}\mu^{2N+1}\}. \tag{22}$$

Further, noticing that

$$\pi_{\mathscr{L}_S^2}(Y_{(1)}^{S,1}) = -b_{200}^{(0)}X_{20} + a_{200}^{(0)}Y_{20} + b_{110}^{(0)}X_{11} - a_{110}^{(0)}Y_{11} + \frac{b_{020}^{(0)}}{3}X_{02} - \frac{a_{020}^{(0)}}{3}Y_{02},$$

and

$$\begin{aligned}\pi_{\mathrm{span}_{\mathbb{F}}\{X_{21}\}}(\mathrm{ad}_{Y_{(1)}^{S,1}}(\pi_{\mathscr{L}_S^2}(V^{(0)})) &= 2(-a_{200}^{(0)}b_{110}^{(0)} - a_{110}^{(0)}b_{200}^{(0)})X_{21} \\ &= -\pi_{\mathrm{span}_{\mathbb{F}}\{X_{21}\}}(\mathrm{ad}_{Y_{(1)}^{S,1}}{}^2(\pi_{\mathscr{L}_S^2}(V^{(0)})),\end{aligned}$$

by Eq. (2) we have

$$a_{210}^{(0),S} = a_{210}^{(0)} - (a_{200}^{(0)}b_{110}^{(0)} + a_{110}^{(0)}b_{200}^{(0)}). \tag{23}$$

Let $Y^{P,2} = (a_{102}/-a_{101})\mu^2$ and $\hat{Y}^{T,2} = -b_{102}^{(1)}\mu^2 + \alpha_2 Z_1 \in \mathscr{R}_2^0 \oplus \mathscr{R}_2^2 = \mathscr{R}_2$. Then, $\mathrm{Im}(L_{(2)}^{P,2}) = \mathrm{span}_{\mathbb{F}}\{X_{10}\mu^2\}$, where $L_{(2)}^{P,2} = L_{(1)}^{P,2}$ and $\pi_{\mathscr{L}_2^2}(\mathrm{Im}(L_{(2)}^{T,2})) = \mathrm{span}_{\mathbb{F}}\{Y_{10}\mu^2\}$. Hence, by Proposition 2.3 this leads to

$$\mathscr{L}_2^2 = \big(\mathrm{Im}(L_{(2)}^{S,2}) + \mathrm{Im}(L_{(2)}^{P,2}) + \pi_{\mathscr{L}_2^2}(\mathrm{Im}(L_{(2)}^{T,2}))\big) \oplus \mathcal{N}_2^{S,P,T},$$

where

$$\mathcal{N}_2^{S,P,T} = \mathrm{span}_{\mathbb{F}}\{X_{21}, Y_{21}\},$$

$$\pi_{\mathscr{L}_2^2}(\mathrm{Im}(L_{(2)}^{T,2})) = \mathscr{R}_2^0 Y_{10}, \quad \text{and} \quad \mathscr{R}_2^U = \mathscr{R}_1^U = \{0\}.$$

Therefore, $a_{210}^{(0),S}X_{21}$ is not effected by the state, time and parameter maps due to the degrees of 1 and 2 (see Proposition 2.6). This implies that $a_{210}^{(2)} = a_{210}^{(0),S}$, and then by Eq. (23) we obtain $a_{210}^{(2)} = a_{210}^{(0)} - (a_{200}^{(0)}b_{110}^{(0)} + a_{110}^{(0)}b_{200}^{(0)})$. So, $V_2^{(2)} = a_{210}^{(2)}X_{21} + b_{210}^{(2)}Y_{21}$.

Now let $K_2^N = \pi_{\mathscr{L}_{N-1}^2}(\ker(L_{(2)}^{S,N})) = \mathrm{span}_{\mathbb{F}}\{X_{10}\mu^{N-1}, Y_{10}\mu^{N-1}\}$. Then,

$$[K_2^{N-1}, V_2^{(2)}] = \mathrm{span}_{\mathbb{F}}\{X_{21}\mu^{N-2}, Y_{21}\mu^{N-2}\} \subseteq L_{(2)}^{S,N},$$

$$\begin{aligned}K_3^N = \pi_{\mathscr{L}_{N-2}^2}(\ker(L_{(3)}^{S,N})) &= \ker(\mathrm{ad}_{K_2^{N-1}}(V_2^{(2)})) \\ &= \mathrm{span}_{\mathbb{F}}\{Y_{10}\mu^{N-2}\},\end{aligned}$$

and $[K_3^{N-1}, Y_{(i+1)i}] = \mathbf{0} \subseteq L_{(2)}^{S,N}$ $(\forall\, i)$. Therefore, $\mathrm{Im}(L_{(2)}^{S,N}) = \mathrm{Im}(L_{(N)}^{S,N})$ $(\forall\, N \geq 2)$, and $\pi_{\mathscr{L}_{N-i}^2}(\ker(L_{(k)}^{S,N})) = \{\mathbf{0}\}$ $(\forall\, i > 2)$.

The parameter operator at degree N is defined by

$$L_{(1)}^{P,N} : \mathscr{P}_N^m \to \mathscr{P}_N^m \quad (\forall\, N, N \geq 1),$$

where

$$\begin{aligned}L_{(1)}^{P,N}(P_N\mu^N) &= a_{101}^{(0)}X_{10}D_\mu(\mu)P_N\mu^N \\ &= a_{101}^{(0)}P_N X_{10}\mu^N.\end{aligned}$$

Thus,

$$\mathrm{Im}(L_{(1)}^{P,N}) = \mathrm{span}_{\mathbb{F}}\{X_{10}\mu^N\} \tag{24}$$

and the parameter solution can be chosen as $Y^{P,N} = (a_{10N}^{(N)}/-a_{101}^{(0)})\mu^N$. Note that $L_{(1)}^{P,N} = L_{(N)}^{P,N}$ $(\forall\, N \in \mathbb{N})$.

Also note that Eq. (23) implies $a_{210}^{(2)} \neq 0$, which leads to a generic Hopf singularity case. By the time decomposition (18)–(19), we obtain $L_{(N)}^{T,N} = L_{(2)}^{T,N}$ $(\forall\, N \geq 2)$ and

$$L_{(2)}^{T,N} : \mathscr{R}_N \to \mathscr{L}_N^2 \oplus \mathscr{L}_{N+1}^2 \oplus \mathscr{L}_{N+2}^2,$$

$$Y^{T,N} \mapsto Y^{T,N}(Y_{10} + a_{101}^{(0)}X_{10}\mu + a_{210}^{(2)}X_{21} + b_{210}^{(2)}Y_{21}).$$

Thus, we choose

$$Y^{T,2k+1} = -b_{10(2k+1)}^{(2k)}\mu^{2k+1} \in \mathscr{R}_{2k+1}^0, \quad k \geq 0,$$

and (when $N = 2k > 1$)

$$\hat{Y}^{T,N} = -b_{10N}^{(N-1)}\mu^N + \alpha_{2k}Z_k \in \mathscr{R}_{2k}^0 \oplus \mathscr{R}_{2k}^2. \quad (25)$$

Now by recalling the formulas

$$[Y_{(k+1)k}, V_2^{(2)}] = 2ka_{210}^{(2)}Y_{(k+2)(k+1)},$$

and

$$[X_{(k+1)k}, V_2^{(2)}] = 2(k-1)a_{210}^{(2)}X_{(k+2)(k+1)} - 2b_{210}^{(2)}Y_{(k+2)(k+1)},$$

we have (for $N = 2k+1$)

$$\pi_{\mathscr{L}_S^2}(Y_{(2)}^{S,2k+1}) = \frac{a_{(k+1)k1}^{(2k)}}{2ka_{101}^{(0)}}X_{(k+1)k} + \frac{b_{(k+1)k1}^{(2k)}}{2ka_{101}^{(0)}}Y_{(k+1)k},$$

and

$$\pi_{\mathscr{L}_X^2} \circ \pi_{\mathscr{L}_{S,2k+2}^2}([Y_{(2)}^{S,N}, V_2^{(2)}]) = \frac{k-1}{ka_{101}^{(0)}}a_{(k+1)k1}^{(2k-1)}a_{210}^{(2)}X_{(k+2)(k+1)}.$$

Since $\pi_{\mathscr{L}_S^2}(Y_{(2)}^{S,N}) = \mathbf{0}$ (when $N \in \mathbb{N}_e$), we obtain

$$\pi_{\mathscr{L}_X^2} \circ \pi_{\mathscr{L}_{S,N+1}^2}\left([Y_{(2)}^{S,N}, V_2^{(2)}]\right) = \mathbf{0}.$$

Furthermore, Proposition 2.6 implies that $\pi_{\mathscr{L}_H^2}([Y_{(1)}^{S,N}, V_2^{(2)}]) = \mathbf{0}$ $(\forall\, N \in \mathbb{N})$. So by Eq. (25), $a_{(k+1)k1}^{(2k)} = \alpha_{2k}a_{101}^{(0)} + a_{(k+1)k1}^{(2k-1),S}$. On the other hand,

$$\pi_{\mathscr{L}_X^2} \circ \pi_{\mathscr{L}_S^2}(Y_{(2)}^{S,2k+1}) = -\frac{\alpha_{2k}a_{101}^{(0)} + a_{(k+1)k1}^{(2k-1),S}}{2ka_{101}^{(0)}}X_{(k+1)k}$$

and thus

$$\pi_{\mathscr{L}_X^2} \circ \pi_{\mathscr{L}_{S,2k+2}^2}\left([Y_{(2)}^{S,2k+1}, V_1^{(2)}]\right) = -\frac{k-1}{ka_{101}^{(0)}}(\alpha_{2k}a_{101}^{(0)} + a_{(k+1)k1}^{(2k-1),S})a_{210}^{(2)}X_{(k+2)(k+1)}.$$

Therefore, we have

$$a_{(k+2)(k+1)0}^{(2k+1)} = -\frac{k-1}{ka_{101}^{(0)}}(\alpha_{2k}a_{101}^{(0)} + a_{(k+1)k1}^{(2k-1),S})a_{210}^{(2)} + \alpha_{2k}a_{210}^{(2)} + a_{(k+2)(k+1)0}^{(2k-1),S}$$
$$= \frac{\alpha_{2k}a_{101}^{(0)}a_{210}^{(2)} - (k-1)a_{(k+1)k1}^{(2k-1),S}a_{210}^{(2)} + ka_{101}^{(0)}a_{(k+2)(k+1)0}^{(2k-1),S}}{ka_{101}^{(0)}}.$$

Since $a_{(k+2)(k+1)0}^{(2k+2)} = a_{(k+2)(k+1)0}^{(2k),S}$, in order to have $a_{(k+2)(k+1)0}^{(2k+2)} = 0$, we only need to set $\alpha_{2k} = (((k-1)a_{(k+1)k1}^{(2k-1),S})/a_{101}^{(0)}) - ((ka_{(k+2)(k+1)0}^{(2k-1),S})/a_{210}^{(2)})$, namely

$$Y^{T,2k} = \hat{Y}^{T,2k} = -b_{10(2k)}^{(2k-1)}\mu^{2k} + \left(\frac{(k-1)a_{(k+1)k1}^{(2k-1),S}}{a_{101}^{(0)}} - \frac{ka_{(k+2)(k+1)0}^{(2k-1),S}}{a_{210}^{(2)}}\right)Z_k \in \mathscr{R}_{2k}^0 \oplus \mathscr{R}_{2k}^2.$$

Thereby, the following holds:

$$\mathscr{L}_{2k}^2 = \left(\mathrm{Im}(L_{(2)}^{S,2k}) + \mathrm{Im}(L_{(1)}^{P,2k}) + \mathscr{R}_{2k}^0V_0^{(0)} + \mathscr{R}_{2k-2}^2V_2^{(2)}\right) \oplus \mathcal{N}_{2k}^{S,P,T} \quad (N = 2k > 2),$$

where $\mathcal{N}_{2k}^{S,P,T} = \mathrm{span}_{\mathbb{F}}\{Y_{(k+1)k}\}$, $\mathscr{R}_{2k}^0Y_{10} = \pi_{\mathscr{L}_{2k}^2}(\mathrm{Im}(L_{(2)}^{T,N}))$ and $\mathscr{R}_{2k-2}^2V_2^{(2)} = \pi_{\mathscr{L}_{2k}^2}(\mathrm{Im}(L_{(2)}^{T,2k-2}))$. By Eqs. (21) and (24) we have

$$\mathcal{N}_{2k}^{S,P} = \mathrm{span}_{\mathbb{F}}\left\{Y_{10}\mu^{2k}, X_{(k+1)k}, Y_{(k+1)k}\right\},$$

and thus $\dim_{\mathbb{F}}\mathcal{N}_{2k}^{S,P} - \dim_{\mathbb{F}}\mathcal{N}_{2k}^{S,P,T} = \dim_{\mathbb{F}}\mathscr{R}_{2k}^0 + \dim_{\mathbb{F}}\mathscr{R}_{2k}^2 = 2$, and

$$\mathscr{R}_n^U = \bigoplus_{k=1}^{[\frac{n-1}{2}]}\mathbb{F}Z_k\mu^{n-2k} \quad (\forall\, n).$$

Then,

$$\mathscr{R}_{2k}^UY_{10} \subseteq \mathrm{Im}(L_{(2)}^{S,2k}) + \mathscr{R}_{2k}^0V_0^{(0)},$$
$$\mathscr{R}_{2k}^UX_{10}\mu \subseteq \mathrm{Im}(L_{(2)}^{S,2k}) + \mathrm{Im}(L_{(1)}^{P,2k}),$$

$\mathscr{R}_{2k}^UV_2^{(2)} \subseteq \mathrm{Im}(L_{(2)}^{S,2k})$, and $\mathscr{R}_{2k}^UY_{(i+1)i} \subseteq \mathrm{Im}(L_{(2)}^{S,2k})$ $(\forall\, i, i > 1)$.

When $N = 2k+1 > 1$, by Eqs. (22) and (24) we have

$$\mathscr{L}_{2k+1}^2 = \mathrm{Im}(L_{(2)}^{S,2k+1}) \oplus \mathrm{Im}(L_{(1)}^{P,2k+1}) \oplus \mathscr{R}_{2k+1}^0V_0^{(0)},$$

and $\dim_{\mathbb{F}}\mathcal{N}_{2k+1}^{S,P} - \dim_{\mathbb{F}}\mathcal{N}_{2k+1}^{S,P,T} = \dim_{\mathbb{F}}\mathscr{R}_{2k}^0 = 1$.

This completes the proof. ∎

Now we are ready to prove that the obtained infinite order parametric normal form of Hopf singularity [see Eq. (20)] is unique, see also Theorem 4.2 in [Kokubu *et al.*, 1996].

Theorem 5.2. *Let* $V, W \in \mathscr{L}^2$ *be two infinite order parametric normal forms associated with the parametric time space decomposition (18)–(19), where*

$$\begin{aligned} V &= V^{(\infty)} \\ &= \sum_{k=0}^{\infty} V_k^{(k)} \\ &= Y_{10} + a_{101}^{(0)} X_{10}\mu + a_{210}^{(2)} X_{21} \\ &\quad + \sum_{i=1}^{\infty} b_{(i+1)i}^{(2i)} Y_{(i+1)i}, \end{aligned} \tag{26}$$

and $a_{101}^{(0)} a_{210}^{(2)} \neq 0$. *If there exists a permutation* σ *on* $\{S, P, T\}$ *with*

$$Y^S = \sum_{n=1}^{\infty} Y_n^S \quad (Y_n^S \in \mathscr{L}_n^2),$$

$$Y^P = \sum_{n=2}^{\infty} Y_n^P \quad (Y_n^P \in \mathscr{P}_n^m),$$

and

$$Y^T = \sum_{n=1}^{\infty} Y_n^{T,0} + Y_{2n}^{T,1} \quad (Y_n^{T,0} = c_n\mu^n, Y_{2n}^{T,1} = d_n Z_n),$$

such that $W = \phi^{\sigma(T)}_{Y^{\sigma(T)}*} \circ \phi^{\sigma(P)}_{Y^{\sigma(P)}*} \circ \phi^{\sigma(S)}_{Y^{\sigma(S)}*}(V)$. *Then,* $V = W$, $Y^T = Y^P = 0$, *and* $Y^S \in \ker(\mathrm{ad}_V)$.

Proof. Let $\mathcal{I}_N^V = \pi_{S,P,T}(\mathscr{L}_N^2)$, where the right-hand side is associated with V. For our convenience, consider $V = \sum_{k=0}^{\infty} V_k^{(k)}$ and $W = \sum_{k=k_0}^{\infty} W_k^{(k)}$, where $V_k^{(k)}, W_k^{(k)} \in \mathscr{L}_k^2$. It is easy to see that $k_0 = 0$, $W_0^{(0)} = Y_{10}$ and $\mathcal{I}_1^V = \mathcal{I}_1^W$. Let us consider the case when σ is the identity permutation. Since $W = \phi^T_{Y^T*} \circ \phi^P_{Y^P*} \circ \phi^S_{Y^S*}(V)$, we have

$$\begin{aligned} W_N^{(N)} &= V_N^{(N)} + \sum_{k=1}^{N} \sum_{i=1}^{[\frac{N}{k}]} \frac{1}{k!} \mathrm{ad}^k_{Y_i^S}(V_{N-ki}^{(N-ki)}) + \sum_{n=1}^{N} \sum_{i=2}^{[\frac{N}{n}+1]} \frac{1}{n!} D_\mu^n V_{N-(i-1)n}^{(N-(i-1)n)} (Y_i^p)^n \\ &\quad + \sum_{i=1}^{N} \sum_{r=0}^{1} Y_i^{T,r} V_{N-i}^{(N-i)} + \sum_{k=1}^{N-1} \sum_{i=1}^{[\frac{N-1}{k}]} \sum_{j=1}^{[N-ki]} \sum_{r=0}^{1} \frac{1}{k!} Y_j^{T,r} \mathrm{ad}^k_{Y_i^S}(V_{N-ki-j}^{(N-ki-j)}) \\ &\quad + \sum_{n=1}^{N} \sum_{k=1}^{N-n} \sum_{i=1}^{[\frac{N-n}{k}]} \sum_{j=2}^{[\frac{N-ki}{n}+1]} \frac{1}{n!k!} D_\mu^n (\mathrm{ad}^k_{Y_i^S}(V_{N-ki-(j-1)n}^{(N-ki-(j-1)n)}))(Y_j^p)^n \\ &\quad + \sum_{n=1}^{N} \sum_{i=2}^{[\frac{N-1}{n}-1]} \sum_{j=1}^{N-(i-1)n} \sum_{r=0}^{1} \frac{1}{n!} Y_j^{T,k_0+r} D_\mu^n V_{N-(i-1)n-j}^{(N-(i-1)n-j)} (Y_i^p)^n \\ &\quad + \sum_{n=1}^{N} \sum_{k=1}^{N-n-1} \sum_{i=1}^{[\frac{N-n-1}{k}]} \sum_{j=1}^{[\frac{N-ki-1}{n}]} \sum_{m=ki+jn+k_0}^{N-1} \sum_{r=0}^{1} \frac{1}{n!k!} Y_{N-m}^{T,k_0+r} D_\mu^n (\mathrm{ad}^k_{Y_i^S}(V_{m-ki-jn}^{(m-ki-jn)}))(Y_{j+1}^p)^n, \end{aligned}$$

while $Y_{2k+1}^{T,1} = 0 \ \forall\, k \in \mathbb{N}_0$. Therefore,

$$W_1^{(1)} - V_1^{(1)} = \mathrm{ad}_{Y_1^S}(Y_{10}) + Y_1^{T,0} Y_{10} \in \mathcal{N}_1^{S,P,T} \cap \mathcal{I}_1^V.$$

Thus, $W_1^{(1)} = V_1^{(1)}, Y_1^{T,0} = 0$, $Y_1^S \in \ker(L_{(1)}^{S,1})$, and $\mathcal{I}_2^V = \mathcal{I}_2^W$. Since

$$\begin{aligned} W_2^{(2)} - V_2^{(2)} &= \sum_{i=1}^{2} \mathrm{ad}_{Y_i^S}(V_{2-i}^{(2-i)}) + a_{101}^{(0)} X_{10}(Y_2^p) \\ &\quad + Y_2^{T,0} Y_{10} + d_1 Y_{21}, \end{aligned}$$

and $W_2^{(2)} - V_2^{(2)} - d_1 Y_{21} \in \mathcal{N}_2^{S,P,T} \cap \mathcal{I}_2^V$, we have $Y_2^{T,0} = Y_2^P = 0$, $(Y_1^S, Y_2^S) \in \ker(L_2^{S,2})$, and $W_2^{(2)} = V_2^{(2)} + d_1 Y_{21}$.

Now by induction on $l \in \mathbb{N}$, consider

$$Y_j^{T,0} = Y_j^p = 0, \quad (Y_1^S, Y_2^S, \ldots, Y_j^S) \in \ker(L_j^{S,j}) \quad (\forall\, j, j \le 2l),$$

$$W_i^{(i)} = V_i^{(i)},\ Y_i^{T,1} = 0 \quad (\forall\, i, i < 2l), \quad \text{and}$$

$$W_{2l}^{(2l)} = V_{2l}^{(2l)} + d_l Y_{(l+1)(l)}.$$

Due to induction hypothesis, $\mathcal{N}^{S,P,T}_{2l+1}$ is the same for both W and V. Thus, $Y^{T,1}_{2l}V^{(1)}_1 = a^{(0)}_{101}d_l X_{(l+1)l}\mu$, $D_\mu V^{(1)}_1(Y^p_{2l+1}) = a^{(0)}_{101}X_{10}Y^p_{2l+1}$, and

$$\begin{aligned} W^{(2l+1)}_{2l+1} - V^{(2l+1)}_{2l+1} &= \left(\sum_{i=1}^{2l+1} \mathrm{ad}_{Y^S_i}(V^{(2l+1-i)}_{2l+1-i}) + a^{(0)}_{101}d_l X_{(l+1)l}\mu\right) \\ &\quad + a^{(0)}_{101}X_{10}Y^p_{2l+1} + Y^{T,0}_{2l+1}Y_{10}. \end{aligned}$$

Then, $W^{(2l+1)}_{2l+1} - V^{(2l+1)}_{2l+1} \in \mathcal{I}^{V^\infty}_{2l+1} \cap \mathcal{N}^{S,P,T}_{2l+1} = \{\mathbf{0}\}$. This shows that $Y^p_{2l+1} = Y^{T,0}_{2l+1} = 0$, $W^{(2l+1)}_{2l+1} = V^{(2l+1)}_{2l+1}$, and

$$\left(Y^S_1, \ldots, Y^S_{2l-1}, \hat{Y}^S_{2l}, Y^S_{2l+1}\right) \in \ker(L^{S,2l+1}_{(2l+1)}),$$

where $\hat{Y}^S_{2l} = Y^S_{2l} - (d_l/2l)X_{(l+1)l}$. On the other hand,

$$\begin{aligned} W^{(2l+2)}_{2l+2} = V^{(2l+2)}_{2l+2} &+ \sum_{i=1}^{2l+2} \mathrm{ad}_{Y^S_i} V^{(2l+2-i)}_{2l+2-i} \\ &+ d_l a^{(0)}_{101}X_{(l+1)l}Y^p_2 + Y^{T,0}_{2l+2}Y_{10} \\ &+ (d_{l+1} + d_l b^{(2)}_{21})Y_{(l+2)(l+1)} \\ &+ a^{(1)}_{101}X_{10}Y^p_{2l+2} + d_l a^{(2)}_{21}X_{(l+2)(l+1)}, \end{aligned}$$

which implies that

$$\begin{aligned} W^{(2l+2)}_{2l+2} - V^{(2l+2)}_{2l+2} - (d_{l+1} + d_l b^{(2)}_{21})Y_{(l+2)(l+1)} \\ \in \mathcal{I}^V_{2l+2} \cap \mathcal{N}^{S,P,T}_{2l+2} = \{\mathbf{0}\}. \end{aligned}$$

Therefore,

$$d_l = Y^{T,0}_{2l+2} = Y^p_{2l+2} = 0, \quad \hat{Y}^S_{2l} = Y^S_{2l},$$
$$W^{(2l)}_{2l} = V^{(2l)}_{2l}, \quad W^{(2l+2)}_{2l+2} = V^{(2l+2)}_{2l+2} + d_{l+1}Y_{(l+2)(l+1)},$$

and $(Y^S_1, \ldots, Y^S_{2l+2}) \in \ker(L^{S,2l+2}_{2l+2})$. This implies $Y^S \in \ker(\mathrm{ad}_V)$, since $\ker(\mathrm{ad}_V)$ is $\tau_{\mathscr{B}}$-closed.

The proof is complete. ■

The following corollary states that the simplest normal form obtained in Theorem 3 of [Yu & Leung, 2003] is an infinite order normal form defined in this paper.

Corollary 5.3. *Consider the system*

$$\begin{aligned} \frac{dx}{dt}\partial_x + \frac{dy}{dt}\partial_y &= f_1(x,y,\mu)\partial_x + f_2(x,y,\mu)\partial_y \\ &= y\partial_x - x\partial_y + \sum_{i+j+k=2, i+j\geq 1}^{\infty} \\ &\qquad \times (\alpha_{ijk}x^i y^j \mu^k \partial_x + \beta_{ijk}x^i y^j \mu^k \partial_y), \end{aligned} \tag{27}$$

where $\mu \in \mathbb{R}$, *and assume* $A_{101}A_{210} \neq 0$, *where*

$$\begin{aligned} A_{210} = \frac{1}{8}[&3(\alpha_{300} + \beta_{030}) + \alpha_{120} + \beta_{210} \\ &- (\alpha_{110}\alpha_{200} + \alpha_{110}\alpha_{020}) \\ &+ 2(\beta_{200}\alpha_{200} - \beta_{020}\alpha_{020}) \\ &+ \beta_{020}\beta_{110} + \beta_{200}\beta_{110}] \end{aligned}$$

and $A_{101} = (\alpha_{101} + \beta_{011})/2$. *Then, the system is a codimension-*1 *generic Hopf singularity and by a sequence of near-identity change of state variable, time rescaling and reparametrization maps, system* (27) *can be transformed to an infinite order parametric normal form*:

$$\begin{aligned} \frac{d\rho}{dt}\partial_\rho + \frac{d\theta}{dt}\partial_\theta &= \rho(A_{101}\mu + A_{210}\rho^2)\partial_\rho \\ &\quad + \left(1 + \sum_{k=1}^{\infty} b^{(2k)}_{(k+1)k}\rho^{2k}\right)\partial_\theta, \end{aligned}$$

where the coefficients $b^{(2n)}_{(n+1)n}$ *are uniquely expressed in terms of* α_{ijk} *and* β_{ijk}.

Proof. Let $g = f_2 + if_1$, $z = y + ix$, $X_{ij} = z^i\bar{z}^j\partial_1 + \bar{z}^i z^j\partial_2$ and $Y_{ij} = iz^i\bar{z}^j\partial_1 - i\bar{z}^i z^j\partial_2$, $Z_k = z^k\bar{z}^k$. Then, system (27) is equivalent to the system associated with (17), where $a^{(0)}_{ijk}, b^{(0)}_{ijk}$ are uniquely determined in terms of α_{ijk} and β_{ijk}. In particular, $a^{(0)}_{101} = (\alpha_{101} + \beta_{011})/2 = A_{101}$. Thus, system (27) can be transformed to the system generated by (26) via near-identity state, parameter and time maps. Recalling the formulas $a^{(0)}_{210} = (3\alpha_{300} + \alpha_{120} + \beta_{210} + 3\beta_{030})/8$, $a^{(0)}_{200} = (\alpha_{110} + \beta_{020} - \beta_{200})/4$, $b^{(0)}_{110} = (\alpha_{200} + \alpha_{020})/2$, $a^{(0)}_{110} = (\beta_{020} + \beta_{200})/2$, and $b^{(0)}_{200} = (\alpha_{020} - \alpha_{200} - \beta_{110})/4$, we have $a^{(2)}_{210} = A_{210}$. So, system (27) is equivalent to

$$\dot{z} = iz + A_{101}z\mu + A_{210}z^2\bar{z} + i\sum_{k=1}^{\infty} b^{(2k)}_{(k+1)k}z^{(k+1)}\bar{z}^k.$$

Further, with the polar coordinates $z = \rho e^{i\theta}$, we obtain $z^{(k+1)}\overline{z}^k = \rho^{2k+1}e^{i\theta}$, $\dot{z} = \dot{\rho}e^{i\theta} + i\dot{\theta}\rho e^{i\theta}$, and

$$\dot{\rho} + i\dot{\theta}\rho = A_{101}\rho\mu + A_{210}\rho^3 + i\rho + i\sum_{k=1}^{\infty} b_{(k+1)k}^{(2k)}\rho^{2k+1}.$$

This completes the proof. ■

We have developed an efficient Maple program to carry out all the algebraic structures described in this paper. We have also implemented the formulas and results presented in Theorem 5.1 and Corollary 5.3. To demonstrate our results, we execute our program for the example in Sec. 4.1 of [Yu & Leung, 2003]:

$$\frac{dx}{dt}\partial_x + \frac{dy}{dt}\partial_y = (y + \mu x + \mu x^2 + 2xy - x^3 + x^2 y)\partial_x - x\partial_y,$$

which is equivalent to

$$Y_{1,0} + \frac{\mu X_{1,0} - \mu X_{0,1}}{2} - \frac{\mu Y_{2,0} + \mu Y_{0,2}}{4} + \frac{\mu Y_{1,1} + X_{2,0} - X_{0,2}}{2} + \frac{1}{2}X_{3,0} + \frac{3X_{1,2} - 3X_{2,1}}{8} + \frac{Y_{2,1} + Y_{1,2}}{8} - \frac{X_{0,3} + Y_{3,0} + Y_{0,3}}{8}$$

in our notations. Our Maple output for the normal form up to degree 8 (equivalent to degree 9 in [Yu & Leung, 2003] is

$$Y_{1,0} - \frac{3}{8}X_{2,1} + \frac{1}{2}\mu X_{1,0} - \frac{5}{72}Y_{2,1} - \frac{1}{288}Y_{3,2} - \frac{1699877}{37324800}Y_{4,3} + \frac{62586677}{10749542400}Y_{5,4}.$$

This shows that the system can be transformed to

$$\frac{d\rho}{dt}\partial_\rho + \frac{d\theta}{dt}\partial_\theta = \rho\left(\frac{1}{2}\mu - \frac{3}{8}\rho^2\right)\partial_\rho + \left(1 - \frac{5}{72}\rho^2 - \frac{1}{288}\rho^4 - \frac{1699877}{37324800}\rho^6 + \frac{62586677}{10749542400}\rho^8 + \sum_{k=5}^{\infty} b_{(k+1)k}^{(2k)}\rho^{2k}\right)\partial_\theta,$$

which is exactly the same as that obtained in [Yu & Leung, 2003]. This reaffirms Theorem 5.2 and Corollary 5.3.

6. Conclusions

A suitable algebraic structure has been introduced, leading to development of a new method for computing infinite order parametric normal forms which are convergent in filtration topology. The theory has been applied to obtain an infinite order normal form of Hopf singularity which agrees with the existing result. Maple programs have also been developed to affiliate applications of the results obtained in this paper. The method presented in this paper can be extended to consider degenerate Hopf bifurcation and other singularities.

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