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FORMAL DECOMPOSITION METHOD AND PARAMETRIC NORMAL FORMS

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We introduce a formal decomposition method for efficiently computing the parametric normal form of nonlinear dynamical systems with multiple parameters. Recently introduced notions of formal basis style and costyle are applied through formal decomposition method to obtain the simplest parametric normal form for degenerate nonlinear parametric center. The necessary formulas are derived and implemented using Maple to compute the simplest parametric normal form of degenerate and nondegenerate nonlinear centers. Our program computes the order of any planar parametric systems associated with this singularity.

Keywords: Parametric normal form; degenerate nonlinear center; formal decomposition; formal basis style.

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1. Introduction

One of the main tools in the study of nonlinear differential equations is normal form theory. The idea is to simplify a system via change of state variables such that we can understand the topological behavior of the original system by exploring the stability and bifurcation analysis of the obtained normal form. Since most real life problems involve some parameters, it is important to directly deal with parametric systems. The classical approach for analyzing parametric systems is to first exclude the parameters from the system (to obtain a simplified system) by setting the parameters zero. Then, compute the normal form of the simplified system via a change of state variables and then add some unfolding to the normal form to obtain a parametric normal form for the original system, see [Chow & Hale, 1982; Chow *et al.*, 1994; Kuznetsov, 2004; Murdock, 2005; Liao *et al.*, 2007; Sanders *et al.*, 2007]. The main drawback of this approach is that the transformation between the original system and the parametric normal form cannot be obtained. This is why some researchers have recently paid attention to develop methodologies for directly computing the parametric normal form and its associated transformation from the original parametric system, see [Yu & Leung, 2003; Liao *et al.*, 2007; Yu & Chen, 2007].

A formal vector field may have different normal forms even when using the same approach and the

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same grading structure. This has two reasons: one comes from the style used and the other stems from the *level* at which we simplify the system. Therefore, by using the same approach, grading structure and style, we just need to simplify the system as much as possible to obtain a unique (parametric) normal form for a given formal vector field. Many significant contributions have been made in recent years in computing the simplest (or unique) nonparametric normal forms, see e.g. [Ashkenazi & Chow, 1988; Baider & Churchill, 1988; Chua & Kokubu, 1988; Baider, 1989; Baider & Sanders, 1991, 1992; Kokubu *et al.*, 1996; Murdock, 1998, 2003, 2004; Yu, 1999; Yu & Yuan, 2000, 2003a, 2003b; Chen & Della Dora, 2000; Chen *et al.*, 2000; Wang *et al.*, 2000; Algaba *et al.*, 2001; Yuan & Yu, 2001; Sanders, 2003, 2005; Peng & Wang, 2004; Murdock & Malonza, 2009]. However, there are only a few results on the simplest parametric normal forms [Yu & Leung, 2003; Yu & Chen, 2007; Gazor & Yu, 2008, 2010].

Computing parametric normal forms is much more difficult than computing the normal forms of systems without parameters. Recently, we extended the method of cohomology spectral sequence from the existing results on nonparametric systems [Sanders, 2003, 2005; Murdock, 2004; Benderesky & Churchill, 2006; Sanders *et al.*, 2007] to consider parametric normal forms [Gazor, 2008; Gazor & Yu, 2010]. This method has been applied on parametric normal forms of systems associated with Hopf, generalized Hopf and Bogdanov–Takens singularity to compute the infinite level cohomology spectral sequence of their associated cochain complexes, see [Gazor, 2008; Gazor & Yu, 2010]. We believe that the method of spectral sequences may potentially lay some grounds for efficiently computing normal forms in an abstract sense. However, as one can see in Sec. 5 from [Gazor & Yu, 2010] the method of spectral sequence has been modified to obtain a suitable parametric normal form for stability and bifurcation analysis. In order to implement those results on the computer using a computer algebra system such as Maple, we need the formal decomposition computational method. We introduced the formal decomposition method in [Gazor & Yu, 2008], where a chess-like computation was the main streamline of the method. We improve this method via notion of invariant spaces.

Our original idea of formal *decomposition* method came from the fact that time transformations are injective and thus, there are no kernel terms for time transformations. However, some parametric time terms must be used to simplify the grade terms higher than the grades, which are normally targeted at the first level time transformation. In formal decomposition method, we *decompose* the transformation spaces into some specific subspaces aimed to simplify certain grades of the system. This way we do not necessarily follow the common approach which uses kernel terms for simplifying higher grades; this overcomes the common concern of recreating already eliminated terms. On the contrary, we instead anticipate all changes that will appear in high enough levels of normal form of the system. Then, based on this calculation we determine our (near-identity) transformation solutions. Thus, some terms of the system may not be necessarily simplified at the time when their designated transformation responsible for simplifying them is applied to the system. Indeed, their coefficients may be changed in levels afterward and then, after going through high enough levels, they turn to be a zero coefficient and remain so. This is sort of similar to playing chess; one anticipates all possible future changes that can be made in the game and based on those calculations one decides one's own move. Similarly to our calculation, one makes a move expecting to attain its goal after a while, not instantly. This is why we call it a *chess-like* computation. In [Gazor & Yu, 2008], we used this kind of chess-like computation for generic nonlinear center with single parameter. The same procedure of chess-like computation will readily work for obtaining the parametric normal form of generalized nonlinear center with multiple parameters. One, however, may expect that this kind of calculation becomes very hard, if it is meant to be literally applied on generalized nonlinear center or more complex singularities. This is why we generalize this method with the notion of invariant spaces to reduce the complexity and necessity of an *excessive* chess-like computation.

A notion of invariant spaces is our key to new improvements on this method, compared with what we did in [Gazor & Yu, 2008]. The advantage here is to only look after those changes that have some effect on our decision making for transformation solutions rather than all possible changes of the system. To illustrate this point, assume there exist some terms of the system that are neither the intended system terms for elimination nor having impact on choosing transformation solutions. Yet they may only change through our calculations within the space of complement spaces. Then,

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keeping track of such terms makes our task very difficult and we avoid it via the notion of invariant spaces. We present this notion by introducing certain invariant subspaces of complement spaces called degenerate spaces. Degenerate spaces consist of vector space span of the corresponding terms from all vector fields satisfying certain degeneracy conditions in their normal form such that they are invariant (descending sequence of subspaces) under our normal form computation. Degeneracy conditions make certain terms of the system zero for certain level of normal form onward while the order of the system (which is related with codimension) guarantees that some other (specifically known) terms will never be zero. Here, there may exist some terms where their coefficients may change in the normal form computation while they may have no impact on our normal form computation. Then, we may simply ignore computing the changes in their coefficients as long as certain invariant conditions are satisfied. This is the case considered in Secs. 3 and 4 for phase terms of the system in the first and higher than first level normal forms. This way we are able to reduce work in chess-like computations. One observes that the notion of invariant spaces helps us (see Sec. 3) so that we will not need to do any chess-like computation. Yet we will do a minor and simple (same grade, see the end of Sec. 2) calculation of this kind in Sec. 4, which aims to implement the results in (and also enhancing the efficiency of) computer programs. Therefore, we present this method in a general sense in Sec. 2, which uses degenerate spaces as the invariant spaces within the framework of formal decomposition method without having to do any chess-like computation (beyond the same garde). One may expect a need for a further modification with a combination of both invariant spaces and chess-like computations within the formal decomposition method for more complicated singularities. We, however, will not deal with it in this paper.

The rest of this paper is organized as follows. We first present our style and some necessary algebraic structures in Sec. 2. Degenerate invariant spaces and our method are also discussed. Further, injective level transformations of state, parameter and time are introduced in this section. Section 3 is devoted to apply the general theory and methodology to degenerate nonlinear center with multiple parameters. In Sec. 4, some intricate concepts are introduced for enhancing the efficiency of computer programs and additional formulas are derived for systematically developing computer programs. The simplest normal form and orbital equivalence of nonparametric nonlinear center are briefly discussed in Sec. 5. Several examples are presented in Sec. 6 to demonstrate the applicability of our theoretical results and Maple programs. Finally, Sec. 7 concludes this paper.

2. Formal Basis Style, Costyle and Formal Decomposition Method

In this section, we first briefly present the concepts of formal basis style (costyle) and formal decomposition, and then, describe the algebraic structures of transformation spaces without the details (detailed explanation on these and on formal basis style and costyle can be found in [Gazor, 2008; Gazor & Yu, 2010]). Note that following [Murdock, 2004; Sanders *et al.*, 2007] the terminology *level* normal form is used instead of *order* normal form in this paper. The reason stems from the context of cohomology spectral sequences [Gazor & Yu, 2010]. This also avoids some possible confusion with the order imposed on a *formal basis* and the *order of a nonlinear center singularity*.

Murdock *et al.* [Murdock, 2003, 2004, 2005; Murdock & Malonza, 2009] introduced the notion of *costyle* (and *style*) of normal forms of vector fields, a rule stating how to uniquely choose a complement space for a vector subspace (in the transformation spaces). Churchill and Kummer [1999] used the phrase *splitting convention* for the case of style. Style of the classical normal form is the rule to specify a unique complement space for the image of the homological operators, while costyle determines a unique complement space for the kernel of the state transformation, leading to a unique choice for transformation maps. In principle, a finite level normal form is not unique even if the same approach, grading structure, and style are used. The reason behind this lies with the notion of costyle; indeed, a fixed costyle for a finite level normal form makes it a *unique* normal form despite being far from the *simplest* normal form. In this paper, we apply a newly introduced style and costyle called *formal basis style* and *costyle*. In order to obtain unique transformations, we apply the formal basis costyle which is compatible with formal basis style and our approach. We first recall the notion of formal basis for introducing the formal basis style and to do this, we need to revisit filtration topology. One should distinguish formal decomposition method (which

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decomposes transformation spaces into some subspaces, each of them is designated for simplifying certain terms of the system) from formal basis style (a rule for uniquely choosing complement spaces) and from chess-like computation.

Let $\mathbb{F}$ be the field of real or complex numbers. Assume V is a (locally finite) graded vector space over $\mathbb{F}$, that is, $V = \prod_{n=1}^{\infty} V_n$ and V_n is a finite dimensional vector space and its elements are called homogeneous vectors of grade n. Denote $\mathscr{B}_n$ for the vector space of V_n and fix an order on $\mathscr{B} = \cup \mathscr{B}_n$, i.e. $\mathscr{B} = \{e_i \,|\, i \in \mathbb{N}\}$. The grading on V gives rise to the filtration $\mathscr{F} = \{\mathscr{F}^N V\}$, where $\mathscr{F}^N V = \prod_{n=N}^{\infty} V_n$ for any natural number N. $\mathscr{F}$ establishes a local base for zero and thus, its associated topology τ is called filtration topology. Then, any sequence of $\{v_i\}_{i=1}^{\infty} \subset V$ is τ-convergent to $v \in V$ if and only if for any $N \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that $v - v_i \in \mathscr{F}^N V = \prod_{n=N}^{\infty} V_n$ for all $i \geq k_0$, see also [Baider & Churchill, 1988; Baider, 1989; Baider & Sanders, 1991, 1992; Gazor & Yu, 2008]. Using linearly isomorphic spaces exchangeably, we have $V = \prod_{n=1}^{\infty} V_n = \overline{\bigoplus_{n=1}^{\infty} V_n^{\tau}}$, i.e. V is the τ-closure of direct sum of V_n. Thus, any element $v \in V$ can be uniquely represented by $v = \sum_{n=1}^{\infty} v_n (= \sum_{i=1}^{\infty} a_i e_i)$ in which $v_n \in V_n (e_i \in \mathscr{B}, a_i \in \mathbb{F})$. Then, $\mathscr{B}$ (together with its order) is called a *formal basis* of V. The common notations used for V varies in the literature; for example, Murdock [2003], Gazor and Yu [2010], and Murdock and Malonza [2009] used $\bigoplus_{n=1}^{\infty} V_n$ (which should not be confused with direct sum of vector spaces); Baider and Churchill [1988] and Gazor [2008] used the notation $\hat{\bigoplus}_{n=1}^{\infty} V_n$; while direct product is frequently used in the literature, see e.g. [Wang *et al.*, 2000; Peng & Wang, 2004; Gazor & Yu, 2008]. We use the notation $\hat{\bigoplus}$ but only when V is associated with a fixed formal basis (and thus, it is inherited to all of its formal decompositions). In other words, the usage of $\hat{\bigoplus}$ means that a fixed formal basis is chosen for the space and all of its formal decompositions as described below.

Our idea of defining formal decomposition originated from thinking of two examples; the grading $\hat{\bigoplus}_{n=1}^{\infty} V_n$ as a formal decomposition for V and $\hat{\bigoplus}_{i=1}^{\infty} \mathbb{F} e_i$ as the other (one-dimensional) formal decomposition. Then, extending the idea to other decompositions is gained through any new grading structure while the associated topology and the formal basis are not changed. The definition and notations are as follows. When a fixed formal basis $\mathscr{B}$ is chosen and $\mathscr{B} \cap V_i$ is a (finite) basis for V_i, then we denote $V = \hat{\bigoplus}_{i=1}^{\infty} V_i$. An infinite sequence vector subspaces of $\{U_k \,|\, k \in \mathbb{N}\}$ is called a formal decomposition for V when the following conditions are satisfied:

(1) U_k is finite dimensional and $U_k \cap U_l = 0$ for any $k \neq l$.
(2) $\mathscr{B}$ remains the fixed chosen formal basis for V.
(3) $\mathscr{B} \cap U_k$ is an ordered basis for U_k and $\mathscr{B} = \bigcup_{k=1}^{\infty} (\mathscr{B} \cap U_k)$.
(4) Every $v \in V$ has a unique representation in the form of $v = \sum_{k=1}^{\infty} w_k$, $w_k \in U_k$ for every k, i.e. $V = \overline{\bigoplus_{k=1}^{\infty} U_k^{\tau}}$.

Then, we denote $V = \hat{\bigoplus}_{k=1}^{\infty} U_k$. Formal basis plays a role in defining a unique projection on a subspace W from V. Assume $W \cap \mathscr{B} = \{e_{i_k} \,|\, k \in \mathbb{N}\}$ is a formal basis for W, that is, $W = \hat{\bigoplus}_{k=0}^{\infty} \mathbb{F} e_{i_k}$. Then, π_W denotes a unique linear projection from V onto W, defined by $\pi(\sum_{i=1}^{\infty} a_i e_i) = \sum_k a_{i_k} e_{i_k}$ (Gazor and Yu [2010] presented this in a more general sense by using the notion of formal basis style). The formal basis plays the key role in determining a unique complement space (style and costyle), see also [Gazor & Yu, 2008, Proposition 2.3]. We will further discuss this in Proposition 2.3 when formal basis for transformation spaces are all defined and also avoid confusion with the notion of formal decompositions.

In the following, we first describe the algebraic structure of parametric state space, parameter space and parametric time space. The algebraic structure associated with parametric state space $\mathscr{L}$ represents the space of all formal parametric 2-vector fields of Hopf singularity (nonlinear center) with complex state variables, see Eq. (1). A reader familiar with normal form theory knows that this is a more convenient format for obtaining the normal forms of Hopf and generalized Hopf singularities. Since any element of $\mathscr{L}$ may represent the logistic function of a differential equation or a vector field, we call any such element a vector field or a parametric system whenever it is appropriate. We present the algebraic structures in an abstract sense as if they are merely considered as algebraic objects. They, however, are merely well known algebraic structures carried out from formal vector fields and normal form theory, see [Peng & Wang, 2004; Gazor, 2008; Gazor & Yu, 2008, 2010] for more details. We do not elaborate

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on how these algebraic structures are obtained. This helps us avoid turning to the concepts of differential equations, making the problem look like an abstract algebraic problem which is more convenient to work with. It is well-known that a near-identity (nonparametric) change of state variable is a group (called Campbell–Hausdorff group) acting on the vector state space. This is not only true for near-identity parametric change of state variable acting on parametric state space but also true for near-identity parametric time rescaling and near-identity reparametrization. Indeed they, altogether, comprise a subgroup of filtration preserving vector space automorphisms of the parametric state space, see [Gazor & Yu, 2010, Lemma 3.5]. Therefore, the problem is to find the simplest element of any orbit associated with a given formal vector field.

Assume the system under consideration is already reduced to a two-dimensional central manifold and is presented by the formal vector field in terms of a complex variable z and p parameters, i.e. μ_i for $1 \leq i \leq p$. Let X_{ij} and Y_{ij}, respectively denote $z^i\bar{z}^j\partial_1 + \bar{z}^i z^j\partial_2$ and $\mathbf{i}z^i\bar{z}^j\partial_1 - \mathbf{i}\bar{z}^i z^j\partial_2$ $\forall\, i, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, i + j > 0$. Then, any formal vector V can be simply represented by a summation on X and Y terms; $V = \sum Y_{ij}\mu^{\mathbf{m}} + \sum X_{ij}\mu^{\mathbf{m}}$, $\mu = (\mu_1, \mu_2, \ldots, \mu_p), \mathbf{m} = (m_1, m_2, \ldots, m_p) \in \mathbb{N}_0^p$. Since we only work with the systems of Hopf singularity, our systems can be described as

$$\left\{bY_{10} + \sum b_{ij\mathbf{m}}Y_{ij}\mu^{\mathbf{m}} + \sum a_{ij\mathbf{m}}X_{ij}\mu^{\mathbf{m}}\right\}, \qquad (1)$$

where sums are over all $i, j \in \mathbb{N}_0, \mathbf{m} \in \mathbb{N}_0^p$ that $i + j \geq 1$, $i + j + |\mathbf{m}| > 1$, with $|\mathbf{m}| = \sum m_i, \mu^{\mathbf{m}} = \mu_1^{m_1}\mu_2^{m_2}\cdots\mu_p^{m_p}$. It is common to call any such element $X_{ij}\mu^{\mathbf{m}}$ or $Y_{ij}\mu^{\mathbf{m}}$ a term and it is our mission to simplify as many terms as possible from the system. Obviously, we can assume $b = 1$ in our computation. $\mathscr{L}$ is a Lie algebra where the continuous bilinear Lie bracket follows

(1) $[X_{ij}, X_{kl}] = (i-k)X_{(i+k-1)(j+l)} + jX_{(i+l)(j+k-1)} - lX_{(k+j)(l+i-1)}$;
(2) $[Y_{ij}, X_{kl}] = (i-k)Y_{(i+k-1)(j+l)} + lY_{(k+j)(i+l-1)} + jY_{(l+i)(j+k-1)}$;
(3) $[Y_{ij}, Y_{kl}] = (k-i)X_{(i+k-1)(j+l)} - lX_{(k+j)(i+l-1)} + jX_{(l+i)(j+k-1)}$;
(4) $[v\mu^{\mathbf{m}}, w\mu^{\mathbf{n}}] = [v, w]\mu^{\mathbf{m}+\mathbf{n}}$,

for any $i, j, k, l \in \mathbb{N}_0$, $\mathbf{m}, \mathbf{n} \in \mathbb{N}_0^p$ and $v, w \in \mathscr{L}$, see [Peng & Wang, 2004; Gazor, 2008; Gazor & Yu, 2008]. Furthermore, we denote by $\mathscr{L}_S$ the state space without parameters, that is, $\mathscr{L}_S = \hat{\bigoplus}\mathbb{F}Y_{ij} \oplus \hat{\bigoplus}\mathbb{F}X_{ij} \subset \mathscr{L}$. The space $\mathscr{L}_H = \{bY_{10} + \sum b_{i\mathbf{m}}Y_{(i+1)i}\mu^{\mathbf{m}} + \sum a_{i\mathbf{m}}X_{(i+1)i}\mu^{\mathbf{m}}\}$ is the space of all classical parametric normal forms associated with a nonlinear center. Indeed, this space is invariant under our time rescaling and reparametrization (in our style and approach), thus $\mathscr{L}_H$ includes all the first level normal forms. $\mathscr{L}_S$ and $\mathscr{L}_H$ are τ-closed graded Lie subalgebras of $\mathscr{L}$, see also [Gazor & Yu, 2008, Theorem 1.5]. In order to define a grading function, let

$$\mathscr{B} = \{X_{ij}\mu^{\mathbf{n}}, Y_{ij}\mu^{\mathbf{n}} \mid \mathbf{n} \in \mathbb{N}_0^p, i, j \in \mathbb{N}_0,$$
$$i + j \geq 1, i + j + |\mathbf{n}| > 1\} \cup \{Y_{10}\}.$$

For any natural number α (fixed throughout this paper), we define a grading function $\delta : \mathscr{B} \to \mathbb{N}_0$ by

$$\delta(X_{ij}\mu^{\mathbf{n}}) = \delta(Y_{ij}\mu^{\mathbf{n}}) = i + j - 1 + |\,\mathbf{n}\,|\,\alpha.$$

A subindex with a natural number, say k, is always used to denote the homogenous space of grade k. This is applied to all graded spaces defined throughout this paper, e.g. $\mathscr{L}_{S,k}, \mathscr{R}_k$, and $\mathscr{P}_k^p$, etc. This grading gives rise to a grading structure on $\mathscr{L}$, i.e. $[\mathscr{L}_m, \mathscr{L}_n] \subseteq \mathscr{L}_{m+n}$. We further fix an order on $\mathscr{B}$, then $\mathscr{B}$ is a formal basis for $\mathscr{L}$ and

$$\mathscr{L} = \hat{\bigoplus_{k=0}^{\infty}}\mathscr{L}_k.$$

The fixed order on $\mathscr{B}$ must follow three conditions; firstly (first priority) the terms with lower grades are in lower orders in a sequence, secondly $Y_{ij}\mu^{\mathbf{m}}$ is before $X_{mn}\mu^{\mathbf{n}}$ when $\delta(Y_{ij}\mu^{\mathbf{m}}) = \delta(X_{mn}\mu^{\mathbf{n}})$, and finally (last priority) the terms without parameter are before terms with parameter when they have the same grade. Note that these rules are not sufficient to provide a *unique* order on $\mathscr{B}$, but yet are enough that any fixed order (satisfying the rules) on $\mathscr{B}$ would lead to the results obtained in Sec. 3. Indeed, our conditions mean that if we have some alternatives for elimination, the ones in higher orders have to be eliminated in priority. In other words, the terms in each grade, X terms (in a sense, amplitude terms) rather than Y terms (phase terms) and then terms with parameters compared to terms without parameters, are preferred for not appearing in the parametric normal form of the system, see Proposition 2.3.

Now we briefly describe our method and define state transformations. Let $v = v^{(0)} = \sum_{k=k_0}^{\infty} v_k^{(0)}$ ($v_k \in \mathscr{L}_k$) denote a parametric vector field before

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any normal form computation and $v^{(n)} = \sum_{k=k_0}^{\infty} v_k^{(n)}$ denote its nth level parametric normal form. So, we have an updated sequence

$$v_{k_0}^{(0)} = v_{k_0}^{(n)}, v_{k_0+1}^{(n)}, v_{k_0+2}^{(n)}, \ldots, v_{k_0+k-1}^{(n)}, v_{k_0+k}^{(n)}, \ldots,$$
$$\text{where } v_k^{(n)} \in \mathscr{L}_k \ \forall\, k, \quad n \in \mathbb{N}_0, \tag{2}$$

which are computed in the normal form process. It is well-known that any element $Y^S \in \mathscr{F}^1\mathscr{L} = \hat{\bigoplus}_{i=1}^{\infty}\mathscr{L}_i$ generates a near-identity transformation $\phi_{Y^S*}^S$ sending a system v into $\phi_{Y^S*}^S(v)$,

$$\begin{aligned}\phi_{Y^S*}^S(v) &= \exp(\mathrm{ad}_{Y^S})(v)\\ &= v + \mathrm{ad}_{Y^S}v + \cdots + \frac{1}{n!}\mathrm{ad}_{Y^S}^n v + \cdots,\end{aligned} \tag{3}$$

where $\mathrm{ad}_{Y^S}v = [Y^S, v]$ and $\mathrm{ad}_{Y^S}^n = \mathrm{ad}_Y \circ \mathrm{ad}_{Y^S}^{n-1}$ $\forall\, n \in \mathbb{N}$, see e.g. [Baider & Churchill, 1988; Baider & Sanders, 1991; Gazor & Yu, 2008; Kokubu *et al.*, 1996] for more details. Therefore, $\phi_{\mathscr{F}^1\mathscr{L}*}^S$ comes equipped with the Campbell–Baker–Hausdorff formula and constitutes a subgroup of filtration preserving automorphisms of $\mathscr{L}$, i.e. $\Phi_{\mathscr{F}^1\mathscr{L}*}^S \le \mathrm{Aut}_{\mathscr{L}}\mathscr{L} \le \mathrm{Aut}_{\mathbb{F}}\mathscr{L}$. Note that Sanders [2003] (Proposition 1) provided a straightforward proof for the formula $\Phi_{Y^S*}^S[v, w] = [\Phi_{Y^S*}^S(v), \Phi_{Y^S*}^S(w)]$ $\forall\, v, w \in \mathscr{L}$ and $Y^S \in \mathscr{F}^1\mathscr{L}$, see also [Chua & Kokubu, 1988; Peng & Wang, 2004].

Once grading structures, approach and the style are fixed, there is a many-to-one map from the set of all systems to their simplest normal form. This map is onto because each simplest normal form is a system and maps to itself. So, if we consider a subset of the space of all systems, it may not map onto the set of all simplest normal forms. In other words, a condition or a set of conditions (degenerate conditions) imposed on a system may single out a subset of the systems such that the vector space span of their simplest normal forms (called infinite level degenerate space) would not yet be the space of all simplest normal forms [Murdock, 2003]. This can be applied to the rth level parametric normal form as long as our sequence of the rth level degenerate space establishes a descending sequence of vector spaces. Therefore, degenerate conditions of $v^{(0)}$ may further restrict the complement space $\mathscr{N}_k^{(N)}$ into a subspace $\mathfrak{D}_k^{(N)}$ where $\mathfrak{D}_k^{(N)} \subseteq \mathscr{N}_k^{(N)}$. Succinctly stated, we call $\mathfrak{D}_k^{(N)}$ the Nth level degenerate space at grade k under some degenerate conditions (i.e. it satisfies a certain set of conditions) when $\mathfrak{D}_k^{(N)}$ is the vector space span of all homogenous vectors of grade k which can appear in the Nth level parametric normal forms of any system satisfying those conditions.

The order of a formal system of nonlinear center becomes explicitly apparent when it is transformed into its first level normal form. Example 3.6 demonstrates our claim where degenerate nonlinear center of order N_0 is treated, showing that its normal form is required for computing its order (N_0) via Lemma 3.4. In this paper, we focus on the degenerate spaces representing the set of all parametric degenerate nonlinear centers of order N_0, see Eq. (26) and [Chow *et al.*, 1994, pp. 384–385]. The first (and higher) level normal form of a system associated with order N_0 of degenerate nonlinear center (generalized Hopf singularity) without parameter (in polar coordinates) does not have an amplitude term of grade *less* than $2N_0 + 1$ (i.e. ρ^i for $i < 2N_0 + 1$). This is because of its order. The first level complement spaces of such systems, however, have amplitude terms of all odd orders. So, the first level degenerate spaces of nonparametric normal form associated with grades less than $2N_0 + 1$ do not have amplitude terms in the classical normal form theory, i.e. without using time rescaling.

The common idea of normal form theory is to use kernel terms of lower level state transformations to define higher level state transformations. We, however, use some terms beyond the kernel terms under which the degenerate subspaces are invariant; i.e. $\mathrm{ad}_{\mathcal{L}_n^i}(\mathfrak{D}_k^{(m+n)}) \subseteq \mathfrak{D}_{k+n}^{(m+n)}$ and as we will see later in Eqs. (19) and (15) for time and parameter. As a consequence, the condition $v_k^{(N)} = v_k^{(k)}$ $\forall\, N \ge k$ is not generally true and thus, in order to derive the necessary formulas for implementing them on a computer, we need a *chess-like* computation. This means that we have to calculate all possible changes of $v_k^{(n)}$ for up to a sufficiently high level (for each grade k we denote m_k, i.e. $m_1 \le m_2 \le m_3 \cdots$, as one of such sufficiently high level and of course we do not make any attempt to exactly compute the smallest of such numbers) and then determine the state, time and parametric solutions such that $v_k^{(m_k)}$ is as simple as possible. Therefore, we have $v_k^{(m_k)} = v_k^{(\infty)}$ throughout this paper. Note that

$$v_k^{(m_k)} = v_k^{(\infty)} \;\Rightarrow\; \mathfrak{D}_k^{(m_k)} = \mathfrak{D}_k^{(n)} \quad \text{for any } n \ge m_k, \tag{4}$$

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but the converse is not true. Indeed, we have $m_k = k$ for any $k \neq 2N_0$ and $m_{2N_0} = 4N_0$ in Sec. 3. However, $\mathfrak{D}_k^{(m_k)} = \mathfrak{D}_k^{(k)}$ for any $k \in \mathbb{N}$.

A new idea used here is that the common Nth level state transformation is modified to be an injective linear transformation. We do this job via defining a formal decomposition of the parametric state space. Indeed, a space decomposition

$$\mathscr{L}_k = \mathcal{L}_k^U \oplus \bigoplus_{i=0}^{n_k} \mathcal{L}_k^{i+k_0} \quad \text{for some } n_k \in \mathbb{N}_0, \tag{5}$$

where

$$\pi_{\mathfrak{D}_{n+k}^{(n+k)}}[\mathcal{L}_k^U, \mathfrak{D}_n^{(n)}] = \{\mathbf{0}\} \quad \text{for any } k \in \mathbb{N},\ n \in \mathbb{N}_0,$$

and

$$[\mathcal{L}_n^i, \mathfrak{D}_k^{(k)}] \subseteq \mathfrak{D}_{k+n}^{(k+n)} \quad \text{for any } k < i \tag{6}$$

leads to a formal decomposition for $\mathscr{L}$. $\mathcal{L}_k^U$ stands for the parametric state terms not used in the process of normal form while $\mathcal{L}_n^i$ denotes the homogenous parametric state terms of grade n, applied to simplify the homogenous terms of grade $n+i$ of the system. Define the nth level linear state transformation $L_{(n)}^{S,N} = L_{(n)}^{S,N}(v_{k_0}^{(N-1)}, v_{k_0+1}^{(N-1)}, \ldots, v_{k_0+n-1}^{(N-1)}) : \mathcal{L}_{N-k_0-n+1}^{k_0+n-1} \oplus \cdots \oplus \mathcal{L}_{N-k_0-1}^{k_0+1} \oplus \mathcal{L}_{N-k_0}^{k_0} \to \mathscr{L}_N$ by

$$L_{(n)}^{S,N}(Y_{N-k_0-n+1}^{k_0+n-1}, \ldots, Y_{N-k_0-1}^{k_0}, Y_{N-k_0}^{k_0}) = \sum_{i=0}^{n-1} \mathrm{ad}_{Y_{N-k_0-i}^{k_0+i}} v_{k_0+i}^{(N-1)} \quad \text{when } n \leq N - k_0, \tag{7}$$

and $L_{(n)}^{S,N} = L_{(n-1)}^{S,N}$ for $n > N - k_0$. Our approach is to choose the state decomposition (5) such that $L_{(n)}^{S,N}$ is an injective linear transformation. Then, $L_{(n)}^{S,N}$ is called an nth-level state transformation at grade N. One can simply infer from our definition, Lemma 3.6 in [Kokubu *et al.*, 1996] and Lemma 2.10 in [Gazor & Yu, 2008] that for a parametric state decomposition, associated with Eq. (5), we have $\mathrm{Im}(L_{(n)}^{S,m}) \subseteq \mathrm{Im}(L_{(n+1)}^{S,m})$ for any $m, n \in \mathbb{N}$.

The following example (with extra conditions on degenerate spaces) illustrates the idea of how we practically choose the state decomposition (5) to make $L_{(n)}^{S,N}$ an injective linear transformation in Sec. 3.

Example 2.1. Recall the state transformation $\hat{L}_{(n)}^{S,N} : \ker(\hat{L}_{(n-1)}^{S,N-1}) \times (\mathscr{L}_{N-k_0}) \to \mathscr{L}_N$ defined inductively by

$$\hat{L}_{(n)}^{S,N}(Y_{N-k_0-n+1}, \ldots, Y_{N-k_0-1}, Y_{N-k_0}) = \sum_{i=0}^{n-1} \mathrm{ad}_{Y_{N-k_0-i}} v_{k_0+i}^{(N-1)} \quad \text{for } n \leq N - k_0, \tag{8}$$

see e.g. [Gazor & Yu, 2008]. Following the formal basis costyle we denote $\mathcal{L}_{N-k_0}^{k_0}$ for the unique complement space associated with $\ker(\hat{L}_{(1)}^{S,N})$, that is,

$$\mathcal{L}_{N-k_0}^{k_0} \oplus \ker(\hat{L}_{(1)}^{S,N}) = \mathscr{L}_{N-k_0}.$$

Then,

$$\mathcal{L}_{N-k_0}^{k_0} + \ker(\hat{L}_{(1)}^{S,N}) = \frac{\mathscr{L}_{N-k_0}}{\ker(\hat{L}_{(1)}^{S,N})} \quad \text{and}$$

$$\begin{aligned}\mathrm{Im}\, \hat{L}_{(1)}^{S,N} &= [\mathcal{L}_{N-k_0}^{k_0}, v_{k_0}^{(k_0)}] \\ &= [\mathscr{L}_{N-k_0}, v_{k_0}^{(k_0)}] \\ &= \mathrm{Im}\, L_{(1)}^{S,N},\end{aligned}$$

where $L_{(1)}^{S,N}$ follows Eq. (7). Note that V/W stands for the quotient vector space associated with all the cosets $v + W$, $v \in V$, where W is a subspace of V. Now assume $v_m^{(n)} = v_m^{(m)}$ for any natural numbers m and n, $n \geq m$. (This is only assumed in this example and is not generally satisfied elsewhere in this paper.) Thus, we have

$$\mathrm{span}_{\mathbb{F}}\{v_m^{(m)}\} \subseteq \mathfrak{D}_m^{(n)} \quad \text{for } n \geq m.$$

Note that $\mathfrak{D}_m^{(m)}$ is the vector space span of all $v_m^{(m)}$ which satisfy a certain degeneracy condition. Further assume $\mathrm{Im}\,\mathrm{ad}_{v_m^{(m)}} \mathcal{L}_q^p = [\mathfrak{D}_m^{(m)}, \mathcal{L}_q^p]$ for all p, q. Then, there similarly exists a subspace $\mathcal{L}_{N-k_0-1}^{k_0+1}$ in $\ker(\hat{L}_{(1)}^{S,N-1})$ satisfying

$$[v_{k_0+1}^{(k_0+1)}, \mathcal{L}_{N-k_0-1}^{k_0+1}] = [v_{k_0+1}^{(k_0+1)}, \ker(\hat{L}_{(1)}^{S,N-1})]$$

such that $L_{(2)}^{S,N}$ is injective, i.e.

$$\mathcal{L}_{N-k_0-1}^{k_0+1} \oplus \pi_1 \ker(\hat{L}_{(2)}^{S,N}) = \ker(\hat{L}_{(1)}^{S,N-1}),$$

where π_1 denotes the projection on the first component. Hence, $\mathrm{Im}(\hat{L}_{(2)}^{S,N}) = \mathrm{Im}(L_{(2)}^{S,N})$, and

$$\begin{aligned}&[\mathfrak{D}_{k_0}^{(k_0)}, \mathcal{L}_{N-k_0-1}^{k_0+1}] \\ &\quad = \mathrm{ad}_{v_{k_0}^{(k_0)}}(\mathcal{L}_{N-k_0-1}^{k_0+1}) = \{\mathbf{0}\} \\ &\quad \subseteq \mathrm{span}_{\mathbb{F}}\{v_{N-1}^{(N-1)}\} \subseteq \mathfrak{D}_{N-1}^{(N-1)} \subseteq \mathscr{N}_{N-1}^{(N-1)}.\end{aligned}$$

Thereby, the spaces $\mathcal{L}^{k_0+i}_{N-k_0-i} \subseteq \pi_1(\ker \hat{L}^{S,N}_{(i)})\ \forall\, i \geq 1$ can be inductively chosen to satisfy

$$\mathcal{L}^{k_0+i}_{N-k_0-i} \oplus \pi_1(\ker \hat{L}^{S,N}_{(i+1)}) = \pi_1(\ker \hat{L}^{S,N-1}_{(i)}).$$

Therefore, $[\mathcal{L}^{k_0+n}_{N-k_0-n}, \mathfrak{D}^{(k_0+m)}_{k_0+j}] = \{\mathbf{0}\}$ for any $j < n$, $m \geq j$,

$$\operatorname{Im}(\hat{L}^{S,N}_{(n)}) = \operatorname{Im}(L^{S,N}_{(n)}) \quad \text{and} \quad \ker(L^{S,N}_{(n)}) = \{\mathbf{0}\}$$
$$\text{for all natural numbers } n \text{ and } N.$$

This implies that the injective state transformation $L^{S,N}_{(n)}$ does the job in which $\hat{L}^{S,N}_{(n)}$ is expected to do ($\hat{L}^{S,N}_{(n)}$ in a sense facilitates the elimination of terms crossing its image) and thus, it is a good transformation adopted from $\hat{L}^{S,N}_{(n)}$.

We may use the notations

$$\mathscr{L}_{S,H} = \hat{\bigoplus} \operatorname{span}_{\mathbb{F}}\{X_{(i+1)i}, Y_{(i+1)i}\},$$

$$\mathscr{L}_{S,H^c} = \hat{\bigoplus_{i \neq j+1}} \operatorname{span}_{\mathbb{F}}\{X_{ij}, Y_{ij}\} \quad \text{and}$$

$$\mathscr{L}_{H^c} = \mathscr{L}_{S,H^c}[[\mu]].$$

Thus,

$$\mathscr{L} = \mathscr{L}_H \oplus \mathscr{L}_{H^c} = \hat{\bigoplus_{k=0}^{\infty}} \mathscr{L}_{H,k} \oplus \hat{\bigoplus_{k=0}^{\infty}} \mathscr{L}_{H^c,k}.$$

Furthermore, for any $X_{ij} \in \mathscr{B}_S$, we denote $\mathscr{L}_{X_{ij}} = \hat{\bigoplus}_{r=0}^{\infty} \operatorname{span}_{\mathbb{F}}\{X_{ij}\mu^{\mathbf{m}_r} \mid \mathbf{m}_r \in \mathbb{N}_0^p\}$ and $\mathscr{L}_X = \hat{\bigoplus}_{k=0}^{\infty} \operatorname{span}_{\mathbb{F}}\{v = X_{ij}\mu^{\mathbf{m}_r} \mid \mathbf{m}_r \in \mathbb{N}_0^p, \delta(v) = k\}$. Similarly the notations $\mathscr{L}_{Y_{ij}}$ and $\mathscr{L}_Y$ are also used. The reader should note that $\mathscr{L}_{S,H}$ denotes the well-known first level (classical) normal form space of vector fields without parameters associated with Hopf singularity. This is why $\mathscr{L}_H$ holds some properties similar to $\mathscr{L}_{S,H}$. Thus, the extension of Proposition 2.6 in [Gazor & Yu, 2008] into multiple parameters is valid here.

Note that Peng and Wang [2004] introduced a tricky grading function on $\mathscr{L}_{S,H}$ (in our notations) to manage the computation of normal form. Their grading function can be expanded for our parametric normal form. This, however, should be implemented on $\mathscr{L}_H$ rather than $\mathscr{L}$. In other words, we have to carry out the computations in two phases. As a consequence, it reduces the efficiency of computation (for systems with parameters).

Remark 2.2. We now explain the analog of the concepts between the near-identity maps of time and parameter and that of the state. This also justifies the notation of push-forward $\phi^T_{Y^T*}$ and $\phi^S_{Y^S_*}$. Let $v(x)$ be a smooth vector field on $M = \mathbb{R}^2$ (or $= \mathbb{C}^2$), $t = (1 + Y^T(x))\tau$ a smooth near-identity time rescaling, and Φ_v the flow generated by $\dot{x} = v$.

For any diffeomorphism $\phi^T_{Y^T} : \mathbb{R} \times M \to \mathbb{R} \times M$ defined by $\phi^T_{Y^T}(t, x) = (t + tY^T(x), x)$, we associate a map $\phi^T_{Y^T_*}$ which sends the vector field $v(t, x) = v(x)$ to the smooth vector field $\phi^T_{Y^T_*}(v)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R} \times M & \xrightarrow{\phi^T_{Y^T}} & \mathbb{R} \times M \\ \Phi_{v(x)} \downarrow & & \downarrow \Phi_{\phi^T_{Y^T*}(v)} \\ M & \xrightarrow[id_M]{} & M. \end{array} \tag{9}$$

Obviously, $\phi^T_{Y^T_*}(v) = (1 + Y^T)v$ and $\Phi_{\phi^T_{Y^T_*}(v)}$ is the flow generated by $\dot{x} = (1 + Y^T)v$.

Now we unify the approach here with what is well-known and common for the state maps and then, express our notion for parametric maps. First recall that the time one mapping $\Phi^S_{Y^S}$ generated from the system $\dot{x} = Y^S$ (Y^S being a smooth vector field without any constant or linear terms) can be considered as a diffeomorphism on M, i.e. $\Phi^S_{Y^S}(x) = \Phi_{Y^S}(t = 1, x)\ \forall\, x \in M$. Then, $\Phi^S_{Y^S_*}(v)$ satisfies $\Phi_v(t, x) = \Phi_{\Phi^S_{Y^S_*}(v)}(t, \Phi^S_{Y^S}(x))$ for any $t \in \mathbb{R}$. However, we now define a new map $\phi^S_{Y^S} : \mathbb{R} \times M \to \mathbb{R} \times M$ given by $\phi^S_{Y^S}(t, x) = (t, \Phi^S_{Y^S}(x))$ as our diffeomorphism (to unify our approach with time map) and thus, the diagram

$$\begin{array}{ccc} \mathbb{R} \times M & \xrightarrow{\phi^S_{Y^S}} & \mathbb{R} \times M \\ \Phi_{v(x)} \downarrow & & \downarrow \Phi_{\phi_{Y^S_*}(v)} \\ M & \xrightarrow[id_M]{} & M \end{array} \tag{10}$$

is commutative for any $v(x) = v(t, x)$. It is easy to see that $\Phi^S_{Y^S_*}(v) = \phi^S_{Y^S_*}(v)$. Thus, $\phi^S_{Y^S_*}(v)$ follows Eq. (3) and we use it throughout this paper instead of $\Phi^S_{Y^S_*}(v)$.

In all the above we could also consider v, Y^T and Y^S as a formal parametric vector field $v = v(x, \mu)$, associated with a parametric time rescaling and a parametric state change of state variable, respectively. So, for the parameter maps we assume M is a manifold of parametric state space, that is, $M = \mathbb{R}^p \times \mathbb{R}^2$. Thus, for any smooth vector field $Y^P(\mu)$ with no constant or

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linear terms, $\Phi^P_{Y^P}: M \to M$, given by $\Phi^P_{Y^P}(x,\mu) = (x, \mu + Y^P(\mu))$, is a diffeomorphism. Thus, we similarly consider $\phi^P_{Y^P}(t,(x,\mu)) = (t, \Phi^P_{Y^P}(x,\mu))$ and define $\phi^P_{Y^P *}(v)$ such that

$$\begin{array}{ccc} \mathbb{R}\times M & \xrightarrow{\phi^P_{Y^P}} & \mathbb{R}\times M \\ \Phi_{v(x,\mu)} \downarrow & & \downarrow \Phi_{\phi^P_{Y^P*}(v)} \\ M & \xrightarrow[id_M]{} & M \end{array} \tag{11}$$

is a commutative diagram. One should note that we could instead (but we do not) define $\Phi^P_{Y^P}$ as the time one mapping generated from the system $\dot{\mu} = Y^P(\mu)$. This is a different near-identity parameter map from what we just defined above. We, however, believe that this does not help us in any way on computation of parametric normal forms and may even be sometimes misleading for the reader. This is why we do not follow the later definition of parameter maps.

In order to present the parameter space, let $\mathscr{P}^1 = \mathbb{F}[[\mu]]$ be the integral domain of formal power series in terms of $\mu = (\mu_1, \mu_2, \ldots, \mu_p)^T$. Then, the parameter space is defined and denoted by

$$\mathscr{P}^p = \mathbb{F}^p[[\mu]] = \{v \mid v = (v_1, v_2, \ldots, v_m)^T \\ \text{where } v_j \in \mathscr{P}^1 \ \forall\, j, 1 \le j \le p\}.$$

Let $\delta_{\mathscr{P}^p}(\mu^{\mathbf{n}}) = \sum_{i=1}^p n_i\alpha$ for any $\mathbf{n} = (n_1, n_2, \ldots, n_p)^T \in \mathbb{F}^p$ and fix an order on $\mathscr{B}_{\mathscr{P}^p} = \{\mu^{\mathbf{n}}\mathbf{e}_k \mid \mathbf{n} \in \mathbb{N}_0^p, k \le p\}_{r=0}^\infty$ such that the lower grade terms (associated with $\delta_{\mathscr{P}^p}$) are ordered in a sequence before higher grade terms. Then, $\mathscr{B}_{\mathscr{P}^p}$ is a formal basis for $\mathscr{P}^p$ and $\mathscr{P}^p = \hat{\bigoplus}_{n=0}^\infty \mathscr{P}^p_n$ where $\mathscr{P}^p_0 = \mathbb{F}^p$. Thus, any element from the space $\mu + \hat{\bigoplus}_{n=2}^\infty \mathscr{P}^p_n$ is uniquely represented from (indeed, we only work with) the vector space $\hat{\bigoplus}_{n=2}^\infty \mathscr{P}^p_n$ and thus we denote $\mathscr{P}^p_I = \hat{\bigoplus}_{n=2}^\infty \mathscr{P}^p_n$. Recall that reparametrization associated with $\mu = \nu + Y^P$ sends $v(\mu)$ to

$$\phi^P_{Y^P *}(v(\mu)) \\ = \sum_{k=0}^\infty \frac{1}{k!} D^k_\mu(v, Y^P) \quad (Y^P \in \mathscr{P}^p, v(\mu) \in \mathscr{L}), \tag{12}$$

where $D^k_\mu(v, Y^P)$ denotes the kth-order formal Frechet derivative of $v(\mu)$ with respect to

$$\mu = (\mu_1, \ldots, \mu_p)^T \quad \text{at } (\overbrace{Y^P, Y^P, \ldots, Y^P}^{k\,\text{times}})^T,$$

see also [Kuznetsov, 2004; Liao *et al.*, 2007; Gazor & Yu, 2008].

Parametric normal forms require reparametrization and parametric time rescaling alongside with parametric change of state variable. Since the state transformation is linear, the kernel terms can be used in the computation of the simplest normal forms via defining higher level state transformations. However, parametric transformations associated with multiple parameters can be nonlinear, see e.g. Remark 3.4 in [Gazor & Yu, 2010], and thus the computation of parametric normal forms is much more complicated. Indeed, for the case of one parameter, $D_\mu(v_k(\mu), Y^P) = \mathbf{0}$ (for $Y^P \in \mathscr{P}^p$ and $v_k(\mu) \in \mathscr{L}_k$) implies $D^n_\mu(v_k(\mu), Y^P) = \mathbf{0}$ for any $n \ge 1$ and thus the parameter transformation is linear. Therefore, we assume that the degenerate spaces for any $Y^P \in \mathscr{P}^p_I$ satisfy the condition:

$$D^n_\mu\left(\sum_{i=1}^k \mathfrak{D}^{(i)}_i, Y^P\right) = \mathbf{0} \\ \Rightarrow \quad D^n_\mu\left(\sum_{i=1}^k \mathfrak{D}^{(i)}_i, Y^P\right) = \mathbf{0} \quad \text{for all } k, \ \ n \in \mathbb{N}. \tag{13}$$

Equation (13) can be satisfied by adding some unfolding into the system wherever they are needed, see [Murdock, 2008; Murdock & Malonza, 2009]. However adding unfolding to parametric systems is not the purpose of this paper. Therefore, this task is instead fulfilled by a smart choice of grading function (i.e. a value for α) and assuming a certain condition with respect to parameters. Then, this condition is satisfied.

Assume there exists a parametric space decomposition:

$$\mathscr{P}^p_I = \mu + \hat{\bigoplus_{k=2}^{\infty}} \mathscr{P}^p_{k\alpha}, \\ \mathscr{P}^p_{k\alpha} = \mathcal{P}^U_{k\alpha} \oplus \bigoplus_{i=0}^{l_k} \mathcal{P}^{i+k_0}_{k\alpha} \quad \text{for some } l_k \in \mathbb{N}_0, \tag{14}$$

such that

$$D_\mu(\mathfrak{D}^{(n)}_n, \mathcal{P}^i_{k\alpha}) \subseteq \mathfrak{D}^{(k\alpha+n-\alpha)}_{k\alpha+n-\alpha} \ \ \forall\, n < i \quad \text{and} \\ \pi_{\mathfrak{D}^{(k\alpha+n)}_{k\alpha+n}} \circ D_\mu(\mathfrak{D}^{(n)}_n, \mathcal{P}^U_{(k+1)\alpha}) = \{\mathbf{0}\}, \tag{15}$$

$\forall\, k \in \mathbb{N}, n \in \mathbb{N}_0$. Here, $\mathcal{P}^i_n$ denotes the space of parameter terms of grade n intended to simplify

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system terms of grade $n+i$, while $\mathcal{P}_n^U$ denotes the space of not usable parameter terms. Then for any natural number k ($k \le ((N-k_0)/\alpha)$) define

$$L_{(k)}^{P,N} : \mathcal{P}_{[\frac{N-k_0}{\alpha}]\alpha-(k-2)\alpha}^{N-[\frac{N-k_0}{\alpha}]\alpha+(k-1)\alpha} \oplus \cdots \oplus \mathcal{P}_{[\frac{N-k_0}{\alpha}]\alpha}^{N-[\frac{N-k_0}{\alpha}]\alpha+\alpha} \oplus \mathcal{P}_{[\frac{N-k_0}{\alpha}]\alpha+\alpha}^{N-[\frac{N-k_0}{\alpha}]\alpha} \to \mathscr{L}_N,$$

by

$$L_{(k)}^{P,N}\left(\mathbf{Y}^P_{[\frac{N-k_0}{\alpha}-k+2]\alpha}, \ldots, \mathbf{Y}^P_{[\frac{N-k_0}{\alpha}]\alpha}, \mathbf{Y}^P_{[\frac{N-k_0}{\alpha}]\alpha+\alpha}\right) = \sum_{i=0}^{k-1} D_\mu\left(v^{(N-1)}_{N-[\frac{N-k_0}{\alpha}-i]\alpha}, \mathbf{Y}^P_{[\frac{N-k_0}{\alpha}+1-i]\alpha}\right), \quad (16)$$

and for $k > (N-k_0)/\alpha$ let $L_{(k)}^{P,N} = L_{(k-1)}^{P,N}$. $L_{(k)}^{P,N}$ is defined as a zero transformation for all $N < k_0 + \alpha$. We choose the parametric space decomposition (14) such that $L_{(k)}^{P,N}$ is an injective linear transformation and call it the kth level injective parameter transformation at grade N.

We now construct the parametric time space $\mathscr{R}$ similar to the parametric state space $\mathscr{L}$ as a merely abstract concept (ring) without referring to differential equations or even $\mathscr{L}$. The link between $\mathscr{R}$ and $\mathscr{L}$ is, then, represented via an $\mathscr{R}$-module operation on $\mathscr{L}$ which indeed can carry out the necessary changes made by a time rescaling into the vector field representing the parametric system, see [Gazor & Yu, 2010] for details. Let $\mathscr{B}_{\mathscr{R}_S} = \{Z_i \mid i \in \mathbb{N}_0\}$ be an infinite sequence of distinct algebraic objects (distinct from the elements of $\mathscr{L}$) and $\mathscr{B}_{\mathscr{R}} = \{Z_i\mu^{\mathbf{m}} \mid i \in \mathbb{N}_0, \mathbf{m} \in \mathbb{N}_0^p\}$. Note that Z_i stands for $z^i\bar{z}^i$ where z is a complex state variable, see [Gazor & Yu, 2010]. Define a grading function $\delta_{\mathscr{R}} : \mathscr{B}_{\mathscr{R}} \to \mathbb{Z}$ by

$$\delta_{\mathscr{R}}(Z_i\mu^{\mathbf{m}}) = 2i + |\mathbf{m}|\alpha, \quad i \in \mathbb{N}_0, \ \mathbf{m} \in \mathbb{N}_0^p.$$

We fix an order in a sequence on $\mathscr{B}_{\mathscr{R}}$ such that the terms of lower grades are in lower orders, and for the terms with the same grade, the terms without parameters are before the terms with parameters. Then, define the parametric time space $\mathscr{R} = \hat{\bigoplus}_{k=0}^{\infty}\mathbb{F}e_k$ (where $\mathscr{B}_{\mathscr{R}} = \{e_k\}$) and time space without parameters $\mathscr{R}_S = \hat{\bigoplus}_{i=0}^{\infty}\mathbb{F}Z_i$. $\mathscr{R}$ constitutes an integral domain structure by the formula

$$z_1z_2 = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{1(n-k)}a_{2k}e_n \quad \text{where} \quad z_i = \sum a_{ik}e_k, \ i=1,2, \ \text{and} \ \mathscr{B}_{\mathscr{R}} = \{e_k\}. \quad (17)$$

Furthermore, $\mathscr{R}$ is a graded ring, that is, $\mathscr{R} = \hat{\bigoplus}_{k=0}^{\infty}\mathscr{R}_k$ and $\mathscr{R}_{N_1}\mathscr{R}_{N_2} \subseteq \mathscr{R}_{N_1+N_2}$. We naturally define an $\mathscr{R}$-module product via the formulas $Z_i\mu^{\mathbf{m}_1}X_{mn}\mu^{\mathbf{m}_2} = X_{(i+m)(i+n)}\mu^{\mathbf{m}_1+\mathbf{m}_2}$ and $Z_i\mu^{\mathbf{m}_1}Y_{mn}\mu^{\mathbf{m}_2} = Y_{(i+m)(i+n)}\mu^{\mathbf{m}_1+\mathbf{m}_2}$. Then, $\mathscr{L}$ is a torsion-free graded $\mathscr{R}$-module of type $\mathbb{Z}$ over the graded ring $\mathscr{R}$, i.e. $\mathscr{R}_{N_1}\mathscr{L}_{N_2} \subseteq \mathscr{L}_{N_1+N_2}$, see also [Gazor & Yu, 2008, 2010]. Let $\mathscr{R}_I = \hat{\bigoplus}_{k=1}^{\infty}\mathscr{R}_k$. By Eq. (9), we have

$$\phi^T_{Y^T*}(v) = v + Y^Tv \quad \text{for any } Y^T \in \mathscr{R}_I \text{ and } v \in \mathscr{L}.$$

In this paper, the parametric time solution $Y^{T,N}$ is designed for eliminating system terms of grade N, while parametric time terms of different grades are used for this purpose (simplifying system terms of grade N). Thus, we consider a fixed parametric time space decomposition

$$\mathscr{R}_I = \hat{\bigoplus_{i=1}^{\infty}}\mathscr{R}_i \quad \text{and} \quad \mathscr{R}_i = \mathcal{R}_i^U \bigoplus_{j=0}^{m_i} \mathcal{R}_i^{k_0+j} \quad (\text{for some } m_i \in \mathbb{N}_0). \quad (18)$$

Note that we denote $\mathcal{R}_i^n$ for parametric time subspace of grade i designed for eliminating system terms of grade $i+n$ while $\mathcal{R}_i^U$ stands for the parametric time subspace which is not usable in our normal form computation. We assume $\mathcal{R}_i^n$ and $\mathcal{R}_i^U$ follow the conditions

$$\mathcal{R}_i^n\mathfrak{D}_k^{(k)} \subseteq \mathfrak{D}_{i+k}^{(i+k)} \quad \text{for } k < n, \quad \text{and}$$
$$\pi_{\mathcal{R}_i^U\mathfrak{D}_k^{(k)}}(\mathfrak{D}_{i+k}^{(i+k)}) = \{\mathbf{0}\} \quad \text{for any } i \in \mathbb{N}, \quad \text{and} \quad k \in \mathbb{N}_0. \quad (19)$$

Now, we define the nth level injective linear time transformation $L_{(n)}^{T,N}: \mathcal{R}_{N-k_0}^{k_0} \oplus \mathcal{R}_{N-k_0-1}^{k_0+1} \cdots \oplus \mathcal{R}_{N-k_0-n+1}^{k_0+n-1} \to \mathscr{L}_N$ by

$$L_{(n)}^{T,N}(Y^{T,k_0}_{N-k_0}, Y^{T,k_0+1}_{N-k_0-1}, \ldots, Y^{T,k_0+n-1}_{N-k_0-n+1}) = \sum_{i=0}^{n-1} Y^{T,k_0+i}_{N-k_0-i}v^{(N-1)}_{k_0+i} \quad \text{for } n \le N-k_0, \quad (20)$$

and $L_{(n)}^{T,N} = L_{(n-1)}^{T,N}$ for $n > N-k_0$, see also [Algaba *et al.*, 2001; Gazor & Yu, 2008; Yu & Leung, 2003; Yu & Chen, 2007]. For any parametric time space decomposition (18), $L_{(n)}^{T,N}$ is an injective linear transformation and $\mathrm{Im}(L_{(n)}^{T,N}) \subseteq \mathrm{Im}(L_{(n+1)}^{T,N})$ $\forall n, N \in \mathbb{N}$. This, in a sense, implies that higher level normal forms are

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simpler than lower level normal forms. Furthermore, $\mathrm{Im}(L_{(n)}^{T,N}) = \mathrm{Im}(L_{(n)}^{T,n})\ \forall N \geq n$ and $\mathrm{Im}(L_{(n)}^{T,n}) = \sum_{i=0}^{[\frac{n-k_0}{2}]} \mathcal{R}_{n-k_0-i}^{k_0+i} v_{k_0+i}^{(n-1)}$.

Now we are equipped with all the structures that we need to establish our method. Let us recall the fact that parametric normal form theory may require a chess-like computation and therefore, we anticipate all the future changes in high enough levels (but yet finite) and based on this fact we determine our transforation solutions. This, however, contradicts with the fact that the finite level normal forms, and also that transformation solutions are not unique in general. Furthermore, computing transformation solutions are fundamental in parametric normal form theory. Thus, we need to establish a way to compute transformation solutions uniquely. This signifies the recently introduced notion of *costyle* by Murdock and Malonza [2009] which determines a rule on how to uniquely determine transformation solutions. Then, finite level normal forms are also unique although they are yet far from being the *simplest* normal form. In fact, we also used formal basis costyle (and style) in our paper [Gazor & Yu, 2008] without notice. The importance of costyle is indeed manifested in parametric normal form theory. Therefore, we turn to defining formal basis style and costyle here. We call our style (costyle) *formal basis style* (*costyle*) throughout this paper. In order to determine a unique complement space $\mathscr{N}$ for $W \subseteq V$, we instead choose the basis of $\mathscr{N}$. First choose an element e_{n_1} from $\mathscr{B}$ with the least index which is not an element of W. Then, inductively choose elements from $\mathscr{B}$ with the least index n_k where e_{n_k} is not in $W + \mathrm{span}_{\mathbb{F}}\{e_{n_i}\,|\,i < k\}$. Therefore, $\mathscr{N} = \mathrm{span}_{\mathbb{F}}\{e_{n_k}\,|\,k \leq N\}$ if it terminates at step N; and otherwise $\mathscr{N} = \mathrm{span}_{\mathbb{F}}\{e_{n_k}\,|\,k \in \mathbb{N}\}$. This procedure leads to the following proposition which presents the formal basis style and costyle.

Proposition 2.3. *Let $L : V \to \mathcal{V}$ be a linear transformation on vector spaces V and $\mathcal{V}$.*

(1) *Costyle: Assume V has a finite basis or a finite formal basis, say $\mathscr{B}_V = \{f_i\}$. Then, there exists a unique vector subspace $\mathcal{C}$ satisfying the following:*
   - *For any $v \in V$ there exist unique vectors $w \in V$ and $v_{\mathcal{C}} \in \mathcal{C}$ such that $v = w + v_{\mathcal{C}}$ and $L(w) = 0$, i.e. $V = \mathcal{C} \oplus \ker L$.*
   - *$\mathscr{B}_V \cap \mathcal{C} = \{f_{i_j}\}$ is either a finite ordered basis or a formal basis for $\mathcal{C}$.*
   - *For any $f_i \in \mathscr{B}$ there exist a unique vector $w \in V$ and unique scalars $a_{i_1}, a_{i_2}, \ldots, a_{i_N}$ $(n_N \leq i)$ such that $f_i = w + \sum_{j=1}^{N} a_{i_j} f_{i_j}$ and $L(w) = 0$.*

(2) *Style: If $\mathcal{V}$ has a formal basis $\mathscr{B} = \{e_n\}_{n=1}^{\infty}$ or a finite ordered basis $\mathscr{B} = \{e_n\}_{n=1}^{\dim_{\mathbb{F}} \mathcal{V}}$, then, there exists a unique vector subspace $\mathscr{N} \subseteq \mathcal{V}$ such that*
   - *$\mathcal{V} = L(V) \oplus \mathscr{N}$ and $\mathscr{B} \cap \mathscr{N} = \{e_{n_k}\}$ is a finite or a formal basis for $\mathscr{N}$.*
   - *For any $e_n \in \mathscr{B}$ there exist a unique vector $\hat{w} = L(w)$ (for some $w \in V$) and unique scalars $b_{n_1}, b_{n_2}, \ldots, b_{n_M} (n_M \leq n)$ satisfying $e_n = \hat{w} + \sum_{k=1}^{M} b_{n_k} e_{n_k}$.*

In particular, when $V \subseteq \mathcal{V}$ and L is the inclusion map, $\mathscr{N}$ represents a unique complement space for V in $\mathcal{V}$.

Now we describe the approach applied in the next section, which is essentially different from that of generic Hopf singularity presented in [Gazor & Yu, 2008]. By mathematical induction hypothesis, assume that the $(n-1)$th level parametric normal form $v^{(n-1)}$ is obtained from a given formal vector field $v = v^{(0)} \in \mathscr{L}$. Now let

$$\mathscr{L}_n = \left(\mathrm{Im}(L_{(n)}^{S,n}) + \mathrm{Im}(L_{(n)}^{P,n}) + \mathrm{Im}(L_{(n)}^{T,n})\right) \oplus \mathscr{N}_n^{(n)}, \tag{21}$$

where $\mathscr{N}_n^{(n)}$ is the unique complement space obtained from formal bases style, see Proposition 2.3. (Note that we can similarly define $\mathscr{N}_n^{(m)}$ via $L_{(n)}^{S,m}, L_{(n)}^{P,m}$ and $L_{(n)}^{T,m}$ for any m; thus we always have $\mathscr{N}_n^{(n)} \subseteq \mathscr{N}_n^{(k)} \subseteq \mathscr{N}_n^{(l)}$ for any $k \geq l$.) Then, there exist state solution $Y^{S,n}$, parameter solution $Y^{P,n}$ and time solution $Y^{T,n}$ such that the combination of transformation maps $\Phi_n = \phi^T_{Y^{T,n}*} \circ \phi^P_{Y^{P,n}*} \circ \phi^S_{Y^{S,n}*}$ sends $v^{(n-1)}$ into

$$\begin{aligned} v^{(n)} &= \Phi_n(v^{(n-1)}) \\ &= \sum_{k=k_0}^{\infty} v_k^{(n)}, \quad v_k^{(n)} \in \mathfrak{D}_k^{(n)} \subseteq \mathscr{N}_k^{(n)} \\ &\qquad \text{for any } k \leq n. \end{aligned} \tag{22}$$

Our approach here should not be confused with that given in [Gazor & Yu, 2008]. The latter uses chess-like computations beyond the grades while we, here, may only permit *the same grade* chess-like computation. Indeed, we need to compute all the changes that may appear by applying the state,

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parameter and time solutions associated with a specific grade and then, determine our transformation solutions. Therefore, in Sec. 3 we do not need to compute any chess-like computation at all as we only play with spaces, while in Sec. 4 we need to use the same grade of such computations to derive the necessary formulas for systematic development of computer programs. Thereby, here compared with [Gazor & Yu, 2008], the time solution $Y^{T,n}$ is just aimed at system terms of grade n, while the method used in [Gazor & Yu, 2008] is to eliminate not only system terms of grade n but also grades higher than n. Therefore, the time solutions $Y^{T,n}$ and $v^{(n)}$ in our method are completely determined at this stage while in [Gazor & Yu, 2008] they still depend on some unknowns at step n to be computed at the later steps associated with higher grades than n. These are very essential and one should deal with them properly. It looks imperative to remark that the system terms intended for elimination of grade n are here eliminated in the nth level parametric normal forms while the remaining terms may be changed in the later steps, i.e. higher level parametric normal forms than n. Further, let us restate that one may need to combine all these and apply them to a more complicated singularity.

In order that already removed terms from nth level normal form (nth level degenerate spaces) do not reappear in later steps and also to obtain the simplest parametric normal form, we ensure that the following conditions are satisfied throughout this paper:

(1) Each of the state, parameter and time transformations have a clearly known and specified responsibility in simplifying certain terms of the system in light of the formal basis style, i.e. $\mathscr{L}_n = \mathrm{Im}(L^{S,n}_{(n)}) \oplus \mathrm{Im}(L^{P,n}_{(n)}) \oplus \bigoplus_{i=0}^{[\frac{n-k_0}{2}]} \mathcal{R}^{k_0+i}_{n-k_0-i} v^{(k_0+i)}_{k_0+i} \oplus \mathscr{N}^{(n)}_n$ following Proposition 2.3 (note that $\oplus$ here denotes the direct sum of vector spaces, i.e. $V = V_1 \oplus V_2$ if and only if $V_1 \cap V_2 = 0$ and $V = V_1 + V_2$);
(2) Transformation solutions associated with grade m simplify all the intended system terms for elimination at the mth level, i.e. $\mathfrak{D}^{(k)}_m = \mathfrak{D}^{(m)}_m$, for any $k \geq m$;
(3) The remaining system terms of grade m may yet be changed even in levels that are higher than m. Furthermore, and most importantly, degenerate spaces stay invariant under our computations, i.e. for any $n, k, i\,(k < i)$, $m = n + k$ and $l = n\alpha + k - \alpha$, we have $[\mathcal{L}^i_n, \mathfrak{D}^{(k)}_k] \subseteq \mathfrak{D}^{(m)}_m$, $\mathcal{R}^i_n \mathfrak{D}^{(k)}_k \subseteq \mathfrak{D}^{(m)}_m$, $D_\mu(\mathfrak{D}^{(k)}_k, \mathcal{P}^i_{n\alpha}) \subseteq \mathfrak{D}^{(l)}_l$ and $\mathfrak{D}^{(i)}_n \subseteq \mathfrak{D}^{(k)}_n \subseteq \mathscr{N}^{(k)}_n$;
(4) Unused transformation terms could not be used to simplify any of the remaining system terms, i.e. $\pi_{\mathfrak{D}^{(n+k)}_{n+k}}[\mathcal{L}^U_k, \mathfrak{D}^{(n)}_n] = \{\mathbf{0}\}$, $\pi_{\mathcal{R}^U_k \mathfrak{D}^{(n)}_n}(\mathfrak{D}^{(n+k)}_{n+k}) = \{\mathbf{0}\}$ and $\pi_{\mathfrak{D}^{(k\alpha+n)}_{k\alpha+n}} \circ D_\mu(\mathfrak{D}^{(n)}_n, \mathcal{P}^U_{(k+1)\alpha}) = \{\mathbf{0}\}$, for any non-negative integers n and k. Note that these are, indeed, assumed for obtaining the simplest normal forms.

Note that $\{v^{(n)}\}_{n=0}^\infty \subset \mathscr{L}$ is a convergent sequence to a vector field $v^{(\infty)} \in \mathscr{L}$ with respect to filtration topology. Furthermore, we use a unique costyle and thus the transformation maps are unique and according to Baider [1989], they can be combined together and are indeed convergent with respect to filtration topology imposed on the transformation spaces. $v^{(\infty)}$ is called the infinite level parametric normal form of v. The above arguments give rise to the following theorem.

Theorem 2.4. *Assume $v \in \mathscr{L}$ and there exist degenerate spaces and accordingly the formal decompositions (5), (6), (18), (19) and (14) such that they altogether satisfy the above conditions. Then, there exist the formal near-identity maps $\Phi^S_{YS,\infty}, \Phi^P_{YP,\infty}$ and $\Phi^T_{YT,\infty}$ which transform v into $v^{(\infty)} = \Phi^T_{YT,\infty} \circ \Phi^P_{YP,\infty} \circ \Phi^S_{YS,\infty}(v) \in \mathscr{L}$. Besides, $v^{(\infty)} = \sum_{n=k_0}^\infty v^{(\infty)}_n$, $v^{(\infty)}_n \in \mathscr{L}_k$, implies that $v^{(\infty)}_n \in \mathfrak{D}^{(n)}_n \subseteq \mathscr{N}^{(n)}_n$.*

3. Parametric Normal Form for Degenerate Nonlinear Center

We apply the method described in Sec. 3 to obtain the parametric normal form for nonlinear center of order N_0. Let $v = v^{(0)}$ and

$$v^{(k)} = Y_{10} + \sum_{i+j+r=2, |\mathbf{m}|=r, i+j\geq 1}^{\infty} a^{(k)}_{ij\mathbf{m}} X_{ij} \mu^{\mathbf{m}} + \sum_{i+j+r=2, |\mathbf{m}|=r, i+j\geq 1}^{\infty} b^{(k)}_{ij\mathbf{m}} Y_{ij} \mu^{\mathbf{m}}, \quad (23)$$

where $v^{(k+1)} = \Phi_{k+1}(v^{(k)})\ \forall\, k \in \mathbb{N}_0$, and Φ_{k+1} is a combination of near-identity transformations of parametric changes of state variable, time rescaling and reparametrization, see Eq. (22).

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We say that the nonlinear center v has an order N_0 at the origin ($N_0 \in \mathbb{N}$) when

$$a^{(N)}_{(k+1)k\mathbf{0}} = 0 \quad \text{and} \quad a^{(N)}_{(N_0+1)N_0\mathbf{0}} \neq 0 \quad \forall\, k < N_0,\ \forall\, N \in \mathbb{N}. \tag{24}$$

Lemma 3.4 implies that the condition (24) holds for $N = 1$ if and only if it is valid for all $N \in \mathbb{N}$. When $N_0 = 1$, the system is associated with generic nonlinear center while $N_0 > 1$ associates the system with a degenerate singularity. Now consider the coefficient matrix

$$A^{(k)} = [a^{(k)}_{ij}]_{p\times N_0}, \quad \text{where } a^{(k)}_{ij} = a^{(k-1)}_{j(j-1)\mathbf{e}_i}. \tag{25}$$

When $\mathrm{rank}_{\mathbb{F}}(A^{(N_0)})$ is the same as the system's order (N_0), we say $v^{(0)}$ has parametric dimension N_0. Furthermore, v is said to have full parametric dimension if $p = N_0$. From now on, set $\alpha = 2N_0 + 1$. Note that the nth level (and higher level) degenerate space of grade n throughout this paper is as follows

$$\mathfrak{D}^{(N)}_N = \begin{cases} \mathrm{span}_{\mathbb{F}}\{Y_{10}\} & \text{when } N = 0,\\ \mathrm{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}, Y_{(N_0+1)N_0}\} & \text{when } N = 2N_0,\\ \mathrm{span}_{\mathbb{F}}\{Y_{(N_0+1)N_0}\mu^{\mathbf{m}} \,|\, r = |\mathbf{m}|\} & \text{when } N = 2N_0 + r\alpha,\ \ r \neq 0,\\ \mathrm{span}_{\mathbb{F}}\{X_{i(i-1)}\mu_k\} & \text{when } N = 2i + \alpha - 2,\ \ i \le N_0,\ \ k \le p,\\ \{\mathbf{0}\} & \text{otherwise.} \end{cases} \tag{26}$$

Consider the parametric state space formal decomposition

$$\mathscr{L}_n = \mathcal{L}^U_n \oplus \bigoplus_{i=0}^{2N_0} \mathcal{L}^i_n \quad (\text{for any } n \in \mathbb{N}), \tag{27}$$

where

$$\mathcal{L}^i_n = \{\mathbf{0}\} \quad (\forall\, i, 0 \neq i \neq 2N_0), \quad \mathcal{L}^0_n = \mathscr{L}_{H^c,n}, \tag{28}$$

$$\mathcal{L}^{2N_0}_n = \{v \in \mathscr{L}_{H,n} \,|\, \pi_{\mathscr{L}_{Y_{10}} \oplus \mathscr{L}_{X_{(N_0+1)N_0}}}(v) = \mathbf{0}\}, \tag{29}$$

and

$$\mathcal{L}^U_n = \pi_{\mathscr{L}_n}(\mathscr{L}_{Y_{10}} \oplus \mathscr{L}_{X_{(N_0+1)N_0}}). \tag{30}$$

Now everything is ready for describing injective state transformations.

Lemma 3.1. *Let $N > 2N_0, v^{(0)}_0 = Y_{10}$, $v^{(n)}_k \in \mathfrak{D}^{(k)}_k$ $\forall\, k \le n$, and in particular for any $n \ge 2N_0$,*

$$v^{(n)}_{2N_0} = a^{(n)}_{(N_0+1)N_0\mathbf{0}} X_{(N_0+1)N_0} + b^{(n)}_{(N_0+1)N_0\mathbf{0}} Y_{(N_0+1)N_0} \in \mathfrak{D}^{(2N_0)}_{2N_0}$$

$$\text{where } a^{(n)}_{(N_0+1)N_0\mathbf{0}} \neq 0.$$

That is, $v = v^{(0)}$ is of order N_0. According to Eqs. (27)–(30), $L^{S,N}_{(N)}$ is an injective linear parametric state transformation and $\mathrm{Im}(L^{S,N}_{(N)}) = \mathscr{L}_{H^c,N} \oplus \bigoplus_{r=1}^{[\frac{N}{\alpha}]} \mathbb{S}_{N,r}$ where

$$\mathbb{S}_{N,r} = \begin{cases} \mathrm{span}_{\mathbb{F}}\{X_{(i+1)i}\mu^{\mathbf{m}}, Y_{(i+1)i}\mu^{\mathbf{m}} \,|\, r = |\mathbf{m}|\} & \textit{if } N - r\alpha = 2i,\ \ N_0 < i \neq 2N_0,\\ \mathrm{span}_{\mathbb{F}}\left\{\left(X_{(N_0+1)N_0} + \dfrac{b^{(N-1)}_{(N_0+1)N_0}}{a^{(N-1)}_{(N_0+1)N_0}} Y_{(N_0+1)N_0}\right)\mu^{\mathbf{m}} \,\middle|\, r = |\mathbf{m}|\right\} & \textit{if } N - r\alpha = 2N_0,\ \ r \neq 0,\\ \mathrm{span}_{\mathbb{F}}\{Y_{(2N_0+1)2N_0}\mu^{\mathbf{m}} \,|\, r = |\mathbf{m}|\} & \textit{if } N - r\alpha = 4N_0,\\ \{\mathbf{0}\} & \textit{otherwise.} \end{cases}$$

Furthermore, $[\mathcal{L}^m_k, \mathfrak{D}^{(n)}_n] = \{\mathbf{0}\} \subseteq \mathfrak{D}^{(n+k)}_{n+k}$ $\forall\, n < m$, and $\pi_{\mathfrak{D}^{(n+k)}_{n+k}}[\mathcal{L}^U_k, \mathfrak{D}^{(n)}_n] = \{\mathbf{0}\}$ for any natural number k and any $n \in \mathbb{N}_0$. Finally, we have $\mathrm{Im}(\hat{L}^{S,N}_{(N)}) \subseteq \mathrm{Im}(L^{S,N}_{(N)})$ where $\hat{L}^{S,N}_{(N)}$ is described in Eq. (8).

Proof. For any $N \le 2N_0$,

$$\mathrm{Im}(L^{S,N}_{(1)}) = [\mathcal{L}^0_N, v^{(0)}_0] = \mathscr{L}_{H^c,N}.$$

Since $\mathscr{L}_{H,N} = \pi_{\mathscr{L}_N}(\ker(\mathrm{ad}_{v^{(0)}_0})) = \ker(\hat{L}^{S,N}_{(1)})$, we have $\ker(L^{S,N}_{(1)}) = \{\mathbf{0}\}$. Besides, by Proposition 2.3

and Eq. (21), we obtain $\pi_{\mathscr{L}_{H^c,N}}(v_N^{(N)}) = \mathbf{0}$. Since $\mathcal{L}^i_{N-i} = \{\mathbf{0}\}$ for any i $(0 \neq i \neq 2N_0)$, $L^{S,N}_{(N)} = L^{S,N}_{(1)}$ when $N < 2N_0$, and for any $N \geq 2N_0$

$$\operatorname{Im}(L^{S,N}_{(N)}) = \mathscr{L}_{H^c,N} \oplus [\mathcal{L}^{2N_0}_{N-2N_0}, v^{(N-1)}_{2N_0}].$$

Now by the formulas:

$$[X_{(k+1)k}, v^{(N-1)}_{2N_0}] = 2(k-N_0)a^{(N-1)}_{(N_0+1)N_0\mathbf{0}} X_{(k+N_0+1)(k+N_0)} - 2N_0 b^{(N-1)}_{(N_0+1)N_0\mathbf{0}} Y_{(k+N_0+1)(k+N_0)}$$

and

$$[Y_{(k+1)k}, v^{(N-1)}_{2N_0}] = 2ka^{(N-1)}_{(N_0+1)N_0\mathbf{0}} Y_{(k+N_0+1)(k+N_0)},$$

and $a^{(N-1)}_{(N_0+1)N_0\mathbf{0}} \neq 0$, we conclude that $L^{S,N}_{(N)}$ is an injective linear transformation, and it is a bijective transformation if and only if $N \neq r\alpha + 2N_0$ and $N \neq r\alpha + 4N_0$ for any $r \in \mathbb{N}_0$ (indeed, this results from $0 \neq k \neq N_0$). Therefore, for any $N \in \mathbb{N}$,

$$\operatorname{Im}(L^{S,N}_{(N)}) = \{v \in \mathscr{L}_N \,|\, \pi_{\mathscr{L}_{X_{(2N_0+1)2N_0}} \oplus \mathscr{L}_{X_{(N_0+1)N_0}} \oplus \mathscr{L}_{Y_{(N_0+1)N_0}}}(v) = 0\} \oplus \pi_{\mathscr{L}_N}(\mathscr{L}_{\delta_{N,2N_0} v^{(N-1)}_{2N_0}}).$$

This proves the claimed equation $\operatorname{Im}(L^{S,N}_{(N)}) = \mathscr{L}_{H^c,N} \oplus \bigoplus_{r=1}^{[\frac{N}{\alpha}]} \mathbb{S}_{N,r}$.

Next, it is easy to verify that $\operatorname{ad}_{\mathcal{L}^{2N_0}_k}(\mathfrak{D}^{(n)}_n) = \{\mathbf{0}\} \subseteq \mathfrak{D}^{(n+k)}_{n+k}$ $\forall\, n < 2N_0$, and $\pi_{\mathfrak{D}^{(n+k)}_{n+k}} \circ \operatorname{ad}_{\mathcal{L}^U_k}(\mathfrak{D}^{(n)}_n) = \{\mathbf{0}\}$ $\forall\, k, n \in \mathbb{N}_0$. Since $\operatorname{ad}_{Y_{10}} \mathscr{L}_H = \{\mathbf{0}\}$,

$$\ker(\operatorname{ad}_{v^{(N-1)}_{2N_0}}) = \hat{\bigoplus_{n=0}^{\infty}} \operatorname{span}\{X_{(N_0+1)N_0}\mu^{\mathbf{m}} \,|\, b^{(N-1)}_{(N_0+1)N_0\mathbf{0}} = 0, n = |\mathbf{m}|\alpha + 2N_0, \mathbf{m} \in \mathbb{N}_0^p\}$$
$$\oplus \hat{\bigoplus_{n=0}^{\infty}} \operatorname{span}\{Y_{10}\mu^{\mathbf{m}} \,|\, n = |\mathbf{m}|\alpha, \mathbf{m} \in \mathbb{N}_0^p\}$$

and

$$\operatorname{ad}_{X_{10}}(w) = \mathbf{0} \quad \text{if and only if } w \in \hat{\bigoplus_{|\mathbf{m}|=0}^{\infty}} \operatorname{span}\{X_{10}\mu^{\mathbf{m}}, X_{01}\mu^{\mathbf{m}}, Y_{10}\mu^{\mathbf{m}}, Y_{01}\mu^{\mathbf{m}} \,|\, \mathbf{m} \in \mathbb{N}_0^p\},$$

we have $\hat{L}^{S,N}_{(N)} = \hat{L}^{S,N}_{(2N_0+2)}$ $\forall\, N \geq 2N_0 + 2$. Recalling that

$$\operatorname{ad}_{cX_{(N_0+1)N_0}\mu^{\mathbf{m}}} \sum_{k=1}^{p} a^{(N-1)}_{10\mu_k} X_{10}\mu_k$$
$$= \sum_{k=1}^{m} ca^{(N-1)}_{10\mu_k}(2N_0 - 1) X_{(N_0+1)N_0}\mu^{\mathbf{m}+\mathbf{e}^p_k}$$
$$= \operatorname{ad}_{\sum_{k=1}^{p} \frac{-a^{(N-1)}_{10\mu_k}}{a^{(N-1)}_{(N_0+1)N_0\mathbf{0}}} X_{10}\mu^{\mathbf{m}+\mathbf{e}^p_k}} X_{(N_0+1)N_0}$$

implies

$$\operatorname{Im}(\hat{L}^{S,N}_{(2N_0+2)}) = \operatorname{Im}(\hat{L}^{S,N}_{(2N_0+1)}) \quad \forall\, N > 2N_0,$$

and noticing that

$$\operatorname{ad}_{cX_{(N_0+1)N_0}} v^{(N-1)}_{2N_0} = cb^{(N-1)}_{(N_0+1)N_0\mathbf{0}} Y_{(2N_0+1)(2N_0)}$$
$$= \operatorname{ad}_{\frac{cb^{(N-1)}_{(N_0+1)N_0\mathbf{0}}}{2N_0 a^{(N-1)}_{(N_0+1)N_0\mathbf{0}}} Y_{(N_0+1)N_0}} v^{(N-1)}_{2N_0},$$

we finally obtain

$$\operatorname{Im}(\hat{L}^{S,N}_{(2N_0+1)}) = \mathscr{L}_{H^c,N} \oplus \operatorname{ad}_{\mathcal{L}^{2N_0}_{N-2N_0}} v^{(N-1)}_{2N_0}$$
$$= \operatorname{Im}(L^{S,N}_{(2N_0+1)}) \quad \forall\, N > 2N_0.$$

The rest of the proof is straightforward. ■

Now its time to describe the contribution of time rescaling. For this purpose, we need to provide a parametric time space decomposition to

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clearly specify the contribution of each time term in simplifying system terms. Therefore, we consider the parametric time space decomposition

$$\mathscr{R}_I = 1 + \hat{\bigoplus_{N=1}^{\infty}} \mathscr{R}_N, \quad \mathscr{R}_N = \mathcal{R}_N^U \oplus \bigoplus_{i=0}^{N-1} \mathcal{R}_N^i, \tag{31}$$

where

$$\mathcal{R}_N^0 = \text{span}_{\mathbb{F}} \left\{ Z_i \mu^{\mathbf{m}} \middle| r = \frac{N-2i}{2N_0+1} \text{ where } i, r \in \mathbb{N}_0, i < N_0 \right\}, \tag{32}$$

$$\mathcal{R}_{N-2N_0}^{2N_0} = \text{span}_{\mathbb{F}} \left\{ Z_{N_0} \mu^{\mathbf{m}} \middle| r = \frac{N-4N_0}{2N_0+1}, r \in \mathbb{N}_0 \right\}, \quad \mathcal{R}_N^k = \{0\} \quad \forall\, k,\ \ 0 \neq k \neq 2N_0, \tag{33}$$

and

$$\mathcal{R}_N^U = \text{span}_{\mathbb{F}} \left\{ Z_i \mu^{\mathbf{m}} \middle| r = \frac{N-2i}{2N_0+1} \text{ where } i, r \in \mathbb{N}_0, i > N_0 \right\} \quad \forall\, N \in \mathbb{N}. \tag{34}$$

The above decomposition leads to the following lemma, stating how further time rescaling can simplify the system along with state transformations.

Lemma 3.2. *Assume the hypothesis in Lemma 3.1 holds and $v_n^{(n)}$ is obtained via formal basis style as an element of the complement space to $L_{(n)}^{S,n}$. Then, we have*

$$\begin{aligned}\text{Im}(L_{(N)}^{T,N}) = &\hat{\bigoplus_{r\in\mathbb{N}_0,\mathbf{m}}} \text{span}_{\mathbb{F}} \left\{ X_{(2N_0+1)2N_0}\mu^{\mathbf{m}} + b^{(N)}_{(N_0+1)N_0\mathbf{0}} Y_{(2N_0+1)2N_0}\mu^{\mathbf{m}} \middle| r = \frac{N-4N_0}{2N_0+1} \right\} \\ &\oplus \hat{\bigoplus_{\{r,k,\mathbf{m}\,|\, k<N_0, k\,\&\, r\in\mathbb{N}_0, \mathbf{m}\in\mathbb{N}_0^p\}}} \text{span}_{\mathbb{F}} \left\{ Y_{(k+1)k}\mu^{\mathbf{m}} \middle| r = \frac{N-2k}{2N_0+1} \right\}.\end{aligned}$$

Further, assume $\mathfrak{D}_N^{(N)}$ follows Eq. (26). Then, $\mathcal{R}_k^{2N_0}\mathfrak{D}_n^{(n)} = \{\mathbf{0}\} \quad \forall\, n, 0<n<2N_0, \mathcal{R}_k^{2N_0}\mathfrak{D}_0^{(0)} \subseteq \mathfrak{D}_k^{(k)}$ and $\pi_{\mathfrak{D}_{k+n}^{(k+n)}}(\mathcal{R}_k^U \mathfrak{D}_n^{(n)}) = \{\mathbf{0}\}\ \forall\, k, n \in \mathbb{N}_0$.

Proof. The proof is straightforward based on the $\mathscr{R}$-module structure of $\mathscr{L}$ and the definition of $L_{(N)}^{T,N}$ and thus, is omitted here. ■

Denote $\mathbf{e}_i^n$ for $(0, \ldots, \overbrace{1}^{i\text{th}}, \ldots, 0)^T \in \mathbb{F}^n$. The following lemma presents the parameter space decomposition.

Lemma 3.3. *Assume*

$$v^{(N)}_{2N_0-1+2k} = \sum_{i=1}^{p} a^{(N_0)}_{k(k-1)\mathbf{e}_i} X_{k(k-1)}\mu_i$$

for all $N \geq N_0$ and $k < N_0$, and $\text{rank}_{\mathbb{F}}(A^{(N_0)}) = N_0$, where $A^{(N_0)}$ is defined by Eq. (25). Then, there exist matrices $B_{N_0\times p}, C_{p\times p}$ and a permutation $\theta \in S_p$ associated with a parameter space decomposition

$$\mathscr{P}^p_{2kN_0+k} = \mathcal{P}^U_{2kN_0+k} \oplus \bigoplus_{i=0}^{N_0-1} \mathcal{P}^{2N_0+1+2i}_{2kN_0+k}, \quad k \geq 2, \tag{35}$$

for which

$$\mathcal{P}^U_{2kN_0+k} = \bigoplus_{i=1}^{p-N_0} \mathscr{P}^1_{2kN_0+k} C^T \mathbf{e}^p_{\theta(i)} \quad \text{and}$$

$$\mathcal{P}^{2N_0+1+2i}_{2kN_0+k} = \mathscr{P}^1_{2kN_0+k} B^T \mathbf{e}^{N_0}_{i+1}, \quad 0 \leq i < N_0, \tag{36}$$

such that the following conclusions hold:

(1) $\text{Im}(L^{P,k}_{(k)}) = \{\mathbf{0}\}$ *for any $k \leq 4N_0$, and $L^{P,k}_{(k)}$ is an injective linear transformation for any natural number k.*
(2) *Let $k = 2i + 2rN_0 + r$, $r > 1$ and $0 \leq i < N_0$. Then, we have $\text{Im}(L^{P,k}_{(k)}) = \text{span}_{\mathbb{F}}\{X_{(i+1)i}\mu^{\mathbf{m}} \mid \mathbf{m} \in \mathbb{N}_0^p\}$.*
(3) $D_\mu(v^{(N-1),S}_{2N_0-1+2i}, \mathcal{P}^U_{2kN_0+k}) = \{\mathbf{0}\}\ \forall\, i, 0 \leq i < N_0$. *If $\mathfrak{D}_n^{(n)}$ follows Eq. (26) and $p = N_0$, then*

$D_\mu(\mathfrak{D}_n^{(n)}, \mathcal{P}^U_{2kN_0+k}) = \{\mathbf{0}\}\ \forall\, k, n \in \mathbb{N}_0$. (*This condition is essential for obtaining the simplest parametric normal forms.*)

(4) $\mathscr{P}^p_{2kN_0+k} = \bigoplus_{i=1}^{N_0} \mathcal{P}^{2N_0-1+2i}_{2kN_0+k} \oplus \mathcal{P}^U_{2kN_0+k}$.

Proof. Since $\alpha = 2N_0+1$, for any $k \leq 4N_0+1$ we have $\mathrm{Im}(L^{P,k}_{(k)}) = \{\mathbf{0}\}$. Note that $\mathscr{P}^p_N = \mathbf{0}$, when $N \neq 2kN_0 + k\ \forall\, k \in \mathbb{N}_0$. Let $B_{N_0\times p} = [b^{(N_0)}_{ij}]$ be the Moore–Penrose pseudoinverse of matrix A and thus, $BA^{(N_0)} = I_{N_0\times N_0}$, where $I_{N_0\times N_0}$ denotes the identity matrix and $A^{(N_0)}_{p\times N_0} = [a^{(N_0)}_{ij}]$ in which $a^{(N_0)}_{j(j-1)\mathbf{e}_i} = a^{(N_0)}_{ij}$. Since the system has parametric dimension of N_0, there exists a matrix $C_{p\times p}$ such that $CA^{(N_0)} = 0_{p\times N_0}$ while $\mathrm{rank}_{\mathbb{F}}(C) = p - N_0$. Therefore, there exists a permutation $\theta \in S_p$ such that $\mathrm{rank}_{\mathbb{F}}(\mathrm{span}_{\mathbb{F}}\{Ce^p_{\theta(i)} \,|\, i \leq N_0\}) = p - N_0$. Consider $N = 2rN_0 + 2k + r - 2$, where $1 \leq k \leq N_0$ and $r > 1$. According to parametric space decomposition (35), only $\mathcal{P}^{2N_0-1+2k}_{2rN_0+r}$ contributes to $L^{P,N}_{(N)}$. Therefore,

$$L^{P,N}_{(N)} : \mathcal{P}^{2N_0-1+2k}_{2rN_0+r} \to \mathscr{L}_N,$$

where $\mathcal{P}^{2N_0-1+2k}_{2rN_0+r} = \mathscr{P}^1_{2rN_0+r} B^T \mathbf{e}^{N_0}_k$. Then,

$$\begin{aligned} L^{P,N}_{(N)}(\mathbf{Y}^P_{2rN_0+r}) &= D_\mu(v^{(N-1)}_{2N_0-1+2k}, \mathbf{Y}^P_{2rN_0+r}) && \text{(by Eq. (16))} \\ &= \langle [a^{(N_0)}_{k(k-1)\mathbf{e}_i}]^p_{i=1}, \mathbf{Y}^P_{2rN_0+r}\rangle X_{k(k-1)}, && \text{(by Eq. (25))} \end{aligned}$$

where $\langle (a_i)^p_{i=1}, (b_i)^p_{i=1}\rangle = \sum_{i=1}^p a_i b_i\ \forall\, a_i, b_i \in \mathbb{F}$. Thus, $L^{P,N}_{(N)}(\mathbf{Y}^P_{2rN_0+r}) = 0$ if and only if

$$\begin{aligned} 0 &= \langle [a^{(N_0)}_{k(k-1)\mathbf{e}_i}]^p_{i=1}, \mathbf{Y}^P_{2rN_0+r}\rangle \\ &= \langle [a^{(N_0)}_{k(k-1)\mathbf{e}_i}]^p_{i=1}, Y_{2rN_0+r}B^T\mathbf{e}^{N_0}_k\rangle \quad (\text{since } \mathbf{Y}^P_{2rN_0+r} = Y_{2rN_0+r}B^T\mathbf{e}^{N_0}_k) \\ &= \langle [a^{(N_0)}_{k(k-1)\mathbf{e}_i}]^p_{i=1}, B^T\mathbf{e}^{N_0}_k\rangle Y_{2rN_0+r} \\ &= \sum_{i=1}^p a^{(N_0)}_{k(k-1)\mathbf{e}_i} b^{(N_0)}_{ki} Y_{2rN_0+r} \\ &= Y_{2rN_0+r} \qquad\qquad (\text{due to } BA^{(N_0)} = I). \end{aligned}$$

Thus, $L^{P,N}_{(N)}$ is an injective transformation. On the other hand,

$$\begin{aligned} \sum_{\mathbf{m}\in\mathbb{N}_0^p} \alpha_{k(k-1)\mathbf{m}} X_{k(k-1)}\mu^{\mathbf{m}} &= \sum_{\mathbf{m}\in\mathbb{N}_0^p} \alpha_{k(k-1)\mathbf{m}} X_{k(k-1)}\mu^{\mathbf{m}}\langle \mathbf{e}^{N_0}_k, \mathbf{e}^{N_0}_k\rangle \quad (BA^{(N_0)} = I) \\ &= \sum_{\mathbf{m}\in\mathbb{N}_0^p} \alpha_{k(k-1)\mathbf{m}}\langle \mathbf{e}^{N_0}_k, {A^{(N_0)}}^T B^T \mathbf{e}^{N_0}_k \mu^{\mathbf{m}}\rangle X_{k(k-1)} \\ &= \sum_{\mathbf{m}\in\mathbb{N}_0^p} \langle A^{(N_0)}\mathbf{e}^{N_0}_k, \alpha_{k(k-1)\mathbf{m}} B^T\mathbf{e}^{N_0}_k\mu^{\mathbf{m}}\rangle X_{k(k-1)} \\ &= \left\langle [a^{(N_0)}_{k(k-1)\mathbf{e}_i}]^p_{i=1}, \sum_{\mathbf{m}\in\mathbb{N}_0^p} \alpha_{k(k-1)\mathbf{m}} B^T\mathbf{e}^{N_0}_k\mu^{\mathbf{m}}\right\rangle X_{k(k-1)} \\ &= D_\mu\left(v^{(N-1),S}_{2N_0-1+2k}, \sum_{\mathbf{m}\in\mathbb{N}_0^p} \alpha_{k(k-1)\mathbf{m}} B^T\mathbf{e}^{N_0}_k\mu^{\mathbf{m}}\right) \\ &= L^{P,N}_{(N)}\left(\sum_{\mathbf{m}\in\mathbb{N}_0^p} \alpha_{k(k-1)\mathbf{m}} B^T\mathbf{e}^{N_0}_k\mu^{\mathbf{m}}\right). \end{aligned}$$

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Then, $\mathrm{span}_{\mathbb{F}}\{X_{k(k-1)}\mu^{\mathbf{m}} \,|\, \mathbf{m} \in \mathbb{N}_0^p\} \subseteq \mathrm{Im}(L_{(N)}^{P,N})$. Further, since

$$\dim_{\mathbb{F}}(\mathcal{P}_{2rN_0+r}^{2N_0-1+2k}) = \binom{r+p-1}{r} = \dim_{\mathbb{F}}(\mathrm{span}_{\mathbb{F}}\{X_{k(k-1)}\mu^{\mathbf{m}} \,|\, \mathbf{m} \in \mathbb{N}_0^p\}),$$

we have $\mathrm{Im}(L_{(N)}^{P,N}) = \mathrm{span}_{\mathbb{F}}\{X_{k(k-1)}\mu^{\mathbf{m}} \,|\, \mathbf{m} \in \mathbb{N}_0^p\}$. Moreover,

$$\begin{aligned} D_\mu\left(\sum_{i=1}^{N_0} v_{2N_0-1+2i}^{(N-1)}, \mathcal{P}_{2kN_0+k}^U\right) &= \sum_{i=1}^{N_0} \left\langle [a_{i(i-1)\mathbf{e}_l}^{(N_0)}]_{l=1}^p, \bigoplus_{j=1}^{p-N_0} \mathscr{P}_{2kN_0+k}^1 C^T \mathbf{e}_{\theta(j)}^p \right\rangle X_{i(i-1)} \\ &= \sum_{i=1}^{N_0}\sum_{j=1}^{p-N_0} \langle [a_{i(i-1)\mathbf{e}_l}^{(N_0)}]_{l=1}^p, C^T\mathbf{e}_{\theta(j)}^p\rangle \mathscr{P}_{2kN_0+k}^1 X_{i(i-1)} \\ &= \{\mathbf{0}\}. \end{aligned}$$

In particular, $D_\mu(\mathfrak{D}_n^{(n)}, \mathcal{P}_{2kN_0+k}^U) = \{\mathbf{0}\}$ when $p = N_0$ ($\forall\, k, n \in \mathbb{N}_0$), and

$$D_\mu(\mathfrak{D}_{2N_0+1+2k}^{(2N_0+1+2k)}, \mathcal{P}_{2rN_0+r}^{2N_0+1+2i}) = \delta_{ik} X_{k(k-1)} \mathscr{P}_{2rN_0+r}^1 \subseteq \mathfrak{D}_{2rN_0+r+2k}^{(2rN_0+r+2k)} \quad \forall\, k \neq i.$$

Thus $D_\mu(\mathfrak{D}_n^{(n)}, \mathcal{P}_{k\alpha}^i) \subseteq \mathfrak{D}_{k\alpha+n-\alpha}^{(k\alpha+n-\alpha)}$ $\forall\, n < i$. Finally,

$$\begin{aligned} \bigoplus_{i=1}^{N_0} \mathcal{P}_{2kN_0+k}^{2N_0-1+2i} \oplus \mathcal{P}_{2kN_0+k}^U &= \bigoplus_{i=1}^{N_0} \mathscr{P}_{2kN_0+k}^1 B^T \mathbf{e}_i^{N_0} \oplus \bigoplus_{i=1}^{p-N_0} \mathscr{P}_{2kN_0+k}^1 C^T \mathbf{e}_{\theta(i)}^p \\ &= \sum_{i=1}^{p} \mathscr{P}_{2kN_0+k}^1 \mathbf{e}_i^p \\ &= \mathscr{P}_{2kN_0+k}^p, \end{aligned}$$

since

$$v = \sum_{i=1}^{N_0} a_i B^T \mathbf{e}_i^{N_0} + \sum_{j=1}^{p-N_0} b_j C^T \mathbf{e}_{\theta(j)}^p = \mathbf{0}$$
$$\text{if and only if } a_i = b_j = 0 \quad \forall\, i, j.$$

The proof is complete. ■

The following two lemmas provide us with a tool for determining the order of a parametric nonlinear center and its parametric dimension.

Lemma 3.4. *Let $\pi_{\mathscr{L}_0}(v^{(0)}) = Y_{10}$, and assume for any $k < N_0$,*

$$\pi_{\mathscr{L}_k}(v^{(0)}) \in \mathscr{L}_{H,k}, \pi_{\mathrm{span}_{\mathbb{F}}\{X_{(k+1)k}\}}(v) = \mathbf{0}, \quad \textit{and}$$
$$\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\}}(v^{(0)}) = aX_{(N_0+1)N_0},$$

where $a \neq 0$. Then, the order of the parametric system $v^{(0)}$ is N_0. In addition, we have

$$\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\}}(v^{(\infty)}) = aX_{(N_0+1)N_0}.$$

Proof. By Lemmas 3.1 and 3.2, we know that under these conditions $a_{(k+1)k\mathbf{0}}^{(N)}$ and $a_{(N_0+1)N_0\mathbf{0}}^{(N)}$ do not change in the normal form computation. Thus, for any $N \in \mathbb{N}$, $a_{(k+1)k\mathbf{0}}^{(N)} = 0$ $\forall\, k < N_0$ and $a_{(N_0+1)N_0=\mathbf{0}}^{(N)} = a \neq 0$. ■

Lemma 3.5. *Let N_0-order nonlinear center $v^{(0)} \in \mathscr{L}$ represent a system with parameter dimension N_0. Then, for any $k \geq N_0$, $A^{(\infty)} = A^{(k)} = A^{(N_0)}$. In particular, if $\pi_{\mathscr{L}_k}(v^{(0)}) \in \mathscr{L}_{H,k}$ $\forall\, k < N_0$, then $N_0 = \mathrm{rank}_{\mathbb{F}}(A^{(\infty)}) = \mathrm{rank}_{\mathbb{F}}(A^{(k)}) = \mathrm{rank}_{\mathbb{F}}(A^{(1)})$ $\forall\, k \in \mathbb{N}$.*

Proof. Since $v^{(1)} \in \mathscr{L}_H$, $v^{(n)} \in \mathscr{L}_H$ and $a_{(k+1)k\mathbf{0}}^{(n)} = 0$ $\forall\, n$ and $k < N_0$. Then, based on our method, Lemmas 3.1 and 3.3, $a_{ij}^{(k)}$ may only be changed via time rescaling. It is evident that the impact of time rescaling associated with time space decomposition (31)–(34) on $v^{(1)}$ may only add a multiple scalar of the columns $j = 1, 2, \ldots, N_0 - N$ to the columns $j = N+1, N+2, \ldots, N_0$ at each step associated

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with N. Obviously, this does not change the rank of $A^{(k)}$.

The proof is complete. ■

Roughly speaking, Lemma 3.4 indicates how to determine the order of a system while Lemma 3.5 provides a tool to verify whether or not a parametric system has parameter dimension N_0. However, if the hypothesis $\pi_{\mathscr{L}_k}(v) \in \mathscr{L}_{H,k}\ \forall\, k < N_0$ fails, the claims in these two lemmas are not necessarily true. In the following, we present two counter examples to illustrate this.

Example 3.6. For the case of Lemma 3.4 in which the hypothesis does not hold, consider the vector field $v^{(0)} = Y_{10} - X_{11} + Y_{20} - X_{21} + Y_{22}$. Since $\pi_{\mathrm{span}_{\mathbb{F}}\{X_{21}\}}(v^{(0)}) \neq \{\mathbf{0}\}$, one may naively expect $v^{(0)}$ to be associated with generic Hopf singularity. However, executing our Maple program we obtain the following infinite order normal form:

$$v^{(\infty)} = Y_{10} - Y_{21} + \frac{1}{2}Y_{32} - X_{32} + \frac{13}{12}X_{54},$$

which is a degenerate system. Indeed, the origin is a weak focus of order two for $v^{(0)}$ and thus the mentioned hypothesis in the lemma cannot be ignored even for the nonparametric systems.

Example 3.7. Consider the system $w^{(0)} = Y_{10} + X_{10}\mu_1 + X_{11} + Y_{20}\mu_2 + X_{32}$, where the hypothesis in Lemma 3.5 is not satisfied. Executing our Maple program on this system yields the parametric normal form (up to grade 9):

$$\begin{aligned} w^{(\infty)} &= Y_{10} + X_{10}\mu_1 + 6X_{21}\mu_1 - X_{21}\mu_2 + X_{32} \\ &\quad + Y_{32}\left(4 - 34\mu_1 - \frac{214}{9}\mu_2 + \cdots\right) \end{aligned}$$

which is associated with an order-2 nonlinear center. Since $A^{(\infty)} = \begin{pmatrix} 1 & 0 \\ 6 & -1 \end{pmatrix}$ and $\mathrm{rank}_{\mathbb{F}} A^{(\infty)} = 2$, $w^{(0)}$ has full parameter dimension. However, $A^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathrm{rank}_{\mathbb{F}} A^{(1)} = 1$. Thus, it does not lead to the conclusion of $\mathrm{rank}_{\mathbb{F}} A^{(\infty)} = \mathrm{rank}_{\mathbb{F}} A^{(1)}$, because $X_{11} = \pi_{\mathscr{L}_1}(w^{(0)})$ being in $\mathscr{L}_{H^c,1}$ violates the hypothesis $\pi_{\mathscr{L}_1}(w^{(0)}) \in \mathscr{L}_{H,1}$.

The following Lemma facilitates the computation of N_0 during parametric normal form computation.

Lemma 3.8. *Assume* $a^{(i)}_{(i+1)i\mathbf{0}} = 0\ \forall\, i < k$. *Then*

$$a^{(k-1),S}_{(k+1)k\mathbf{0}} = a^{(k-1)}_{(k+1)k\mathbf{0}} - \sum_{n=0,\, n\neq \frac{k}{2}}^{k} a^{(k-1)}_{(k-n+1)n\mathbf{0}} b^{(k-1)}_{(n+1)(k-n)\mathbf{0}}, \tag{37}$$

and furthermore, $a^{(k-1),S}_{(k+1)k\mathbf{0}} = a^{(N)}_{(k+1)k\mathbf{0}}\ \forall\, N, N \geq k$.

Proof. By Lemma 3.1, we have

$$Y^{S,k} = \sum_{i+j=k+1,\, j+1\neq i} \frac{a^{([\frac{k}{2}])}_{ij\mathbf{0}}}{i-j-1} Y_{ij} + \frac{b^{([\frac{k}{2}])}_{ij\mathbf{0}}}{j+1-i} X_{ij}.$$

Let $\psi^X_{X_{(k+1)k}} = \pi_{\mathrm{span}_{\mathbb{F}}\{X_{(k+1)k}\}} \circ \mathrm{ad}_{X_{(k-n+1)n}} \circ \pi_{\mathscr{L}_k}$ and $\psi^Y_{X_{(k+1)k}} = \pi_{\mathrm{span}_{\mathbb{F}}\{X_{(k+1)k}\}} \circ \mathrm{ad}_{Y_{(k-n+1)n}} \circ \pi_{\mathscr{L}_k}$. It is evident that $\mathrm{Im}(\psi^X_{X_{(k+1)k}}) = \mathrm{Im}(\psi^X_{X_{(k+1)k}} \circ \pi_{\mathrm{span}_{\mathbb{F}}\{X_{ij}\}})$ and

$$\begin{aligned} \mathrm{Im}(\psi^Y_{X_{(k+1)k}}) &= \mathrm{Im}(\psi^Y_{X_{(k+1)k}} \circ \pi_{\mathrm{span}_{\mathbb{F}}\{Y_{ij}\}}) \\ &= \mathrm{span}_{\mathbb{F}}\{X_{(k+1)k}\} \end{aligned}$$

if and only if $i = n+1$ and $j = k-n$, for some $n \neq k/2$. Then

$$\begin{aligned} &\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(k+1)k}\}} \circ \mathrm{ad}_{Y^{S,k}}(v^{(k-1)}) \\ &\quad = \sum_{n=0,\, n\neq\frac{k}{2}}^{k} -2a^{(k-1)}_{(k-n+1)n\mathbf{0}} b^{(k-1)}_{(n+1)(k-n)\mathbf{0}} X_{(k+1)k}. \end{aligned} \tag{38}$$

Further,

$$[Y^{S,k}, Y_{10}] = \sum_{i+j=k+1,\, j+1\neq i} - a^{([\frac{k}{2}])}_{ij\mathbf{0}} X_{ij} - b^{([\frac{k}{2}])}_{ij\mathbf{0}} Y_{ij}$$

and thus

$$\begin{aligned} &\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(k+1)k}\}} \circ \mathrm{ad}^2_{Y^{S,k}}(Y_{10}) \\ &\quad = \sum_{n=0,\, n\neq\frac{k}{2}}^{k} 2a^{(k-1)}_{(n+1)(k-n)\mathbf{0}} b^{(k-1)}_{(k-n+1)n\mathbf{0}} X_{(k+1)k}. \end{aligned} \tag{39}$$

Therefore, by Eqs. (38) and (39) we have

$$a^{(k-1),S}_{(k+1)k\mathbf{0}} = a^{(k-1)}_{(k+1)k\mathbf{0}} - \sum_{n=0,\, n\neq\frac{k}{2}}^{k} a^{(k-1)}_{(k-n+1)n\mathbf{0}} b^{(k-1)}_{(n+1)(k-n)\mathbf{0}}.$$

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Lemmas 3.2 and 3.3 imply that $a^{(k-1),S}_{(k+1)k\mathbf{0}}$ ($k \leq N_0$) may not be changed by time rescaling or reparametrization. The proof is completed. ■

Now we are ready to state our main result of this paper.

Theorem 3.9. *Consider a formal parametric N_0-order nonlinear center $v^{(0)}$ with parametric dimension N_0. Then, $v^{(0)}$ can be transformed to (in terms of the complex variable z)*

$$v^{(\infty)} = \mathbf{i}z + \sum_{i,k} a^{(2i+\alpha)}_{(i+1)i\mathbf{e}_k} z^{i+1}\bar{z}^i \mu_k + a^{(2N_0)}_{N_0} z^{N_0+1}\bar{z}^{N_0} + \sum_{\mathbf{m}} b^{(2N_0+r\alpha)}_{N_0\mathbf{m}} \mathbf{i} z^{N_0+1}\bar{z}^{N_0}\mu^{\mathbf{m}}, \tag{40}$$

where $i < N_0, k \leq p, \mathbf{m} \in \mathbb{N}_0^p$, and the coefficients are expressed in terms of the coefficients of $v^{(0)}$. In particular, $a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} = a^{(N_0-1)}_{(N_0+1)N_0\mathbf{0}} - \sum_{n=0,n\neq\frac{N_0}{2}}^{N_0} a^{(N_0-1)}_{(n+1)(N_0-n)\mathbf{0}} b^{(N_0-1)}_{(N_0-n+1)n\mathbf{0}}$.

Proof. Based on Lemmas 3.1–3.5 and Theorem 2.4, the proof (except the last part) is straightforward and thus, the details are omitted here. Equation (37) in Lemma 3.8 proves the last part of Theorem 3.9. ■

Note that the parametric normal form of nonlinear center for system (40) is essentially different from what is discussed in [Yu & Leung, 2003; Yu & Chen, 2007; Gazor & Yu, 2008]. This, however, is a consistent alternative form. Theorems 3.9 and 1.3 from [Chow *et al.*, 1994] implies the following theorem.

Theorem 3.10. *Assume the hypothesis of Theorem (3.9) holds for v and $p = N_0$. Then, there exist ε and a neighborhood U of the origin such that for $|\mu| < \varepsilon$, $v(\mu)$ has at most N_0 limit cycles in U. Furthermore, for any $j, 1 \leq j \leq N_0$, and a neighborhood U^* of the origin, there are δ and an open subset G from $\{\mu \,|\, 0 < \mu < \delta\}$ such that $0 \in \overline{G}$ and $v(\mu)$ has exactly j limit cycles in U^* for any $\mu \in G$.*

Normal form theory is also concerned with the uniqueness of normal forms, e.g. see [Kokubu *et al.*, 1996, Theorem 4.2] and [Gazor & Yu, 2008]. Indeed, the common notion of uniqueness of normal forms implies that the unique normal form is the simplest normal form. In this regard, we have the following theorem which is similar to that of [Kokubu *et al.*, 1996; Gazor & Yu, 2008].

Theorem 3.11. *Let $p = N_0 > 1$ and*

$$v = Y_{10} + \sum_{i=1}^{N_0}\sum_{j=1}^{p} a_{ij} X_{i(i-1)}\mu_j + A_{N_0} X_{(N_0+1)N_0} + Y_{(N_0+1)N_0}\left(B_{N_0} + \sum_{|\mathbf{m}|=1}^{\infty}\sum_{\mathbf{m}} b_{\mathbf{m}}\mu^{\mathbf{m}}\right), \tag{41}$$

where $\operatorname{rank}_{\mathbb{F}}([a_{ij}]) = N_0$ and $A_{N_0}B_{N_0} \neq 0$. Assume that

$$w = Y_{(i_0+1)i_0} + \sum_{i=1}^{M_0}\sum_{j=1}^{p} \tilde{a}_{ij} X_{i(i-1)}\mu_j + \tilde{A}_{M_0} X_{(M_0+1)M_0} + Y_{(M_0+1)M_0}\left(\tilde{B}_{N_0} + \sum_{|\mathbf{m}|=1}^{\infty}\sum_{\mathbf{m}} \tilde{b}_{\mathbf{m}}\mu^{\mathbf{m}}\right), \tag{42}$$

where $i_0, M_0 \in \mathbb{N}_0$ and that there exists a permutation θ of $\{S, P, T\}$ with

$$Y^S = \sum_{n=1}^{\infty} Y_n^S \quad (Y_n^S \in \mathscr{L}_n),$$

$$Y^P = \sum_{n=2}^{\infty} Y_n^P \quad (Y_n^P \in \mathscr{P}^p_n),$$

and

$$Y^T = \sum_{n=1}^{\infty} Y_n^T \quad (Y_n^T \in \mathscr{R}_n),$$

such that $w = \phi^{\theta(T)}_{Y^{\theta(T)}{}_*} \circ \phi^{\theta(P)}_{Y^{\theta(P)}{}_*} \circ \phi^{\theta(S)}_{Y^{\theta(S)}{}_*}(v)$. Then, $i_0 = 0, M_0 = N_0$ and $\pi_{\mathscr{L}_N}(v - w) = \mathbf{0}\ \forall\, N \leq 4N_0$. In addition, assume $\pi_{\operatorname{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\mu^{\mathbf{m}} \,|\, \mathbf{m}\in\mathbb{N}_0^p\}}(Y^S) = \mathbf{0}$ and $\pi_{\mathcal{R}_n^U}(Y_n^T) = 0\ \forall\, n \in \mathbb{N}$, where $\mathcal{R}_n^U$ is given by Eq. (34). Then, $v = w, Y^T = 0, Y^P = \mathbf{0}$, and $Y^S \in \ker(\operatorname{ad}_v)$.

Proof. Clearly, v is an N_0-order parametric nonlinear center with full parametric dimension and $i_0 = 0$. Proposition 2.6 in [Gazor & Yu, 2008] implies that $\mathscr{L}_{Y_{10}} \subseteq \ker(\operatorname{ad}_v)$, since $v \in \mathscr{L}_H$. Without loss of generality, we assume that σ is an identity permutation. Since $w = \phi^T_{Y^T{}_*} \circ \phi^P_{Y^P{}_*} \circ \phi^S_{Y^S{}_*}(v)$, we have

$$\pi_{\mathscr{L}_1}(w - v) = \operatorname{ad}_{Y_1^S}(Y_{10}) + Y_1^T Y_{10} = \mathbf{0}.$$

Obviously, $Y_1^T = 0$, and thus $Y_1^S \in \ker(\operatorname{ad}_{Y_{10}}) = \mathscr{L}_{H,1}$. Then, $\pi_{\mathscr{L}_1}(w) = Y_1^S = \mathbf{0}$.

Consider $N \leq 2N_0$ and

$$\pi_{\mathscr{L}_N}(w-v) = \sum_{N=ki} \frac{1}{k!}\mathrm{ad}^k_{Y^S_i} Y_{10} + Y^T_N \pi_{\mathscr{L}_0}(v) + \sum_{ki+j=N} Y^T_j \mathrm{ad}^k_{Y^S_i} Y_{10}. \tag{43}$$

Then by induction on N, we easily have $Y^S_i \in \mathscr{L}_{H,i}\ \forall\, i \leq N$, and $Y^T_N = 0$ only if $2M_0 \neq N \neq 2N_0$. Furthermore, $M_0 = N_0$ and $\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\}}(w) = A_{N_0} X_{(N_0+1)N_0}$.

Similarly, for $2N_0 < N \leq 4N_0$, we obtain

$$\begin{aligned}\pi_{\mathscr{L}_N}(w-v) = &\sum_{N=ki} \frac{1}{k!}\mathrm{ad}^k_{Y^S_i} Y_{10} + \sum_{N=ki+2N_0} \frac{1}{k!}\mathrm{ad}^k_{Y^S_i}\pi_{\mathscr{L}_{2N_0}}(v) + \sum_{p=2N_0+1}^{N-1}\sum_{N=ki+p}\frac{1}{k!}\mathrm{ad}^k_{Y^S_i}\pi_{\mathscr{L}_p}(v)\\ &+ Y^T_N Y_{10} + Y^T_{N-2N_0}\pi_{\mathscr{L}_{2N_0}}(v) + \sum_{p=2N_0+1}^{N-1} Y^T_{N-p}\pi_{\mathscr{L}_p}(v) + \sum_{ki+j=N}\frac{1}{k!}Y^T_j \mathrm{ad}^k_{Y^S_i} Y_{10}\\ &+ \sum_{ki+j=N-2N_0}\frac{1}{k!}Y^T_j \mathrm{ad}^k_{Y^S_i}\pi_{\mathscr{L}_{2N_0}}(v) + \sum_{p=2N_0+1}^{N-2}\sum_{ki+j=N-p}\frac{1}{k!}Y^T_j \mathrm{ad}^k_{Y^S_i}\pi_{\mathscr{L}_p}(v).\end{aligned} \tag{44}$$

Therefore, $Y^S_N \in \mathscr{L}_{H,N}\ \forall\, N \leq 4N_0$.

Now based on mathematical induction we claim $Y^S_i \in \pi_{\mathscr{L}_i}\ker(\mathrm{ad}_v)$ for any $i < 2N_0$. Recalling that

$$\pi_{\mathscr{L}_X} \circ \mathrm{ad}_{Y^S_i} \circ \pi_{\mathscr{L}_{2N_0}}(v) = \mathbf{0} \quad \text{if and only if} \quad Y^S_i \in \mathrm{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\mu^{\mathbf{m}}\} \oplus \pi_{\mathscr{L}_Y}\mathscr{L}_{H,i}, \tag{45}$$

and assuming $Y^S_i \in \pi_{\mathscr{L}_Y}\mathscr{L}_{H,i}\ \forall\, i \in \mathbb{N}$, we have

$$\pi_{\mathscr{L}_Y} \circ \mathrm{ad}_{Y^S_i} \circ \pi_{\mathscr{L}_{2N_0}}(v) = \mathbf{0} \quad \text{if and only if} \quad Y^S_i \in \mathrm{span}_{\mathbb{F}}\{Y_{10}\mu^{\mathbf{m}} \mid i = r\alpha, \mathbf{m} \in \mathbb{N}^p_0\}. \tag{46}$$

For $i = N - 2N_0$, since

$$\begin{aligned}\pi_{\mathscr{L}_X} \circ \pi_{\mathscr{L}_{S,N}}(w-v) &= \sum_{kp=N-2N_0}\frac{1}{k!}\pi_{\mathscr{L}_X}\circ \mathrm{ad}^k_{Y^S_p}\circ\pi_{\mathscr{L}_{2N_0}}(v)\\ &\quad + \pi_{\mathscr{L}_X}(Y^T_i \pi_{\mathscr{L}_{2N_0}}(v))\\ &= \pi_{\mathscr{L}_X}\circ \mathrm{ad}_{Y^S_i}\circ \pi_{\mathscr{L}_{2N_0}}(v)\end{aligned} \tag{47}$$

and

$$\begin{aligned}\pi_{\mathscr{L}_Y} \circ \pi_{\mathscr{L}_{S,N}}(w-v) &= \sum_{kp=N-2N_0}\frac{1}{k!}\pi_{\mathscr{L}_Y}\circ \mathrm{ad}^k_{Y^S_p}\circ\pi_{\mathscr{L}_{2N_0}}(v)\\ &\quad + \pi_{\mathscr{L}_X}(Y^T_i \pi_{\mathscr{L}_{2N_0}}(v))\\ &= \pi_{\mathscr{L}_Y}\circ \mathrm{ad}_{Y^S_i}\circ \pi_{\mathscr{L}_{2N_0}}(v),\end{aligned} \tag{48}$$

our claim $Y^S_i \in \pi_{\mathscr{L}_i}\ker(\mathrm{ad}_v)$ holds for any $i < 2N_0$. By Eq. (47), for $N = 4N_0$ we have

$$\begin{aligned}\mathbf{0} &= \pi_{\mathscr{L}_{S,X}} ad_{Y^S_{2N_0}}\pi_{\mathscr{L}_{2N_0}}(v) + \pi_{\mathscr{L}_{S,X}}(Y^T_{2N_0}\pi_{\mathscr{L}_{2N_0}}(v))\\ &= \pi_{\mathscr{L}_X}(Y^T_{2N_0}A_{N_0}X_{(N_0+1)N_0}),\end{aligned}$$

and thus, $Y^T_{2N_0} = 0$. Therefore, $\pi_{\mathscr{L}_N} v = \pi_{\mathscr{L}_N} w$ $\forall\, N \leq 4N_0$.

Assume $\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\mu^{\mathbf{m}}\}}(Y^S_N) = \mathbf{0}$ and $\pi_{\mathcal{R}^U_N}(Y^T_N) = 0$ for any $N, r \in \mathbb{N}_0$ and $\mathbf{m} \in \mathbb{N}^p_0$. Then $Y^T_{4N_0} = 0$, and thus Eq. (48) is satisfied for $N = 4N_0$. This implies $Y^S_{2N_0} \in \ker(\mathrm{ad}_v)$.

Now we prove that $Y^T_N \in \mathrm{span}_{\mathbb{F}}\{Z_{N_0}\mu^{\mathbf{m}} \mid N = 2N_0 + r\alpha\}, Y^T_{N-2N_0} = 0$, $Y^S_N \in \mathscr{L}_{H,N}$, $Y^S_{N-2N_0} \in \ker(\mathrm{ad}_v)$, and $Y^p_{[\frac{N}{\alpha}]\alpha} = \mathbf{0}$. By induction hypothesis, for $N > 4N_0$ we have

$$\begin{aligned}\pi_{\mathscr{L}_N}(w-v) &= \mathrm{ad}_{Y^S_N} Y_{10} + \mathrm{ad}_{Y^S_{N-2N_0}}\pi_{\mathscr{L}_{2N_0}}(v)\\ &\quad + Y^T_N Y_{10} + Y^T_{N-2N_0}\pi_{\mathscr{L}_{2N_0}}(v)\\ &\quad + \pi_{\mathscr{L}_N}\circ D_\mu\left(\sum_{i=1}^{N_0}\sum_{j=1}^{p} a_{ij}X_{i(i-1)}\mu_j, Y^p_{[\frac{N}{\alpha}]\alpha}\right).\end{aligned} \tag{49}$$

Since $\pi_{\mathscr{L}_{H^c,N}}(w-v) = \mathbf{0}$, we have $Y^S_N \in \mathscr{L}_{H,N}$. In addition, $\pi_{\mathrm{span}_{\mathbb{F}}\{Y_{(k+1)k}\mu^{\mathbf{m}} \mid k \neq N_0\}}(w-v) = \mathbf{0}$ implies $Y^T_N \in \mathrm{span}_{\mathbb{F}}\{Z_{N_0}\mu^{\mathbf{m}} \mid N = 2N_0 + r\alpha\}$.

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On the other hand, $\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(2N_0+1)2N_0}\mu^{\mathbf{m}}\}} \circ \pi_{\mathscr{L}_N}(w-v) = \mathbf{0}$ for $\mathbf{m} \in \mathbb{N}_0^p$ and $\forall r \in \mathbb{N}_0$ results in $Y^T_{N-2N_0} = 0$. From the formula $\pi_{\mathscr{L}_X} \circ \pi_{\mathscr{L}_N}(w-v) = \mathbf{0}$ and Eqs. (45) and (49), we obtain $Y^S_{N-2N_0} \in \pi_{\mathscr{L}_Y}(\mathscr{L}_{H,N-2N_0})$. Further, $\pi_{\mathrm{span}_{\mathbb{F}}\{Y_{(n+1)n}\mu^{\mathbf{m}}\}} \circ \pi_{\mathscr{L}_N}(w-v) = \mathbf{0}$ $\forall n \neq N_0$ implies $Y^S_{N-2N_0} \in \ker(\mathrm{ad}_v)$.

Assume $N = k\alpha + 2N_0 - 1$ and $N_i = k\alpha + 2i - 2$ where $i \leq N_0$. Then,

$$
\begin{aligned}
\mathbf{0} &= \pi_{\mathscr{L}_X} \circ \pi_{\mathscr{L}_{N_i}}(w-v) && \text{(by induction hypothesis)}\\
&= \pi_{\mathscr{L}_X}(\mathrm{ad}_{Y^S_{N_i-2N_0}} \circ \pi_{\mathscr{L}_{2N_0}}(v) + Y^T_{N_i-2N_0}\pi_{\mathscr{L}_{2N_0}}(v))\\
&\quad + \pi_{\mathscr{L}_{N_i}} \circ D_\mu\left(\sum_{i=1}^{N_0}\sum_{j=1}^{p} a_{ij}X_{i(i-1)}\mu_j, Y^p_{[\frac{N}{\alpha}]\alpha}\right)\\
&= \pi_{\mathscr{L}_{N_i}} \circ D_\mu\left(\sum_{i=1}^{N_0}\sum_{j=1}^{p} a_{ij}X_{i(i-1)}\mu_j, Y^p_{[\frac{N}{\alpha}]\alpha}\right) && \text{(proved above in induction)}\\
&= \pi_{\mathscr{L}_{N_i}}\left(\sum_{i=1}^{N_0}\langle [a_{ij}]_{j=1}^p, Y^p_{[\frac{N}{\alpha}]\alpha}\rangle X_{i(i-1)}\right) && \text{(by derivative formula)}\\
&= \langle [a_{ij}]_{j=1}^p, Y^p_{[\frac{N}{\alpha}]\alpha}\rangle X_{i(i-1)}.
\end{aligned}
$$

On the other hand,

$$
\mathrm{rank}_{\mathbb{F}}([a_{ij}]) = N_0 \quad \text{and} \quad \langle [a_{ij}]_{j=1}^p, Y^p_{[\frac{N}{\alpha}]\alpha}\rangle = 0 \quad \forall\, i \leq N_0
$$

imply $Y^p_{[\frac{N}{\alpha}]\alpha} = \mathbf{0}$. Thereby, $w = v$.

The proof is complete. ■

4. Computation and Additional Formulas

In this section, we discuss some important points appearing in practical computations, and then derive necessary formulas (not derived in the previous chapter) for implementation.

The first point is that it is imperative to compute all amplitude coefficients of the infinite level parametric normal form of an N_0-order parametric nonlinear center. On the other hand, computation of parametric normal form consumes much more efforts than normal forms without parameter. Therefore, it is at our advantage to postpone computation of terms involved with parameters. This is why we choose $\alpha = 2N_0 + 1$ for an N_0-order parametric nonlinear center. By this approach many of the parametric terms are eliminated by time and state maps in phases before it comes to computation of parameter maps. Thereby, it greatly increases the efficiency of symbolic computation.

The second point is that the choice of $\alpha = 2N_0 + 1$ in theoretical results is based on knowing the order of the system before determining the value for α. This is, however, not a practical approach as we do not know the order of the system in advance of the normal form computation. Thus, we need to evaluate the order in the process of computation; as a result, the value for α has to be computed during the parametric normal form computation. Indeed, we initially set the value $\alpha = 3$ (i.e. as if $N_0 = 1$) and in the process of computations we update the value of α according to the obtained information about the system under parametric normal form computation. Now one should recall that any change in the value of α results in a significant change in grading structure and thus its computation. Therefore, we need to carefully illustrate our approach such that it does not hamper our computations. By Lemma 3.8, N_0 and $a^{(N_0)}_{(N_0+1)N_0\mathbf{0}}$ are determined when $N = N_0$. Then, it is possible to process the normal form computation as usual. It, however, is important to notice that we are recursively seeking for the order of the system via Formula (39), while the grading function depends

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on $\tilde{N}_0$. Thus, the grading structure changes as $\tilde{N}_0$ grows; indeed for any $N_0 > \tilde{N}_0$ we have

$$\left\{v = \sum_{k=0}^{N} v_k \,|\, v_k \in \mathscr{L}_k(\delta_{\mathscr{L}}(\tilde{N}_0))\right\} \supseteq \left\{v = \sum_{k=0}^{N} v_k \,|\, v_k \in \mathscr{L}_k(\delta_{\mathscr{L}}(N_0))\right\}, \quad (50)$$

where $\mathscr{L} = \hat{\bigoplus} \mathscr{L}_k(\delta_{\mathscr{L}}(\tilde{N}_0))$ denotes the new formal decomposition associated with $\tilde{N}_0$, i.e. the grading structure associated with the grading function $\delta_{\mathscr{L}}(\tilde{N}_0)$. This facilitates us to freely truncate unnecessary terms associated with grading function $\delta_{\mathscr{L}}(\tilde{N}_0)$. As a consequence of set inclusion (50), some terms in certain grades may be also associated with a higher grade when $\tilde{N}_0$ increases to higher numbers. However, the most important point is that all of these terms (with increased grades) include parameters and thus they are eliminated at a step N greater than $2N_0 + 1$. Hence, no term reappears in normal form computation once it is eliminated. Therefore, this change of grading structure does not hamper the computation of parametric normal forms in our approach.

Our final point in this section deals with the fact that computing the image of the state transformation consumes much more computational efforts than time rescaling. Hence, it is vital (for an efficient computation) to use time rescaling rather than parametric state change of variable whenever it is feasible. Therefore, in this section the space $\mathcal{L}^{2N_0}_{N-2N_0}$ is further restricted and some formulas are derived accordingly. Note that $v_k^{(n),S}$ denotes the homogenous terms of grade k of the nth level normal form once the state transformation map associated with $n+1$th level is applied, i.e. $v_k^{(n),S} = \pi_{\mathscr{L}_k} \phi^S_{Y^{S,n+1}*} v^{(n)}$. Accordingly, $a^{(n),S}_{ij\mathbf{m}}, b^{(n),S}_{ij\mathbf{m}}, v^{(n),S,P}_{ij\mathbf{m}}$ etc. are denoted. Therefore, we have $v^{(n),S,P,T} = v^{(n+1)}$.

Parametric state variable 4.1 (see Lemma 3.1). Note that $\phi_{Y^{\eta,n}*}(n < 2N_0, \eta \in \{S, P, T\})$ does not change terms of grades less than n. In fact $\phi_{Y^{S,n}*}$ only affects the terms of grade n as well as the grades higher than $2n-1$. Therefore, $a^{([\frac{N}{2}])}_{ij\mathbf{0}} = a^{(N-1)}_{ij\mathbf{0}}, b^{([\frac{N}{2}])}_{ij\mathbf{0}} = b^{(N-1)}_{ij\mathbf{0}}, v^{(N)}_N = \mathbf{0}\,(0 < N < 2N_0)$, and

$$Y^{S,N} = \sum_{i+j=N+1,\, j+1\neq i} \frac{a^{([\frac{N}{2}])}_{ij\mathbf{0}}}{i-j-1} Y_{ij} + \frac{b^{([\frac{N}{2}])}_{ij\mathbf{0}}}{j+1-i} X_{ij} \quad (0 < N \leq 2N_0),$$

while

$$v^{(2N_0)}_{2N_0} = a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} X_{(N_0+1)N_0} + b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} Y_{(N_0+1)N_0}\ (a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} \neq 0).$$

We assume

$$a^{(n)}_{(N_0+1)N_0\mathbf{0}} = a^{(N_0)}_{(N_0+1)N_0\mathbf{0}} \neq 0 \quad (n \geq N_0),$$

which has been proved in Lemma 3.8.

Time rescaling maps are sufficient to eliminate the terms in the form of $Y_{(i+1)i}\mu_k\,(0 \leq i < N_0)$ and $Y_{(i+1)i}\,(N_0 < i)$, see Lemma 3.2. Therefore, $\mathcal{L}^{2N_0}_{N-2N_0} = \pi_X(\mathscr{L}_{H,k-2N_0})$ is defined for the target terms of $X_{(i+1)i}\,(N_0 < i < 2N_0)$. So,

$$Y^{S,N} = \frac{a^{\left(\frac{N}{2}+N_0\right)}_{(k+N_0+1)(N_0+k)\mathbf{0}}\delta_{N,2k+2N_0}}{2(N_0-k)a^{(N_0)}_{(N_0+1)N_0\mathbf{0}}} X_{(k+1)k} + \sum_{i+j=N+1,\, j+1\neq i} \frac{a^{\left(\left[\frac{N}{2}+N_0\right]\right)}_{ij\mathbf{0}}}{i-j-1} Y_{ij} + \frac{b^{\left(\left[\frac{N}{2}+N_0\right]\right)}_{ij\mathbf{0}}}{j+1-i} X_{ij}$$
$$+ \sum_{i+j=N-2N_0,\, j+1\neq i} \sum_{k=1}^{p} \frac{a^{\left(\left[\frac{N}{2}+N_0\right]\right)}_{ij\mathbf{e}_k}}{i-j-1} Y_{ij}\mu_k + \frac{b^{\left(\left[\frac{N}{2}+N_0\right]\right)}_{ij\mathbf{e}_k}}{j+1-i} X_{ij}\mu_k,$$

where $N < 4N_0$, $\delta_{N,N} = 1$, and $\delta_{N,2k+2N_0} = 0$ if $N \neq 2k + 2N_0$.

When $N = 4N_0$, all the terms with grades less than $4N_0$ in $v^{(N-1)}$ include parameters, except Y_{10}, $a^{(N_0)}_{(N_0+1)N_0\mathbf{0}} X_{(N_0+1)N_0}$, and $b^{(N_0)}_{(N_0+1)N_0\mathbf{0}} Y_{(N_0+1)N_0}$, see Lemma 3.2. Since

$$\pi_{\mathrm{span}_{\mathbb{F}}\{X_{(2N_0+1)2N_0}\}} \circ \mathrm{ad}_{\mathscr{L}}(Y_{10} + a^{(N_0)}_{(N_0+1)N_0\mathbf{0}} X_{(N_0+1)N_0} + b^{(N_0)}_{(N_0+1)N_0\mathbf{0}} Y_{(N_0+1)N_0}) = \{\mathbf{0}\},$$

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we have $\pi_{\text{span}_{\mathbb{F}}\{X_{(2N_0+1)2N_0}\}}(\text{Im}(L^{S,4N_0}_{(4N_0)})) = \{\mathbf{0}\}$. Therefore, $L^{S,4N_0}_{(4N_0)} = L^{S,4N_0}_{(1)}$, $\mathcal{L}^i_{4N_0-i} = \{\mathbf{0}\}\,(i \neq 0)$, and

$$\begin{aligned} Y^{S,4N_0} = &\sum_{i+j=2N_0,\, j+1\neq i}\sum_{k=1}^{p}\frac{a^{(3N_0)}_{ij\mathbf{e}_k}}{i-j-1}Y_{ij}\mu_k \\ &+\frac{b^{(3N_0)}_{ij\mathbf{e}_k}}{j+1-i}X_{ij}\mu_k \\ &+\sum_{i+j=4N_0+1,\, j+1\neq i}\frac{a^{(3N_0)}_{ij\mathbf{0}}}{i-j-1}Y_{ij} \\ &+\frac{b^{(3N_0)}_{ij\mathbf{0}}}{j+1-i}X_{ij}, \end{aligned}$$

where $a^{(4N_0-1)}_{ij\mathbf{0}} = a^{(3N_0)}_{ij\mathbf{0}}, b^{(4N_0-1)}_{ij\mathbf{0}} = b^{(3N_0)}_{ij\mathbf{0}}, a^{(4N_0-1)}_{ij\mathbf{e}_k} = a^{(3N_0)}_{ij\mathbf{e}_k}$, and $b^{(4N_0-1)}_{ij\mathbf{e}_k} = b^{(3N_0)}_{ij\mathbf{e}_k}$, see Lemma 3.2. Obviously, $\mathscr{N}^{(4N_0-1),S}_{4N_0} = \text{span}_{\mathbb{F}}\{X_{(2N_0+1)2N_0}, Y_{(2N_0+1)2N_0}\} = \pi_{\mathscr{L}_S}(\mathscr{L}_{H,4N_0})$.

Finally, consider the case in which $N > 4N_0$. Lemma 3.2 indicates that the terms in the form of $X_{(2N_0+1)2N_0}\mu^{\mathbf{m}}, Y_{(2N_0+1)2N_0}\mu^{\mathbf{m}}$ are eliminated with time rescaling associated with grade $N = 4N_0 + 2rN_0 + r$. Thus, it is sufficient to consider

$$\begin{aligned} \mathcal{L}^{2N_0}_{N-2N_0} = \{&v \in \pi_X(\mathscr{L}_{H,N-2N_0})\,| \\ &\pi_{\text{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\mu^{\mathbf{m}}\}}(v) = \mathbf{0}\;\forall\, r \in \mathbb{N}\} \end{aligned}$$

in order to kill the terms $X_{(i+1)i}\mu^{\mathbf{m}}\,(N_0 \le i, N = 2i + 2(r+1)N_0 + r)$ and $X_{(i+1)i}\,(2N_0 < i, N = 2i + 2N_0)$. By Lemma 2.3, the general solution to the homological equations of state variable is

$$\begin{aligned} Y^{S,N} = &\sum_{\{i,j,r\in\mathbb{N}_0\,|\,i+j+r2N_0+r=N+1,\, j+1\neq i\}}\;\sum_{\mathbf{m}\in\mathbb{N}_0^p}\frac{a^{\left(\left[\frac{N}{2}\right]+N_0\right)}_{ij\mathbf{m}}}{i-j-1}Y_{ij}\mu^{\mathbf{m}} + \frac{b^{\left(\left[\frac{N}{2}\right]+N_0\right)}_{ij\mathbf{m}}}{j+1-i}X_{ij}\mu^{\mathbf{m}} \\ &+\sum_{\{k,r\in\mathbb{N}_0\,|\,2(k+rN_0)+r=N-2N_0,\, k\neq N_0\}}\;\sum_{|\mathbf{m}|=r}\frac{a^{\left(\left[\frac{N}{2}\right]+N_0\right)}_{(N_0+k+1)(N_0+k)\mathbf{m}}}{2(N_0-k)a^{(N_0)}_{(N_0+1)N_0\mathbf{0}}}X_{(k+1)k}\mu^{\mathbf{m}}, \end{aligned} \tag{51}$$

where $a^{([\frac{N}{2}]+N_0)}_{ij\mathbf{m}} = a^{(N-1)}_{ij\mathbf{m}}, b^{([\frac{N}{2}]+N_0)}_{ij\mathbf{m}} = b^{(N-1)}_{ij\mathbf{m}}$, see Lemma 3.2. Consequently,

$$\begin{aligned} \mathcal{L}^U_k = &\,\text{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}\mu^{\mathbf{m}}\,|\,k = 2N_0 + 2rN_0 + r\} \\ &\oplus \pi_{\mathscr{L}_Y}(\mathscr{L}_{H,k}). \end{aligned} \tag{52}$$

Time rescaling 4.2 (see Lemma 3.2). Let

$$\begin{aligned} &L^{T,N}_{(N)}: \mathcal{R}^0_N \to \mathscr{L}_N, \\ &Y^{T,0}_N \mapsto Y^{T,0}_N v^{(0)}_0, \end{aligned}$$

where $N < 2N_0$, and $v^{(0)}_0 = Y_{10}$. Then, $\text{Im}(L^{T,N}_{(N)}) = \text{span}_{\mathbb{F}}\{Y_{(k+1)k}\,|\,2k = N\}$ and

$$Y^{T,N} = -b^{(2k-1),S,P}_{(k+1)k\mathbf{0}}Z_k, \tag{53}$$

where $b^{(k)}_{(k+1)k\mathbf{0}} = b^{(2k-1),S,P}_{(k+1)k\mathbf{0}}$ and $N = 2k$. Moreover, $\text{Im}(L^{T,2N_0}_{(2N_0)}) = \{\mathbf{0}\}$ and $Y^{T,2k+1} = Y^{T,2N_0} = 0\;\forall\, k < N_0$. Now assuming $2N_0 < N \le 4N_0$, we have

$$L^{T,N}_{(N)}: \mathcal{R}^0_N \oplus \mathcal{R}^{2N_0}_{N-2N_0} \to \mathscr{L}_N$$

$$(Y^{T,0}_N + Y^{T,2N_0}_{N-2N_0}) \mapsto Y^{T,0}_N v^{(0)}_0 + Y^{T,2N_0}_{N-2N_0} v^{(N-1),S,P}_{2N_0},$$

where $v^{(N-1),S,P}_{2N_0} = a^{(N-1),S,P}_{(N_0+1)N_0\mathbf{0}}X_{(N_0+1)N_0} + b^{(N-1),S,P}_{(N_0+1)N_0\mathbf{0}}Y_{(N_0+1)N_0}$. It is obvious that our reparametrization maps do not directly change $a^{(n)}_{(N_0+1)N_0\mathbf{0}}$ and $b^{(n)}_{(N_0+1)N_0\mathbf{0}}$. Further, $v^{(N)}_k = \mathbf{0}$ for $1 \le k < 2N_0$, $\pi_{\mathscr{L}_{2N_0}}(Y^{S,N}) = 0$ for $N < 4N_0$, and $\pi_{\text{span}_{\mathbb{F}}\{Z_{N_0}\}}(Y^{T,N}) = 0$ for $N < 4N_0$. Therefore, $a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} = a^{(n)}_{(N_0+1)N_0\mathbf{0}} \neq 0\;\forall\, n \ge 2N_0$ and $b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} = b^{(4N_0-1)}_{(N_0+1)N_0\mathbf{0}}$. Then,

$$\begin{aligned} \pi_{\mathscr{L}_Y}&(\text{Im}(L^{T,N}_{(N)})) \\ &= \text{span}_{\mathbb{F}}\left\{Y_{(2N_0+1)2N_0}\,\middle|\,N_0 = \frac{N}{4}\right\} \\ &\quad\oplus \text{span}_{\mathbb{F}}\{Y_{(i+1)i}\,|\,N = 2i\} \\ &\quad\oplus \text{span}_{\mathbb{F}}\{Y_{(i+1)i}\mu^{\mathbf{m}_1}\,|\,\mathbf{m}_1 \in \mathbb{N}_0^p, \\ &\qquad N = 2i + 2N_0 + 1\}, \end{aligned}$$

While

$$\pi_{\mathscr{L}_X}(\text{Im}(L^{T,N}_{(N)})) = \text{span}_{\mathbb{F}}\{X_{(2N_0+1)2N_0}\,|\,N = 4N_0\}.$$

Note that we have assumed $\mathrm{span}_{\mathbb{F}}\emptyset = \{\mathbf{0}\}$. Thus, the time solution is

$$Y^{T,2(i+N_0)+1} = \sum_{k=1}^{p} -b^{(2N_0)}_{(i+1)i\mathbf{e}_k} Z_i \mu_k,$$

$$Y^{T,N} = -b^{(2N_0)}_{(\frac{N}{2}+1)\frac{N}{2}\mathbf{0}} Z_{\frac{N}{2}}, \quad (4N_0 \neq N \in \mathbb{N}_e),$$

and $Y^{T,4N_0} = -(a^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}}/a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}})Z_{N_0} + \beta Z_{2N_0}$. Here, $a^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}} = a^{(4N_0-1),S,P}_{(2N_0+1)2N_0\mathbf{0}}$ and $\beta \in \mathbb{F}$ is to be determined. Because $\mathcal{L}^{2N_0}_{4N_0} = \{\mathbf{0}\}$ and $\mathcal{R}^{2N_0}_{2N_0} = \mathrm{span}_{\mathbb{F}}\{Z_{N_0}\}$, $b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} = b^{(4N_0-1),S,P}_{(N_0+1)N_0\mathbf{0}}$ is not necessarily equal to $b^{(4N_0-1),S,P,T}_{(N_0+1)N_0\mathbf{0}} = b^{(4N_0)}_{(N_0+1)N_0\mathbf{0}}$ (unlike $a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} = a^{(4N_0)}_{(N_0+1)N_0\mathbf{0}}$). Therefore,

$$v^{(4N_0-1),S,P}_{2N_0} = a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} X_{(N_0+1)N_0} + b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} Y_{(N_0+1)N_0},$$

$$b^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}} = b^{(4N_0-1),S,P}_{(2N_0+1)2N_0\mathbf{0}}, \quad v^{(4N_0)} = \phi^T_{Y^{T,4N_0}*}(v^{(4N_0-1),S,P}),$$

and thus

$$\pi_{\mathrm{span}_{\mathbb{F}}\{Y_{(2N_0+1)2N_0}\}} \circ \phi^T_{Y^{T,4N_0}*}(v^{(4N_0)}) = \left(\beta + b^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}} - \frac{a^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}} b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}}{a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}}\right) Y_{(2N_0+1)2N_0}.$$

Hence,

$$Y^{T,4N_0} = -\frac{a^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}}}{a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}} Z_{N_0} - \left(b^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}} - \frac{a^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}}}{a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}} b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}\right) Z_{2N_0}, \tag{54}$$

leading to $b^{(4N_0)}_{(N_0+1)N_0\mathbf{0}} = b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} - (a^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}}/a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}})$. Thus, we have that for $N > 4N_0$,

$$L^{T,N}_{(N)} : \mathcal{R}^0_N \oplus \mathcal{R}^{2N_0}_{N-2N_0} \to \mathscr{L}_N$$

$$(Y^{T,0}_N + Y^{T,2N_0}_{N-2N_0}) \mapsto Y^{T,0}_N v^{(0)}_0 + Y^{T,2N_0}_{N-2N_0} v^{(N-1)}_{2N_0},$$

where $v^{(m)}_{2N_0} = v^{(4N_0)}_{2N_0} = a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}} X_{(N_0+1)N_0} + b^{(4N_0)}_{(N_0+1)N_0\mathbf{0}} Y_{(N_0+1)N_0}$ $\forall\, m, m \geq 4N_0$. So,

$$\pi_{\mathscr{L}_Y}(\mathrm{Im}(L^{T,N}_{(N)})) = \mathrm{span}_{\mathbb{F}}\{b^{(4N_0)}_{(N_0+1)N_0\mathbf{0}} Y_{(2N_0+1)2N_0}\mu^{\mathbf{m}} \,|\, r \in \mathbb{N}, \mathbf{m} \in \mathbb{N}^p_0, N = 4N_0 + 2rN_0 + r\}$$
$$\oplus\, \mathrm{span}_{\mathbb{F}}\{Y_{(i+1)i}\mu^{\mathbf{m}} \,|\, (i+r)(i-N_0) \neq 0, \mathbf{m} \in \mathbb{N}^p_0, N = 2i + 2rN_0 + r\},$$

while

$$\pi_{\mathscr{L}_X}(\mathrm{Im}(L^{T,N}_{(N)})) = \mathrm{span}_{\mathbb{F}}\{X_{(2N_0+1)2N_0}\mu^{\mathbf{m}} \,|\, N = 4N_0 + 2rN_0 + r, \mathbf{m} \in \mathbb{N}^p_0, r \in \mathbb{N}\}.$$

Similar to Eq. (54), one can obtain the solution of time rescaling associated with $N > 4N_0$ by using the following formula:

$$Y^{T,N} = \sum_{r=\frac{N-4N_0}{2N_0+1} \in \mathbb{N}} \sum_{\mathbf{m} \in \mathbb{N}^p_0} \left(\frac{-a^{(N-1),S,P}_{(2N_0+1)2N_0\mathbf{m}}}{a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}} Z_{N_0} + \left(\frac{b^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}}{a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}} a^{([\frac{N}{2}]+N_0)}_{(2N_0+1)2N_0\mathbf{m}} \right.\right.$$
$$\left.\left. - \frac{a^{(3N_0)}_{(2N_0+1)2N_0\mathbf{0}}}{{a^{(2N_0)}_{(N_0+1)N_0\mathbf{0}}}^2} a^{(N-1),S,P}_{(2N_0+1)2N_0\mathbf{m}} - b^{([\frac{N}{2}]+N_0)}_{(2N_0+1)2N_0\mathbf{m}} \right) Z_{2N_0} \right) \mu^{\mathbf{m}}$$
$$+ \sum_{i=0, N_0 \neq i \neq 2N_0}^{[\frac{N}{2}]} \sum_{r=\frac{N-2i}{2N_0+1} \in \mathbb{N}_0} \sum_{\mathbf{m} \in \mathbb{N}^p_0} -b^{(N-1)}_{(i+1)i\mathbf{m}} Z_i \mu^{\mathbf{m}}, \tag{55}$$

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where $a^{(N-1),S,P}_{(2N_0+1)2N_0\mathbf{m}} = a^{([\frac{N}{2}]+N_0)}_{(2N_0+1)2N_0\mathbf{m}}$. Note that $\mathcal{R}^i_n\mathfrak{D}_k \subseteq \mathfrak{D}_{n+k}\ \forall\, k < i\,\&\, n \in \mathbb{N}$.

5. Systems Without Parameters

In this section, we present the simplest normal form for systems without parameters. We will not give a detailed proof on the uniqueness of the simplest normal form (56) since the method presented in the previous sections has to be modified, and Peng and Wang [2004] have already obtained it by a different approach, see also [Gazor & Yu, 2010, Corollary 6.1]. Assume the system is already described in a classical normal form, Peng and Wang [2004] have provided some recursive formulas for practical computation, see Lemma 4.10 from [Peng & Wang, 2004]. Independently, our results obtained in the previous sections are modified and efficiently implemented using Maple to compute the simplest normal form of such systems. Note that the approach described in this section is different from that of other sections for computing the parametric normal forms.

Corollary 5.1. *For any system of nonlinear center with no parameter,*

(1) [*Peng & Wang, 2004, Theorem 4.7*] *there exists a natural number N_0 such that the system can be transformed to the simplest normal from*:

$$v^{(\infty)} = Y_{10} + A_{N_0}X_{(N_0+1)N_0} + A_{2N_0}X_{(2N_0+1)2N_0} + \sum_{i=n}^{N_0} B_iY_{(i+1)i}, \quad A_{N_0} \neq 0, \tag{56}$$

where only state change of variable is used.

(2) *When time rescaling is also used, the simplest orbital equivalence is*

$$v^{(\infty)} = Y_{10} + A_{N_0}X_{(N_0+1)N_0} + A_{2N_0}X_{(2N_0+1)2N_0}, \quad A_{N_0} \neq 0. \tag{57}$$

Proof. Note that the degenerate spaces are different from the parametric system. For part (2) of Corollary 5.1, the degenerate spaces are still invariant and thus Eq. (56) can be easily obtained by excluding parameters from (and some adjustment on) Lemmas 3.1 and 3.2. However, for part (1) of Corollary 5.1 the degenerate space $\mathfrak{D}_n$ is not invariant and thus a modification in the state space decomposition (and as a result on the state solution $Y^{S,N}$) is required to keep them invariant. Indeed, for all $N \in \mathbb{N}$ we have

$$\mathfrak{D}_N = \begin{cases} \mathrm{span}_{\mathbb{F}}\{Y_{(k+1)k}\} & \text{for } N = 2k, 1 \le k < N_0, \\ \mathrm{span}_{\mathbb{F}}\{X_{(N_0+1)N_0}, Y_{(N_0+1)N_0}\} & \text{when } N = 2N_0, \\ \mathrm{span}_{\mathbb{F}}\{X_{(2N_0+1)2N_0}\} & \text{if } N = 4N_0, \\ \{\mathbf{0}\} & \text{otherwise,} \end{cases} \tag{58}$$

and $\mathrm{ad}_{\mathcal{L}^{2N_0}_{2n}}(\mathfrak{D}_{2k}) = \mathrm{ad}_{\mathrm{span}_{\mathbb{F}}\{X_{(n+1)n}\}}(\mathfrak{D}_{2k}) = \mathrm{span}_{\mathbb{F}}\{Y_{(k+n+1)(k+n)}\}$ for $0 < k < N_0$ and $N_0 < k + n$. The latter equation implies that at step $N = 2(N_0 + n)$, when the state solution is used to eliminate the terms of grade N, some already eliminated terms of the form $Y_{(k+n+1)(k+n)}$ at lower grades may appear again. However, noticing the formulas $\mathrm{ad}_{Y_{(k+n-N_0+1)(k+n-N_0)}}\mathfrak{D}_{2l} = \{0\}\ \forall\, l < N_0$ and $\mathrm{ad}_{Y_{(k+n-N_0+1)(k+n-N_0)}}\mathfrak{D}_{2N_0} = \mathrm{span}_{\mathbb{F}}\{Y_{(k+n+1)(k+n)}\}$, we may properly choose the term $Y_{(k+n-N_0+1)(k+n-N_0)}$ in the state solution $Y^{S,N}$ to make the degenerate spaces invariant and thus, the normal form computation is carried out in a forward process.

In order to obtain the results for the unique normal form of N_0-order nonlinear center, without using time rescaling (part (1)), we need to also use the terms in the form of $Y_{(k+1)k}$ in the state map. Therefore, let

$$Y^{S,N}_{(1)} = \sum_{\{i,j\in\mathbb{N}_0\,|\, i+j=N+1,\, j+1\neq i\}} \frac{a^{(N-1)}_{ij\mathbf{0}}}{i-j-1}Y_{ij} + \frac{b^{(N-1)}_{ij\mathbf{0}}}{j+1-i}X_{ij} + \sum_{\{k\in\mathbb{N}\,|\,2k=N-2N_0, k\neq N_0\}} \frac{a^{(N-1)}_{(N_0+k+1)(N_0+k)\mathbf{0}}}{2(N_0-k)a^{(N-1)}_{(N_0+1)N_0\mathbf{0}}}X_{(k+1)k} \tag{59}$$

and

$$Y^{S,N}_{(2)} = \sum_{\{k\in\mathbb{N}\,|\,2k=N-2N_0\}} -\frac{b^{(N-1)}_{(N_0+k+1)(N_0+k)\mathbf{0}}}{2ka^{(N-1)}_{(N_0+1)N_0\mathbf{0}}} Y_{(k+1)k}.$$

Then, $Y^{S,N}_{(1)}$ and $Y^{S,N}_{(2)}$ affect the system in two phases, respectively. Splitting this to two phases in principle is not necessary; however, this phenomenon for systems without parameter does not significantly reduce the efficiency of computation. Therefore, the corresponding formulas for one phase computation are omitted. Finally, uniqueness follows from a similar argument in the proof of Theorem 3.11. ■

Note that the origin is said to be a weak focus of order N_0 for the vector field v when v satisfies Corollary 5.1, see p. 385 in [Chow *et al.*, 1994] for more details.

6. Illustrative Examples

We have efficiently implemented the formulas derived in Secs. 3 and 4 using Maple. We now present the examples taken from [Yu & Leung, 2003; Yu & Chen, 2007] to give a comparison to illustrate the applicability of our results. The following example is the generic nonlinear center which has the order of one. Therefore, it only needs one parameter in the correct place to be a full parametric dimension. Indeed, Yu *et al.* chose the parameters in such a way that the presented system is full parametric dimension and thus our results can be simply applied to this example.

Example 6.1. We execute our Maple code on the example of generic nonlinear center of order one with one parameter taken from [Yu & Leung, 2003, Section 4.1] for $A = 1, B = 2 + \mu$. The system is described by

$$\begin{aligned} &Y_{10} + \frac{\mu X_{10} - \mu X_{01}}{2} - \frac{\mu Y_{20} + \mu Y_{02}}{4} \\ &\quad + \frac{\mu Y_{11} + X_{20} - X_{02}}{2} + \frac{1}{2}X_{30} \\ &\quad + \frac{3X_{12} - 3X_{21}}{8} + \frac{Y_{21} + Y_{12}}{8} \\ &\quad - \frac{X_{03} + Y_{30} + Y_{03}}{8} \end{aligned} \tag{60}$$

in our notations. Our Maple output for the normal form up to grade 8 is

$$\begin{aligned} &Y_{10} + \frac{1}{2}\mu X_{10} - \frac{3}{8}X_{21} - \frac{5}{72}Y_{21} \\ &\quad - \frac{1}{216}\mu Y_{21} - \frac{1699877}{20995200}\mu^2 Y_{21}, \end{aligned} \tag{61}$$

which is an order one nonlinear center with full parametric dimension (i.e. $\text{rank}_{\mathbb{F}}(A^{(\infty)}) = 1$), see Lemmas 3.4–3.5. By Theorem 3.9, this system can be transformed into (in terms of polar coordinates ρ, θ)

$$\begin{aligned} \frac{d\rho}{dt} &= \rho\left(\frac{1}{2}\mu - \frac{3}{8}\rho^2\right) \\ \frac{d\theta}{dt} &= 1 - \frac{5}{72}\rho^2 - \frac{1}{216}\mu\rho^2 \\ &\quad - \frac{1699877}{20995200}\mu^2\rho^2 + \rho^2\sum_{k=3}^{\infty} b^{(3k+2)}_{21k}\mu^k. \end{aligned}$$

Yu and Leung [2003] presented something which only differs with ours in the phase equation, their phase equation is,

$$\begin{aligned} \frac{d\theta}{dt} &= 1 - \frac{5}{72}\rho^2 - \frac{1}{288}\rho^4 \\ &\quad - \frac{1699877}{37324800}\rho^6 + \sum_{k=5}^{\infty} b^{(2k)}_{(k+1)k}\rho^{2k}. \end{aligned} \tag{62}$$

However, this does not affect the bifurcation analysis of the system since the amplitude equations are identical. Therefore, our result is consistent with the result in [Yu & Leung, 2003], where a bifurcation analysis for this system is also given.

Generally parametric systems may have more parameters than required to be full parametric dimension. Our Maple program can obtain the parametric normal form of such systems as well. As mentioned before, the above example only needs one parameter to have full parameter dimension. However, if we add an extra (not necessary) parameter to the system, its parametric normal form can yet be computed with our Maple program. We consider the same example as in [Yu & Leung, 2003] in Sec. 4.1 for the case of $A = 1 + \mu_1$ and $B = 2 + \mu_2$ which is a generic nonlinear center (order one) with two parameters (one extra than required to be full parametric dimension). For this case our code yields the normal form up to

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grade 8 as

$$Y_{10} + \left(\frac{1}{2}\mu_1 - \mu_2\right) X_{10} - \frac{3}{8}X_{21} - \left(\frac{5}{72} + \frac{\mu_1 + 19\mu_2}{216} + \frac{1699877}{20995200}\mu_1^2 + \frac{3112603}{5248800}\mu_1\mu_2 + \frac{6896297}{5248800}\mu_2^2\right) Y_{21}.$$

Thus, by Lemma 3.5 we have $\mathrm{rank}_{\mathbb{F}}(A^{(\infty)}) = 1$. If we set $\mu_2 = 0$ in B of the original system and in the above normal form (i.e. before and after normal form computation), either of these, lead to the parametric normal form and the original system for one parameter case (see Eq. (61)), as expected. This shows that the output of our code for two parameters is consistent with the single parameter. Succinctly stated, if we have more parameters in the system than its parametric dimension, then there is no problem in computing parametric normal form of the system with our Maple code.

In order to test our Maple program for systems of nonlinear center of higher order with multiple parameters, we again take an example from [Yu & Chen, 2007].

Example 6.1. Consider the nonlinear electric circuit described by Yu and Chen [2007]. It is easy to see that their system, by a time rescaling, (see Eq. (57) in Sec. 4.2 of [Yu & Chen, 2007]) can be transformed to:

$$\begin{aligned}\dot{x}_1 &= x_2 - \frac{5}{3}\mu_1 x_1 - \frac{20}{9}(\mu_1 + \mu_2)x_2 \\ &\quad - \frac{5}{9}\beta_1(3x_1 + 4x_2)^2 \\ &\quad + \frac{5}{9}\beta_2(3x_1 + 4x_2)^3 + \frac{500}{9}\beta_3 x_2^3, \\ \dot{x}_2 &= -x_1 + \frac{5}{3}\mu_2 x_2 - \frac{125}{3}\beta_3 x_2^3,\end{aligned} \tag{63}$$

or equivalently

$$\begin{aligned}&Y_{10} - \frac{5}{6}\mu_1 X_{10} + \frac{5}{6}\mu_2 X_{10} + \frac{5}{6}\mu_1 X_{01} - \frac{10}{9}(\mu_1 + \mu_2)Y_{10} - \frac{10}{9}(\mu_1 + \mu_2)Y_{01} + \frac{5}{6}\mu_2 X_{01} \\ &\quad + \frac{1625}{324}Y_{21} + \frac{625}{432}X_{21} - \frac{125}{72}Y_{11} - \frac{5}{6}X_{20} + \frac{5}{6}X_{02} - \frac{35}{144}Y_{20} - \frac{35}{144}Y_{02} - \frac{1625}{432}X_{12} \\ &\quad + \frac{1625}{324}Y_{1,2} + \frac{5}{972}Y_{03} - \frac{2255}{1296}X_{03} + \frac{1255}{1296}X_{30} + \frac{5}{972}Y_{30},\end{aligned} \tag{64}$$

in our notation (where $\beta_1 = 1/4, \beta_2 = 2/27$ and $\beta_3 = 1/6$). This system is nonlinear center with two parameters.

By executing our Maple code we obtain the parametric normal form up to grade 8, given by

$$Y_{10} - \frac{5}{6}(\mu_1 - \mu_2)X_{10} + \left(\frac{9557795}{2343168}\mu_1 - \frac{86585885}{7029504}\mu_2\right) X_{21} + \frac{44140625}{2519424}X_{32} + \frac{399447272359079}{1372609069056}Y_{32}.$$

Let

$$B = \begin{bmatrix} -\dfrac{5}{6} & \dfrac{5}{6} \\ \dfrac{9557795}{2343168} & -\dfrac{86585885}{7029504} \end{bmatrix} \quad \text{and}$$

$$\begin{pmatrix}\nu_1 \\ \nu_2\end{pmatrix} = B\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}.$$

Then the above system is equivalent to (since $\det B \neq 0$)

$$Y_{10} + \nu_1 X_{10} + \nu_2 X_{21} + \frac{44140625}{2519424}X_{32} + \frac{399447272359079}{1372609069056}Y_{32}.$$

Therefore, $\mathrm{rank}_{\mathbb{F}}(A^{(\infty)}) = 2$ infers from Lemma 3.5. This and Lemma 3.4 confirms the fact that the system is of order two and is of full parametric dimension. By Theorem 3.9 and using a time rescaling and then representing it in polar coordinates to compare our result with that of [Yu & Chen, 2007], we obtain

$$\begin{aligned}\frac{d\rho}{dt} &= \frac{3}{5}\rho\left(\nu_1 + \nu_2\rho^2 + \frac{44140625}{2519424}\rho^4\right) \\ \frac{d\theta}{dt} &= \frac{3}{5}\left(1 + \frac{399447272359079}{1372609069056}\rho^4 + \cdots\right)\end{aligned} \tag{65}$$

The normal form given by Eq. (61) in [Yu & Chen, 2007] is

$$\frac{d\rho}{dt}\partial_\rho + \frac{d\theta}{dt}\partial_\theta = \frac{3}{5}\rho\left(\nu_1 + \nu_2\rho^2 + \frac{44140625}{2519424}\rho^4\right)\partial_\rho + \left(\frac{3}{5} + \frac{2811217217}{439344000}\rho^2 + \cdots\right)\partial_\theta. \quad (66)$$

Since the amplitude equations in (65) and (66) are identical, these two system of equations result in the same bifurcation and stability analysis.

7. Conclusions

Formal decomposition method is presented and applied through a notion of invariant spaces. This method intends to compute the simplest parametric normal form of systems with multiple parameters. The simplest parametric normal form of any system of nonlinear center with full parametric dimension is obtained via this method. Maple programs have been developed to compute the simplest parametric normal form and also the order of such systems. Examples are provided to illustrate the applicability of our results.

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