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Global and non-global center conditions of generalized Liénard systems[☆]

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ABSTRACT

In this paper, we study global dynamics of generalized Liénard systems, and establish the necessary and sufficient conditions for generalized polynomial Liénard systems to have a center, a global center and an unbounded non-global center, respectively. As a corollary, we improve some known results and confirm a conjecture on global center conditions of a polynomial Liénard system.

1. Introduction

It is known that a differential equation of the form,

$$x'' + f(x)x' + g(x) = 0, \tag{1.1}$$

is called a Liénard equation. The equation plays an important role in the qualitative theory of dynamical systems. It is also an important model in applied sciences. The above 2nd-order equation can be transferred to the following differential system of first order,

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{1.2}$$

Let $F(x) = \int_0^x f(x)dx$. Then system (1.2) is equivalent to the following system,

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x) \tag{1.3}$$

which is called Liénard system. A more general form is given by

$$\dot{x} = h(y) - F(x), \quad \dot{y} = -g(x), \tag{1.4}$$

which is called generalized Liénard system.

There have been many studies on the local or global dynamical behaviors of the systems (1.2), (1.3) and (1.4). Cherkas [1], Christiopher [2], Coll, Prohens and Gasull [3] studied the center problem of polynomial or discontinuous Liénard differential equations. Chen, Li, and Yu [4], Han and Romanovski [5] studied the nilpotent center problem of certain class of Liénard systems.

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Chen, Li and Zhang [6], Garcia and Llibre [7], He, Llibre and Xiao [8], He and Xiao [9], Llibre and Valls [10] studied the global center problem of Liénard systems and some Hamiltonian polynomial differential systems. Christopher and Lynch [11], Han [12], Han, Tian and Yu [13], Jiang and Han [14], Liu and Han [15], Lloyd and Lynch [16], Tian, Han and Xu [17], Xiong and Han [18,19], Yu and Han [20] studied Hopf bifurcations and the number of limit cycles appearing in certain classes of Liénard systems.

For polynomial Liénard systems, an important topic is to find the conditions under which a global center appears. A singular point of the system (1.4) is called a center if all nontrivial orbits near the point are closed. A center is said to be global if all nontrivial orbits on the plane are closed. In [10] the centers are classified in three types: if the Jacobian of the system evaluated at a center has a pair of purely imaginary eigenvalues, then it is a linear type center; if both eigenvalues are zero but its linear part is not identically zero, then it is a nilpotent type center; if its linear part of the system is identically zero, then it is a degenerate center. The authors of [10] established a necessary and sufficient condition for the origin of the polynomial Liénard system (1.2) to be a global linear type center. In [7] the authors considered the following polynomial Liénard system,

$$\dot{x} = y, \quad \dot{y} = -x^{2n-1} - yf(x), \tag{1.5}$$

where

$$f(x) = \sum_{j=k}^{m} b_j x^j, \quad f(-x) = -f(x),\ n \geq 2,\ m \geq k \geq 1.$$

The following theorem can be found in [7].

Theorem 1.1 (*[7]*). *The polynomial differential system (1.5) has a center at the origin. Then the following statements hold:*

(i) The center is not global if $2n < m + 2$.
(ii) The center is global if either $f(x) \equiv 0$, *or* $k = n - 1 = m$ *with* $b_k^2 - 4n < 0$ *and* $2n > m + 2$.

Based on Theorem 1.1, the authors of [7] posed the conjecture given below.

Conjecture (*[7]*). The center at the origin of (1.5) is global if and only if either $f(x) \equiv 0$, or $k = n - 1 = m$ with $b_k^2 - 4n < 0$.

Later, the authors of [6] considered the system (1.2) with

$$g(x) = \sum_{j=r}^{s} a_j x^j, \quad f(x) = \sum_{j=k}^{m} b_j x^j,\ a_s a_r b_k b_m \neq 0, \tag{1.6}$$

and established the following theorem.

Theorem 1.2 (*[6]*). *The system (1.2) satisfying (1.6) exhibits a global center at the origin if and only if the following four conditions are satisfied:*

(i) $xg(x) > 0$, *for all* $x \neq 0$;
(ii) s *is odd,* $s > 2m + 1$, $a_s > 0$; *or* $s = 2m + 1$ *and* $4(m + 1)a_s b_m^{-2} > 1$:
(iii) $F(x_1) = F(x_2)$ *if* $G(x_1) = G(x_2)$ *for all* $x_1 < 0 < x_2$, *where* $G(x) = \int_0^x g(x)dx$, $F(x) = \int_0^x f(x)dx$;
(iv) either r *is odd,* $1 \leq r < 2k + 1$, $k \geq 1$, $a_r > 0$, *or* $r = 2k + 1 \geq 3$ *and* $b_k^2 - 2(r + 1)a_r < 0$.

Moreover, the global center is of linear type if $r = 1$, *and nilpotent type if* $r > 1$.

In this paper, we consider the generalized Liénard system (1.4), in which h, F and g are all polynomials. We establish the necessary and sufficient conditions for the system to have a center, an unbounded non-global center or a global center, respectively. The notation used for the unbounded non-global center is first introduced in the present work. Our results show that the origin of (1.5) is always not a center. We will present an improvement to Theorems 1.1 and 1.2, and confirm the conjecture stated above (see the conclusion (II) of Theorem 3.5). We will also revisit the system considered in [21] and obtain some new results on the conditions of the existence of a non-global center.

In the next section, we present certain preliminary theorems which are needed in Section 3 for proving our main results. In Section 4, we classify the center types for some lower degree generalized polynomial Liénard systems.

2. Preliminaries

In this section, we consider the generalized Liénard system (1.4), where h, F, and g are C^1 functions on the plane. For a general system of the form (1.4), we provide several preliminary theorems which will be used in Section 3. Some of them are relative to local properties of orbits, while the others are about global properties of orbits.

2.1. Local behavior of the origin

In order to study local properties of orbits for the system (1.4), we need to propose certain basic assumptions. More precisely, suppose that there exists a constant $\delta > 0$ such that

$$F(0) = h(0) = g(0) = 0, \quad xg(x) > 0, \quad yh(y) > 0, \quad 0 < |x| < \delta, \quad 0 < |y| < \delta. \tag{2.1}$$

It is clear that system (1.4) has a singularity at the origin. Define two curves as follows:

$$C^+ = \{(x, y) \mid h(y) = F(x),\ x > 0\},$$

and

$$C^- = \{(x, y) \mid h(y) = F(x),\ x < 0\}.$$

Also, for the sake of convenience, let

$$b(p) = (p+1)\left(\frac{p+1}{p}\right)^{\frac{p}{p+1}}, \quad Q(x) = \frac{F(x)}{[G(x)]^{\frac{p}{p+1}}}, \quad x \neq 0, \tag{2.2}$$

where $p > 0$, $G(x) = \int_0^x g(u)du$.

On local behavior near the origin of system (1.4), Han [22] obtained the following two theorems, the first of which was proved by applying Theorem 1.3 in [23], and can be found in Theorem 8.5 of the book [24].

Theorem 2.1 ([22,23]). *Suppose that (2.1) holds and that the function $h(y)$ is monotonically increasing with respect to y for $|y| < \delta$. Further, assume that there exist constants $p > 0$ and $\epsilon > 0$ such that*

$$|h(y)| = |y|^p(1 + O(|y|^\epsilon)) \quad \text{for } 0 < |y| \ll 1. \tag{2.3}$$

Let $Q(x)$ and $b(p)$ be given by (2.2). Then, the following conclusions are true.

(i) If $\lim_{x\to 0^+} \sup Q(x) < b(p)$, then there exists a neighborhood U_1 of the origin such that for any point $P_1 \in U_1 \cap C^+$, the positive semi-orbit $\gamma^+_{P_1}$ of system (1.4) passing through the point P_1 must intersect the negative y-axis.

(ii) If $\lim_{x\to 0^-} \inf Q(x) > -b(p)$, then there exists a neighborhood U_2 of the origin such that for any point $P_2 \in U_2 \cap C^-$, the positive semi-orbit $\gamma^+_{P_2}$ of system (1.4) passing through the point P_2 must intersect the positive y-axis.

(iii) If $\lim_{x\to 0^+} \inf Q(x) > -b(p)$, then there exists a neighborhood U_3 of the origin such that for any point $P_3 \in U_3 \cap C^+$, the negative semi-orbit $\gamma^-_{P_3}$ of system (1.4) passing through the point P_3 must intersect the positive y-axis.

(iv) If $\lim_{x\to 0^-} \sup Q(x) < b(p)$, then there exists a neighborhood U_4 of the origin such that for any point $P_4 \in U_4 \cap C^-$, the negative semi-orbit $\gamma^-_{P_4}$ of system (1.4) passing through the point P_4 must intersect the negative y-axis.

Theorem 2.2 ([22]). *Suppose that (2.1) and (2.3) are satisfied and that the function $h(y)$ is monotonically increasing with respect to y for $|y| < \delta$. Let q be a constant satisfying $q > \frac{p}{p+1}$. Then the following results hold.*

(i) If $F(x) \geq b(p)[G(x)]^{\frac{p}{p+1}} - [G(x)]^q$ for $0 < x \ll 1$, then there exists a point $P \in C^+$ near the origin such that the positive semi-orbit γ^+_P of system (1.4) approaches the origin through only first quadrant.

(ii) If $F(x) \leq -b(p)[G(x)]^{\frac{p}{p+1}} + [G(x)]^q$ for $0 < -x \ll 1$, then there exists a point $P \in C^-$ near the origin such that the positive semi-orbit γ^+_P of system (1.4) approaches the origin through only third quadrant.

(iii) If $F(x) \leq -b(p)[G(x)]^{\frac{p}{p+1}} + [G(x)]^q$ for $0 < x \ll 1$, then there exists a point $P \in C^+$ near the origin such that the negative semi-orbit γ^-_P of system (1.4) approaches the origin through only fourth quadrant.

(iv) If $F(x) \geq b(p)[G(x)]^{\frac{p}{p+1}} - [G(x)]^q$ for $0 < -x \ll 1$, then there exists a point $P \in C^-$ near the origin such that the negative semi-orbit γ^-_P of system (1.4) approaches the origin through only second quadrant.

In order to derive further results which will be applied conveniently by using the above theorems, we assume that the following two limits:

$$Q_0^+ = \lim_{x\to 0^+} Q(x), \quad Q_0^- = \lim_{x\to 0^-} Q(x) \tag{2.4}$$

exist, where $Q(x)$ is given in (2.2). Note that it is possible to have that $|Q_0^\pm| = +\infty$. Also, suppose that there exist constants $h_0 > 0$, $p > 0$, and $\sigma > 0$ such that

$$|h(y)| = h_0|y|^p(1 + O(|y|^\sigma)), \quad \text{for } 0 < |y| \ll 1. \tag{2.5}$$

Theorem 2.3. *Suppose that (2.1), (2.4) and (2.5) hold and that the function $h(y)$ is monotonically increasing with respect to y for $|y| < \delta$. Further, if $-h_0^{\frac{1}{p+1}} b(p) < Q_0^+ < h_0^{\frac{1}{p+1}} b(p)$, where $b(p)$ and Q_0^+ are given by (2.2) and (2.4), then there exists $\bar{y}_0 > 0$ such that*

(i) for any $y_0 \in (0, \bar{y}_0)$, the positive semi-orbit $\gamma^+_{y_0}$ of system (1.4) passing through the point $(0, y_0)$ must intersect the negative y-axis;

(ii) for any $y_0 \in (-\bar{y}_0, 0)$, the negative semi-orbit $\gamma^-_{y_0}$ of system (1.4) passing through the point $(0, y_0)$ must intersect the positive y-axis.

Proof. By introducing the change of variables $x \to x$, $y \to h_0^{-\frac{1}{p}} y$, we can transform the system (1.4) to

$$\dot{x} = \tilde{h}(y) - F(x), \quad \dot{y} = -\tilde{g}(x), \tag{2.6}$$

where $\tilde{h}(y) = h(h_0^{-\frac{1}{p}} y) = \mathrm{sgn}(y)|y|^p(1 + O(|y|^\sigma))$, $\tilde{g}(x) = h_0^{\frac{1}{p}} g(x)$. Denote by $\tilde{C}^+$ the following curve

$$\tilde{C}^+ = \{(x, y) \mid \tilde{h}(y) = F(x),\ x > 0\}.$$

Now, we apply the conclusions (i) and (iii) of Theorem 2.1 to (2.6) to get a neighborhood U of the origin of system (2.6) such that for any point $P \in U \cap \tilde{C}^+$, the positive semi-orbit γ_P^+ of (2.6) passing through the point intersects the negative y-axis at a point $(0, -y_1)$ and the negative semi-orbit γ_P^- passing through the point intersects the positive y-axis at a point $(0, y_2)$. See Fig. 2.1.(a).

Next, we fix the point P and let $\bar{y}_0 = \min\{y_1, y_2\}$. Then, by (2.1) it is easy to see that for any $y_0 \in (0, \bar{y}_0)$, the positive semi-orbit $\gamma_{y_0}^+$ of (2.6) passing through the point $(0, y_0)$ intersects the negative y-axis, while the negative semi-orbit $\gamma_{-y_0}^-$ of (2.6) passing through the point $(0, -y_0)$ intersects the positive y-axis. This completes the proof. □

Theorem 2.4. *Suppose that (2.1), (2.4) and (2.5) hold and that the function $h(y)$ is monotonically increasing with respect to y for $|y| < \delta$.*

- *(i) If either $Q_0^+ > h_0^{\frac{1}{p+1}} b(p)$, or $Q_0^+ = h_0^{\frac{1}{p+1}} b(p)$ and there exists a constant $\varepsilon > 0$ such that $Q(x) = Q_0^+ + O([G(x)]^\varepsilon)$ for $0 < x \ll 1$, then there exists $\bar{y}_0 > 0$ such that for any $y_0 \in (0, \bar{y}_0)$, the positive semi-orbit $\gamma_{y_0}^+$ of system (1.4) passing through the point $(0, y_0)$ approaches the origin through only first quadrant.*
- *(ii) If either $Q_0^+ < -h_0^{\frac{1}{p+1}} b(p)$, or $Q_0^+ = -h_0^{\frac{1}{p+1}} b(p)$ and there exists a constant $\varepsilon > 0$ such that $Q(x) = Q_0^+ + O([G(x)]^\varepsilon)$ for $0 < x \ll 1$, then there exists $\bar{y}_0 > 0$ such that for any $y_0 \in (-\bar{y}_0, 0)$, the negative semi-orbit $\gamma_{y_0}^-$ of system (1.4) passing through the point $(0, y_0)$ approaches the origin through only fourth quadrant.*

Proof. Because of the similarity, we only prove conclusion (i) of the theorem. First consider the case $Q_0^+ > h_0^{\frac{1}{p+1}} b(p)$, where $b(p)$ and Q_0^+ are given by (2.2) and (2.4). In this case, we have that

$$Q_0^+ = \lim_{x\to 0^+} Q(x) = \lim_{x\to 0^+} \frac{F(x)}{[G(x)]^{\frac{p}{p+1}}} > h_0^{\frac{1}{p+1}} b(p),$$

which implies that

$$\frac{F(x)}{[\tilde{G}(x)]^{\frac{p}{p+1}}} = \frac{F(x)}{h_0^{\frac{1}{p+1}}[G(x)]^{\frac{p}{p+1}}} > b(p), \quad 0 < x \ll 1,$$

where $\tilde{G}(x) = \int_0^x \tilde{g}(u)du = h_0^{\frac{1}{p}} G(x)$. Therefore, there exists a constant $q > \frac{p}{p+1}$ such that

$$F(x) > b(p)[\tilde{G}(x)]^{\frac{p}{p+1}} - [\tilde{G}(x)]^q, \quad \text{for } 0 < x \ll 1.$$

By applying the conclusion (i) in Theorem 2.2 to (2.6), there exists a point $P \in \tilde{C}^+$ near the origin such that the positive semi-orbit γ_P^+ of system (2.6) approaches the origin through only first quadrant. See Fig. 2.1.(b). Thus, the first part of conclusion (i) is proved.

Now, suppose that $Q_0^+ = h_0^{\frac{1}{p+1}} b(p)$ and there exists a constant $\varepsilon > 0$ such that $Q(x) = Q_0^+ + O([G(x)]^\varepsilon)$, for $0 < x \ll 1$. For the given $\varepsilon > 0$, there exists $x_1 > 0$ and $N > 0$ such that when $0 < x < x_1$,

$$|Q(x) - Q_0^+| < N[G(x)]^\varepsilon.$$

Then, we have $Q(x) > Q_0^+ - N[G(x)]^\varepsilon = Q_0^+ - N[G(x)]^{\frac{\varepsilon}{2}}[G(x)]^{\frac{\varepsilon}{2}}$, for $0 < x \ll 1$. Hence, by the definition of the function Q in (2.2), we obtain that

$$\frac{F(x)}{[\tilde{G}(x)]^{\frac{p}{p+1}}} = \frac{F(x)}{h_0^{\frac{1}{p+1}}[G(x)]^{\frac{p}{p+1}}} > b(p) - h_0^{-\frac{1}{p+1}} N[G(x)]^{\frac{\varepsilon}{2}}[G(x)]^{\frac{\varepsilon}{2}} > b(p) - [\tilde{G}(x)]^{\frac{\varepsilon}{2}}, \quad 0 < x \ll 1,$$

which leads to

$$F(x) > b(p)[\tilde{G}(x)]^{\frac{p}{p+1}} - [\tilde{G}(x)]^q, \quad \text{for } 0 < x \ll 1,$$

where $q = \frac{\varepsilon}{2} + \frac{p}{p+1}$. Therefore, applying conclusion (i) in Theorem 2.2 again to (2.6), the second part of conclusion (i) follows. The proof is complete. □

With a similar procedure, we can prove the following two theorems by Theorems 2.1 and 2.2. We omit the proofs due to the similarity.

Theorem 2.5. *Suppose that (2.1), (2.4) and (2.5) hold and that the function $h(y)$ is monotonically increasing with respect to y for $|y| < \delta$. If $-h_0^{\frac{1}{p+1}} b(p) < Q_0^- < h_0^{\frac{1}{p+1}} b(p)$, where $b(p)$ and Q_0^- are given by (2.2) and (2.4), then there exists $\bar{y}_0 > 0$ such that*

- *(i) for any $y_0 \in (0, \bar{y}_0)$, the negative semi-orbit $\gamma_{y_0}^-$ of system (1.4) passing through the point $(0, y_0)$ must intersect the negative y-axis;*

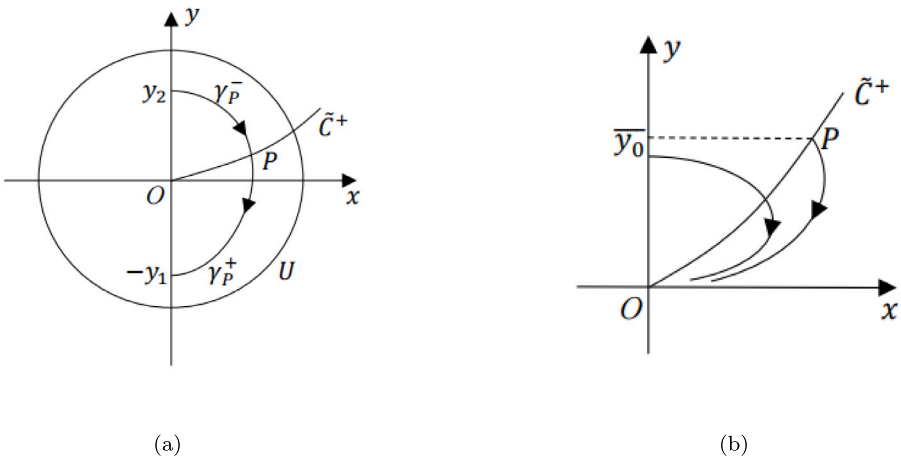


Fig. 2.1. (a) The orbit γ_P of (2.6) in Theorem 2.3; and (b) the orbit γ_P^+ of (2.6) in Theorem 2.4.

(ii) for any $y_0 \in (-\bar{y}_0, 0)$, the positive semi-orbit $\gamma_{y_0}^+$ of system (1.4) passing through the point $(0, y_0)$ must intersect the positive y-axis.

Theorem 2.6. *Suppose that (2.1), (2.4) and (2.5) hold and that the function $h(y)$ is monotonically increasing with respect to y for $|y| < \delta$.*

- *(i) If either $Q_0^- > h_0^{\frac{1}{p+1}} b(p)$, or $Q_0^- = h_0^{\frac{1}{p+1}} b(p)$ and there exists a constant $\epsilon > 0$ such that $Q(x) = Q_0^- + O([G(x)]^\epsilon)$ for $0 < -x \ll 1$, then there exists $\bar{y}_0 > 0$ such that for any $y_0 \in (0, \bar{y}_0)$, the negative semi-orbit $\gamma_{y_0}^-$ of system (1.4) passing through the point $(0, y_0)$ approaches the origin through only second quadrant.*
- *(ii) If either $Q_0^- < -h_0^{\frac{1}{p+1}} b(p)$, or $Q_0^- = -h_0^{\frac{1}{p+1}} b(p)$ and there exists a constant $\epsilon > 0$ such that $Q(x) = Q_0^- + O([G(x)]^\epsilon)$ for $0 < -x \ll 1$, then there exists $\bar{y}_0 > 0$ such that for any $y_0 \in (-\bar{y}_0, 0)$, the positive semi-orbit $\gamma_{y_0}^+$ of system (1.4) passing through the point $(0, y_0)$ approaches the origin through only third quadrant.*

Next, we present a theorem on center-focus problem which was originally proved in [23] and appeared as a corollary in [25]. However, the statement here is different from that given in [23,25]. In fact, the following theorem is a direct corollary of Theorems 2.3 and 2.5 and the relevant results in [23,25].

Theorem 2.7. *[23,25] Consider the system (1.4) satisfying (2.1), (2.4) and (2.5). Suppose that the function $h(y)$ is monotonically increasing with respect to y for $|y| < \delta$. Also, assume that $-h_0^{\frac{1}{p+1}} b(p) < Q_0^\pm < h_0^{\frac{1}{p+1}} b(p)$, where $b(p)$, Q_0^+ and Q_0^- are given by (2.2) and (2.4). Let $\alpha(x) < 0$ be the unique function satisfying $G(\alpha(x)) = G(x)$ for $0 < x \ll 1$, where $G(x) = \int_0^x g(u)du$. Then, the system (1.4) has a stable focus (resp., a center, an unstable focus) at the origin if*

$$F(\alpha(x)) - F(x) < 0 \quad (\text{resp.}, \equiv 0, > 0) \tag{2.7}$$

for $0 < x \ll 1$.

2.2. Global behavior of orbits

On global behavior of the system (1.4), Han [22] gave the following two theorems. The first one is a direct corollary of Lemmas 1.3 and 1.4 and Corollary 1.1, and its proof can be found in [23]. A similar result can also be found in Lemmas 3.1.5 and 3.1.6 in the book [26].

Theorem 2.8. *[22,23] Suppose that $xg(x) > 0$ for $|x| \gg 1$ and that there exist constants $0 < \epsilon < p$ such that*

$$h(y) = \mathrm{sgn}(y)|y|^p(1 + O(|y|^{-\epsilon})), \quad \text{for } |y| \gg 1. \tag{2.8}$$

Let $Q(x)$ and $b(p)$ be given by (2.2). Then, the following holds.

- *(i) If $G(+\infty) = \infty$, $\lim_{x\to+\infty} \inf Q(x) > -b(p)$, then there does not exist an unbounded positive semi-orbit γ^+ of system (1.4) that lies entirely in the right half-plane.*
- *(ii) If $G(-\infty) = \infty$, $\lim_{x\to-\infty} \inf Q(x) > -b(p)$, then there does not exist an unbounded negative semi-orbit γ^- of system (1.4) that lies entirely in the left half-plane.*
- *(iii) If $G(-\infty) = \infty$, $\lim_{x\to-\infty} \sup Q(x) < b(p)$, then there does not exist an unbounded positive semi-orbit γ^+ of system (1.4) that lies entirely in the left half-plane.*

(iv) *If* $G(+\infty) = \infty$, $\lim_{x\to+\infty}\sup Q(x) < b(p)$, *then there does not exist an unbounded negative semi-orbit* γ^- *of system* (1.4) *that lies entirely in the right half-plane.*

Theorem 2.9. *[22] Suppose that* $xg(x) > 0$ *for* $|x| \gg 1$ *and* (2.8) *holds. Let* q *be a constant satisfying* $0 < q < \frac{p}{p+1}$.

(i) *If* $G(+\infty) = \infty$, $F(x) \le -b(p)[G(x)]^{\frac{p}{p+1}} + [G(x)]^q$ *for* $x \gg 1$, *then there exists an unbounded positive semi-orbit* γ^+ *of system* (1.4) *that lies entirely in the right half-plane.*
(ii) *If* $G(-\infty) = \infty$, $F(x) \le -b(p)[G(x)]^{\frac{p}{p+1}} + [G(x)]^q$ *for* $-x \gg 1$, *then there exists an unbounded negative semi-orbit* γ^- *of system* (1.4) *that lies entirely in the left half-plane.*
(iii) *If* $G(-\infty) = \infty$, $F(x) \ge b(p)[G(x)]^{\frac{p}{p+1}} - [G(x)]^q$ *for* $-x \gg 1$, *then there exists an unbounded positive semi-orbit* γ^+ *of system* (1.4) *that lies entirely in the left half-plane.*
(iv) *If* $G(+\infty) = \infty$, $F(x) \ge b(p)[G(x)]^{\frac{p}{p+1}} - [G(x)]^q$ *for* $x \gg 1$, *then there exists an unbounded negative semi-orbit* γ^- *of system* (1.4) *that lies entirely in the right half-plane.*

Similarly, in order to derive further results by using the above theorems, we assume that the following two limits:

$$Q_1^+ = \lim_{x\to+\infty} Q(x), \quad Q_1^- = \lim_{x\to-\infty} Q(x) \tag{2.9}$$

exist with $|Q_1^\pm| \le +\infty$, where $Q(x)$ is given in (2.2). Moreover, suppose that there exist constants $h_1 > 0$, $p > 0$, and $\sigma > 0$ such that

$$h(y) = \mathrm{sgn}(y)h_1|y|^p(1 + O(|y|^{-\sigma})), \quad \text{for } |y| \gg 1. \tag{2.10}$$

Theorem 2.10. *Suppose that* $G(+\infty) = \infty$, $g(x) > 0$ *for* $x \gg 1$ *and that* (2.9) *and* (2.10) *hold. If* $-h_1^{\frac{1}{p+1}} b(p) < Q_1^+ < h_1^{\frac{1}{p+1}} b(p)$, *where* $b(p)$ *and* Q_1^+ *are given by* (2.2) *and* (2.10), *then there exists* $\bar{y}_0 > 0$ *such that*

(i) *for any* $y_0 \ge \bar{y}_0$, *the positive semi-orbit* $\gamma_{y_0}^+$ *of system* (1.4) *passing through the point* $(0, y_0)$ *must intersect the negative* y*-axis in the right half-plane;*
(ii) *for any* $y_0 \le -\bar{y}_0$, *the negative semi-orbit* $\gamma_{y_0}^-$ *of system* (1.4) *passing through the point* $(0, y_0)$ *must intersect the positive* y*-axis in the right half-plane.*

Proof. By making the change of variables $x \to x$, $y \to h_1^{-\frac{1}{p}} y$, we can obtain from system (1.4)

$$\dot{x} = \hat{h}(y) - F(x), \quad \dot{y} = -\hat{g}(x), \tag{2.11}$$

where $\hat{h}(y) = h(h_1^{-\frac{1}{p}} y) = \mathrm{sgn}(y)|y|^p(1 + O(|y|^{-\sigma}))$, $\hat{g}(x) = h_1^{\frac{1}{p}} g(x)$. It is easy to know that there exists a constant $x_0 \gg 1$ such that $\hat{g}(x) > 0$ for $x \ge x_0$, $\hat{G}(x_0) > 0$, where $\hat{G}(x_0) = \int_0^{x_0} \hat{g}(u)du$. Denote by $\hat{C}^+$ the following curve,

$$\hat{C}^+ = \{(x, y) \mid \hat{h}(y) = F(x), \quad x \ge x_0\}.$$

By applying the conclusion (i) of Theorem 2.8 to (2.11), we can conclude that for any point $P_1(x_0, y_3)$ with $y_3 > \hat{y}_0 = \max\{y \mid \hat{h}(y) = F(x_0)\}$, the trajectory starting from point P_1 intersects the curve $\hat{C}^+$ at a point P and then intersects the line $x = x_0$ at a point $P_2(x_0, y_4)$, where $y_4 < \hat{y}_0$. Moreover, we apply the conclusion (iv) of Theorem 2.8 to (2.11) to get $y_4 \to -\infty$, as $y_3 \to +\infty$. For system (2.11), we have

$$\frac{dy}{dx} = \left.\frac{-\hat{g}(x)}{\hat{h}(y) - F(x)}\right|_{0\le x\le x_0} \to 0 \ (|y| \to +\infty).$$

Hence, when $|y| \gg 1$, the negative semi-orbit $\gamma_{P_1}^-$ must intersect the positive y-axis at a point $P_1^*(0, y_3^*)$, while the positive semi-orbit $\gamma_{P_2}^+$ must intersect the negative y-axis at a point $P_2^*(0, y_4^*)$, as shown in Fig. 2.2.

Next, we fix the point $P_1(x_0, y_3)$ and let $\bar{y}_0 = \max\{y_3^*, |y_4^*|\}$, then by the above proof one can see that for any $y_0 \ge \bar{y}_0$, the positive semi-orbit $\gamma_{y_0}^+$ of (2.11) passing through the point $(0, y_0)$ intersects the negative y-axis, while the negative semi-orbit $\gamma_{-y_0}^-$ of (2.11) passing through the point $(0, -y_0)$ intersects the positive y-axis. This completes the proof. □

Theorem 2.11. *Suppose that* $G(+\infty) = \infty$, $g(x) > 0$ *for* $x \gg 1$ *and that* (2.9) *and* (2.10) *hold.*

(i) *If either* $Q_1^+ > h_1^{\frac{1}{p+1}} b(p)$, *or* $Q_1^+ = h_1^{\frac{1}{p+1}} b(p)$ *and there exists a constant* $\varepsilon > 0$ *such that* $Q(x) = Q_1^+ + O([G(x)]^{-\varepsilon})$ *for* $x \gg 1$, *then there exists an unbounded negative semi-orbit* γ^- *of system* (1.4) *that lies entirely in the right half-plane.*
(ii) *If either* $Q_1^+ < -h_1^{\frac{1}{p+1}} b(p)$, *or* $Q_1^+ = -h_1^{\frac{1}{p+1}} b(p)$ *and there exists a constant* $\varepsilon > 0$ *such that* $Q(x) = Q_1^+ + O([G(x)]^{-\varepsilon})$ *for* $x \gg 1$, *then there exists an unbounded positive semi-orbit* γ^+ *of system* (1.4) *that lies entirely in the right half-plane.*

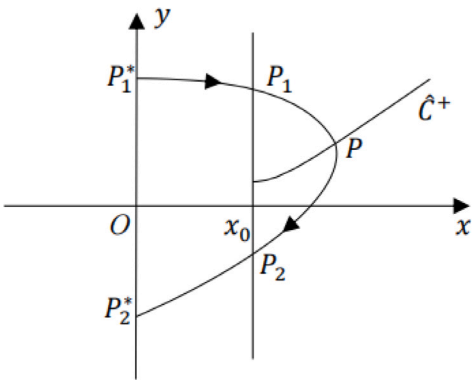


Fig. 2.2. The orbit passing through P_1 of (2.11).

Proof. Because of the similarity, we only prove the conclusion (i) of the theorem. When $Q_1^+ > h_1^{\frac{1}{p+1}} b(p)$, where $b(p)$ and Q_1^+ are given by (2.2) and (2.10), we have

$$Q_1^+ = \lim_{x\to+\infty} Q(x) = \lim_{x\to+\infty} \frac{F(x)}{[G(x)]^{\frac{p}{p+1}}} > h_1^{\frac{1}{p+1}} b(p),$$

which implies that

$$\frac{F(x)}{[\hat{G}(x)]^{\frac{p}{p+1}}} = \frac{F(x)}{h_1^{\frac{1}{p+1}}[G(x)]^{\frac{p}{p+1}}} > b(p), \quad x \gg 1,$$

where $\hat{G}(x) = \int_0^x \hat{g}(u)du = h_1^{\frac{1}{p}} G(x)$. Therefore, there exists a constant $0 < q < \frac{p}{p+1}$ such that

$$F(x) > b(p)[\hat{G}(x)]^{\frac{p}{p+1}} - [\hat{G}(x)]^q, \quad \text{for } x \gg 1.$$

By applying the conclusion (iv) in Theorem 2.9 to (2.11), we know that there exists an unbounded negative semi-orbit γ^- of system (2.11) that lies entirely in the right half-plane. Then the first part of conclusion (i) is proved.

Suppose $Q_1^+ = h_1^{\frac{1}{p+1}} b(p)$ and there exists a constant $\epsilon > 0$ such that $Q(x) = Q_1^+ + O([G(x)]^{-\epsilon})$ for $x \gg 1$. For the given $\epsilon > 0$, there exists $x_2 > 0$ and $M > 0$ such that when $x > x_2$,

$$|Q(x) - Q_1^+| < M[G(x)]^{-\epsilon}.$$

Thus, we have $Q(x) > Q_1^+ - M[G(x)]^{-\epsilon} = Q_1^+ - M[G(x)]^{-\frac{\epsilon}{2}}[G(x)]^{-\frac{\epsilon}{2}}$, for $x \gg 1$. Hence, by the definition of the function Q in (2.2) and $G(+\infty) = +\infty$, we obtain

$$\frac{F(x)}{[\hat{G}(x)]^{\frac{p}{p+1}}} = \frac{F(x)}{h_1^{\frac{1}{p+1}}[G(x)]^{\frac{p}{p+1}}} > b(p) - h_1^{-\frac{1}{p+1}} M[G(x)]^{-\frac{\epsilon}{2}}[G(x)]^{-\frac{\epsilon}{2}} > b(p) - [\hat{G}(x)]^{-\frac{\epsilon}{2}}, \quad x \gg 1,$$

which in turn yields

$$F(x) > b(p)[\hat{G}(x)]^{\frac{p}{p+1}} - [\hat{G}(x)]^q, \quad \text{for } x \gg 1,$$

where $q = \frac{p}{p+1} - \frac{\epsilon}{2}$. Applying the conclusion (iv) in Theorem 2.9 again to (2.11), the second part of conclusion (i) follows. This ends the proof. □

Using the same method as applied for the right half-plane of system (1.4), we can derive the following two theorems for the left half-plane of the system.

Theorem 2.12. *Suppose that $G(-\infty) = \infty$, $g(x) < 0$ for $-x \gg 1$ and that (2.9) and (2.10) hold. If $-h_1^{\frac{1}{p+1}} b(p) < Q_1^- < h_1^{\frac{1}{p+1}} b(p)$, where $b(p)$ is given by (2.2), then there exists $\bar{y}_0 > 0$ such that*

(i) for any $y_0 \geq \bar{y}_0$, the negative semi-orbit $\gamma_{y_0}^-$ of system (1.4) passing through the point $(0, y_0)$ must intersect the negative y-axis in the left half-plane;
(ii) for any $y_0 \leq -\bar{y}_0$, the positive semi-orbit $\gamma_{y_0}^+$ of system (1.4) passing through the point $(0, y_0)$ must intersect the positive y-axis in the left half-plane.

Theorem 2.13. *Suppose that $G(-\infty)=\infty$, $g(x)<0$ for $-x\gg 1$ and that (2.9) and (2.10) hold.*

(i) If either $Q_1^- > h_1^{\frac{1}{p+1}}b(p)$, or $Q_1^- = h_1^{\frac{1}{p+1}}b(p)$ and there exists a constant $\varepsilon>0$ such that $Q(x)=Q_1^-+O([G(x)]^{-\varepsilon})$ for $-x\gg 1$, then there exists an unbounded positive semi-orbit γ^+ of system (1.4) that lies entirely in the left half-plane.

(ii) If either $Q_1^- < -h_1^{\frac{1}{p+1}}b(p)$, or $Q_1^- = -h_1^{\frac{1}{p+1}}b(p)$ and there exists a constant $\varepsilon>0$ such that $Q(x)=Q_1^-+O([G(x)]^{-\varepsilon})$ for $-x\gg 1$, then there exists an unbounded negative semi-orbit γ^- of system (1.4) that lies entirely in the left half-plane.

3. Main results

In this section, we consider the generalized Liénard system (1.4), where h, F and g are all polynomials. We will present a necessary and sufficient condition for (1.4) to have a center (local or nonlocal) at the origin. Recall that the singular point O at the origin is a center if there is a neighborhood S of the point O such that for any $P\in S$ with $P\neq O$ the orbit passing through P is periodic and that the center O to be a global center if we can take the set S as the plane $\mathbb{R}^2$. We call the center O to be an unbounded non-global center if it is not a global center and we can take the set S to be unbounded. Now, we suppose that (1.4) is a polynomial system with a singular point at the origin. In this case, we can assume that h, F and g are given by

$$h(y)=\sum_{j=k_1}^{k} h_j y^j,\quad F(x)=\sum_{j=m_1}^{m} a_j x^j,\quad g(x)=\sum_{j=n_1}^{n} b_j x^j, \tag{3.1}$$

where $1\le k_1\le k$, $1\le m_1\le m$, $1\le n_1\le n$. If $F(x)\equiv 0$, then (1.4) is a Hamiltonian system and has a center at the origin if and only if (2.1) holds for some $\delta>0$. Thus, we suppose that $F(x)$ is not identically equal to zero. Then, without loss of generality, we can further assume that

$$h_{k_1}h_k\neq 0,\quad a_{m_1}a_m\neq 0,\quad b_{n_1}b_n\neq 0. \tag{3.2}$$

Let $G(x)=\int_0^x g(u)du$. Then, for any x and y satisfying $0<|x|\ll 1$ and $0<|y|\ll 1$, the polynomial functions h, F and G can be written in the following forms:

$$h(y)=h_{k_1}y^{k_1}(1+O(y)),\quad F(x)=a_{m_1}x^{m_1}(1+O(x)),\quad G(x)=\frac{b_{n_1}}{n_1+1}x^{n_1+1}(1+O(x)). \tag{3.3}$$

The following theorem presents a necessary and sufficient condition for the singular point at the origin to be a center.

Theorem 3.1. *Consider the system (1.4) satisfying (3.1) and (3.2). Then, the singular point at the origin is a center if and only if the following conditions (1)-(3) are satisfied:*

(1) both k_1 and n_1 are odd with $h_{k_1}b_{n_1}>0$;

(2) $m_1-\frac{k_1(n_1+1)}{k_1+1}>0$, or $m_1-\frac{k_1(n_1+1)}{k_1+1}=0$ and $|a_{m_1}|<\left(\frac{|b_{n_1}|}{n_1+1}\right)^{\frac{k_1}{k_1+1}}|h_{k_1}|^{\frac{1}{k_1+1}}b(k_1)$;

(3) $F(\alpha(x))-F(x)\equiv 0$ for $0<x\ll 1$, where $G(\alpha(x))=G(x)$ with $\alpha(x)<0<x$,

where $b(k_1)=(k_1+1)\left(\frac{k_1+1}{k_1}\right)^{\frac{k_1}{k_1+1}}$.

Proof. We first prove that the conditions (1)-(3) will be satisfied if the singular point at the origin is a center. Let the singular point at the origin be a center. When (1) is not true, either k_1 or n_1 is even, or $h_{k_1}b_{n_1}<0$. Then, there are totally 6 possible cases described as follows:

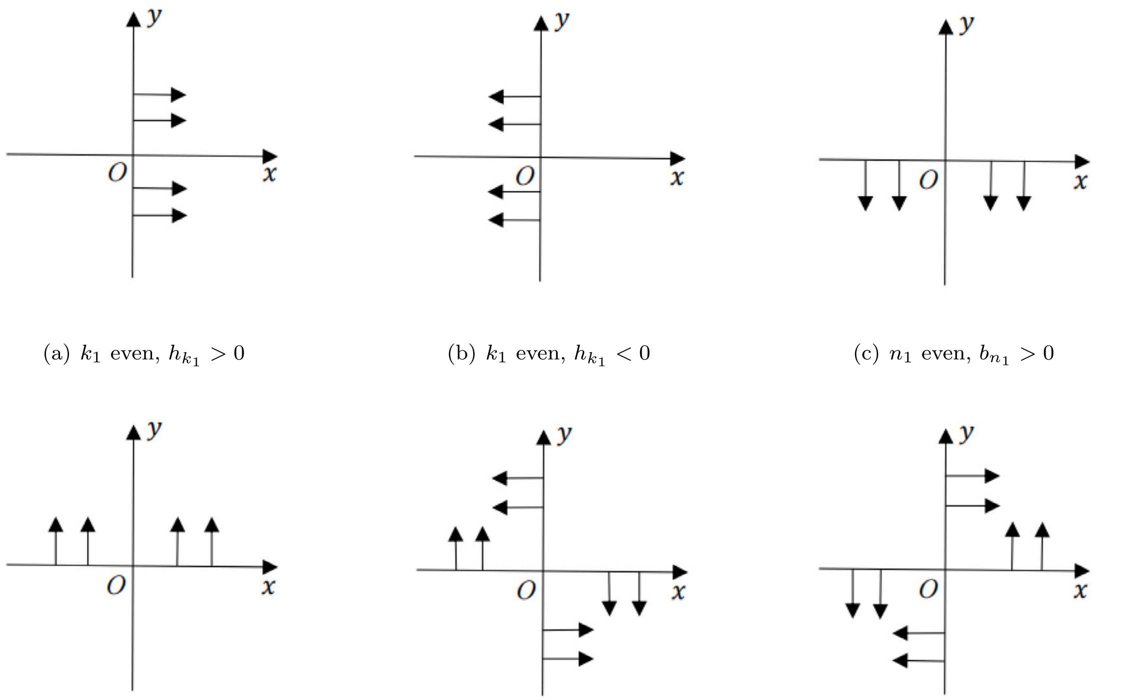
(a) k_1 is even with $h_{k_1}>0$;
(b) k_1 is even with $h_{k_1}<0$;
(c) n_1 is even with $b_{n_1}>0$;
(d) n_1 is even with $b_{n_1}<0$;
(e) both k_1 and n_1 are odd with $h_{k_1}<0$ and $b_{n_1}>0$;
(f) both k_1 and n_1 are odd with $h_{k_1}>0$ and $b_{n_1}<0$.

Note that for the system (1.4),

$$\dot{x}|_{x=0}=h(y),\quad \dot{y}|_{y=0}=-g(x).$$

In these 6 cases, the directions of the vector field of (1.4) on the lines $x=0$ and $y=0$ near the origin are shown in Fig. 3.1.(a)–(f), which indicate that the origin is not a center for all the six cases, leading to a contradiction. Thus, the condition (1) holds. Then, we have either $h_{k_1}>0$ and $b_{n_1}>0$, or $h_{k_1}<0$ and $b_{n_1}<0$. Without loss of generality, we suppose $h_{k_1}>0$ and $b_{n_1}>0$. Otherwise, one can change the sign of the time by $t\to -t$. With these discussions, let

$$Q_0(x)=\frac{F(x)}{[G(x)]^{\frac{k_1}{k_1+1}}},\quad x\neq 0. \tag{3.4}$$



(a) k_1 even, $h_{k_1} > 0$ (b) k_1 even, $h_{k_1} < 0$ (c) n_1 even, $b_{n_1} > 0$

(d) n_1 even, $b_{n_1} < 0$ (e) k_1, n_1 odd, $h_{k_1} < 0$, $b_{n_1} > 0$ (f) k_1, n_1 odd, $h_{k_1} > 0$, $b_{n_1} < 0$

Fig. 3.1. The vector field of (1.4) near the origin corresponding to the six cases (a)–(f).

Then, by (3.1), (3.2) and (3.4) we have

$$Q_0(x) = \begin{cases} Bx^{\beta}(1 + O(x)), \ 0 < x \ll 1, \\ (-1)^{m_1} B|x|^{\beta}(1 + O(|x|)), \ 0 < -x \ll 1. \end{cases}$$

where

$$B = a_{m_1}\left(\frac{b_{n_1}}{n_1 + 1}\right)^{\frac{-k_1}{k_1+1}} \neq 0, \quad \beta = m_1 - \frac{k_1(n_1 + 1)}{k_1 + 1}. \tag{3.5}$$

It follows that

$$Q_0(0+) = \begin{cases} 0, \ \beta > 0, \\ B, \ \beta = 0, \\ +\infty, \ \beta < 0, B > 0, \\ -\infty, \ \beta < 0, B < 0, \end{cases} \tag{3.6}$$

and

$$Q_0(0-) = \begin{cases} 0, \ \beta > 0, \\ (-1)^{m_1} B, \ \beta = 0, \\ +\infty, \ \beta < 0, (-1)^{m_1} B > 0, \\ -\infty, \ \beta < 0, (-1)^{m_1} B < 0. \end{cases} \tag{3.7}$$

By Theorems 2.4 and 2.6, we can conclude that the origin is not a center if

$$|Q_0(0+)| \geq h_{k_1}^{\frac{1}{k_1+1}} b(k_1)$$

or

$$|Q_0(0-)| \geq h_{k_1}^{\frac{1}{k_1+1}} b(k_1).$$

Thus, we have the necessary condition,

$$|Q_0(0\pm)| < h_{k_1}^{\frac{1}{k_1+1}} b(k_1), \tag{3.8}$$

since the origin is a center. By (2.2), (3.5), (3.6) and (3.7), one can see that (3.8) holds if and only if

$$m_1 - \frac{k_1(n_1+1)}{k_1+1} > 0,$$

or

$$m_1 - \frac{k_1(n_1+1)}{k_1+1} = 0,\ |a_{m_1}| < \left(\frac{|b_{n_1}|}{n_1+1}\right)^{\frac{k_1}{k_1+1}} |h_{k_1}|^{\frac{1}{k_1+1}} b(k_1).$$

Hence, the condition (2) follows. Further, by Theorem 2.7, the condition (3) also holds.

Now, we suppose that conditions (1)-(3) hold. Then, by Theorem 2.7 the singular point at the point is a center. The proof is complete. □

Remark 3.1. Let $u = \text{sign}(x)\sqrt{2G(x)}$ which has the inverse $x = X(u)$ and let $F^*(u) = F(X(u))$. Then, by Theorems?? Can give more details here, e.g., some special theorems? [12,20], $F(\alpha(x)) - F(x) \equiv 0$ if and only if $F^*(-u) - F^*(u) \equiv 0$.

Now, we consider the conditions for the existence of a global center.

Theorem 3.2. *Consider the system (1.4) satisfying (3.1) and (3.2). Then, the singular point at the origin is a global center if and only if the following conditions (1)-(4) are satisfied:*

(1) $xg(x) > 0\ (x \neq 0)$ *and* $yh(y) > 0\ (y \neq 0)$, *or* $xg(x) < 0\ (x \neq 0)$ *and* $yh(y) < 0\ (y \neq 0)$;

(2) $m_1 - \frac{k_1(n_1+1)}{k_1+1} > 0$, *or* $m_1 - \frac{k_1(n_1+1)}{k_1+1} = 0$ *and* $|a_{m_1}| < \left(\frac{|b_{n_1}|}{n_1+1}\right)^{\frac{k_1}{k_1+1}} |h_{k_1}|^{\frac{1}{k_1+1}} b(k_1)$;

(3) $F(\alpha(x)) - F(x) \equiv 0$ *for* $0 < x \ll 1$, *where* $G(\alpha(x)) = G(x)$ *with* $\alpha(x) < 0 < x$;

(4) $m - \frac{k(n+1)}{k+1} < 0$, *or* $m - \frac{k(n+1)}{k+1} = 0$ *and* $|a_m| < \left(\frac{|b_n|}{n+1}\right)^{\frac{k}{k+1}} |h_k|^{\frac{1}{k+1}} b(k)$.

Proof. Let us first suppose that the singular point at the origin is a global center. Then, by Theorem 3.1, the conditions (1)-(3) of Theorem 3.2 hold. Therefore, we need only to prove for the conditions (1) and (4). First, by the condition (1) of Theorem 3.1 there exists a constant $\delta > 0$ such that

$$xg(x) > 0\ (0 < |x| < \delta),\quad yh(y) > 0\ (0 < |y| < \delta), \tag{3.9}$$

or

$$xg(x) < 0\ (0 < |x| < \delta),\quad yh(y) < 0\ (0 < |y| < \delta).$$

Because of the similarity, we can assume that (3.9) holds. Since the origin is a global center, it is the only singular point of system (1.4) on the plane. Hence, $h(y) \neq 0$ for all $y \neq 0$. Then, it, together with (3.9), implies that $yh(y) > 0$ for all $|y| > 0$. Thus, by (3.1) and (3.2) one can see that k is odd with $h_k > 0$, which yields that $h(\pm\infty) = \pm\infty$. Hence, if $g(x_0) = 0$ for some $x_0 \neq 0$, then the equation $h(y) = F(x_0)$ has at least one solution $y = y_0$ in y, leading to a singular point (x_0, y_0) not at the origin, leading to a contradiction. Therefore, by (3.9) we have $xg(x) > 0$ for all $|x| > 0$. Thus, the condition (1) follows. Let

$$Q_1(x) = \frac{F(x)}{[G(x)]^{\frac{k}{k+1}}},\ x \neq 0.$$

Then, noticing that

$$F(x) = a_m x^m (1 + O(x^{-1})),\ G(x) = \frac{b_n}{n+1} x^{n+1} (1 + O(x^{-1}))$$

by (3.1), we have

$$Q_1(x) = \begin{cases} B_1 x^{\beta_1} (1 + O(x^{-1})),\ x \gg 1, \\ (-1)^m B_1 |x|^{\beta_1} (1 + O(|x|^{-1})),\ -x \gg 1, \end{cases}$$

where

$$B_1 = a_m \left(\frac{b_n}{n+1}\right)^{\frac{-k}{k+1}} \neq 0,\quad \beta_1 = m - \frac{k(n+1)}{k+1}. \tag{3.10}$$

Then, similar to (3.6) and (3.7), we obtain

$$Q_1(+\infty) = \begin{cases} 0,\ \beta_1 < 0, \\ B_1,\ \beta_1 = 0, \\ +\infty,\ \beta_1 > 0, B_1 > 0, \\ -\infty,\ \beta_1 > 0, B_1 < 0, \end{cases} \tag{3.11}$$

and

$$Q_1(-\infty) = \begin{cases} 0, \ \beta_1 < 0, \\ (-1)^m B_1, \ \beta_1 = 0, \\ +\infty, \ \beta_1 > 0, (-1)^m B_1 > 0, \\ -\infty, \ \beta_1 > 0, (-1)^m B_1 < 0. \end{cases} \tag{3.12}$$

Therefore, by Theorems 2.11 and 2.13, we obtain the necessary condition,

$$|Q_1(\pm\infty)| < h_k^{\frac{1}{k+1}} b(k).$$

Hence, the condition (4) follows from (3.10), (3.11) and (3.12).

Now, we suppose that conditions (1)-(4) hold. We want to prove that the singular point at the origin of system (1.4) is a global center. First, we know that the origin is a center by Theorem 3.1. Moreover, by the condition (1), the origin is the only singular point of (1.4) on the plane. For definiteness, we can suppose that (3.9) holds for any $\delta > 0$. Then by Theorems 2.10 and 2.12, for any $y_0 > 0$ the positive semi-orbit $\gamma_{y_0}^+$ and the negative semi-orbit $\gamma_{y_0}^-$ of system (1.4) passing through the point $(0, y_0)$ must intersect the negative y-axis. Similarly, for any $y_0 < 0$, the negative semi-orbit $\tilde{\gamma}_{y_0}^-$ and the positive semi-orbit $\tilde{\gamma}_{y_0}^+$ of system (1.4) passing through the point $(0, y_0)$ must intersect the positive y-axis.

Note that the function $\alpha(x)$ satisfying $G(\alpha(x)) = G(x)$ for $x \geq 0$ is well defined and analytic for all $x \geq 0$ with $\alpha(x) = -x + O(x^2)$ for $x > 0$ small. The condition (3) implies that

$$F(\alpha(x)) \equiv F(x) \text{ for } x \geq 0. \tag{3.13}$$

Now as in [12] we make a change of variables $T: \ x = \alpha(u), \ y = y$ with $u \geq 0$ under which the system (1.4) with $x \leq 0$ becomes

$$\dot{u} = \frac{g(\alpha(u))}{g(u)}[h(y) - F(\alpha(u))], \quad \dot{y} = -g(\alpha(u)), \ u \geq 0$$

which is equivalent to

$$\frac{du}{d\tau} = h(y) - F(\alpha(u)), \quad \frac{dy}{d\tau} = -g(u), \ u \geq 0, \tag{3.14}$$

where $d\tau = \frac{g(\alpha(u))}{g(u)} dt$. This clearly show that for any $y_0 > 0$ the image of the negative semi-orbit $\gamma_{y_0}^-$ of the system (1.4) under the change T is a positive semi-orbit of system (3.14), denoted by $\hat{\gamma}_{y_0}^+$, passing through the point $(0, y_0)$. Noticing (3.13) we see that $\hat{\gamma}_{y_0}^+ = \gamma_{y_0}^+$, which implies that $\gamma_{y_0}^+$ is closed. In the same way, we can prove that for any $y_0 < 0$, the orbit $\tilde{\gamma}_{y_0}$ of system (1.4) passing through the point $(0, y_0)$ is also closed. Therefore, the singular point at the origin is a global center of system (1.4). The proof is complete. □

From the proof of Theorem 3.2 it is easy to obtain the following result.

Theorem 3.3. *Consider system (1.4) under (3.1) and (3.2). The origin of this system is a unique singular point on the plane, and is an unbounded non-global center if and only if the following conditions (1)-(4) are satisfied:*

(1) $xg(x) > 0$ $(x \neq 0)$ *and* $yh(y) > 0$ $(y \neq 0)$, *or* $xg(x) < 0$ $(x \neq 0)$ *and* $yh(y) < 0$ $(y \neq 0)$;

(2) $m_1 - \frac{k_1(n_1+1)}{k_1+1} > 0$, *or* $m_1 - \frac{k_1(n_1+1)}{k_1+1} = 0$ *and* $|a_{m_1}| < \left(\frac{|b_{n_1}|}{n_1+1}\right)^{\frac{k_1}{k_1+1}} |h_{k_1}|^{\frac{1}{k_1+1}} b(k_1)$;

(3) $F(\alpha(x)) - F(x) \equiv 0$ *for* $0 < x \ll 1$, *where* $G(\alpha(x)) = G(x)$ *with* $\alpha(x) < 0 < x$;

(4) $m - \frac{k(n+1)}{k+1} > 0$, *or* $m - \frac{k(n+1)}{k+1} = 0$ *and* $|a_m| \geq \left(\frac{|b_n|}{n+1}\right)^{\frac{k}{k+1}} |h_k|^{\frac{1}{k+1}} b(k)$.

We now apply Theorem 3.1, 3.2 and 3.3 to consider systems (1.2) and (1.5). First consider (1.2), where the functions f and g satisfy (1.6) with $s \geq r \geq 1$ and $m \geq k \geq 1$. Let

$$G(x) = \int_0^x g(x)dx, \ F(x) = \int_0^x f(x)dx = \sum_{j=k+1}^{m+1} \tilde{b}_j x^j, \quad \tilde{b}_j = \frac{b_{j-1}}{j}.$$

Noticing that (1.2) is equivalent to (1.3), applying Theorem 3.1, 3.2 and 3.3 to the equivalent form (1.3) of (1.2) by taking $k_1 = k = 1, h_1 = 1$, replacing n, n_1, m, m_1 by $s, r, m+1, k+1$, respectively, and using $a_j, \tilde{b}_j = \frac{b_{j-1}}{j}$ for b_j, a_j, respectively, we obtain the following theorem.

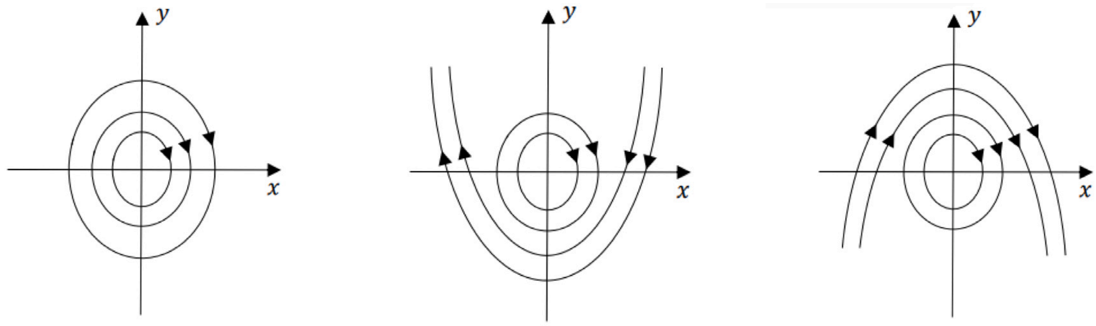
Theorem 3.4. *Consider the system (1.2) under (1.6) with* $s \geq r \geq 1$ *and* $m \geq k \geq 1$. *Then,*

(I) The singular point at the origin is a center of (1.2) if and only if

(I-1) the integer r *is odd with* $a_r > 0$;

(I-2) $2k + 1 > r$, *or* $2k + 1 = r$ *and* $|b_k| < \sqrt{2(r+1)a_r}$;

(I-3) $F(\alpha(x)) - F(x) \equiv 0$ *for* $0 < x \ll 1$, *where* $G(\alpha(x)) = G(x)$ *with* $\alpha(x) < 0 < x$.



(a) $|b| < 2\sqrt{2}$ (b) $b \geq 2\sqrt{2}$ (c) $b \leq -2\sqrt{2}$

Fig. 3.2. The phase portraits of (3.15).

(II) The singular point at the origin is a global center of (1.2) if and only if

- *(II-1)* $xg(x) > 0$ *for* $x \neq 0$*;*
- *(II-2) the conditions (I-2) and (I-3) hold;*
- *(II-3)* $2m+1 < s$*, or* $2m+1 = s$ *and* $|b_m| < \sqrt{2(s+1)a_s}$.

(III) The singular point at the origin is an unbounded non-global center of (1.2) if and only if the conditions (II-1) and (II-2) hold and $2m+1 > s$*, or* $2m+1 = s$ *and* $|b_m| \geq \sqrt{2(s+1)a_s}$.

Further, by taking $r = s = 2n-1$ with $a_r = 1$ and applying Theorem 3.4 to (1.5) we immediately obtain the following result.

Theorem 3.5. *Consider system (1.5). Then the following conclusions hold.*

(I) The singular point at the origin is a center of (1.5) if and only if either $f(x) \equiv 0$*, or* $f(x)$ *is not zero identically and satisfies* $b_k \neq 0$, $k > n-1$*, or* $k = n-1$ *and* $b_k^2 < 4n$.

(II) The singular point at the origin is a global center of (1.5) if and only if either $f(x) \equiv 0$*, or* $f(x)$ *is not zero identically and satisfies* $b_k b_m \neq 0$, $m = k = n-1$ *and* $b_k^2 < 4n$.

(III) The singular point at the origin is an unbounded non-global center of (1.5) if and only if $f(x)$ *is not zero identically with* $b_k b_m \neq 0$ *and satisfies*

- *(III-1)* $k > n-1$*, or* $k = n-1$ *and* $b_k^2 < 4n$*;*
- *(III-2)* $m > n-1$*, or* $m = n-1$ *and* $b_m^2 \geq 4n$.

Remark 3.2. Theorem 3.4 contains more conclusions than Theorem 1.2 in [6], and the condition (I-3) here is easer to check than the condition (iii) there. The conclusion (I) of Theorem 3.5 gives a correction to the conclusion in Theorem 1.1 in [7]. The conclusion (II) of Theorem 3.5 shows that the conjecture in Section 1 posed in [7] is true. Also, in the special case of $h(y) = y$, comparing with the main result Theorem 1 in [10], the conditions of Theorem 3.4 are quite easy to check.

For example, for the case of $n = 2$, $m = k = 1$, the system (1.5) is equivalent to a cubic system of the form,

$$\dot{x} = y - \frac{1}{2}bx^2, \quad \dot{y} = -x^3. \tag{3.15}$$

By Theorem 3.5, we can easily obtain the phase portraits of (3.15), as shown in Fig. 3.2.(a)–(c).

Recently, a system of the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(y)y, \tag{3.16}$$

called Rayleigh–Duffing system or Rayleigh–Duffing oscillator, has been extensively investigated, see [21,27,28] and the references therein. We suppose that the functions g and f in (3.16) are both polynomials with the form,

$$g(x) = \sum_{j=r}^{s} a_j x^j, \quad f(y) = \sum_{j=k}^{m} b_j y^j, \quad a_r a_s b_k b_m \neq 0, \ s \geq r \geq 1, \ m \geq k \geq 1. \tag{3.17}$$

Then, note that (3.16) is equivalent to

$$\dot{x} = g(y) + f(x)x, \quad \dot{y} = -x,$$

which is in the form of (1.4). Noticing $\frac{2r}{r+1} < 2$, we can obtain the following theorem by directly applying Theorem 3.1, 3.2 and 3.3.

Theorem 3.6. *Consider system (3.16) under (3.17). The following conclusions hold.*

(I) *The singular point at the origin is a center of (3.16) if and only if the integer r is odd with $a_r > 0$ and $f(-y) = -f(y)$.*
(II) *System (3.16) does not have a global center at the origin.*
(III) *The singular point at the origin is an unbounded non-global center of (3.16) if and only if $xg(x) > 0$ for $x \neq 0$ and $f(-y) = -f(y)$.*

We remark that the conclusion (II) in the above theorem has been noted in [21].

4. Applications to systems of lower degrees

In this section, we consider two generalized polynomial Liénard systems of degree 5, and give the necessary and sufficient conditions for the origin of these system to be a center or a global center. The first system has the form

$$\dot{x} = y(h_1 + h_2 y + h_3 y^2) - F(x), \quad \dot{y} = -x(b_1 + b_2 x + b_3 x^2), \tag{4.1}$$

where

$$F(x) = \sum_{j=2}^{5} a_j x^j.$$

For this system, we consider two separate cases: $b_1 = 0$ and $b_1 \neq 0$. For the first case, by applying Theorems 3.1 and 3.2 we can easily obtain the following theorem.

Theorem 4.1. *Consider the system (4.1) with $b_1 = 0$. The following conclusions hold.*

(I) *The singular point at the origin is a center of (4.1) if and only if*

(I-1) $b_2 = a_3 = a_5 = 0$;
(I-2) $h_1 b_3 > 0$ *and* $|a_2| < \sqrt{2h_1 b_3}$, *or* $h_1 = h_2 = 0$, $h_3 b_3 > 0$ *and* $a_2 = 0$.

(II) *The singular point at the origin is a global center of (4.1) if and only if the conditions (I1) holds and either*

$$h_1 b_3 > 0, \; h_2^2 < 4h_1 h_3, \quad |a_2| < \sqrt{2h_1 b_3}, \quad a_4 = 0,$$

or

$$h_1 > 0, h_2 = h_3 = 0, \quad |a_2| < \sqrt{2h_1 b_3}, \quad a_4 = 0,$$

or

$$h_1 = h_2 = 0, \quad h_3 b_3 > 0, \quad a_2 = a_4 = 0.$$

Similar to the conclusion (II) of Theorem 4.1, by Theorem 3.2 one can prove that for any $m \geq 2$, the system

$$\dot{x} = y^3 - \sum_{j=2}^{m} a_j x^j, \; \dot{x} = -x^3$$

does not has a global center at the origin unless $a_j = 0$ for all $j = 2, \dots, m$.

Now, we consider the case $b_1 \neq 0$. Without loss of generality, we may assume $b_1 = 1$. Then, (4.1) becomes

$$\dot{x} = y(h_1 + h_2 y + h_3 y^2) - \sum_{j=2}^{5} a_j x^j, \quad \dot{y} = -x(1 + b_2 x + b_3 x^2). \tag{4.2}$$

As a preliminary, we first present a lemma as follows.

Lemma 4.1. *Let $\alpha(x) = -x + O(x^2)$ satisfy $G(\alpha(x)) = G(x)$, where*

$$G(x) = \int_0^x g dx, \; g(x) = x(1 + b_2 x + b_3 x^2), \quad b_2 \neq 0, \; b_3 > 0.$$

Then, for $F(x) = \sum_{j=2}^{5} a_j x^j$, $F(\alpha(x)) - F(x) \equiv 0$ if and only if

$$a_3 - \frac{2}{3} a_2 b_2 = a_4 - \frac{1}{2} a_2 b_3 = a_5 = 0. \tag{4.3}$$

Proof. Let $F(\alpha(x)) - F(x) \equiv 0$. Using Remark 3.1, we suppose

$$F^*(u) = A_2 u^2 + A_3 u^3 + A_4 u^4 + A_5 u^5 + A_6 u^6 + A_7 u^7 + A_8 u^8 + A_9 u^9 + \cdots.$$

Then, by (22) and (23) in [20], solving $A_3 = A_5 = A_7 = A_9 = 0$ gives

$$\begin{aligned} a_3 &= \frac{2}{3}a_2b_2, \\ a_5 &= \frac{2}{3}b_2(2a_4 - a_2b_3), \\ a_7 &= 2b_2a_6 - \frac{2}{9}(2b_2^2 + 3b_3)b_2(2a_4 - a_2b_3), \\ a_6 &= \frac{B_2(2a_4 - a_2b_3)}{9B_1}, \end{aligned} \tag{4.4}$$

where

$$B_1 = 135b_3 + 220b_2^2 > 0, \quad B_2 = 405b_3^2 + 1260b_2^2b_3 + 440b_2^4.$$

Hence, we further have

$$a_7 = \frac{2b_2B_3(2a_4 - a_2b_3)}{9B_1},$$

where

$$B_3 = B_2 - B_1(2b_2^2 + 3b_3) = 330b_3b_2^2.$$

Further, note that

$$2G(x) = x^2 + \frac{2b_2}{3}x^3 + \frac{b_3}{2}x^4, \quad a_6 = a_7 = 0.$$

Under (4.4) we have $2a_4 - a_2b_3 = 0$ and hence, $a_5 = 0$. Thus, (4.3) follows.

If (4.3) is satisfied, then

$$F(x) = 2a_2G(x),$$

which yields $F(\alpha(x)) - F(x) \equiv 0$. This completes the proof. □

Then, by Theorems 3.1, 3.2 and Lemma 4.1, we can prove the following theorem.

Theorem 4.2. *Consider system (4.2). The following conclusions hold.*

(I) The singular point at the origin is a center of (4.2) if and only if

(I-1) $a_3 - \frac{2}{3}a_2b_2 = a_4 - \frac{1}{2}a_2b_3 = a_5 = 0.$

(I-2) $h_1 > 0$, *or* $h_1 = h_2 = 0,\ h_3 > 0$;

(II) The singular point at the origin is a global center of (4.2) if and only if $a_2 = a_3 = a_4 = a_5 = 0$ *and one of the following six conditions hold:*

(1) $h_1 > 0,\ h_2^2 < 4h_1h_3,\ b_2^2 < 4b_3$;
(2) $h_1 > 0,\ h_2^2 < 4h_1h_3,\ b_2 = b_3 = 0$;
(3) $h_1 > 0,\ h_2 = h_3 = 0,\ b_2^2 < 4b_3$;
(4) $h_1 > 0,\ h_2 = h_3 = 0,\ b_2 = b_3 = 0$;
(5) $h_1 = h_2 = 0,\ h_3 > 0,\ b_2^2 < 4b_3$;
(6) $h_1 = h_2 = 0,\ h_3 > 0,\ b_2 = b_3 = 0.$

Proof. For the conclusion (I), there are two possible cases to consider: (I-a) $h_1 > 0$ with $k_1 = 1$ and (I-b) $h_1 = h_2 = 0,\ h_3 > 0$ with $k_1 = 3$. Noticing $n_1 = 1$ and $m_1 \geq 2$ in each of the cases, we have

$$m_1 - \frac{2k_1}{k_1 + 1} > 0.$$

Hence, the conclusion (I) follows directly from Theorem 3.1 and Lemma 4.1.

For the conclusion (II) there are three possible cases to ensure that $h_1 + h_2y + h_3y^2 > 0$:

(II-a) $h_1 > 0,\ h_2^2 < 4h_1h_3$ with $k_1 = 1$ and $k = 3$,
(II-b) $h_1 > 0,\ h_2 = h_3 = 0$ with $k_1 = k = 1$ and
(II-c) $h_1 = h_2 = 0$ and $h_3 > 0$ with $k_1 = k = 3$.

Also, there are two possible cases to ensure $1 + b_2x + b_3x^2 > 0$:

(II-i) $b_2^2 < 4b_3$ with $n_1 = 1$ and $n = 3$ and
(II-ii) $b_2 = b_3 = 0$ with $n_1 = n = 1$.

Thus, there are totally six possible cases (1)-(6), as listed in the theorem. Then, for each of these cases, an application of Theorem 3.2 to (4.2) together with Lemma 4.1 yields $a_2 = a_3 = a_4 = a_5 = 0$. This ends the proof. □

Next, we consider the following system with degree 5

$$\dot{x} = y(1 + hy^{2k}) - \sum_{j=2}^{5} a_j x^j, \quad \dot{y} = -x(1 + b_1 x^2 + b_2 x^4), \tag{4.5}$$

where $k = 1, 2$, $h \geq 0$, and $b_2 > 0$. It is easy to prove the following theorem by Theorems 3.1 and 3.2.

Theorem 4.3. *Consider system (4.5). The following conclusions hold.*

(I) The singular point at the origin is a center of (4.2) if and only if $a_3 = a_5 = 0$.
(II) The singular point at the origin is a global center of (4.2) if and only if $a_3 = a_5 = 0$, $b_1 > -2\sqrt{b_2}$ and either $h = 0$ and $a_4 = 0$ with a_2 arbitrary, or $h > 0$, $k = 1$ with a_2 and a_4 arbitrary, or $h > 0$, $k = 2$ with a_2 and a_4 arbitrary.

Clearly, both Theorem 2 and Proposition 3 in [10] can be directly obtained from Theorems 4.3.

5. Conclusion

In this paper, we generalize the existing theorems to study the existence of a center, a global center or an unbounded non-global center of the polynomial system in the form of (1.4), as shown in Theorems 3.1–3.3. We especially improve some known results and confirm a conjecture on the global center conditions of a polynomial Liénard system, as shown in Theorems 3.4–3.5, and also obtain the conditions for Rayleigh–Duffing system of the form (3.16) to have a center, a global center or an unbounded non-global center at the origin, as shown in Theorems 3.6 by applying Theorems 3.1–3.3. For the two generalized polynomial Liénard systems (4.1) and (4.5) of degree 5, we derive explicit and concrete necessary and sufficient conditions for the origin of these systems to be a center or a global center in terms of the system coefficients.

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