International Journal of Bifurcation and Chaos, Vol. 17, No. 11 (2007) 4117–4144

BIFURCATION OF LIMIT CYCLES IN A FOURTH-ORDER NEAR-HAMILTONIAN SYSTEM*

MAOAN HAN
Department of Mathematics, Shanghai Normal University,
Shanghai 200234, P. R. China
mahan@shnu.edu.cn

DESHENG SHANG† and WANG ZHENG‡
School of Mathematics and Information Science,
Shandong University of Technology,
Zibo 255049, P. R. China
†*shangdesheng@sdut.edu.cn*
‡*wangzhengxx@sdut.edu.cn*

PEI YU
Department of Applied Mathematics,
The University of Western Ontario,
London, Ontario, Canada N6A 5B7
pyu@pyu1.apmaths.uwo.ca

Received October 12, 2005; Revised September 1, 2006

This paper is concerned with bifurcation of limit cycles in a fourth-order near-Hamiltonian system with quartic perturbations. By bifurcation theory, proper perturbations are given to show that the system may have 20, 21 or 23 limit cycles with different distributions. This shows that $H(4) \geq 20$, where $H(n)$ is the Hilbert number for the second part of Hilbert's 16th problem. It is well known that $H(2) \geq 4$, and it has been recently proved that $H(3) \geq 12$. The number of limit cycles obtained in this paper greatly improves the best existing result, $H(4) \geq 15$, for fourth-degree polynomial planar systems.

Keywords: Near-Hamiltonian system; perturbation; bifurcation; limit cycle; homoclinic loop; heteroclinic orbit.

1. Introduction and Main Result

One of the problems posed by Smale [1988] in his Mathematical Problems for the 21th Century is Hilbert's 16th Problem [Hilbert, 1902]. The second part of the problem is concerned with the number and relative distributions of limit cycles of planar polynomial systems. There have been many studies on this aspect. If we denote by $H(n)$ the maximal number of limit cycles of a given system of degree n, then, up to now, we only know that $H(n)$ is finite, and $H(2) \geq 4$ [Chen & Wang, 1979; Shi, 1980] and $H(3) \geq 12$ [Yu & Han, 2004, 2005a, 2005b]. Zhang *et al.* [2004] studied quartic perturbations to a cubic

*Project supported by National Natural Science Foundation of China (#10671127), Shanghai Outstanding Discipline Leader Project (06XD14034) and the Science Foundation of Shandong University of Technology (2006KJM01).

Hamiltonian system and obtained $H(4) \geq 15$. Li *et al.* [2002] considered a Z_6-equivalent Hamiltonian system to show that $H(5) \geq 23$ by using the method of detection function. In this paper, we prove $H(4) \geq 20$, which improves the best existing result $H(4) \geq 15$.

In 1986, Roussarié proposed a method to study the number of limit cycles appearing in the vicinity of homoclinic loops with the aid of computing coefficients in the first-order Melnikov function. Subsequently, the idea was further developed to obtain more results (e.g. see [Han, 1997; Han *et al.*, 1999; Han & Zhang, 1999; Han & Chen, 2000; Han *et al.*, 2003; Han *et al.*, 2004; Zhang *et al.*, 2004a; Zhang *et al.*, 2004b; Han & Yang, 2005]).

A new approach was originated by Han [1997] to find limit cycles near homoclinic loops, and was further generalized in [Han & Zhang 1999; Han & Chen, 2000] to study the cases of double homoclinic loops or heteroclinic loops. The method consists of three steps:

(i) find discriminate values to determine the stability of homoclinic loops or double-homoclinic loops;
(ii) vary parameters to change the stability of these loops in order to produce limit cycles; and
(iii) break the homoclinic loops to find limit cycles.

In this paper, we use this method to study the following perturbed system:

$$\begin{cases} \dot{x} = y(1-y^2)(2-x) + \varepsilon \displaystyle\sum_{i+j=1}^{4} a_{ij}x^i y^j, \\ \dot{y} = -x(1-2x^2)(2-x) + \varepsilon \displaystyle\sum_{i+j=1}^{4} b_{ij}x^i y^j, \end{cases} \tag{1}$$

where ε is a small positive perturbation parameter, and the coefficients a_{ij} and b_{ij} are treated as parameters.

Our main result is stated in the following theorem:

Theorem 1. *System (1) can have at least* 20 *limit cycles, i.e.* $H(4) \geq 20$.

2. Proof of the Main Theorem

To prove Theorem 1, first rescale time by $t \to t/(2-x)$ in the region $x < 2$; then it follows from (1) that

$$\begin{cases} \dot{x} = y(1-y^2) + \varepsilon \dfrac{P_4(x,y)}{2-x}, \\ \dot{y} = -x(1-2x^2) + \varepsilon \dfrac{Q_4(x,y)}{2-x}, \end{cases} \tag{2}$$

where $P_4(x,y) = \sum_{i+j=1}^{4} a_{ij}x^i y^j$ and $Q_4(x,y) = \sum_{i+j=1}^{4} b_{ij}x^i y^j$.

When $\varepsilon = 0$, (2) is reduced to

$$\dot{x} = y(1-y^2), \quad \dot{y} = -x(1-2x^2), \tag{3}$$

which has nine finite singular points:

$$O(0,0), \quad S_1(0,1), \quad S_2(0,-1),$$

$$S_3\left(\frac{1}{\sqrt{2}}, 0\right), \quad S_4\left(-\frac{1}{\sqrt{2}}, 0\right),$$

$$A_1\left(\frac{\sqrt{2}}{2}, 1\right), \quad A_2\left(-\frac{1}{\sqrt{2}}, 1\right),$$

$$A_3\left(\frac{1}{\sqrt{2}}, -1\right), \quad A_4\left(-\frac{1}{\sqrt{2}}, -1\right),$$

among them S_i, $i = 1,2,3,4$ are saddles, while O and A_i, $i = 1,2,3,4$ are centers. The Hamiltonian function of system (3) has the form

$$H(x,y) = \frac{x^2}{2} + \frac{y^2}{2} - \frac{y^4}{4} - \frac{x^4}{2} \tag{4}$$

from which we obtain

$$H(O) = 0, \quad H(S_i) = \frac{1}{4}, \quad i = 1,2,$$

$$H(S_j) = \frac{1}{8}, \quad j = 3,4,$$

$$H(A_k) = \frac{3}{8}, \quad k = 1,2,3,4,$$

implying that

$$H(O) < H(S_j) < H(S_i) < H(A_k),$$
$$i = 1,2; \; j = 3,4; \; k = 1,2,3,4.$$

The level curves of the function H defined by (4), $H(x,y) = h$, are divided into six categories, described as follows (see Fig. 1 for the notations):

(1) Γ^h $(-\infty < h < 1/8)$: the family of closed orbits surrounding all nine singular points;
(2) $\Gamma^{\frac{1}{8}} = L_1 \cup L_2$ with $L_1 \cup L_2 = \lim_{h \to \frac{1}{8}^-} \Gamma^h$: the heteroclinic loop surrounding the singular points $O,\ S_1,\ S_2,\ A_i,\ i = 1,2,3,4$;

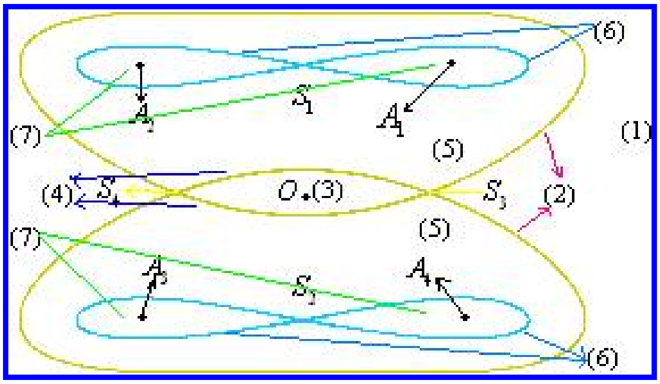


Fig. 1. The structure of the unperturbed system of (1) when $\varepsilon = 0$.

(3) Γ_1^h $(0 \le h < 1/8)$: the family of closed orbits which surrounds only the center O;

(4) $\Gamma_1^{\frac{1}{8}} = L_3 \cup L_4$ with $L_3 \cup L_4 = \lim\limits_{h \to \frac{1}{8}^-} \Gamma_1^h$: the heteroclinic loop surrounding the singular point O;

(5) Γ_j^h $(1/8 < h < 1/4)$, $(-1)^j y > 0$, $j = 2, 3$: two families of closed orbits;

(6) L_5, L_6, L_7, L_8: four homoclinic loops, where L_{4+i} surrounding A_i, $i = 1, 2, 3, 4$, and $L_5 \cup L_6 = \lim_{h \to \frac{1}{4}^-} \Gamma_2^h$, $L_7 \cup L_8 = \lim_{h \to \frac{1}{4}^-} \Gamma_3^h$;

(7) Γ_{ij}^h $(1/4 < h < 3/8)$, $(-1)^i x > 0$, $(-1)^j y > 0$, $i, j = 1, 2$: four families of closed orbits.

Here, Γ^h is defined by

$$y = \pm\sqrt{1 + \sqrt{1 + 2x^2 - 2x^4 - 4h}}, \quad x \in \left[-\sqrt{\frac{1+\sqrt{3-8h}}{2}}, \sqrt{\frac{1+\sqrt{3-8h}}{2}}\right],$$

$$y = \pm\sqrt{1 - \sqrt{1 + 2x^2 - 2x^4 - 4h}}, \quad x \in \left[-\sqrt{\frac{1+\sqrt{3-8h}}{2}}, -\sqrt{\frac{1+\sqrt{1-8h}}{2}}\right] \bigcup \left[\sqrt{\frac{1+\sqrt{1-8h}}{2}}, \sqrt{\frac{1+\sqrt{3-8h}}{2}}\right]$$

or

$$x = \pm\sqrt{\frac{1 + \sqrt{1 - 8h + 4y^2 - 2y^4}}{2}}, \quad y \in \left[-\sqrt{\sqrt{\frac{3}{2} - 4h} + 1}, \sqrt{\sqrt{\frac{3}{2} - 4h} + 1}\right],$$

$$x = \pm\sqrt{\frac{1 - \sqrt{1 - 8h + 4y^2 - 2y^4}}{2}}, \; y \in \left[-\sqrt{\sqrt{\frac{3}{2} - 4h} + 1}, -\sqrt{\sqrt{1-4h} + 1}\right] \bigcup \left[\sqrt{\sqrt{1-4h} + 1}, \sqrt{\sqrt{\frac{3}{2} - 4h} + 1}\right];$$

and L_k, $k = 1, 2, \ldots, 8$, are given below.

$$L_1 : \left\{ y = \sqrt{1 + \sqrt{2x^2 - 2x^4 + \frac{1}{2}}}, \; x \in \left[-\sqrt{\frac{1+\sqrt{2}}{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right] \right\}$$
$$\bigcup \left\{ y = \sqrt{1 - \sqrt{2x^2 - 2x^4 + \frac{1}{2}}}, \; x \in \left[-\sqrt{\frac{1+\sqrt{2}}{2}}, -\sqrt{\frac{1}{2}}\right] \bigcup \left[\sqrt{\frac{1}{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right] \right\},$$

$$L_2 : \left\{ y = -\sqrt{1 + \sqrt{2x^2 - 2x^4 + \frac{1}{2}}}, \; x \in \left[-\sqrt{\frac{1+\sqrt{2}}{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right] \right\}$$
$$\bigcup \left\{ y = -\sqrt{1 - \sqrt{2x^2 - 2x^4 + \frac{1}{2}}}, \; x \in \left[-\sqrt{\frac{1+\sqrt{2}}{2}}, -\sqrt{\frac{1}{2}}\right] \bigcup \left[\sqrt{\frac{1}{2}}, \sqrt{\frac{1+\sqrt{2}}{2}}\right] \right\},$$

$$L_3 : y = \sqrt{1 - \sqrt{\frac{1}{2} + 2x^2 - 2x^4}}, \quad x \in \left[-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right],$$

$$L_4 : y = -\sqrt{1 - \sqrt{\frac{1}{2} + 2x^2 - 2x^4}}, \quad x \in \left[-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}\right],$$

$$
\begin{aligned}
L_5 &: y = \sqrt{1 \pm x\sqrt{2(1-x^2)}}, \quad x \in [0,1], \\
L_6 &: y = \sqrt{1 \pm x\sqrt{2(1-x^2)}}, \quad x \in [-1,0], \\
L_7 &: y = -\sqrt{1 \pm x\sqrt{2(1-x^2)}}, \quad x \in [-1,0], \\
L_8 &: y = -\sqrt{1 \pm x\sqrt{2(1-x^2)}}, \quad x \in [0,1].
\end{aligned} \tag{5}
$$

Let L_k^s and L_k^u respectively denote the stable and unstable manifolds near L_k after perturbation. Then, as we know, the directed distance from L_k^s to L_k^u can be measured as

$$d_k(\varepsilon, \theta) = \varepsilon\, N_k M_k(\theta) + O(\varepsilon^2), \tag{6}$$

where $N_k > 0$, $k = 1, 2, \ldots, 8$, and $\theta = (\theta_1, \ldots, \theta_{13})$ is a vector parameter with

$$
\begin{gathered}
\theta_1 = a_{04}, \quad \theta_2 = b_{40}, \quad \theta_3 = a_{40}, \quad \theta_4 = b_{04}, \\
\theta_5 = a_{31}, \quad \theta_6 = b_{31}, \quad \theta_7 = a_{22}, \quad \theta_8 = b_{22}, \\
\theta_9 = b_{03}, \quad \theta_{10} = a_{21}, \quad \theta_{11} = a_{03}, \\
\theta_{12} = b_{13}, \quad \theta_{13} = a_{10} + b_{01},
\end{gathered}
$$

and

$$
\begin{aligned}
M_k(\theta) &= \int_{L_k} \frac{Q_4(x,y)dx - P_4(x,y)dy}{2-x} \\
&= \sum_{i+j=1}^{4} (A_{ij}^{(k)} a_{ij} + B_{ij}^{(k)} b_{ij}),
\end{aligned} \tag{7}
$$

where

$$A_{ij}^{(k)} = -\int_{L_k} \frac{x^i y^j}{2-x} dy, \quad B_{ij}^{(k)} = \int_{L_k} \frac{x^i y^j}{2-x} dx,$$
$$1 \le i + j \le 4, \; k = 1, 2, \ldots, 8. \tag{8}$$

With integration by parts, we have

$$\int_{L_k} \frac{x^i y^j}{2-x} dy = -\frac{1}{j+1} \int_{L_k} \left[\frac{i x^{i-1}}{2-x} + \frac{x^i}{(2-x)^2} \right] \times y^{j+1} dx \quad \text{for } j \ge 0.$$

Furthermore, by (5), (8) and using a computer algebra system such as Mathematica or Maple, we obtain the explicit expressions for the coefficients $A_{ij}^{(1)}$ and $B_{ij}^{(1)}$ by computing the integrals along the curve L_1. The accuracy in the numerical calculation takes 20 decimal digits, as listed in Appendix A. For $k = 2$, it is easy to show by using (8) and noticing from (5) the opposite signs in the variable y of L_1 and L_2 that

$$A_{ij}^{(2)} = (-1)^j A_{ij}^{(1)}, \quad B_{ij}^{(2)} = (-1)^{j+1} B_{ij}^{(2)}, \quad 1 \le i + j \le 4.$$

Similarly, one can obtain the coefficients $A_{ij}^{(k)}$ and $B_{ij}^{(k)}$ for $k = 3, 4$, which are also given in Appendix A. Again, one may show that

$$A_{ij}^{(4)} = (-1)^j A_{ij}^{(3)}, \quad B_{ij}^{(4)} = (-1)^{j+1} B_{ij}^{(3)}, \quad 1 \le i + j \le 4.$$

The coefficients $A_{ij}^{(k)}$ and $B_{ij}^{(k)}$ for $k = 5, 6$ are similarly obtained, as given in Appendix A.

By noticing the signs of the variable y, we have

$$
\begin{aligned}
A_{ij}^{(7)} &= (-1)^j A_{ij}^{(6)}, \quad B_{ij}^{(7)} = (-1)^{j+1} B_{ij}^{(6)}, \\
A_{ij}^{(8)} &= (-1)^j A_{ij}^{(5)}, \quad B_{ij}^{(8)} = (-1)^{j+1} B_{ij}^{(5)}.
\end{aligned}
$$

Having obtained the above coefficients, we apply the implicit function theorem to obtain the following result:

Lemma 1. *There exist functions ϕ_i, $i = 1, 2, \ldots, 8$, given by*

$$
\begin{aligned}
\phi_1(\varepsilon) = {} & -0.94821452555803331 86a_{01} - 0.88453526867695672 93a_{02} - 0.91305636465790894 34a_{03} \\
& - 2.5539763858870095 61a_{10} - 1.8964290511160666 373a_{11} - 1.7690705373539134 586a_{12} \\
& - 1.8261127293158178 869a_{13} - 1.3106946977600911 045a_{20} - 1.0535243596755620 864\theta_{10} \\
& - 1.0058295031590574 797\theta_7 - 2.621389395520182209\theta_{11} - 2.10704871935112 4173\theta_5 \\
& - 1.3759829182011716 388\theta_3 - 2.091009654617619136b_{01} - 3.046359946483204281b_{02} \\
& - 4.237547958778344212b_{03} - 5.821818198275195096\theta_4 - 0.08995697023324826691b_{10} \\
& - 0.3847612352213102558b_{11} - 0.6140524078532661855b_{12} - 0.8781612029103801109\theta_{12} \\
& - 0.1799139404664965 3382b_{20} - 0.7695224704426205115b_{21} - 1.2281048157065323709\theta_8
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 0.027399404666561945866b_{30} - 0.25011298327217676632\theta_6 - 0.054798809333123891720\theta_2 + O(\varepsilon) \\
&= \phi_1^* + O(\varepsilon), \\
\phi_2(\varepsilon) &= -17.303560736033584701a_{01} - 34.607121472067169402a_{11} - 19.225314792355257269\theta_{10} \\
&\quad - 16.661974516771106261a_{03} - 38.450629584710514538\theta_5 - 33.323949033542212521a_{13} \\
&\quad - 1.6415862192624784410b_{10} - 3.2831724385249568820b_{20} - 55.591717841244960017b_{02} \\
&\quad - 11.205579378931756961b_{12} - 22.411158757863513922\theta_8 - 106.23986668914934354\theta_4 \\
&\quad - 0.5b_{30} + O(\varepsilon) \\
&= \phi_2^* + O(\varepsilon), \\
\phi_3(\varepsilon) &= -21.029432426536066268a_{10} + 51.096535044613654366a_{01} - 3.0408381251205152994a_{20} \\
&\quad + 102.19307008922730873a_{11} - 0.58784092819649298548a_{02} - 6.0816762502410305934\theta_{11} \\
&\quad + 58.771041489446332281\theta_{10} - 1.1756818563929859717a_{12} + 51.096535044613654370a_{03} \\
&\quad + 117.54208297889266456\theta_5 - 0.013191132128485927905\theta_7 + 102.19307008922730874a_{13} \\
&\quad - 19.998612940556157271b_{01} - 0.97919915316069730616b_{11} + 162.72765009272586546b_{02} \\
&\quad - 1.9583983063213946123b_{21} + 34.225102807435160540b_{12} - 3.5185626993750559749\theta_9 \\
&\quad - 0.19567911248210215975\theta_6 + 68.450205614870321080\theta_8 - 0.022607656777653907303\theta_{12} \\
&\quad + 325.45530018545173092\theta_4 + O(\varepsilon) \\
&= \phi_3^* + O(\varepsilon), \\
\phi_4(\varepsilon) &= -0.15700016258914112771a_{01} - 0.31400032517828225540a_{11} - 0.18058099362940862982\theta_{10} \\
&\quad - 0.15700016258914112771a_{03} - 0.36116198725881725963\theta_5 - 0.31400032517828225542a_{13} \\
&\quad - 0.5b_{02} - 0.10516068654568823797b_{12} - 0.21032137309137647593\theta_8 + O(\varepsilon) \\
&= \phi_4^* + O(\varepsilon), \\
\phi_5(\varepsilon) &= 77.147930708533360667a_{10} - 0.72395927570724844192a_{01} + 6.6246357994301813215a_{20} \\
&\quad - 1.4479185514144968980a_{11} + 1.8189585851884235854a_{02} + 23.676092650885969462\theta_{11} \\
&\quad - 5.4245752925359223056\theta_{10} + 3.6379171703768471702a_{12} - 0.72395927570724845643a_{03} \\
&\quad - 4.8249265474282921725\theta_7 - 1.4479185514144968980a_{13} + 74.471007297766483464b_{01} \\
&\quad + 1.2707889778964269618b_{11} + 7.7549884818056573310b_{21} - 5.0574763794138570121b_{12} \\
&\quad + 14.663212728801531240\theta_9 - 0.27408480364599831192\theta_6 - 0.26580217375586943440\theta_8 \\
&\quad - 6.9758572069428970621\theta_{12} + O(\varepsilon) \\
&= \phi_5^* + O(\varepsilon), \\
\phi_6(\varepsilon) &= -57.161330941360084203a_{10} - 1.4973284076028837810a_{01} - 15.599632407329294122a_{20} \\
&\quad - 2.9946568152057673922a_{11} - 5.6096858434823165209a_{02} - 5.2815643221879658798\theta_{11} \\
&\quad - 12.240899392799274673\theta_{10} - 11.219371686964633429a_{12} - 1.4973284076028837290a_{03} \\
&\quad - 14.633878639637842750\theta_7 - 2.9946568152057673909a_{13} - 53.166357659436601899b_{01} \\
&\quad - 7.6096858434823163689b_{11} - 2.2605214407293206620b_{21} - 12.503171524775478995b_{12} \\
&\quad - 22.092782936885708745\theta_9 - 0.52454426395240859958\theta_8 - 20.770971670897143467\theta_{12} + O(\varepsilon) \\
&= \phi_6^* + O(\varepsilon),
\end{aligned}
$$

$$\begin{aligned}\phi_7(\varepsilon) &= 114.02292912364458808a_{10} + 0.31457094046003529412a_{01} + 13.390934363342234056a_{20}\\&\quad + 0.62914188092007052541a_{11} + 4.1234357587619612664a_{02} + 30.864377074909867575\theta_{11}\\&\quad + 2.5716677881201287525\theta_{10} + 8.2468715175239227008a_{12} + 0.31457094046003525850a_{03}\\&\quad + 0.62914188092007051974a_{13} + 109.38917982135445238b_{01} + 4.1234357587619612991b_{11}\\&\quad + 10.288125691636622197b_{21} + 2.6267680525599752142b_{12} + 25.452626154622797330\theta_9\\&\quad + 0.11020052887969286178\theta_8 - 1.5759767958979418521\theta_{12} + O(\varepsilon)\\&= \phi_7^* + O(\varepsilon),\\\phi_8(\varepsilon) &= -2.8545320395281925080a_{01} - 5.7090640790563848388a_{11} - 23.336256316225540329\theta_{10}\\&\quad - 2.8545320395281925883a_{03} - 5.7090640790563851553a_{13} - 23.836256316225541297b_{12} + O(\varepsilon)\\&= \phi_8^* + O(\varepsilon).\end{aligned}$$

These functions satisfy $d_1 = \cdots = d_k = 0$, $d_{k+1} \geq 0$ *for* $0 \leq k \leq 7$ *if and only if* $\theta_j = \phi_j$, $j = 1, \ldots, k$, $\theta_{k+1} \leq \phi_{k+1}$ *for* $k = 0, 5$, *and* $\theta_{k+1} \geq \phi_{k+1}$ *for* $k = 1, 2, 3, 4, 6, 7$.

Proof. The conclusion follows from (7) and the implicit function theorem with the help of symbolic computation. Details are omitted for brevity. ■

It is clear that for $1 \leq k \leq 8$ there exist connections $L_1^*, \ldots, L_k^*$ near $L_1, \ldots, L_k$ respectively if and only if $d_1 = \cdots = d_k = 0$ or equivalently $\theta_j = \phi_j$, $j = 1, \ldots, k$. Here, L_1^*, L_2^*, L_3^* and L_4^* (resp. L_5^*, L_6^*, L_7^* and L_8^*) are heteroclinic (resp. homoclinic) orbits.

Next, we investigate the stability of the homoclinic loops L_i^*, $i = 5, 6, 7, 8$. Let $S_{j\varepsilon}$ denote the saddle points of system (1) near S_j, and σ_{0j} represent the divergence at $S_{j\varepsilon}$. Then, it is easy to show that

$$S_{1\varepsilon} = \left(\frac{\varepsilon}{2}(b_{01} + b_{02} + \theta_9 + \theta_4) + O(\varepsilon^2),\ 1 + \frac{\varepsilon}{4}(a_{01} + a_{02} + a_{03} + \theta_1) + O(\varepsilon^2)\right),$$

$$S_{2\varepsilon} = \left(\frac{\varepsilon}{2}(-b_{01} + b_{02} - \theta_9 + \theta_4) + O(\varepsilon^2),\ -1 + \frac{\varepsilon}{4}(-a_{01} + a_{02} - a_{03} + \theta_1) + O(\varepsilon^2)\right)$$

and

$$\begin{aligned}\sigma_{01} &= \frac{\varepsilon}{4}(a_{01} + a_{02} + a_{03} + \theta_1 + 2a_{10} + 2a_{11} + 2a_{12}\\&\quad + 2a_{13} + 2b_{01} + 4b_{02} + 6\theta_9 + 8\theta_4) + O(\varepsilon^2),\\\sigma_{02} &= \frac{\varepsilon}{4}(-a_{01} + a_{02} - a_{03} + \theta_1 + 2a_{10} - 2a_{11}\\&\quad + 2a_{12} - 2a_{13} + 2b_{01} - 4b_{02} + 6\theta_9 - 8\theta_4)\\&\quad + O(\varepsilon^2).\end{aligned} \tag{9}$$

Lemma 2. *Under the conditions* $\theta_i = \phi_i$, $i = 1, 2, \ldots, 8$, *there is a function* ϕ_9 *such that*

$$\sigma_{01} \geq 0\ (< 0) \text{ if and only if } \theta_9 \leq \phi_9 \quad (> \phi_9); \tag{10}$$

and under the conditions $\theta_i = \phi_i$, $i = 1, 2, \ldots, 9$, *there is a function* ϕ_{10} *such that*

$$\sigma_{02} \geq 0\ (< 0) \text{ if and only if } \theta_{10} \leq \phi_{10}\ (> \phi_{10}), \tag{11}$$

where

$$\begin{aligned}\phi_9 &= -4.4952218902927897264a_{10} + 0.17045188144336646370a_{01} - 0.55206683405617142902a_{20}\\&\quad + 0.34090376288673292301a_{11} - 0.16907330695155856893a_{02} - 1.2835213209183248050\theta_{11}\\&\quad + 1.3636150515469317173\theta_{10} - 0.33814661390311716574a_{12} + 0.17045188144336646939a_{03}\\&\quad + 0.34090376288673293721a_{13} - 4.3037251267404824963b_{01} - 0.16907330695155862003b_{11}\\&\quad - 0.42784044030610832070b_{21} + 1.3636150515469317679b_{12} + 0.028879683418702851550\theta_{12}\\&\quad + O(\varepsilon)\\&= \phi_9^* + O(\varepsilon),\end{aligned}$$

$$\begin{aligned}\phi_{10} &= -0.125a_{01} - 0.125a_{03} - 0.25a_{11} - 0.25a_{13} - b_{12} + O(\varepsilon)\\ &= \phi_{10}^* + O(\varepsilon).\end{aligned}$$

Proof. Substituting $\theta_i = \phi_i,\ i = 1, 2, \ldots, 8$, into the first equation of (9), we have

$$\begin{aligned}\sigma_{01} = \frac{\varepsilon}{4}(&4.44645730737001193376a_{01} - 4.41049541260565445072a_{02} + 4.44645730737001193377a_{03}\\ &- 117.26366440245523199368a_{10} + 8.89291461474002386753a_{11} - 8.82099082521130890144a_{12}\\ &+ 8.89291461474002386753a_{13} - 14.40137584671537582611a_{20} + 35.57165845896009547013\theta_{10}\\ &- 33.48231012902440154802\theta_{11} - 112.26822418540037130599b_{01} - 26.08629056903294412862\theta_9\\ &- 4.41049541260565445072b_{11} + 35.57165845896009547013b_{12} + 0.75336381320196515140\theta_{12}\\ &- 11.16077004300813384934b_{21}) + O(\varepsilon^2),\end{aligned}$$

which directly yields the function ϕ_9. Then, substituting $\theta_i = \phi_i,\ i = 1, 2, \ldots, 9$, into the second equation of (9) results in

$$\begin{aligned}\sigma_{02} = \frac{\varepsilon}{4}(&-8.89291461474002386753a_{01} - 8.89291461474002386753a_{03} - 17.78582922948004773506a_{11}\\ &- 17.78582922948004773506a_{13} - 71.14331691792019094026\theta_{10} - 71.14331691792019094026b_{12})\\ &+ O(\varepsilon^2).\end{aligned}$$

Therefore, ϕ_{10} can be obtained by a direct application of the implicit function theorem.

This completes the proof. ■

Furthermore, according to [Han & Chen, 2000] or [Han *et al.*, 2003], we can easily obtain the following lemma. (The detailed proof can be found in [Han & Chen, 2000].)

Lemma 3. *The stabilities of the homoclinic and heteroclinic loops related to* L_k, $k = 5, 6, 7, 9$, *are determined as follows.*

(i) *If* $\sigma_{0i} > 0$ (<0), $i = 1, 2$, *then the homoclinic loops* L^*_{3+2i} *and* L^*_{4+2i}, $i = 1, 2$, *are unstable* (*stable*) *inside, while the double-homoclinic loops* $L^*_{3+2i} \cup L^*_{4+2i}$, $i = 1, 2$, *are unstable* (*stable*) *outside.*

(ii) *When* $\theta_i = \phi_i$, $i = 1, 2, \ldots, 10$, *the integral* $\oint_{L_k^*}[(P_{4x} + Q_{4y})/(2 - x) + P_4/(2 - x)^2]dt = \sigma_{1,k}(\theta, \varepsilon)$ *converges finitely to* $\sigma_{1,k}(\theta, 0) = \oint_{L_k}[(P_{4x} + Q_{4y})/(2 - x) + P_4/(2 - x)^2]dt$.

(iii) L_k^* *is stable* (*unstable*) *inside if* $\sigma_{1k}(\theta, \varepsilon) < 0$ (>0), $k = 5, 6, 7, 8$, *and the double-homoclinic loop* $L_5^* \cup L_6^*$ (*resp.* $L_7^* \cup L_8^*$) *is unstable* (*stable*) *outside if* $\sigma_{1,5}(\theta, \varepsilon) + \sigma_{1,6}(\theta, \varepsilon) > 0$ (*resp.* $\sigma_{1,7}(\theta, \varepsilon) + \sigma_{1,8}(\theta, \varepsilon) < 0$).

If $\sigma_{0k} = 0$, $k = 5, 6$ (*resp.* $k = 7, 8$), *we obtain*

$$\begin{aligned}\sigma_{1,k}(\theta, 0) = &C_{01}^{(k)}a_{01} + C_{02}^{(k)}a_{02} + C_{03}^{(k)}a_{03} + C_{04}^{(k)}a_{04}\\ &+ C_{05}^{(k)}a_{11} + C_{06}^{(k)}b_{02} + C_{07}^{(k)}a_{12}\\ &+ C_{08}^{(k)}b_{03} + C_{09}^{(k)}a_{13} + C_{10}^{(k)}b_{04}\\ &+ C_{11}^{(k)}b_{01} + C_{12}^{(k)}a_{20} + C_{13}^{(k)}\theta_{11} + C_{14}^{(k)}a_{21}\\ &+ C_{15}^{(k)}a_{40} + C_{16}^{(k)}a_{31} + C_{17}^{(k)}a_{22}\\ &+ C_{18}^{(k)}(2a_{20} + b_{11}) + C_{19}^{(k)}(3a_{30} + b_{21})\\ &+ C_{20}^{(k)}(4a_{40} + b_{31}) + C_{21}^{(k)}(2a_{21} + 2b_{12})\\ &+ C_{22}^{(k)}(3a_{31} + 2b_{22}) + C_{23}^{(k)}(2a_{22} + 3b_{13}),\end{aligned}$$

where the explicit expressions of the coefficients $C_{ij}^{(k)}$ *are given in Appendix A.*

Under the conditions $\theta_i = \phi_i$, $i = 1, 2, \ldots, 10$, we then have

$$\begin{aligned}\sigma_{1,8} = &0.00638703115173034319a_{02} - 0.49870153362917956314a_{10} + 0.01277406230346068638a_{12}\\ &- 0.01303028330817343931a_{20} - 0.15482607366980475449\theta_{11} - 0.48899287639922767186b_{01}\\ &+ 0.00638703115173034322b_{11} - 0.07664437382076411797\theta_{12} - 0.05160869122326825150b_{21}\\ &+ O(\varepsilon),\end{aligned}$$

$$\begin{aligned}\sigma_{1,5} = {} & 0.00638703115173034319a_{02} - 0.49870153362917956314a_{10} + 0.01277406230346068638a_{12} \\ & - 0.01303028330817343931a_{20} - 0.15482607366980475449\theta_{11} - 0.48899287639922767186b_{01} \\ & + 0.00638703115173034322b_{11} - 0.07664437382076411797\theta_{12} - 0.05160869122326825150b_{21} \\ & + O(\varepsilon).\end{aligned}$$

It is noted from the above expressions that the dominant parts of $\sigma_{1,5}$ and $\sigma_{1,8}$ are identical. However, they are not necessarily the same. More precisely, there exist functions

$$\begin{aligned}\phi_{11} = {} & -3.22104359949573857 03a_{10} - 0.08416078118704331193 2a_{20} + 0.04125294273981164669 6a_{02} \\ & + 0.08250588547962348412 7a_{12} - 3.15833673753231007 55b_{01} + 0.04125294273981154845 1b_{11} \\ & - 0.33333333333333388736b_{21} - 0.49503531287774157713\theta_{12} + O(\varepsilon) \\ = {} & \phi_{11}^* + O(\varepsilon)\end{aligned}$$

and

$$\tilde{\phi}_{11} = \phi_{11}^* + O(\varepsilon)$$

such that

$$\begin{cases}\sigma_{1,8} \geq 0\ (<0) & \text{if and only if } \theta_{11} \leq \phi_{11} \quad (>\phi_{11}), \\ \sigma_{1,5} \geq 0\ (<0) & \text{if and only if } \theta_{11} \leq \tilde{\phi}_{11} \quad (>\tilde{\phi}_{11}).\end{cases} \tag{12}$$

Then, substituting $\theta_{11} = \phi_{11}$ into $\sigma_{1,5}$ yields

$$\sigma_{1,5} = O(\varepsilon) \Rightarrow \begin{cases}\sigma_{1,5} \equiv 0 & \text{if } \phi_{11} = \tilde{\phi}_{11}, \\ \sigma_{1,5} < 0 & \text{if } \phi_{11} > \tilde{\phi}_{11}, \\ \sigma_{1,5} > 0 & \text{if } \phi_{11} < \tilde{\phi}_{11}.\end{cases}$$

Similarly, under the conditions $\theta_i = \phi_i,\ i = 1, 2, \ldots, 11$, we obtain

$$\begin{aligned}\sigma_{1,7} = {} & -0.00192480494518933452a_{02} + 0.72785265305612861333a_{10} - 0.00384960989037866905a_{12} \\ & - 0.00384960989037866904a_{20} + 0.72881505552872328059b_{01} - 0.00192480494518933452b_{11} \\ & + 0.02309765934227201418\theta_{12} + O(\varepsilon), \\ \sigma_{1,6} = {} & -0.00192480494518933452a_{02} + 0.72785265305612861333a_{10} - 0.00384960989037866905a_{12} \\ & - 0.00384960989037866904a_{20} + 0.72881505552872328059b_{01} - 0.00192480494518933452b_{11} \\ & + 0.02309765934227201418\theta_{12} + O(\varepsilon),\end{aligned}$$

and there exist functions

$$\begin{aligned}\phi_{12} = {} & -31.51196587803401478 7a_{10} + 0.16666666666666741067a_{20} + 0.08333333333333151562a_{02} \\ & + 0.16666666666666659498a_{12} - 31.55363254470067332 9b_{01} + 0.08333333333333336594 27b_{11} + O(\varepsilon) \\ = {} & \phi_{12}^* + O(\varepsilon)\end{aligned}$$

and

$$\tilde{\phi}_{12} = \phi_{12}^* + O(\varepsilon)$$

such that

$$\begin{cases}\sigma_{1,7} \geq 0\ (<0) & \text{if and only if } \theta_{12} \leq \phi_{12}\ (>\phi_{12}), \\ \sigma_{1,6} \geq 0\ (<0) & \text{if and only if } \theta_{12} \leq \tilde{\phi}_{12}\ (>\tilde{\phi}_{12}).\end{cases} \tag{13}$$

A similar treatment by substituting $\theta_{12} = \phi_{12}$ into $\sigma_{1,6}$, we have

$$\sigma_{1,6} = O(\varepsilon) \Rightarrow \begin{cases}\sigma_{1,6} \equiv 0 & \text{if } \phi_{12} = \tilde{\phi}_{12}, \\ \sigma_{1,6} < 0 & \text{if } \phi_{12} < \tilde{\phi}_{12}, \\ \sigma_{1,6} > 0 & \text{if } \phi_{12} > \tilde{\phi}_{12}.\end{cases}$$

When the conditions $\theta_i = \phi_i, i = 1, 2, \ldots, 12$ hold, we can apply the same method described above to find the divergence after perturbation at the saddle point $S_{3\varepsilon}$ near $S_3(1/\sqrt{2}, 0)$ as

$$\begin{aligned}\mathrm{div}(S_{3\varepsilon}) &= \frac{\varepsilon}{(4-\sqrt{2})^2}[8a_{10} + (8\sqrt{2}-2)a_{20} \\ &\quad + (12-2\sqrt{2})\theta_{11} + (8\sqrt{2}-3)\theta_3 \\ &\quad + (8-2\sqrt{2})b_{01} + (4\sqrt{2}-2)b_{11} \\ &\quad + (4-\sqrt{2})b_{21} + (2\sqrt{2}-1)\theta_6] \\ &= 20.96728078517308628507\varepsilon\theta_{13} + O(\varepsilon^2).\end{aligned}$$

Similarly, the divergence at the saddle point $S_{4\varepsilon}$ near $S_4(-(1/\sqrt{2}), 0)$ after perturbation is

$$\begin{aligned}\mathrm{div}(S_{4\varepsilon}) &= \frac{\varepsilon}{(4+\sqrt{2})^2}[8a_{10} - (8\sqrt{2}+2)a_{20} \\ &\quad + (12+2\sqrt{2})\theta_{11} - (8\sqrt{2}+3)\theta_3 \\ &\quad + (8+2\sqrt{2})b_{01} - (4\sqrt{2}+2)b_{11} \\ &\quad + (4+\sqrt{2})b_{21} - (2\sqrt{2}+1)\theta_6] \\ &= -16.54967115762177314318\varepsilon\theta_{13} \\ &\quad + O(\varepsilon^2),\end{aligned}$$

which is negative for fixed $\theta_{13} > 0$ and small $\varepsilon > 0$.

Denote the eigenvalues of the system at the saddle points $S_{i\varepsilon}, i = 1, 2, 3, 4$ by $\lambda_{i1}, \lambda_{i2}$ (where $\lambda_{i1} < 0 < \lambda_{i2}$), and let $\gamma_i = |\lambda_{i1}|/\lambda_{i2}$. When $\theta_{13} > 0$ and $\varepsilon > 0$, it is not difficult to show that

$$\begin{aligned}\gamma_3\gamma_4 &= \frac{|\lambda_{31}|}{\lambda_{32}}\frac{|\lambda_{41}|}{\lambda_{42}} \\ &= 1 - 3.12372172427651208787\theta_{13}\varepsilon \\ &\quad + O(\varepsilon^2) < 1.\end{aligned}$$

Next, by the result given in [Han & Zhang, 1999] or [Han & Yang, 2005], we obtain

Lemma 4. *If $\theta_i = \phi_i$, $i = 1, 2, \ldots, 12$, then for $\theta_{13} > 0$ and $\varepsilon > 0$ small, the heteroclinic loop $L_1^* \cup L_2^*$ is unstable outside, while the heteroclinic loops $L_2^* \cup L_4^*$, $L_1^* \cup L_3^*$ and $L_3^* \cup L_4^*$ are unstable inside.*

This is a direct result of Cherkas [1968], and so the proof is omitted.

Now we consider the boundedness of the positive orbit ρ_B^+ of system (2) starting from $B(b_0, 0)$ with $b_0 \in (1/\sqrt{2}, 2)$. Let $B^*(b^*, 0)$ be the first intersection point of ρ_B^+ with the line $y = 0$, $x > 0$ after B. We may fix B such that $H(B) = -3$, and thus we have

$$H(B^*) - H(B) = \varepsilon M + O(\varepsilon^2), \tag{14}$$

in which

$$\begin{aligned}M &= \oint_{\Gamma^{-3}} \frac{Q_4(x,y)dx - P_4(x,y)dy}{2-x} \\ &= D_{01}a_{20} + D_{02}a_{11} + D_{03}\theta_{11} + D_{04}\theta_{10} \\ &\quad + D_{05}a_{12} + D_{06}a_{03} + D_{07}\theta_3 + D_{08}\theta_5 \\ &\quad + D_{09}\theta_7 + D_{10}a_{13} + D_{11}\theta_1 + D_{12}b_{10} \\ &\quad + D_{13}b_{20} + D_{14}b_{11} + D_{15}b_{02} + D_{16}b_{30} \\ &\quad + D_{17}b_{21} + D_{18}b_{12} + D_{19}\theta_9 + D_{20}\theta_2 \\ &\quad + D_{21}\theta_6 + D_{22}\theta_8 + D_{23}\theta_{12} + D_{24}\theta_4,\end{aligned}$$

where D_{ij} are listed in Appendix A.

Therefore, under the conditions $\theta_i = \phi_i$, $i = 1, 2, \ldots, 12$, we use symbolic computation to obtain

$$M = 25505.61712996293187830693\theta_{13} + O(\varepsilon),$$

$$\begin{aligned}\mathrm{div}(O) &= \frac{\varepsilon}{2}(a_{10} + b_{01}) + O(\varepsilon^2) \\ &= \frac{\varepsilon}{2}\theta_{13} + O(\varepsilon^2),\end{aligned}$$

$$\begin{aligned}\mathrm{div}(A_{1\varepsilon}) &= \frac{\varepsilon}{(4-\sqrt{2})^2}\Bigg[4a_{01} + 4a_{02} + 4a_{03} + 4\theta_1 \\ &\quad + 8a_{10} + 8a_{11} + 8a_{12} + 8a_{13} \\ &\quad + (8\sqrt{2}-2)a_{20} + (8\sqrt{2}-2)\theta_{10} \\ &\quad + (8\sqrt{2}-2)\theta_7 + (12-2\sqrt{2})\theta_{11} \\ &\quad + (12-2\sqrt{2})\theta_5 + (8\sqrt{2}-3)\theta_3 \\ &\quad + 2(4-\sqrt{2})\bigg(b_{01} + 2b_{02} + 3\theta_9 + 4\theta_4 \\ &\quad + \frac{b_{11}}{\sqrt{2}} + \sqrt{2}b_{12} + \frac{3\theta_{12}}{\sqrt{2}} + \frac{b_{21}}{2} \\ &\quad + \theta_8 + \frac{\theta_6}{2\sqrt{2}}\bigg)\Bigg] \\ &= 0.33876078460172253203\varepsilon\theta_{13} + O(\varepsilon^2),\end{aligned}$$

$$\begin{aligned}\mathrm{div}(A_{2\varepsilon}) &= \frac{\varepsilon}{(4-\sqrt{2})^2}\Big[-4a_{01} + 4a_{02} - 4a_{03} \\ &\quad + 4\theta_1 + 8a_{10} - 8a_{11} + 8a_{12} - 8a_{13} \\ &\quad + (8\sqrt{2}-2)a_{20} - (8\sqrt{2}-2)\theta_{10} \\ &\quad + (8\sqrt{2}-2)\theta_7 + (12-2\sqrt{2})\theta_{11} \\ &\quad - (12-2\sqrt{2})\theta_5 + (8\sqrt{2}-3)\theta_3\end{aligned}$$

$$
\begin{aligned}
&+2(4-\sqrt{2})\Bigg(b_{01}-2b_{02}+3\theta_9-4\theta_4\\
&+\frac{b_{11}}{\sqrt{2}}-\sqrt{2}b_{12}+\frac{3\theta_{12}}{\sqrt{2}}+\frac{b_{21}}{2}\\
&-\theta_8+\frac{\theta_6}{2\sqrt{2}}\Bigg)\Bigg]\\
&=0.33876078460172253203\varepsilon\theta_{13}+O(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{div}(A_{3\varepsilon})=&\frac{\varepsilon}{(4+\sqrt{2})^2}\Bigg[4a_{01}+4a_{02}+4a_{03}+4\theta_1\\
&+8a_{10}+8a_{11}+8a_{12}+8a_{13}\\
&-(8\sqrt{2}+2)a_{20}-(8\sqrt{2}+2)\theta_{10}\\
&-(8\sqrt{2}+2)\theta_7+(12+2\sqrt{2})\theta_{11}\\
&+(12+2\sqrt{2})\theta_5-(8\sqrt{2}+3)\theta_3\\
&+2(4+\sqrt{2})\Bigg(b_{01}+2b_{02}+3\theta_9+4\theta_4\\
&-\frac{b_{11}}{\sqrt{2}}-\sqrt{2}b_{12}-\frac{3\theta_{12}}{\sqrt{2}}+\frac{b_{21}}{2}\\
&+\theta_8-\frac{\theta_6}{2\sqrt{2}}\Bigg)\Bigg]\\
=&-0.00453974461065986637\varepsilon\theta_{13}+O(\varepsilon^2),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{div}(A_{4\varepsilon})=&\frac{\varepsilon}{(4+\sqrt{2})^2}\Bigg[-4a_{01}+4a_{02}-4a_{03}+4\theta_1\\
&+8a_{10}-8a_{11}+8a_{12}-8a_{13}\\
&-(8\sqrt{2}+2)a_{20}+(8\sqrt{2}+2)\theta_{10}\\
&-(8\sqrt{2}+2)\theta_7+(12+2\sqrt{2})\theta_{11}\\
&-(12+2\sqrt{2})\theta_5-(8\sqrt{2}+3)\theta_3\\
&+2(4+\sqrt{2})\Bigg(b_{01}-2b_{02}+3\theta_9-4\theta_4\\
&-\frac{b_{11}}{\sqrt{2}}+\sqrt{2}b_{12}-\frac{3\theta_{12}}{\sqrt{2}}+\frac{b_{21}}{2}\\
&-\theta_8-\frac{\theta_6}{2\sqrt{2}}\Bigg)\Bigg]\\
=&-0.00453974461065986637\varepsilon\theta_{13}\\
&+O(\varepsilon^2),
\end{aligned}
\tag{15}
$$

where $A_{i\varepsilon}$ are the focus points of system (2), located near the centers A_i, $i = 1, 2, 3, 4$ of system (3).

From the results given in [Han & Chen, 2000] or [Han & Hu, 2003], we know that the stability of the homoclinic loops L_k^*, $k = 5, 6, 7, 8$, are determined by the first saddle value when $\sigma_{1,k} = 0$. In the following, we give the first-order saddle value obtained by using the method of [Han & Hu, 2003] with a straightforward computation. Denote by R_{1j} the first saddle values of the saddle points $S_{j\varepsilon}$ of the system (2), $j = 1, 2$. Then, we have the following results.

Lemma 5. *Assume $\theta_i = \phi_i$, $i = 1, 2, \ldots, 12$. Then,*

(i) $R_{1j} = R_0\,\varepsilon + O(\varepsilon^2)$, $j = 1, 2$, *where* $R_0 = -9.10505\theta_{13}$; *and*
(ii) *further assume $\sigma_{1,k} = 0$, $k = 5, 6$, then if L^*_{3+2j} and L^*_{4+2j} ($j = 1, 2$) are oriented clockwise, they are stable (unstable) inside while the double-homoclinic loop $L^*_{3+2j} \cup L^*_{4+2j}$ is unstable (stable) outside when $R_{1j} > 0$ (< 0); if L^*_{3+2j} and L^*_{4+2j} are oriented counterclockwise, then the conclusion is reversed. (See [Han & Chen, 2000; Han & Hu, 2003].)*

Proof. We only prove for R_{11}, since a similar proof can be applied to obtain R_{12}. Let T be a reversible matrix such that $\det(T) = 1$, and $TDT^{-1} = \mathrm{diag}(\lambda_{11}, \lambda_{12})$, where $D = (\partial(P_4, Q_4)/\partial(x, y))(S_{1\varepsilon})$, and $\lambda_{11} > 0 > \lambda_{12}$ are the eigenvalues of D. Let

$$T = \begin{pmatrix} 1 & \alpha \\ \beta & 1+\alpha\beta \end{pmatrix}.$$

Then, it follows from (2) that

$$
\begin{aligned}
P_{4x}(S_{1\varepsilon}) &= \frac{1}{4}\bigg[(a_{01}+a_{02}+a_{03}+a_{04})\\
&\qquad +\frac{1}{2}(a_{10}+a_{11}+a_{12}+a_{13})\bigg]\varepsilon+O(\varepsilon^2),\\
P_{4y}(S_{1\varepsilon}) &= -2+\left(-a_{01}-\frac{1}{2}a_{02}+\frac{1}{2}a_{04}\right)\varepsilon+O(\varepsilon^2),\\
Q_{4x}(S_{1\varepsilon}) &= -1+\frac{1}{4}\bigg[(b_{01}+b_{02}+b_{03}+b_{04})\\
&\qquad +\frac{1}{2}(b_{10}+b_{11}+b_{12}+b_{13})\bigg]\varepsilon+O(\varepsilon^2),\\
Q_{4y}(S_{1\varepsilon}) &= \left(\frac{1}{2}b_{01}+b_{02}+\frac{3}{2}b_{03}+2b_{04}\right)\varepsilon+O(\varepsilon^2),\\
\lambda_{11} &= \sqrt{2}+\frac{1}{8}[a_{01}+a_{02}+a_{03}+a_{04}+2a_{10}\\
&\qquad +2a_{11}+2a_{12}+2a_{13}+2b_{01}+4b_{02}\\
&\qquad +6b_{03}+8b_{04}+\sqrt{2}(2a_{01}+a_{02}-a_{04}\\
&\qquad -b_{01}-b_{02}-b_{03}-b_{04}-2b_{10}\\
&\qquad -2b_{11}-2b_{12}-2b_{13})]\varepsilon+O(\varepsilon^2),
\end{aligned}
$$

$$\begin{aligned}\lambda_{12} = \sqrt{2} + \frac{1}{8}[&a_{01} + a_{02} + a_{03} + a_{04} + 2a_{10} \\ &+ 2a_{11} + 2a_{12} + 2a_{13} + 2b_{01} + 4b_{02} \\ &+ 6b_{03} + 8b_{04} - \sqrt{2}(2a_{01} + a_{02} - a_{04} \\ &- b_{01} - b_{02} - b_{03} - b_{04} - 2b_{10} \\ &- 2b_{11} - 2b_{12} - 2b_{13})]\varepsilon + O(\varepsilon^2).\end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}\alpha = \frac{1}{2\sqrt{2}} - \frac{\varepsilon}{16\sqrt{2}}\,&(2a_{01} + a_{02} - a_{04} + b_{01} \\ &+ b_{02} + b_{03} + b_{04} + 2b_{10} + 2b_{11} \\ &+ 2b_{12} + 2b_{13}) + O(\varepsilon^2),\end{aligned}$$

$$\begin{aligned}\beta = -\sqrt{2} + \Bigg(&\frac{a_{01}}{8} - \frac{a_{01}}{2\sqrt{2}} + \frac{a_{02}}{8} - \frac{a_{02}}{4\sqrt{2}} \\ &+ \frac{a_{03}}{8} + \frac{a_{04}}{8} + \frac{a_{04}}{4\sqrt{2}} + \frac{a_{10}}{4} + \frac{a_{11}}{4} \\ &+ \frac{a_{12}}{4} + \frac{a_{13}}{4} - \frac{b_{01}}{4} - \frac{b_{01}}{4\sqrt{2}} - \frac{b_{02}}{2} \\ &- \frac{b_{02}}{4\sqrt{2}} - \frac{3b_{03}}{4} - \frac{b_{03}}{4\sqrt{2}} - b_{04} - \frac{b_{04}}{4\sqrt{2}} \\ &- \frac{b_{10}}{2\sqrt{2}} - \frac{b_{11}}{2\sqrt{2}} - \frac{b_{12}}{2\sqrt{2}} - \frac{b_{13}}{2\sqrt{2}}\Bigg)\varepsilon + O(\varepsilon^2),\end{aligned}$$

$$\begin{aligned}1 + \alpha\beta = \frac{1}{2} + \frac{\sqrt{2}\varepsilon}{32}&(a_{01} + a_{02} + a_{03} + a_{04} + 2a_{10} \\ &+ 2a_{11} + 2a_{12} + 2a_{13} - 2b_{01} \\ &- 4b_{02} - 6b_{03} - 8b_{04}) + O(\varepsilon^2).\end{aligned}$$

Now, employing a linear transformation in the form of

$$\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix},$$

where

$$x_1 = \frac{\varepsilon}{2}(b_{01} + b_{02} + b_{03} + b_{04}) + O(\varepsilon^2),$$

$$y_1 = 1 + \frac{\varepsilon}{4}(a_{01} + a_{02} + a_{03} + a_{04}) + O(\varepsilon^2),$$

into system (2) yields

$$\dot{u} = \lambda_{11}\left(u + \sum_{k=2}^{3}\sum_{j+l=k} m_{jl}u^j v^l + O(|u,v|^4)\right),$$

$$\dot{v} = -\lambda_{12}\left(-v + \sum_{k=2}^{3}\sum_{j+l=k} n_{jl}u^j v^l + O(|u,v|^4)\right),$$

which describes the dynamic behavior of the system near the origin in the (u, v)-plane. Here,

$$\begin{aligned}m_{20} = -\frac{3}{8\sqrt{2}} - \frac{3\varepsilon}{128}(&-a_{01} - 2\sqrt{2}a_{01} - a_{02} \\ &- \sqrt{2}a_{02} - a_{03} - a_{04} + \sqrt{2}a_{04} - 2a_{10} - 2a_{11} \\ &- 2a_{12} - 2a_{13} - 2b_{01} + \sqrt{2}b_{01} - 4b_{02} \\ &+ \sqrt{2}b_{02} - 6b_{03} + \sqrt{2}b_{03} - 8b_{04} + \sqrt{2}b_{04} \\ &+ 2\sqrt{2}b_{10} + 2\sqrt{2}b_{11} + 2\sqrt{2}b_{12} + 2\sqrt{2}b_{13}) \\ &+ \frac{\varepsilon}{32\sqrt{2}}(4a_{01} - \sqrt{2}a_{01} + 3a_{02} - 2\sqrt{2}a_{02} \\ &+ 4a_{03} - 3\sqrt{2}a_{03} + 7a_{04} - 4\sqrt{2}a_{04} + 2a_{10} \\ &+ 2a_{11} - 2\sqrt{2}a_{11} + 2a_{12} - 4\sqrt{2}a_{12} + 2a_{13} \\ &- 6\sqrt{2}a_{13} + 4a_{20} + 4a_{21} + 4a_{22} + 5b_{01} \\ &- 25\sqrt{2}b_{01} + 7b_{02} - 27\sqrt{2}b_{02} + 9b_{03} \\ &- 31\sqrt{2}b_{03} + 11b_{04} - 37\sqrt{2}b_{04} + 6b_{10} \\ &- 2\sqrt{2}b_{10} + 10b_{11} - 2\sqrt{2}b_{11} + 14b_{12} \\ &- 2\sqrt{2}b_{12} + 18b_{13} - 2\sqrt{2}b_{13} - 4\sqrt{2}b_{20} \\ &- 4\sqrt{2}b_{21} - 4\sqrt{2}b_{22}) + O(\varepsilon^2),\end{aligned}$$

$$\begin{aligned}m_{11} = \frac{3}{2} - \frac{\varepsilon}{32}[&(8 + 3\sqrt{2})a_{01} + (4 + 3\sqrt{2})a_{02} \\ &+ 8a_{03} + 3\sqrt{2}a_{03} + 20a_{04} + 3\sqrt{2}a_{04} - 8a_{10} \\ &+ 6\sqrt{2}a_{10} - 8a_{11} + 6\sqrt{2}a_{11} - 8a_{12} + 6\sqrt{2}a_{12} \\ &- 8a_{13} + 6\sqrt{2}a_{13} - 16a_{20} - 16a_{21} - 16a_{22} \\ &+ 106\sqrt{2}b_{01} + 104\sqrt{2}b_{02} + 94\sqrt{2}b_{03} \\ &+ 76\sqrt{2}b_{04} + 8\sqrt{2}b_{10} + 8\sqrt{2}b_{11} + 8\sqrt{2}b_{12} \\ &+ 8\sqrt{2}b_{13} + 16\sqrt{2}b_{20} + 16\sqrt{2}b_{21} \\ &+ 16\sqrt{2}b_{22}] + O(\varepsilon^2),\end{aligned}$$

$$\begin{aligned}m_{21} = -\frac{27}{16}\sqrt{2} + \frac{\varepsilon}{128}[&(33 - 32\sqrt{2})a_{01} \\ &+ (29 - 19\sqrt{2})a_{02} + 21a_{03} + 6\sqrt{2}a_{03} + 9a_{04} \\ &+ 55\sqrt{2}a_{04} + 66a_{10} + 66a_{11} - 4\sqrt{2}a_{11} \\ &+ 58a_{12} - 8\sqrt{2}a_{12} + 42a_{13} - 12\sqrt{2}a_{13} \\ &+ 24a_{20} + 24a_{21} - 8\sqrt{2}a_{21} + 24a_{22} \\ &- 16\sqrt{2}a_{22} + 48a_{30} + 48a_{31} + 58b_{01} \\ &- 75\sqrt{2}b_{01} + 116b_{02} - 71\sqrt{2}b_{02} + 150b_{03} \\ &- 63\sqrt{2}b_{03} + 136b_{04} - 51\sqrt{2}b_{04} - 150\sqrt{2}b_{10} \\ &+ 8b_{11} - 150\sqrt{2}b_{11} + 16b_{12} - 142\sqrt{2}b_{12} \\ &+ 24b_{13} - 126\sqrt{2}b_{13} - 24\sqrt{2}b_{20}\end{aligned}$$

$$+16b_{21}-24\sqrt{2}b_{21}+32b_{22}-24\sqrt{2}b_{22} - 48\sqrt{2}b_{30}-48\sqrt{2}b_{31}]+O(\varepsilon^2),$$

$$\begin{aligned}
n_{11} = {} & \frac{3\sqrt{2}}{8}+\frac{\varepsilon}{64}[(3-10\sqrt{2})a_{01}+(3-5\sqrt{2})a_{02} \\
&+3a_{03}-4\sqrt{2}a_{03}+3a_{04}-7\sqrt{2}a_{04} \\
&+6a_{10}+4\sqrt{2}a_{10}+6a_{11}+4\sqrt{2}a_{11}+6a_{12} \\
&+4\sqrt{2}a_{12}+6a_{13}+4\sqrt{2}a_{13}+8\sqrt{2}a_{20} \\
&+8\sqrt{2}a_{21}+8\sqrt{2}a_{22}+106b_{01}-3\sqrt{2}b_{01} \\
&+104b_{02}-3\sqrt{2}b_{02}+94b_{03}-3\sqrt{2}b_{03} \\
&+76b_{04}-3\sqrt{2}b_{04}+8b_{10}-6\sqrt{2}b_{10}+8b_{11} \\
&-6\sqrt{2}b_{11}+8b_{12}-6\sqrt{2}b_{12}+8b_{13}-6\sqrt{2}b_{13} \\
&+16b_{20}+16b_{21}+16b_{22}]+O(\varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
n_{02} = {} & -\frac{3}{4}+\frac{\varepsilon}{64}[(16+\sqrt{2})a_{01}+(12+5\sqrt{2})a_{02} \\
&+16a_{03}+9\sqrt{2}a_{03}+28a_{04}+13\sqrt{2}a_{04}+8a_{10} \\
&-6\sqrt{2}a_{10}+8a_{11}+2\sqrt{2}a_{11}+8a_{12}+10\sqrt{2}a_{12} \\
&+8a_{13}+18\sqrt{2}a_{13}+16a_{20}+16a_{21}+16a_{22} \\
&+8b_{01}+94\sqrt{2}b_{01}+16b_{02}+96\sqrt{2}b_{02}+24b_{03} \\
&+106\sqrt{2}b_{03}+32b_{04}+124\sqrt{2}b_{04}+8\sqrt{2}b_{10} \\
&+16b_{11}+8\sqrt{2}b_{11}+32b_{12}+8\sqrt{2}b_{12} \\
&+48b_{13}+8\sqrt{2}b_{13}+16\sqrt{2}b_{20}+16\sqrt{2}b_{21} \\
&+16\sqrt{2}b_{22}]+O(\varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
n_{12} = {} & \frac{27}{16}\sqrt{2}+\frac{\varepsilon}{128}[(33+32\sqrt{2})a_{01}+(29+19\sqrt{2})a_{02} \\
&+21a_{03}-6\sqrt{2}a_{03}+9a_{04}-55\sqrt{2}a_{04}+66a_{10} \\
&+66a_{11}+4\sqrt{2}a_{11}+58a_{12}+8\sqrt{2}a_{12}+42a_{13} \\
&+12\sqrt{2}a_{13}+24a_{20}+24a_{21}+8\sqrt{2}a_{21} \\
&+24a_{22}+16\sqrt{2}a_{22}+48a_{30}+48a_{31}+58b_{01} \\
&+75\sqrt{2}b_{01}+116b_{02}+71\sqrt{2}b_{02}+150b_{03} \\
&+63\sqrt{2}b_{03}+136b_{04}+51\sqrt{2}b_{04}+150\sqrt{2}b_{10} \\
&+8b_{11}+150\sqrt{2}b_{11}+16b_{12}+142\sqrt{2}b_{12} \\
&+24b_{13}+126\sqrt{2}b_{13}+24\sqrt{2}b_{20}+16b_{21} \\
&+24\sqrt{2}b_{21}+32b_{22}+24\sqrt{2}b_{22}+48\sqrt{2}b_{30} \\
&+48\sqrt{2}b_{31}]+O(\varepsilon^2).
\end{aligned}$$

Substituting above expressions into the formula for R_{11},

$$R_{11}=m_{21}+n_{12}-m_{20}m_{11}+n_{02}n_{11}$$

(see [Han & Hu, 2003]), we get

$$\begin{aligned}
R_{11} = {} & \frac{\varepsilon}{32}(15a_{01}+16a_{02}+15a_{03}+12a_{04}+24a_{10} \\
&+30a_{11}+32a_{12}+30a_{13}+12a_{20}+12a_{21} \\
&+12a_{22}+24a_{30}+24a_{31}+20b_{01}+52b_{02} \\
&+84b_{03}+104b_{04}+4b_{11}+8b_{12}+12b_{13} \\
&+8b_{21}+16b_{22})+O(\varepsilon^2).
\end{aligned}$$

Hence, when $\theta_i=\phi_i$ $(i=1,2,\ldots,12)$ hold, we have

$$R_{11}=-9.105050725850044526460\theta_{13}\varepsilon+O(\varepsilon^2).$$

With a similar procedure, we obtain

$$R_{12}=-9.105050725850044526460\theta_{13}\varepsilon+O(\varepsilon^2).$$

This completes the proof of Lemma 5. ■

We are now ready to prove Theorem 1.

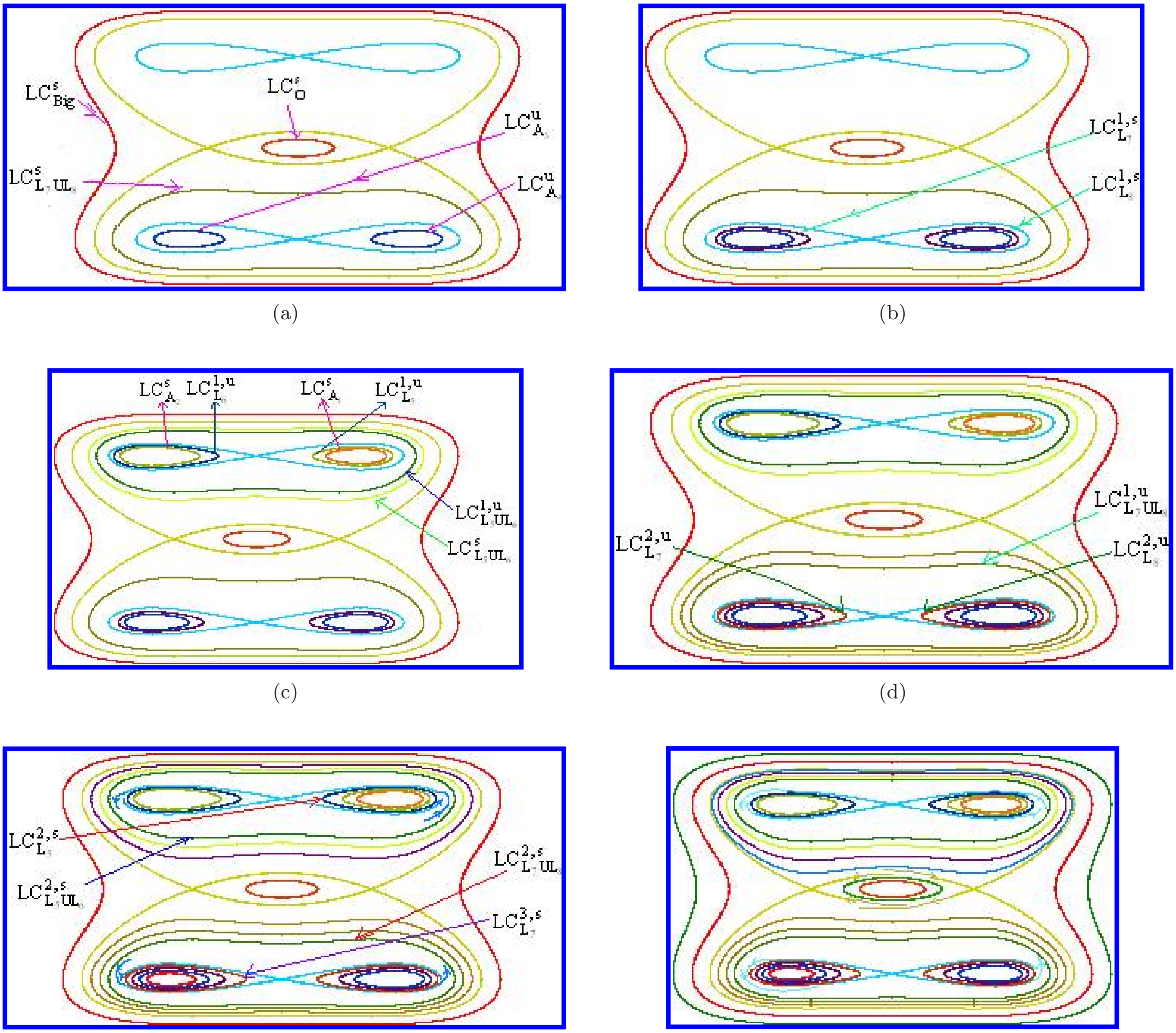
Proof of Theorem 1. The proof includes two main steps: (I) to show the existence of five basic limit cycles; and (II) to generate more limit cycles by employing proper perturbations to change the stabilities of homoclinic or heteroclinic loops.

(I) The existence of five basic limit cycles.
Assume that the conditions $\theta_i=\phi_i$, $i=1,2,\ldots,12$, $\theta_{13}=a_{10}+b_{01}>0$ are satisfied, and $\varepsilon>0$ small.

(1) From Eqs. (14) and (15), we have $M>0$ and so $H(B^*)>H(B)$, which implies that $b^*<b_0$ and hence the trajectory ρ_B^* of system (1) is bounded for $x<2$. Then by Lemma 4, we know that the heteroclinic loop $L_1^*\cup L_2^*$ is unstable outside. Thus, by the Poincaré–Bendixson theorem, there at least exists a big stable limit cycle, denoted by $\mathrm{LC}^S_{L_1\cup L_2}$, which encloses all the 9 singular points.

(2) It follows from Eq. (15) that $\mathrm{div}(O)=(1/2)\,\varepsilon\,\theta_{13}+O(\varepsilon^2)>0$. Further, by Lemma 4, we know that the heteroclinic loop $L_3^*\cup L_4^*$ is unstable inside. Therefore, there at least exists a stable limit cycle inside the heteroclinic loop $L_3^*\cup L_4^*$, named LC^S_O.

(3) It is seen from Lemma 5 that $R_{12}<0$ and thus the double homoclinic loop $L_7^*\cup L_8^*$ is unstable outside. Further, by Lemma 4, the heteroclinic loop $L_2^*\cup L_4^*$ is unstable inside. This indicates that there at least exists a stable limit cycle inside the heteroclinic loop $L_2^*\cup L_4^*$, called $\mathrm{LC}^S_{L_7\cup L_8}$.



(a) (b) (c) (d) (e) (f)

Fig. 2. The distribution of the 23 limit cycles of system (1) for Category (A).

(4) Again by Lemma 5, we know that both the two homoclinic loops L_7^* and L_8^* are stable inside. However, it is seen from Eq. (15) that $\mathrm{div}(A_{3\varepsilon}) < 0$ and $\mathrm{div}(A_{4\varepsilon}) < 0$, implying that both the two focus points $A_{3\varepsilon}$ and $A_{4\varepsilon}$ are stable. This implies the existence of two unstable limit cycles, located inside the homoclinic loops L_7^* and L_8^*, respectively, which are denoted by $\mathrm{LC}_{A_3}^U$ and $\mathrm{LC}_{A_4}^U$.

The above discussions show that there at least exist five limit cycles, with the distributions as shown in Fig. 2(a).

(II) More limit cycles generated by perturbations. Perturbations are employed to generate more limit cycles, in addition to the above five limit cycles. According to the values of $\sigma_{1,5}$ and $\sigma_{1,6}$, there are 13 cases.

(1) $\sigma_{1,5} \equiv 0, \quad \sigma_{1,6} \equiv 0;$
(2) $\sigma_{1,5} \equiv 0, \quad \sigma_{1,6} > 0;$
(3) $\sigma_{1,5} \equiv 0, \quad \sigma_{1,6} < 0;$
(4) $\sigma_{1,5} > 0, \quad \sigma_{1,6} \equiv 0;$
(5) $\sigma_{1,5} < 0, \quad \sigma_{1,6} \equiv 0;$
(6) $\sigma_{1,5} > 0, \quad \sigma_{1,6} > 0;$
(7) $\sigma_{1,5} < 0, \quad \sigma_{1,6} < 0;$
(8) $\sigma_{1,5} > 0, \quad \sigma_{1,6} < 0, \quad \sigma_{1,5} + \sigma_{1,6} \equiv 0;$

(9) $\sigma_{1,5} > 0, \quad \sigma_{1,6} < 0, \quad \sigma_{1,5} + \sigma_{1,6} > 0$;
(10) $\sigma_{1,5} > 0, \quad \sigma_{1,6} < 0, \quad \sigma_{1,5} + \sigma_{1,6} < 0$;
(11) $\sigma_{1,5} < 0, \quad \sigma_{1,6} > 0, \quad \sigma_{1,5} + \sigma_{1,6} \equiv 0$;
(12) $\sigma_{1,5} < 0, \quad \sigma_{1,6} > 0, \quad \sigma_{1,5} + \sigma_{1,6} > 0$;
(13) $\sigma_{1,5} < 0, \quad \sigma_{1,6} > 0, \quad \sigma_{1,5} + \sigma_{1,6} < 0$.

It should be noted that the system must follow one and only one of the above 13 cases.

It is easy to see that the two cases (4) and (5) are "symmetric" with the two cases (2) and (3), producing the same number of limit cycles. Similarly, the three cases (11)–(13) are "symmetric" with the three cases (8)–(10). Furthermore, it can be shown that the three cases (1), (2) and (6) have the same number of limit cycles; the two cases (8) and (9) have the same number of limit cycles; and the three cases (3), (7) and (10) have the same number of limit cycles; So we only have three categories, described as follows.

(A) $(\sigma_{1,5} \equiv 0,\ \sigma_{1,6} \equiv 0)$ or $(\sigma_{1,5} \equiv 0,\ \sigma_{1,6} > 0)$ or $(\sigma_{1,5} > 0,\ \sigma_{1,6} > 0)$
$\Rightarrow$ 23 limit cycles;
(B) $(\sigma_{1,5} > 0,\ \sigma_{1,6} < 0,\ \sigma_{1,5} + \sigma_{1,6} \equiv 0)$ or $(\sigma_{1,5} > 0,\ \sigma_{1,6} < 0,\ \sigma_{1,5} + \sigma_{1,6} > 0)$
$\Rightarrow$ 21 limit cycles;
(C) $(\sigma_{1,5} \equiv 0,\ \sigma_{1,6} < 0)$ or $(\sigma_{1,5} < 0,\ \sigma_{1,6} < 0)$ or $(\sigma_{1,5} > 0,\ \sigma_{1,6} < 0,\ \sigma_{1,5} + \sigma_{1,6} < 0)$
$\Rightarrow$ 20 limit cycles;

This shows that system (1) must at least have 20 limit cycles, i.e. $H(4) \geq 20$.

In the following, we give a detailed analysis for category (A), and then briefly discuss the other two categories (B) and (C).

(A) In order to prove this category, we shall apply perturbations step by step, starting from perturbing the homoclinic loops L_7^* and L_8^*. The details are given in the following eleven steps, where the notation $\Rightarrow$ stands for "becomes" or "implies".

(a) Fix $\theta_{13} > 0$, perturbing θ_{12} from ϕ_{12} such that $0 < \theta_{12} - \phi_{12} \ll \theta_{13}$; then the homoclinic loop $\sigma_{1,7} \equiv 0 \Rightarrow \sigma_{1,7} > 0$ (by Eq. (13)); the limit cycle L_7^* stable inside $\Rightarrow$ unstable inside (by Lemma 3); $\mathrm{LC}_{A_3}^U$ unstable (stability unchanged); $\Rightarrow$ **one** stable limit cycle, denoted by $\mathrm{LC}_{L_7}^{1,S}$, is generated between the homoclinic loop L_7^* and the limit cycle $\mathrm{LC}_{A_3}^U$.

This limit cycle is shown in Fig. 2(b).

(b) Fix $\theta_{13} > 0$ and keep θ_{12} as perturbed as in (a), perturbing θ_{11} from ϕ_{11} such that $0 < \phi_{11} - \theta_{11} \ll \theta_{12} - \phi_{12}$; then $\sigma_{1,8} \equiv 0 \Rightarrow \sigma_{1,8} > 0$ (by Eq. (13)) the homoclinic loop L_8^* stable inside $\Rightarrow$ unstable inside (by Lemma 3) the limit cycle $\mathrm{LC}_{A_4}^U$ unstable (stability unchanged) $\Rightarrow$ **one** stable limit cycle, denoted by $\mathrm{LC}_{L_8}^{1,S}$, is generated between the homoclinic loop L_8^* and the limit cycle $\mathrm{LC}_{A_4}^U$.

This limit cycle is depicted in Fig. 2(b).

(c) First, under the assumptions $\sigma_{1,5} \equiv 0$ and $\sigma_{1,6} \equiv 0$, similar to the discussions for L_7^* and L_8^*, we have, before perturbation, that the double homoclinic loop $L_5^* \cup L_6^*$ is unstable outside due to $R_{11} < 0$ (by Lemma 5). On the other hand, by Lemma 4, the heteroclinic loop $L_1^* \cup L_3^*$ is unstable inside. Thus, there exists **one** stable limit cycle, $\mathrm{LC}_{L_5^* \cup L_6^*}^S$, inside the heteroclinic loop $L_1^* \cup L_3^*$. Furthermore, since before perturbation we assume that $\sigma_{1,5} \equiv 0$ and $\sigma_{1,6} \equiv 0$, thus by (12) and (13) the perturbations given in (a) and (b) simultaneously generate **two** stable limit cycles: $\mathrm{LC}_{A_1}^S$ inside the homoclinic loop L_5^*, and $\mathrm{LC}_{A_2}^S$ inside the homoclinic loop L_6^*, because the two homoclinic loops L_5^* and L_6^* change their stabilities from stable to unstable (while the focus points $A_{1\varepsilon}$ and $A_{2\varepsilon}$ are kept unstable, see Eq. (15)).

Now, fix $\theta_{13} > 0$, and keep θ_{12} and θ_{11} as perturbed above, perturbing θ_{10} from ϕ_{11} such that $0 < \theta_{10} - \phi_{10} \ll |\theta_{11} - \phi_{11}|$; then $\sigma_{0,1} \equiv 0 \Rightarrow \sigma_{0,1} < 0$ (by Lemma 2); the two homoclinic loops L_5^* and L_6^* unstable inside $\Rightarrow$ stable inside (by Lemma 3); the two limit cycles $\mathrm{LC}_{A_1}^S$ and $\mathrm{LC}_{A_2}^S$ stable (stability unchanged); $\Rightarrow$ **two** unstable limit cycles, $\mathrm{LC}_{L_5}^{1,U}$ between the homoclinic loop L_5^* and the limit cycle $\mathrm{LC}_{A_1}^S$, and $\mathrm{LC}_{L_6}^{1,U}$ between the homoclinic loop L_6^* and the limit cycle $\mathrm{LC}_{A_2}^S$. Under the same perturbation for θ_{10}, the double homoclinic loop $L_5^* \cup L_6^*$ unstable outside $\Rightarrow$ stable outside (by Lemma 3); and the limit cycle $\mathrm{LC}_{L_5 \cup L_6}^S$ stable (stability unchanged); thus $\Rightarrow$ **one** unstable limit cycle, $\mathrm{LC}^{1,U}_{L_5\cup L_6}$ between the double homoclinic loop $L_5^* \cup L_6^*$ and the limit cycle $\mathrm{LC}^{S}_{L_5\cup L_6}$.

The six limit cycles generated in this step are shown in Fig. 2(c).

(d) This step is similar to the second part of step (c). Fix $\theta_{13} > 0$, and keep θ_i, $i = 10, 11, 12$ as perturbed above, perturbing θ_9 from ϕ_9 such that $0 < \theta_9 - \phi_9 \ll |\theta_{10} - \phi_{10}|$; then $\sigma_{0,2} \equiv 0 \;\Rightarrow\; \sigma_{0,2} < 0$ (by Lemma 2); the two homoclinic loops L_7^* and L_8^* unstable inside $\Rightarrow$ stable inside (by Lemma 3); the two limit cycles $\mathrm{LC}^{1,S}_{L_7}$ and $\mathrm{LC}^{1,S}_{L_8}$ stable (stability unchanged); $\Rightarrow$ **two** unstable limit cycles, $\mathrm{LC}^{2,U}_{L_7}$ between the homoclinic loop L_7^* and the limit cycle $\mathrm{LC}^{1,S}_{L_7}$, and $\mathrm{LC}^{2,U}_{L_8}$ between the homoclinic loop L_8^* and the limit cycle $\mathrm{LC}^{1,S}_{L_8}$. Under the same perturbation for θ_9, the double homoclinic loop $L_7^* \cup L_8^*$ unstable outside $\Rightarrow$ stable outside (by Lemma 3); the limit cycle $\mathrm{LC}^{S}_{L_7\cup L_8}$ stable (stability unchanged); $\Rightarrow$ **one** unstable limit cycle, $\mathrm{LC}^{1,U}_{L_7\cup L_8}$ between the double homoclinic loop $L_7^* \cup L_8^*$ and the limit cycle $\mathrm{LC}^{S}_{L_7\cup L_8}$. The three limit cycles are shown in Fig. 2(d).

(e) Fix $\theta_{13} > 0$, and keep θ_i, $i = 9, 10, 11, 12$ as perturbed above, perturbing θ_8 from ϕ_8 such that $0 < |\theta_8 - \phi_8| \ll |\theta_9 - \phi_9|$, under which the homoclinic loop L_8^* is broken. There are two subcases:

(α) If $\theta_8 < \phi_8$ (i.e. $M_8 < 0$ or $d_8 < 0$), then as shown in Fig. 3(a), a stable limit cycle, $\mathrm{LC}^{2,S}_{L_7\cup L_8}$, is generated inside the unstable limit cycle $\mathrm{LC}^{1,U}_{L_7\cup L_8}$.

(β) If $\theta_8 > \phi_8$ (i.e. $M_8 > 0$ or $d_8 > 0$), then as shown in Fig. 3(a), a stable limit cycle, $\mathrm{LC}^{3,S}_{L_8}$, is produced outside the unstable limit cycle $\mathrm{LC}^{2,U}_{L_8}$.

Hence, this step generates **one** limit cycle.

(f) Fix $\theta_{13} > 0$, and keep θ_i, $i = 8, 9, \ldots, 12$ as perturbed above, perturbing θ_7 from ϕ_7 such that $0 < |\theta_7 - \phi_7| \ll |\theta_8 - \phi_8|$, under which the homoclinic loop L_7^* is broken. As shown in Fig. 3(b), **one** stable limit cycle, $\mathrm{LC}^{3,S}_{L_7}$, is produced outside the unstable limit cycle $\mathrm{LC}^{2,U}_{L_7}$.

(g) Fix $\theta_{13} > 0$, and keep θ_i, $i = 7, 8, \ldots, 12$ as perturbed above, perturbing θ_6 from ϕ_6 such that $0 < |\theta_6 - \phi_6| \ll |\theta_7 - \phi_7|$, under which the homoclinic loop L_6^* is broken. There are two subcases:

(α) If $\theta_6 > \phi_6$ (i.e. $M_6 < 0$ or $d_6 < 0$), then as shown in Fig. 3(c), a stable limit cycle, $\mathrm{LC}^{2,S}_{L_5\cup L_6}$, is generated inside the unstable limit cycle $\mathrm{LC}^{1,U}_{L_5\cup L_6}$.

(β) If $\theta_6 < \phi_6$ (i.e. $M_6 > 0$ or $d_6 > 0$), then as shown in Fig. 3(c), a stable limit cycle, $\mathrm{LC}^{2,S}_{L_6}$, is produced outside the unstable limit cycle $\mathrm{LC}^{1,U}_{L_6}$.

So **one** limit cycle is generated in this step.

(h) Fix $\theta_{13} > 0$ and keep θ_i, $i = 6, 7, \ldots, 12$ as perturbed above, perturbing θ_5 from ϕ_5 such that $0 < \phi_5 - \theta_5 \ll |\theta_6 - \phi_6|$, under which the homoclinic loop L_5^* is broken. As shown in Fig. 3(d), **one** stable limit cycle, $\mathrm{LC}^{2,S}_{L_5}$, is produced outside the unstable limit cycle $\mathrm{LC}^{1,U}_{L_5}$.

The four limit cycles obtained in the steps (e)–(h) are depicted in Fig. 2(e).

(i) Fix $\theta_{13} > 0$ and keep θ_i, $i = 5, 6, \ldots, 12$ as perturbed above, perturbing θ_4 from ϕ_4 such that $0 < |\theta_4 - \phi_4| \ll |\theta_5 - \phi_5|$, under which the heteroclinic orbit L_4^* is broken. There are two subcases:

(α) If $\theta_4 < \phi_4$ (i.e. $M_4 < 0$ or $d_4 < 0$), then as shown in Fig. 3(e), an unstable limit cycle, $\mathrm{LC}^{U}_{L_4\cup L_2}$, is produced outside the stable limit cycle $\mathrm{LC}^{S}_{L_7\cup L_8}$.

(β) If $\theta_6 > \phi_6$ (i.e. $M_6 > 0$ or $d_6 > 0$), then as shown in Fig. 3(e), an unstable limit cycle, $\mathrm{LC}^{U}_{L_4\cup L_3}$, is produced outside the stable limit cycle LC^{S}_{O}.

This step generates **one** limit cycle.

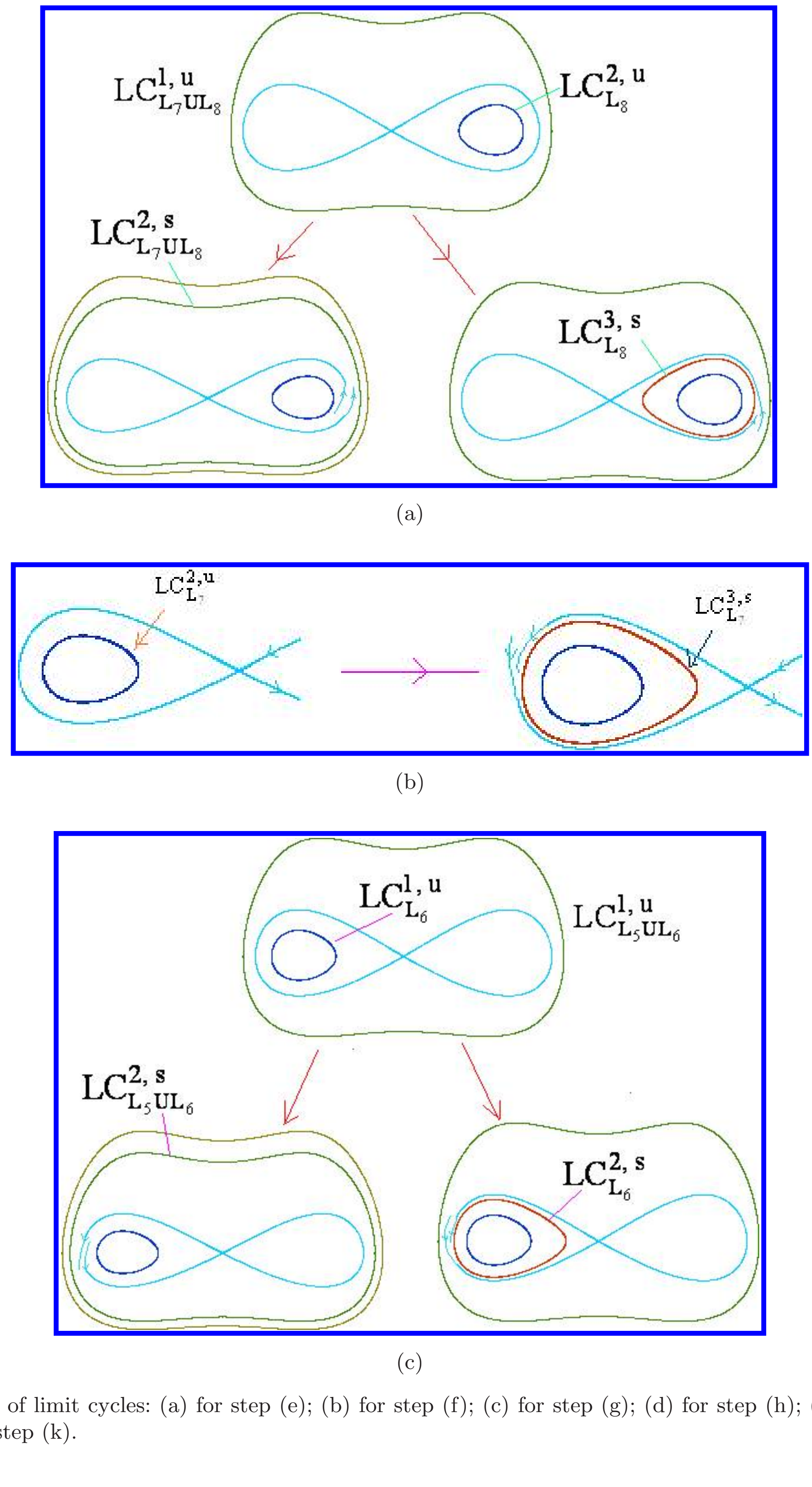
(j) Fix $\theta_{13} > 0$ and keep θ_i, $i = 4, 5, \ldots, 12$ as perturbed above, perturbing θ_3 from ϕ_3 such that $0 < \phi_3 - \theta_3 \ll |\theta_4 - \phi_4|$, under which the heteroclinic orbit L_3^* is broken. As shown in Fig. 3(f), **one** unstable limit cycle, $\mathrm{LC}^{U}_{L_3\cup L_1}$, is produced outside the stable limit cycle $\mathrm{LC}^{S}_{L_5\cup L_6}$.

(k) Fix $\theta_{13} > 0$ and keep θ_i, $i = 3, 4, \ldots, 12$ as perturbed above, perturbing θ_2 from ϕ_2 such that $0 < \theta_2 - \phi_2 \ll |\theta_3 - \phi_3|$, under which the heteroclinic orbit L_2^* is broken.

As shown in Fig. 3(g), **one** unstable limit cycle, $\mathrm{LC}^{U}_{L_2\cup L_1}$, is produced inside the stable limit cycle $\mathrm{LC}^{S}_{L_1\cup L_2}$.

Note that this limit cycle can also be generated by perturbing θ_1, which causes the heteroclinic orbit L_1^* to break. The process is similar and omitted here.

Summarizing the results obtained so far in the two steps (I) and (II) for one case of Category (A) gives

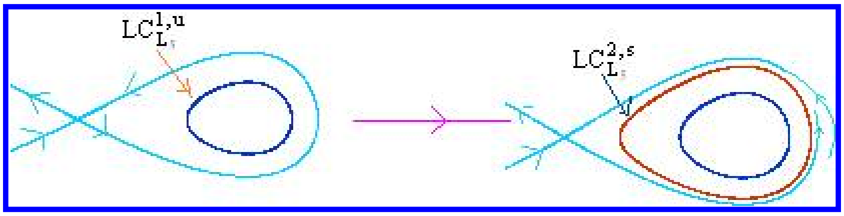


(a)

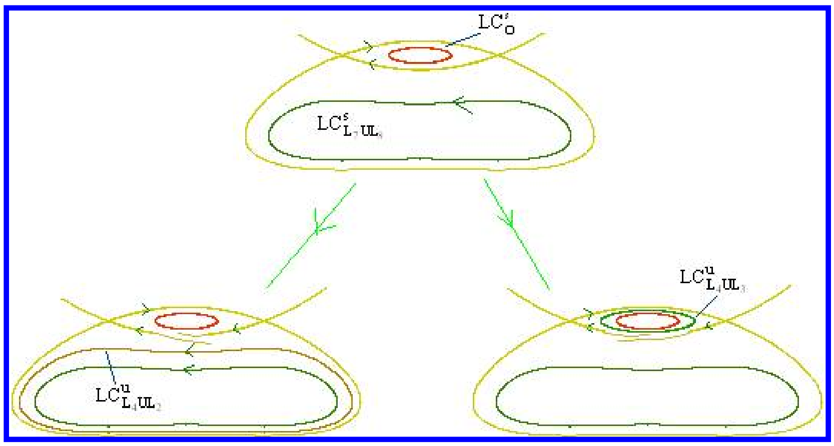
(b)

(c)

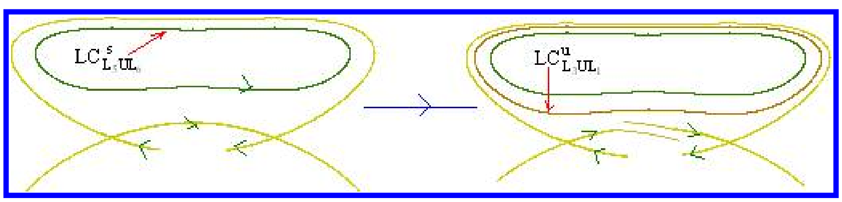
Fig. 3. Distribution of limit cycles: (a) for step (e); (b) for step (f); (c) for step (g); (d) for step (h); (e) for step (i); (f) for step (j); and (g) for step (k).



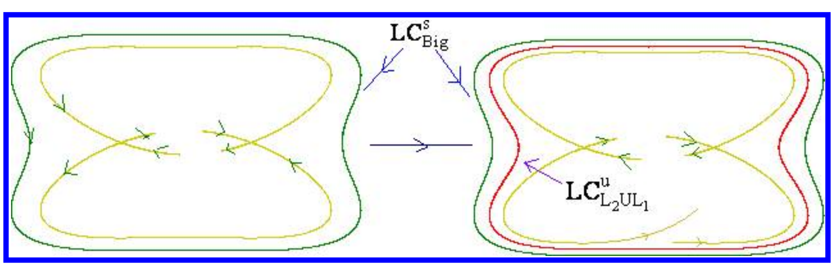
(d)



(e)



(f)



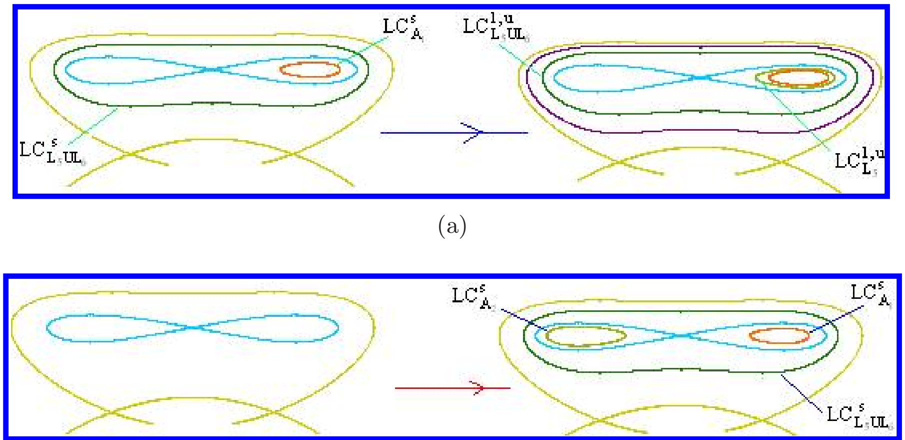
(g)

Fig. 3. (*Continued*)

the following 23 limit cycles when $\sigma_{1,5} \equiv 0$ and $\sigma_{1,6} \equiv 0$: $\mathrm{LC}^U_{A_3}$, $\mathrm{LC}^U_{A_4}$, $\mathrm{LC}^S_{A_7 \cup L_8}$, LC^S_O, $\mathrm{LC}^S_{L_1 \cup L_2}$, $\mathrm{LC}^{1,S}_{L_7}$, $\mathrm{LC}^{2,U}_{L_7}$, $\mathrm{LC}^{3,S}_{L_7}$, $\mathrm{LC}^{1,S}_{L_8}$, $\mathrm{LC}^{2,U}_{L_8}$, $\mathrm{LC}^{1,U}_{L_7 \cup L_8}$, $\mathrm{LC}^{2,S}_{L_7 \cup L_8}$, $\mathrm{LC}^U_{L_4 \cup L_3}$, $\mathrm{LC}^S_{A_1}$, $\mathrm{LC}^{1,U}_{L_5}$, $\mathrm{LC}^{2,S}_{L_5}$, $\mathrm{LC}^S_{A_2}$, $\mathrm{LC}^{1,U}_{L_6}$, $\mathrm{LC}^S_{L_5 \cup L_6}$, $\mathrm{LC}^{1,U}_{L_5 \cup L_6}$, $\mathrm{LC}^{2,S}_{L_5 \cup L_6}$, $\mathrm{LC}^U_{L_3 \cup L_1}$, $\mathrm{LC}^U_{L_2 \cup L_1}$. These 23 limit cycles are shown in Fig. 2(f), where for certain, we choose $\theta_8 < \phi_8$, $\theta_6 > \phi_6$ and $\theta_4 > \phi_4$.

The other two cases of Category (A) can be similarly discussed to obtain 23 limit cycles.

Next, we briefly discuss Categories (B) and (C). Note that the two large limit cycles enclosing all the nine singular points and the 12 limit cycles below



(a)

(b)

Fig. 4. The distributions of the limit cycles of system (1): (a) 21 limit cycles for Category (B); and (b) 20 limit cycles for Category (C).

the heteroclinic orbit L_3^* are not changed for Categories (B) and (C). Therefore, we do not need to repeat the steps (a), (b), (d), (e), (f), (i), (j) and (k). Also, note that the analyses in the steps (g) and (h) are not changed for (B) and (C). Consequently, we only need to consider the step (c) for Categories (B) and (C). The number of limit cycles from Part (I) and all the steps but (c) in (II) is: $5+1+1+3+1+1+1+1+1+1+1=17$.

(B) There are two cases in Category (B). We only discuss the second case: $\sigma_{1,5}>0$, $\sigma_{1,6}<0$, $\sigma_{1,5}+\sigma_{1,6}>0$, since the first case: $\sigma_{1,5}>0$, $\sigma_{1,6}<0$, $\sigma_{1,5}+\sigma_{1,6}\equiv 0$ can be similarly discussed.

First, under the assumptions, we know that before perturbation, the double homoclinic loop $L_5^*\cup L_6^*$ is unstable outside by Lemma 3. On the other hand, by Lemma 4, the heteroclinic loop $L_1^*\cup L_3^*$ is unstable inside. Thus, there exists **one** stable limit cycle, $\mathrm{LC}^S_{L_5\cup L_6}$, inside the heteroclinic loop $L_1^*\cup L_3^*$.

Furthermore, since before perturbation we assume $\sigma_{1,5}>0$, the homoclinic loop L_5^* is unstable inside by Lemma 3, and $\mathrm{div}(A_{1\varepsilon})>0$ indicating that the focus point $A_{1\varepsilon}$ is unstable. This implies that there exists **one** stable limit cycle, $\mathrm{LC}^S_{A_1}$, inside the homoclinic loop L_5^*.

Now, fix $\theta_{13}>0$, and keep θ_{12} and θ_{11} as perturbed above, perturbing θ_{10} from ϕ_{10} such that $0<\theta_{10}-\phi_{10}\ll|\theta_{11}-\phi_{11}|$; then $\sigma_{0,1}\equiv 0\Rightarrow\ \sigma_{0,1}<0$ (by Lemma 2); the homoclinic loop L_5^* unstable inside $\Rightarrow$ stable inside (by Lemma 3); the limit cycle $\mathrm{LC}^S_{A_1}$ stable (stability unchanged); $\Rightarrow$ **one** unstable limit cycle, $\mathrm{LC}^{1,U}_{L_5}$, between the homoclinic loop L_5^* and the limit cycle $\mathrm{LC}^S_{A_1}$. Under the same perturbation for θ_{10}, the double homoclinic loop $L_5^*\cup L_6^*$ unstable outside $\Rightarrow$ stable outside (by Lemma 3); the limit cycle $\mathrm{LC}^S_{L_5\cup L_6}$ stable (stability unchanged); $\Rightarrow$ **one** unstable limit cycle, $\mathrm{LC}^{1,U}_{L_5\cup L_6}$, between the double homoclinic loop $L_5^*\cup L_6^*$ and the limit cycle $\mathrm{LC}^S_{L_5\cup L_6}$.

The number of limit cycles generated in this step is 4, as shown in Fig. 4(a), and thus the total number of limit cycles for the whole system is 21.

(C) Finally we discuss Category (C). There are three cases in this category. We only discuss the second case: $\sigma_{1,5}<0$, $\sigma_{1,6}<0$, which gives $\sigma_{1,5}+\sigma_{1,6}<0$. The other two cases can be similarly discussed.

First, under the assumption, we know that before perturbation, the double homoclinic loop $L_5^*\cup L_6^*$ is stable outside by Lemma 3, and by Lemma 4, the heteroclinic loop $L_1^*\cup L_3^*$ is unstable inside. So we cannot conclude that limit cycles exist inside the heteroclinic loop $L_1^*\cup L_3^*$.

Similarly, before perturbation we assume $\sigma_{1,5}<0$ and $\sigma_{1,6}<0$, by Lemma 3, the two homoclinic loops L_5^* and L_6^* are stable. Further, note that $\mathrm{div}(A_{1\varepsilon}>0$ and $\mathrm{div}(A_{2\varepsilon}>0$, implying that the two focus points $A_{1\varepsilon})$ $A_{2\varepsilon})$ are unstable. Therefore, again we cannot conclude that limit cycles exist inside the homoclinic loops L_5^* and L_6^*.

Now, fix $\theta_{13}>0$ and keep θ_{12} and θ_{11} as perturbed above, perturbing θ_{10} from ϕ_{10} such that $0<\phi_{10}-\theta_{10}\ll|\theta_{11}-\phi_{11}|$; then $\sigma_{0,1}\equiv 0\ \Rightarrow\ \sigma_{0,1}>0$ (by Lemma 2); the two homoclinic loops L_5^* and L_6^* stable inside $\Rightarrow$ unstable inside (by Lemma 3); the two focus points $A_{1\varepsilon}$ and $A_{2\varepsilon}$ are unstable (stability unchanged); $\Rightarrow$ **two** stable limit cycles, $\mathrm{LC}^S_{A_1}$

inside the homoclinic loop L_5^*, and $\mathrm{LC}_{A_2}^S$ inside the homoclinic loop L_6^*.

Under the same perturbation for θ_{10}, the double homoclinic loop $L_5^* \cup L_6^*$ stable outside $\Rightarrow$ unstable outside (by Lemma 3); the heteroclinic loop $L_1^* \cup L_3^*$ unstable (stability unchanged); $\Rightarrow$ **one** stable limit cycle, $\mathrm{LC}_{L_5 \cup L_6}^S$, outside the double homoclinic loop $L_5^* \cup L_6^*$. The number of limit cycles generated in this step is 3, as shown in Fig. 4(b), and thus the total number of limit cycles for the whole system is 20.

The proof of Theorem 1 is complete. ■

3. Conclusion

In this paper we have studied perturbations to a special fourth-order near-Hamiltonian system and shown that the system can have at least 20 limit cycles. That is, $H(4) \geq 20$, which greatly improves the best existing result, $H(4) \geq 15$.

Acknowledgments

This work was supported by Shanghai Outstanding Discipline Leader Project (06XD14034), Shanghai Leading Academic Discipline project (No. T0401), and the Natural Sciences and Engineering Research Council of Canada (No. R2686A02).

References

Cao, H., Liu, Z. & Jing, Z. [2000] "Bifurcation set and distribution of limit cycles for a class of cubic Hamiltonian system with higher-order perturbed terms," *Chaos Solit. Fract.* **11**, 2293–2304.

Chen, L. S. & Wang, M. S. [1979] "The relative position, and the number, of limit cycles of a quadratic differential system," *Acta. Math. Sin.* **22**, 751–758.

Cherkas, L. A. [1968] "On the stability of a singular cycle," *Diff. Uravnenija* **4**, 1012–1017 (in Russian).

Han, M. [1997] "Cyclicity of planar homoclinic loops and quadratic integrable systems," *Sci. China Ser A* **40**, 1247–1258.

Han, M., Ye, Y. & Zhu, D. [1999] "Cyclicity of homoclinic loops and degenerate cubic Hamiltonians," *Sci. China Ser A* **42**, 605–617.

Han, M. & Zhang, Z. [1999] "Cyclicity 1 and 2 conditions for a 2-polycycle of integrable systems on the plane," *J. Diff. Eqns.* **155**, 245–261.

Han, M. & Chen, J. [2000] "On the number of limit cycles in double homoclinic bifurcations," *Sci. China Ser A* **43**, 914–928.

Han, M., Hu, S. & Liu, X. [2003] "On the stability of double homoclinic and heteroclinic cycles," *Nonlin. Anal.* **53**, 701–713.

Han, M., Lin, Y. & Yu, P. [2004] "A study on the existence of limit cycles of a planar system with 3rd-degree polynomials," *Int. J. Bifurcation and Chaos* **14**, 41–60.

Han, M. & Yang, C. [2005] "On the cyclicity of a 2-polycycle for quadratic systems," *Chaos Solit. Fract.* **23**, 1787–1794.

Hilbert, D. [1902] "Mathematical problems," (M. Newton, Transl.) *Bull. Amer. Math.* **8**, 437–479.

Li, J. & Huang, Q. [1987] "Bifurcations of limit cycles forming compound eyes in the cubic system," *Chinese Ann Math B* **8**, 391–403.

Li, J. & Liu, Z. [1991] "Bifurcation set and limit cycles forming compound eyes in a perturbed Hamiltonian system," *Publ. Math.* **35**, 487–506.

Li, J., Chan, H. & Chung, K. [2002] "Bifurcations of limit cycles in a Z_6-equivariant planar vector field of degree 5," *Sci. China Ser A* **45**, 817–826.

Li, J. [2003] "Hilbert's 16th problem and bifurcations of planar polynomial vector fields," *Int. J. Bifurcation and Chaos* **13**, 47–106.

Liu, Z., Qian, T. & Li, J. [2002] "Detection function method and its application to a perturbed quintic Hamiltonian system," *Chaos Solit. Fract.* **13**, 295–310.

Liu, Z., Yang, Z. & Jiang, T. [2003] "The same distribution of limit cycles in five perturbed cubic Hamiltonian systems," *Int. J. Bifurcation and Chaos* **13**, 243–249.

Roussarié, R. [1986] "On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields," *Bol. Soc. Bras. Mat.* **17**, 67–101.

Shi, S. [1980] "A concrete example of the existence of four limit cycles for plane quadratic systems," *Sci. Sinica* **23**, 153–158.

Smale, S. [1998] "Mathematical problems for the next century," *The Math. Intell.* **20**, 7–15.

Yu, P. & Han, M. [2004] "Twelve limit cycles in a cubic order planar system with Z_2 symmetry," *Commun. Pure Appl. Anal.* **3**, 515–525.

Yu, P. & Han, M. [2005a] "Twelve limit cycles in a cubic case of the 16th Hilbert problem," *Int. J. Bifurcation and Chaos* **15**, 2191–2205.

Yu, P. & Han, M. [2005b] "Small limit cycles bifurcating from fine focus points in cubic order Z_2-equivariant vector fields," *Chaos Solit. Fract.* **24**, 329–348.

Zhang, T., Zang, H. & Han, M. [2004a] "Bifurcations of limit cycles in a cubic system," *Chaos Solit. Fract.* **20**, 629–638.

Zhang, T., Han, M., Zang, H. & Meng, X. [2004b] "Bifurcations of limit cycles for a cubic Hamiltonian system under quartic perturbations," *Chaos Solit. Fract.* **22**, 1127–1138.

Zang, H., Chen, W. & Zhang, T. [2004] "Perturbation from a cubic Hamiltonian with three figure eight-loops," *Chaos Solit. Fract.* **22**, 61–74.

Appendix A

In this appendix, the coefficients A_{ij}, B_{ij}, C_{ij} and D_{ij}, used in the proof of Theorem 1 are listed.

A.1. *Coefficients A_{ij} and B_{ij}*

$$A_{10}^{(1)} = \int_{L_1}\left[\frac{1}{2-x}+\frac{x}{(2-x)^2}\right]ydx = -1.8108497019006027028$$

$$A_{01}^{(1)} = \frac{1}{2}\int_{L_1}\frac{y^2}{(2-x)^2}dx = -0.6723139651693518469$$

$$A_{20}^{(1)} = \int_{L_1}\left[\frac{2x}{2-x}+\frac{x^2}{(2-x)^2}\right]ydx = -0.9293238245416440749 1$$

$$A_{11}^{(1)} = \frac{1}{2}\int_{L_1}\left[\frac{1}{2-x}+\frac{x}{(2-x)^2}\right]y^2dx = -1.3446279303387036939$$

$$A_{02}^{(1)} = \frac{1}{3}\int_{L_1}\frac{y^3}{(2-x)^2}dx = -0.62716336629241648628$$

$$A_{30}^{(1)} = \int_{L_1}\left[\frac{3x^2}{2-x}+\frac{x^3}{(2-x)^2}\right]ydx = -1.8586476490832881498$$

$$A_{21}^{(1)} = \frac{1}{2}\int_{L_1}\left[\frac{2x}{2-x}+\frac{x^2}{(2-x)^2}\right]y^2dx = -0.74698195457313706197$$

$$A_{12}^{(1)} = \frac{1}{3}\int_{L_1}\left[\frac{1}{2-x}+\frac{x}{(2-x)^2}\right]y^3dx = -1.2543267325848329726$$

$$A_{03}^{(1)} = \frac{1}{4}\int_{L_1}\frac{y^4}{(2-x)^2}dx = -0.6473857216909956208$$

$$A_{40}^{(1)} = \int_{L_1}\left[\frac{4x^3}{2-x}+\frac{x^4}{(2-x)^2}\right]ydx = -0.97561522926122632068$$

$$A_{31}^{(1)} = \frac{1}{2}\int_{L_1}\left[\frac{3x^2}{2-x}+\frac{x^3}{(2-x)^2}\right]y^2dx = -1.4939639091462741239$$

$$A_{22}^{(1)} = \frac{1}{3}\int_{L_1}\left[\frac{2x}{2-x}+\frac{x^2}{(2-x)^2}\right]y^3dx = -0.71316479902605934491$$

$$A_{13}^{(1)} = \frac{1}{4}\int_{L_1}\left[\frac{1}{2-x}+\frac{x}{(2-x)^2}\right]y^4dx = -1.2947714433819912417$$

$$A_{04}^{(1)} = \frac{1}{5}\int_{L_1}\frac{y^5}{(2-x)^2}dx = -0.70903149767051780534$$

$$B_{10}^{(1)} = \int_{L_1}\frac{x}{2-x}dx = -0.063782325330382208117$$

$$B_{01}^{(1)} = \int_{L_1}\frac{y}{2-x}dx = -1.4825917070570426633$$

$$B_{20}^{(1)} = \int_{L_1}\frac{x^2}{2-x}dx = 2B_{10}^{(1)} = -0.12756465066076441623$$

$$B_{11}^{(1)} = \int_{L_1}\frac{xy}{2-x}dx = -0.27280783485452399596$$

$$B_{20}^{(1)} = \int_{L_1}\frac{y^2}{2-x}dx = -2.1599651552984648022$$

$$B_{30}^{(1)} = \int_{L_1}\frac{x^3}{2-x}dx = -0.019427040926012991001$$

$$B_{21}^{(1)} = \int_{L_1}\frac{x^2y}{2-x}dx = 2B_{11}^{(1)} = -0.54561566970904799191$$

$$B_{12}^{(1)} = \int_{L_1}\frac{xy^2}{2-x}dx = -0.43538249838838895269$$

$$B_{03}^{(1)} = \int_{L_1} \frac{y^3}{2-x}dx = -3.0045549756632550450$$

$$B_{40}^{(1)} = \int_{L_1} \frac{x^4}{2-x}dx = 2B_{30}^{(1)} = -0.038854081852025982002$$

$$B_{31}^{(1)} = \int_{L_1} \frac{x^3 y}{2-x}dx = -0.17733798311631265750$$

$$B_{22}^{(1)} = \int_{L_1} \frac{x^2 y^2}{2-x}dx = 2B_{12}^{(1)} = -0.87076499677677790537$$

$$B_{13}^{(1)} = \int_{L_1} \frac{xy^3}{2-x}dx = -0.62264395289569028935$$

$$B_{04}^{(1)} = \int_{L_1} \frac{y^4}{2-x}dx = -4.1278524762885371586$$

$$A_{10}^{(3)} = \int_{L_3} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] ydx = 0.26819304280651981360$$

$$A_{01}^{(3)} = \frac{1}{2}\int_{L_3} \frac{y^2}{(2-x)^2}dx = 0.027932787954614884840$$

$$A_{20}^{(3)} = \int_{L_3} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] ydx = 0.039882496532439631341$$

$$A_{11}^{(3)} = \frac{1}{2}\int_{L_3} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^2dx = 0.055865575909229769679$$

$$A_{02}^{(3)} = \frac{1}{3}\int_{L_3} \frac{y^3}{(2-x)^2}dx = 0.0084487461287359437330$$

$$A_{30}^{(3)} = \int_{L_3} \left[\frac{3x^2}{2-x} + \frac{x^3}{(2-x)^2}\right] ydx = 0.079764993064879262682$$

$$A_{21}^{(3)} = \frac{1}{2}\int_{L_3} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] y^2dx = 0.0058171599732780772396$$

$$A_{12}^{(3)} = \frac{1}{3}\int_{L_3} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^3dx = 0.016897492257471887466$$

$$A_{03}^{(3)} = \frac{1}{4}\int_{L_3} \frac{y^4}{(2-x)^2}dx = 0.0030045444762586587247$$

$$A_{40}^{(3)} = \int_{L_3} \left[\frac{4x^3}{2-x} + \frac{x^4}{(2-x)^2}\right] ydx = 0.014221803669163429550$$

$$A_{31}^{(3)} = \frac{1}{2}\int_{L_3} \left[\frac{3x^2}{2-x} + \frac{x^3}{(2-x)^2}\right] y^2dx = 0.011634319946556154479$$

$$A_{22}^{(3)} = \frac{1}{3}\int_{L_3} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] y^3dx = 0.0013437954828909060307$$

$$A_{13}^{(3)} = \frac{1}{4}\int_{L_3} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^4dx = 0.0060090889525173174493$$

$$A_{04}^{(3)} = \frac{1}{5}\int_{L_3} \frac{y^5}{(2-x)^2}dx = 0.0011706172823943376490$$

$$B_{10}^{(3)} = \int_{L_3} \frac{x}{2-x}dx = 0.063782325330382208117$$

$$B_{01}^{(3)} = \int_{L_3} \frac{y}{2-x}dx = 0.25465136545416107825$$

$$B_{20}^{(3)} = \int_{L_3} \frac{x^2}{2-x}dx = 2B_{10}^{(3)} = 0.12756465066076441623$$

$$B_{11}^{(3)} = \int_{L_3} \frac{xy}{2-x}dx = 0.012799141827722160637$$

$$B_{02}^{(3)} = \int_{L_3} \frac{y^2}{2-x}dx = 0.10779792633501720054$$

$$B_{30}^{(3)} = \int_{L_3} \frac{x^3}{2-x}dx = 0.01942704092601299100 1$$

$$B_{21}^{(3)} = \int_{L_3} \frac{x^2y}{2-x}dx = 0.02559828365544432127 4$$

$$B_{12}^{(3)} = \int_{L_3} \frac{xy^2}{2-x}dx = 0.00376786897967147685 14$$

$$B_{03}^{(3)} = \int_{L_3} \frac{y^3}{2-x}dx = 0.04933333208194483587 8$$

$$B_{40}^{(3)} = \int_{L_3} \frac{x^4}{2-x}dx = 0.03885408185202598200 2$$

$$B_{31}^{(3)} = \int_{L_3} \frac{x^3y}{2-x}dx = 2B_{11}^{(3)} = 0.00276050649069027727 65$$

$$B_{22}^{(3)} = \int_{L_3} \frac{x^2y^2}{2-x}dx = 2B_{12}^{(3)} = 0.00753573795934295370 28$$

$$B_{13}^{(3)} = \int_{L_3} \frac{xy^3}{2-x}dx = 0.00131309706773106505 15$$

$$B_{04}^{(3)} = \int_{L_3} \frac{y^4}{2-x}dx = 0.02351801836164195532 3$$

$$A_{10}^{(5)} = \int_{L_5} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] ydx = -0.54126512313905968149$$

$$A_{01}^{(5)} = \frac{1}{2}\int_{L_5} \frac{y^2}{(2-x)^2}dx = -0.25627208044780896678$$

$$A_{20}^{(5)} = \int_{L_5} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] ydx = -0.58602665719751936713$$

$$A_{11}^{(5)} = \frac{1}{2}\int_{L_5} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^2dx = -0.51254416089561793355$$

$$A_{02}^{(5)} = \frac{1}{3}\int_{L_5} \frac{y^3}{(2-x)^2}dx = -0.25207487924477713867$$

$$A_{30}^{(5)} = \int_{L_5} \left[\frac{3x^2}{2-x} + \frac{x^3}{(2-x)^2}\right] ydx = -0.58412005874087720430$$

$$A_{21}^{(5)} = \frac{1}{2}\int_{L_5} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] y^2dx = -0.55368380100020418417$$

$$A_{12}^{(5)} = \frac{1}{3}\int_{L_5} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^3dx = -0.50414975848955427734$$

$$A_{03}^{(5)} = \frac{1}{4}\int_{L_5} \frac{y^4}{(2-x)^2}dx = -0.25627208044780896678$$

$$A_{40}^{(5)} = \int_{L_5} \left[\frac{4x^3}{2-x} + \frac{x^4}{(2-x)^2}\right] ydx = -0.56879291632144951064$$

$$A_{31}^{(5)} = \frac{1}{2}\int_{L_5} \left[\frac{3x^2}{2-x} + \frac{x^3}{(2-x)^2}\right] y^2dx = -0.55200723473061258746$$

$$A_{22}^{(5)} = \frac{1}{3}\int_{L_5} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] y^3dx = -0.54424711693056142773$$

$$A_{13}^{(5)} = \frac{1}{4}\int_{L_5} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^4dx = A_{11}^{(5)} = -0.51254416089561793355$$

$$A_{04}^{(5)} = \frac{1}{5}\int_{L_5} \frac{y^5}{(2-x)^2}dx = -0.26798467867216414957$$

$$B_{i0}^{(5)} = \int_{L_5} \frac{x^i}{2-x}dx = 0, \quad i = 1, \ldots, 4$$

$$\begin{aligned} B_{01}^{(5)} &= \int_{L_5} \frac{y}{2-x}dx \\ &= -0.36192377869572237908 \end{aligned}$$

$$\begin{aligned} B_{11}^{(5)} &= \int_{L_5} \frac{xy}{2-x}dx \\ &= -0.22734396831084476230 \end{aligned}$$

$$\begin{aligned} B_{02}^{(5)} &= \int_{L_5} \frac{y^2}{2-x}dx \\ &= -0.68614725045683653106 \end{aligned}$$

$$\begin{aligned} B_{21}^{(5)} &= \int_{L_5} \frac{x^2y}{2-x}dx \\ &= -0.16072130879460875963 \end{aligned}$$

$$\begin{aligned} B_{12}^{(5)} &= \int_{L_5} \frac{xy^2}{2-x}dx \\ &= -0.42948545933160969625 \end{aligned}$$

$$\begin{aligned} B_{03}^{(5)} &= \int_{L_5} \frac{y^3}{2-x}dx \\ &= -1.0126793358961648645 \end{aligned}$$

$$\begin{aligned} B_{31}^{(5)} &= \int_{L_5} \frac{x^3y}{2-x}dx \\ &= -0.12162688386911588660 \end{aligned}$$

$$\begin{aligned} B_{22}^{(5)} &= \int_{L_5} \frac{x^2y^2}{2-x}dx \\ &= -0.30361055139342361162 \end{aligned}$$

$$\begin{aligned} B_{13}^{(5)} &= \int_{L_5} \frac{xy^3}{2-x}dx \\ &= -0.6332014716466883482 \end{aligned}$$

$$\begin{aligned} B_{04}^{(5)} &= \int_{L_5} \frac{y^4}{2-x}dx \\ &= -1.3722945009136730621 \end{aligned}$$

$$\begin{aligned} A_{10}^{(6)} &= \int_{L_6} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] ydx \\ &= -0.15145489041876868053 \end{aligned}$$

$$\begin{aligned} A_{01}^{(6)} &= \frac{1}{2}\int_{L_6} \frac{y^2}{(2-x)^2}dx \\ &= -0.072106210815176617223 \end{aligned}$$

$$\begin{aligned} A_{20}^{(6)} &= \int_{L_6} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] ydx \\ &= 0.19359380824306263479 \end{aligned}$$

$$\begin{aligned} A_{11}^{(6)} &= \frac{1}{2}\int_{L_6} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^2dx \\ &= -0.14421242163035323445 \end{aligned}$$

$$\begin{aligned} A_{02}^{(6)} &= \frac{1}{3}\int_{L_6} \frac{y^3}{(2-x)^2}dx \\ &= -0.071042665976278002421 \end{aligned}$$

$$\begin{aligned} A_{30}^{(6)} &= \int_{L_6} \left[\frac{3x^2}{2-x} + \frac{x^3}{(2-x)^2}\right] ydx \\ &= -0.20074563916803626038 \end{aligned}$$

$$\begin{aligned} A_{21}^{(6)} &= \frac{1}{2}\int_{L_6} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] y^2dx \\ &= 0.18297967753032521404 \end{aligned}$$

$$\begin{aligned} A_{12}^{(6)} &= \frac{1}{3}\int_{L_6} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^3dx \\ &= -0.14208533195255600484 \end{aligned}$$

$$\begin{aligned} A_{03}^{(6)} &= \frac{1}{4}\int_{L_6} \frac{y^4}{(2-x)^2}dx \\ &= -0.072106210815176617223 \end{aligned}$$

$$\begin{aligned} A_{40}^{(6)} &= \int_{L_6} \left[\frac{4x^3}{2-x} + \frac{x^4}{(2-x)^2}\right] ydx \\ &= 0.19795592282423237721 \end{aligned}$$

$$\begin{aligned} A_{31}^{(6)} &= \frac{1}{2}\int_{L_6} \left[\frac{3x^2}{2-x} + \frac{x^3}{(2-x)^2}\right] y^2dx \\ &= -0.18940101220914535279 \end{aligned}$$

$$\begin{aligned} A_{22}^{(6)} &= \frac{1}{3}\int_{L_6} \left[\frac{2x}{2-x} + \frac{x^2}{(2-x)^2}\right] y^3dx \\ &= 0.17988173614343511727 \end{aligned}$$

$$\begin{aligned} A_{13}^{(6)} &= \frac{1}{4}\int_{L_6} \left[\frac{1}{2-x} + \frac{x}{(2-x)^2}\right] y^4dx = A_{11}^{(6)} \\ &= -0.14421242163035323445 \end{aligned}$$

$$A_{04}^{(6)} = \frac{1}{5}\int_{L_6} \frac{y^5}{(2-x)^2}dx$$
$$= -0.07508211590551114943 0$$

$$B_{i0}^{(6)} = \int_{L_6} \frac{x^i}{2-x}dx = 0, \quad i = 1, \ldots, 4$$

$$B_{01}^{(6)} = \int_{L_6} \frac{y}{2-x}dx$$
$$= -0.19309496640180809453$$

$$B_{11}^{(6)} = \int_{L_6} \frac{xy}{2-x}dx$$
$$= 0.11031365627698380680$$

$$B_{02}^{(6)} = \int_{L_6} \frac{y^2}{2-x}dx$$
$$= -0.36716260723301109500$$

$$B_{21}^{(6)} = \int_{L_6} \frac{x^2y}{2-x}dx$$
$$= -0.07333931527311315138 1$$

$$B_{12}^{(6)} = \int_{L_6} \frac{xy^2}{2-x}dx$$
$$= 0.20848382711604117586$$

$$B_{03}^{(6)} = \int_{L_6} \frac{y^3}{2-x}dx$$
$$= -0.54237334438080080947$$

$$B_{31}^{(6)} = \int_{L_6} \frac{x^3y}{2-x}dx$$
$$= 0.05313710317387532989 4$$

$$B_{22}^{(6)} = \int_{L_6} \frac{x^2y^2}{2-x}dx$$
$$= -0.13839271303771342916$$

$$B_{13}^{(6)} = \int_{L_6} \frac{xy^3}{2-x}dx$$
$$= 0.30741051138403976192$$

$$B_{04}^{(6)} = \int_{L_6} \frac{y^4}{2-x}dx$$
$$= -0.73432521446602219001$$

A.2. *Coefficients* C_{ij}

$$C_{01}^{(5)} = \oint_{L_5} \frac{y-1}{(2-x)^2y(1-y^2)}dx$$
$$= -0.27062440691906989025$$

$$C_{02}^{(5)} = \oint_{L_5} \frac{y^2-1}{(2-x)^2y(1-y^2)}dx$$
$$= -0.34408388504791177672$$

$$C_{03}^{(5)} = \oint_{L_5} \frac{y^3-1}{(2-x)^2y(1-y^2)}dx$$
$$= -0.27062440691906989025$$

$$C_{04}^{(5)} = \oint_{L_5} \frac{y^4-1}{(2-x)^2y(1-y^2)}dx$$
$$= -0.07345132347838193597 7$$

$$C_{05}^{(5)} = \oint_{L_5} \frac{2(y-1)}{(2-x)^2y(1-y^2)}dx = 2C_{01}^{(5)}$$
$$= -0.54124881383813978050$$

$$C_{06}^{(5)} = \oint_{L_5} \frac{4(y-1)-2xy}{(2-x)^2y(1-y^2)}dx$$
$$= -3.8339907354217363027$$

$$C_{07}^{(5)} = \oint_{L_5} \frac{2(y^2-1)}{(2-x)^2y(1-y^2)}dx = 2C_{02}^{(5)}$$
$$= -0.68816777009582355345$$

$$C_{08}^{(5)} = \oint_{L_5} \frac{6(y^2-1)-3xy^2}{(2-x)^2y(1-y^2)}dx$$
$$= 6C_{02}^{(5)} = -6.045264459001518869$$

$$C_{09}^{(5)} = \oint_{L_5} \frac{2(y^3-1)}{(2-x)^2y(1-y^2)}dx = 2C_{01}^{(5)}$$
$$= -0.54124881383813978050$$

$$C_{10}^{(5)} = \oint_{L_5} \frac{8(y^3-1)-4xy^3}{(2-x)^2y(1-y^2)}dx$$
$$= -7.6679814708434726054$$

$$C_{11}^{(5)} = \oint_{L_5} \frac{-x}{(2-x)^2y(1-y^2)}dx$$
$$= -1.5568840353872070710$$

$$C_{12}^{(5)} = \oint_{L_5} \frac{x^2}{(2-x)^2 y(1-y^2)} dx = 1.17360803764554678314$$

$$C_{13}^{(5)} = \oint_{L_5} \frac{x^3}{(2-x)^2 y(1-y^2)} dx = 0.98582305584301228711$$

$$C_{14}^{(5)} = \oint_{L_5} \frac{x^2 y}{(2-x)^2 y(1-y^2)} dx = 1.0414266675296379476$$

$$C_{15}^{(5)} = \oint_{L_5} \frac{x^4}{(2-x)^2 y(1-y^2)} dx = 0.86624981433682597440$$

$$C_{16}^{(5)} = \oint_{L_5} \frac{x^3 y}{(2-x)^2 y(1-y^2)} dx = 0.88416192370682143063$$

$$C_{17}^{(5)} = \oint_{L_5} \frac{x^2 y^2}{(2-x)^2 y(1-y^2)} dx = 1.00570640452133258383$$

$$C_{18}^{(5)} = \oint_{L_5} \frac{x}{(2-x) y(1-y^2)} dx = 1.9401600331288673589$$

$$C_{19}^{(5)} = \oint_{L_5} \frac{x^2}{(2-x) y(1-y^2)} dx = 1.36139301944808127917$$

$$C_{20}^{(5)} = \oint_{L_5} \frac{x^3}{(2-x) y(1-y^2)} dx = 1.10539629734919859981$$

$$C_{21}^{(5)} = \oint_{L_5} \frac{xy}{(2-x) y(1-y^2)} dx = 1.7100664402158187941$$

$$C_{22}^{(5)} = \oint_{L_5} \frac{x^2 y}{(2-x) y(1-y^2)} dx = 1.1986914113524544646$$

$$C_{23}^{(5)} = \oint_{L_5} \frac{xy^2}{(2-x) y(1-y^2)} dx = 1.6481343612880328885$$

$$C_{01}^{(6)} = \oint_{L_6} \frac{y-1}{(2-x)^2 y(1-y^2)} dx = -0.073701031786839426197$$

$$C_{02}^{(6)} = \oint_{L_6} \frac{y^2-1}{(2-x)^2 y(1-y^2)} dx = -0.094092034718746070540$$

$$C_{03}^{(6)} = \oint_{L_6} \frac{y^3-1}{(2-x)^2 y(1-y^2)} dx = C_{01}^{(6)} = -0.073701031786839426197$$

$$C_{04}^{(6)} = \oint_{L_6} \frac{y^4-1}{(2-x)^2 y(1-y^2)} dx = -0.018364589509361730273$$

$$C_{05}^{(6)} = \oint_{L_6} \frac{2(y-1)}{(2-x)^2 y(1-y^2)} dx = 2C_{01}^{(6)} = -0.14740206357367885239$$

$$C_{06}^{(6)} = \oint_{L_6} \frac{4(y-1)-2xy}{(2-x)^2 y(1-y^2)} dx = 0.37383564553882314165$$

$$C_{07}^{(6)} = \oint_{L_6} \frac{2(y^2-1)}{(2-x)^2 y(1-y^2)} dx = 2C_{02}^{(6)} = -0.18818406943749214108$$

$$C_{08}^{(6)} = \oint_{L_6} \frac{6(y^2-1)-3xy^2}{(2-x)^2 y(1-y^2)} dx = 0.4044232882769730308$$

$$C_{09}^{(6)} = \oint_{L_6} \frac{2(y^3-1)}{(2-x)^2 y(1-y^2)} dx = 2C_{01}^{(6)} = -0.14740206357367885239$$

$$C_{10}^{(6)} = \oint_{L_6} \frac{8(y^3-1)-4xy^3}{(2-x)^2 y(1-y^2)} dx = 0.7476712910776462833$$

$$C_{11}^{(6)} = \oint_{L_6} \frac{-x}{(2-x)^2 y(1-y^2)} dx = 0.37627030733798153177$$

$$C_{12}^{(6)} = \oint_{L_6} \frac{x^2}{(2-x)^2 y(1-y^2)} dx = 0.21391670802908000746$$

$$C_{13}^{(6)} = \oint_{L6} \frac{x^3}{(2-x)^2 y(1-y^2)} dx$$
$$= -0.15817898534140728165$$

$$C_{14}^{(6)} = \oint_{L6} \frac{x^2 y}{(2-x)^2 y(1-y^2)} dx$$
$$= 0.18639344742172855059$$

$$C_{15}^{(6)} = \oint_{L6} \frac{x^4}{(2-x)^2 y(1-y^2)} dx$$
$$= 0.129006968065014802060$$

$$C_{16}^{(6)} = \oint_{L6} \frac{x^3 y}{(2-x)^2 y(1-y^2)} dx$$
$$= -0.13858813401990722827$$

$$C_{17}^{(6)} = \oint_{L6} \frac{x^2 y^2}{(2-x)^2 y(1-y^2)} dx$$
$$= 0.17901618408429384456$$

$$C_{18}^{(6)} = \oint_{L6} \frac{x}{(2-x) y(1-y^2)} dx$$
$$= -0.96645732270504307100$$

$$C_{19}^{(6)} = \oint_{L6} \frac{x^2}{(2-x) y(1-y^2)} dx$$
$$= 0.58601240139956729658$$

$$C_{20}^{(6)} = \oint_{L6} \frac{x^3}{(2-x) y(1-y^2)} dx$$
$$= -0.44536493874782936537$$

$$C_{21}^{(6)} = \oint_{L6} \frac{xy}{(2-x) y(1-y^2)} dx$$
$$= -0.85503322010790939703$$

$$C_{22}^{(6)} = \oint_{L6} \frac{x^2 y}{(2-x) y(1-y^2)} dx$$
$$= 0.51137502886336432945$$

$$C_{23}^{(6)} = \oint_{L6} \frac{xy^2}{(2-x) y(1-y^2)} dx$$
$$= -0.82499984847726014722$$

$$C_{01}^{(7)} = \oint_{L7} \frac{y+1}{(2-x)^2 y(1-y^2)} dx = -C_{01}^{(6)}$$

$$C_{02}^{(7)} = \oint_{L7} \frac{y^2+1}{(2-x)^2 y(1-y^2)} dx = C_{02}^{(6)}$$

$$C_{03}^{(7)} = \oint_{L7} \frac{y^3+1}{(2-x)^2 y(1-y^2)} dx = -C_{03}^{(6)}$$

$$C_{04}^{(7)} = \oint_{L7} \frac{y^4-1}{(2-x)^2 y(1-y^2)} dx = C_{04}^{(6)}$$

$$C_{05}^{(7)} = \oint_{L7} \frac{2(y+1)}{(2-x)^2 y(1-y^2)} dx = -C_{05}^{(6)}$$

$$C_{06}^{(7)} = \oint_{L7} \frac{4(y+1)-2xy}{(2-x)^2 y(1-y^2)} dx = -C_{06}^{(6)}$$

$$C_{07}^{(7)} = \oint_{L7} \frac{2(y^2-1)}{(2-x)^2 y(1-y^2)} dx = C_{07}^{(6)}$$

$$C_{08}^{(7)} = \oint_{L7} \frac{6(y^2-1)-3xy^2}{(2-x)^2 y(1-y^2)} dx = C_{08}^{(6)}$$

$$C_{09}^{(7)} = \oint_{L7} \frac{2(y^3+1)}{(2-x)^2 y(1-y^2)} dx = -C_{09}^{(6)}$$

$$C_{10}^{(7)} = \oint_{L7} \frac{8(y^3+1)-4xy^3}{(2-x)^2 y(1-y^2)} dx = -C_{10}^{(6)}$$

$$C_{11}^{(7)} = \oint_{L7} \frac{-x}{(2-x)^2 y(1-y^2)} dx = C_{11}^{(6)}$$

$$C_{12}^{(7)} = \oint_{L7} \frac{x^2}{(2-x)^2 y(1-y^2)} dx = C_{12}^{(6)}$$

$$C_{13}^{(7)} = \oint_{L7} \frac{x^3}{(2-x)^2 y(1-y^2)} dx = C_{13}^{(6)}$$

$$C_{14}^{(7)} = \oint_{L7} \frac{x^2 y}{(2-x)^2 y(1-y^2)} dx = -C_{14}^{(6)}$$

$$C_{15}^{(7)} = \oint_{L7} \frac{x^4}{(2-x)^2 y(1-y^2)} dx = C_{15}^{(6)}$$

$$C_{16}^{(7)} = \oint_{L7} \frac{x^3 y}{(2-x)^2 y(1-y^2)} dx = -C_{16}^{(6)}$$

$$C_{17}^{(7)} = \oint_{L7} \frac{x^2 y^2}{(2-x)^2 y(1-y^2)} dx = C_{17}^{(6)}$$

$$C_{18}^{(7)} = \oint_{L7} \frac{x}{(2-x) y(1-y^2)} dx = C_{18}^{(6)}$$

$$C_{19}^{(7)} = \oint_{L7} \frac{x^2}{(2-x) y(1-y^2)} dx = C_{19}^{(6)}$$

$$C_{20}^{(7)} = \oint_{L7} \frac{x^3}{(2-x) y(1-y^2)} dx = C_{20}^{(6)}$$

$$C_{21}^{(7)} = \oint_{L_7} \frac{xy}{(2-x)y(1-y^2)}dx = -C_{21}^{(6)}$$

$$C_{22}^{(7)} = \oint_{L_7} \frac{x^2y}{(2-x)y(1-y^2)}dx = -C_{22}^{(6)}$$

$$C_{23}^{(7)} = \oint_{L_7} \frac{xy^2}{(2-x)y(1-y^2)}dx = C_{23}^{(6)}$$

$$C_{01}^{(8)} = \oint_{L_8} \frac{y+1}{(2-x)^2y(1-y^2)}dx = -C_{01}^{(5)}$$

$$C_{02}^{(8)} = \oint_{L_8} \frac{y^2-1}{(2-x)^2y(1-y^2)}dx = C_{02}^{(5)}$$

$$C_{03}^{(8)} = \oint_{L_8} \frac{y^3+1}{(2-x)^2y(1-y^2)}dx = -C_{03}^{(5)}$$

$$C_{04}^{(8)} = \oint_{L_8} \frac{y^4-1}{(2-x)^2y(1-y^2)}dx = C_{04}^{(5)}$$

$$C_{05}^{(8)} = \oint_{L_8} \frac{2(y+1)}{(2-x)^2y(1-y^2)}dx = -C_{05}^{(5)}$$

$$C_{06}^{(8)} = \oint_{L_8} \frac{4(y+1)-2xy}{(2-x)^2y(1-y^2)}dx = -C_{06}^{(5)}$$

$$C_{07}^{(8)} = \oint_{L_8} \frac{2(y^2-1)}{(2-x)^2y(1-y^2)}dx = C_{07}^{(5)}$$

$$C_{08}^{(8)} = \oint_{L_8} \frac{6(y^2-1)-3xy^2}{(2-x)^2y(1-y^2)}dx = C_{08}^{(5)}$$

$$C_{09}^{(8)} = \oint_{L_8} \frac{2(y^3+1)}{(2-x)^2y(1-y^2)}dx = -C_{09}^{(5)}$$

$$C_{10}^{(8)} = \oint_{L_8} \frac{8(y^3+1)-4xy^3}{(2-x)^2y(1-y^2)}dx = -C_{10}^{(5)}$$

$$C_{11}^{(8)} = \oint_{L_8} \frac{-x}{(2-x)^2y(1-y^2)}dx = C_{11}^{(5)}$$

$$C_{12}^{(8)} = \oint_{L_8} \frac{x^2}{(2-x)^2y(1-y^2)}dx = C_{12}^{(5)}$$

$$C_{13}^{(8)} = \oint_{L_8} \frac{x^3}{(2-x)^2y(1-y^2)}dx = C_{13}^{(5)}$$

$$C_{14}^{(8)} = \oint_{L_8} \frac{x^2y}{(2-x)^2y(1-y^2)}dx = -C_{14}^{(5)}$$

$$C_{15}^{(8)} = \oint_{L_8} \frac{x^4}{(2-x)^2y(1-y^2)}dx = C_{15}^{(5)}$$

$$C_{16}^{(8)} = \oint_{L_8} \frac{x^3y}{(2-x)^2y(1-y^2)}dx = -C_{16}^{(5)}$$

$$C_{17}^{(8)} = \oint_{L_8} \frac{x^2y^2}{(2-x)^2y(1-y^2)}dx = C_{17}^{(5)}$$

$$C_{18}^{(8)} = \oint_{L_8} \frac{x}{(2-x)y(1-y^2)}dx = C_{18}^{(5)}$$

$$C_{19}^{(8)} = \oint_{L_8} \frac{x^2}{(2-x)y(1-y^2)}dx = C_{19}^{(5)}$$

$$C_{20}^{(8)} = \oint_{L_8} \frac{x^3}{(2-x)y(1-y^2)}dx = C_{20}^{(5)}$$

$$C_{21}^{(8)} = \oint_{L_8} \frac{xy}{(2-x)y(1-y^2)}dx = -C_{21}^{(5)}$$

$$C_{22}^{(8)} = \oint_{L_8} \frac{x^2y}{(2-x)y(1-y^2)}dx = -C_{22}^{(5)}$$

$$C_{23}^{(8)} = \oint_{L_8} \frac{xy^2}{(2-x)y(1-y^2)}dx = C_{23}^{(5)}$$

A.3. *Coefficients* D_{ij}

$$D_{01} = \oint_{\Gamma^{-3}} \frac{4y}{2-x}dx + \oint_{\Gamma^{-3}} \frac{\frac{15}{4}x - x^2 + \frac{1}{2}y^2}{2-x}dy = 59.80947180355010918$$

$$D_{02} = \oint_{\Gamma^{-3}} \frac{2y-xy}{2-x}dy = 0$$

$$D_{03} = \oint_{\Gamma^{-3}} \frac{-y}{2-x}dx + \oint_{\Gamma^{-3}} \frac{\frac{1}{2}x + y - 3y^2 - x^3}{2-x}dy = -206.92075515187769842$$

$$D_{04} = \oint_{\Gamma^{-3}} \frac{-x^2y}{2-x}dy = 0$$

$$D_{05} = \oint_{\Gamma^{-3}} \frac{2y^2-xy^2}{2-x}dy = 0$$

$$D_{06} = \oint_{\Gamma^{-3}} \frac{-y^3}{2-x}dy = 0$$

$$D_{07} = \oint_{\Gamma^{-3}} \frac{4y}{2-x}dx + \oint_{\Gamma^{-3}} \frac{\frac{29}{8}x + \frac{3}{4}y^2 - x^4}{2-x}dy = -122.04411604341146734$$

$$D_{08} = \oint_{\Gamma^{-3}} \frac{-x^2y}{2-x}dy = 0$$

$$D_{09} = \oint_{\Gamma^{-3}} \frac{-x^2y^2}{2-x}dy = -90.67342899190338470$$

$$D_{10} = \oint_{\Gamma^{-3}} \frac{2y-xy^3}{2-x}dy = 0$$

$$D_{11} = \oint_{\Gamma^{-3}} \frac{y^2-y^4}{2-x}dy = -39.88149871954244962$$

$$D_{12} = \oint_{\Gamma^{-3}} \frac{x}{2-x}dx = 0$$

$$D_{13} = \oint_{\Gamma^{-3}} \frac{x^2}{2-x}dx = 0$$

$$D_{14} = \oint_{\Gamma^{-3}} \frac{2y+xy}{2-x}dx + \oint_{\Gamma^{-3}} \frac{\frac{7}{4}x+\frac{1}{2}y^2}{2-x}dy = 59.80947180355010918$$

$$D_{15} = \oint_{\Gamma^{-3}} \frac{y^2}{2-x}dx + \oint_{\Gamma^{-3}} \frac{4y}{2-x}dy = 0$$

$$D_{16} = \oint_{\Gamma^{-3}} \frac{x^3}{2-x}dx = 0$$

$$D_{17} = \oint_{\Gamma^{-3}} \frac{-\frac{1}{2}y+x^2y}{2-x}dx + \oint_{\Gamma^{-3}} \frac{-y^2}{2-x}dy = -43.65090577238874003$$

$$D_{18} = \oint_{\Gamma^{-3}} \frac{xy^2}{2-x}dx = 0$$

$$D_{19} = \oint_{\Gamma^{-3}} \frac{y^3}{2-x}dx + 6\oint_{\Gamma^{-3}} \frac{y^2}{2-x}dy = 115.49814630805972235$$

$$D_{20} = \oint_{\Gamma^{-3}} \frac{x^4}{2-x}dx = 0$$

$$D_{21} = \oint_{\Gamma^{-3}} \frac{y+x^3y}{2-x}dx + \oint_{\Gamma^{-3}} \frac{\frac{7}{8}x+\frac{1}{4}y^2}{2-x}dy = 9.96398654200382978$$

$$D_{22} = \oint_{\Gamma^{-3}} \frac{x^2y^2}{2-x}dx = 0$$

$$D_{23} = \oint_{\Gamma^{-3}} \frac{xy^3}{2-x}dx = -41.02399435959070939$$

$$D_{24} = \oint_{\Gamma^{-3}} \frac{y^4}{2-x}dx + \oint_{\Gamma^{-3}} \frac{8y}{2-x}dy = 0$$

This article has been cited by:

1. Maoan Han, Jibin Li. 2012. Lower bounds for the Hilbert number of polynomial systems. *Journal of Differential Equations* **252**:4, 3278-3304. [CrossRef]
2. Tomas Johnson. 2011. A Quartic System with Twenty-Six Limit Cycles. *Experimental Mathematics* **20**:3, 323-328. [CrossRef]
3. Yu Hai Wu, Xue Di Wang, Li Xin Tian. 2010. Bifurcations of limit cycles in a Z 4-equivariant quintic planar vector field. *Acta Mathematica Sinica, English Series* **26**:4, 779-798. [CrossRef]