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Small-amplitude limit cycles of polynomial Liénard systems

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Abstract In this paper, we study the number of limit cycles appeared in Hopf bifurcations of a Liénard system with multiple parameters. As an application to some polynomial Liénard systems of the form $\dot{x} = y$, $\dot{y} = -g_m(x) - f_n(x)y$, we obtain a new lower bound of maximal number of limit cycles which appear in Hopf bifurcation for arbitrary degrees m and n.

Keywords limit cycle, Liénard system, Hopf bifurcation

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1 Introduction and main results

Consider a Liénard system with a vector parameter of the form

$$\dot{x} = y - F(x, a), \quad \dot{y} = -g(x), \tag{1.1}$$

where F and g are C^∞ functions satisfying

$$g(0) = 0, \quad g'(0) > 0, \quad F(0, a) = 0, \quad a \in D \tag{1.2}$$

with $D \subset \mathbb{R}^m$, $m \geqslant 1$. It is easy to see that the origin is a focus or node of (1.1) for all $a \in D$. If $\frac{\partial F}{\partial x}(0, a_0) = 0$ for some $a_0 \in D$, then the origin is a focus or center, and (1.1) may have a limit cycle near the origin for a near a_0. Further, for a near a_0 the Poincaré return map, denoted by $P(r, a)$, can be defined for $|r|$ small and it has a formal expansion of the form

$$P(r, a) - r = \sum_{j\geqslant 1} d_j(a) r^j.$$

For fixed $a \in D$, the origin is called a focus of order k if

$$d_j(a) = 0, \quad j = 1, \ldots, 2k, \quad d_{2k+1}(a) \neq 0.$$

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Let us introduce two numbers H_D^* and $\widehat{H}_D$ for the family (1.1). First, H_D^* is defined as follows:

$$H_D^* = \max_{a\in D}\{\text{the order of the focus at the origin for (1.1)}\}.$$

More precisely, for all $a \in D$, the origin is a focus of order at most H_D^* unless it is a center, and there exists $a^* \in D$ such that the origin is a focus of order H_D^*. Thus, H_D^* is the maximal order of the origin as a focus of (1.1) for all possible $a \in D$.

Then, we define $\widehat{H}_D$ by

$$\widehat{H}_D = \max_{a\in D}\{\text{the number of limit cycles of (1.1) near the origin}\}.$$

In other words, there exists a neighborhood V of the origin such that (1.1) has at most $\widehat{H}_D$ limit cycles in V for all $a \in D$, and for any neighborhood $U \subset V$ of the origin, there exists $a \in D$ such that (1.1) has $\widehat{H}_D$ limit cycles in U. The number $\widehat{H}_D$ is called the cyclicity of the family (1.1) at the origin.

Han [5] gave a way to find H_D^* and $\widehat{H}_D$, obtaining the following theorem.

Theorem 1.1. *Let* (1.2) *be satisfied. Suppose*

$$F(\alpha(x), a) - F(x, a) = \sum_{j\geqslant 1} B_j(a)x^j,$$

where $\alpha(x) = -x + O(x^2)$ *is the solution to the equation* $G(x) = G(y)$ *on* $y < 0 < x$ *with* $G(x) = \int_0^x g(x)dx$. *Then*

(1) *for all* $k \geqslant 1$ $B_{2k} = O(|B_1, B_3, \ldots, B_{2k-1}|)$, *and for fixed* $a \in D$ *the origin is a focus of order* k *if and only if*

$$B_j(a) = 0, \quad j = 1, \ldots, 2k, \quad B_{2k+1}(a) \neq 0;$$

in this case, it is stable (*unstable*) *if* $B_{2k+1}(a) < 0$ (> 0).

(2) *If* (i) *for some* $a_0 \in D$ $B_{2j+1}(a_0) = 0$, $j = 0, \ldots, k$, *and*

$$\operatorname{rank}\frac{\partial(B_1, B_3, \ldots, B_{2k+1})}{\partial(a_1, a_2, \ldots, a_m)}\bigg|_{a=a_0} = k + 1,$$

then (1.1) *has at least* k *limit cycles near the origin for some* a *near* a_0, *each having an odd multiplicity. If further* (ii) $F(\alpha(x), a) - F(x, a) \equiv 0$ *as* $B_{2j+1} = 0$, $j = 0, \ldots, k$, *then the cyclicity of* (1.1) *at the origin is* k *for all* a *near* a_0. *Moreover, when* F *is linear in* a *then the cyclicity of* (1.1) *at the origin is* k *for all* $a \in D$, *and hence we have* $H_D^* = \widehat{H}_D = k$ *in this case.*

An easy corollary of the above theorem is that $H_D^* = \widehat{H}_D = [\frac{n-1}{2}]$ for the system

$$\dot{x} = y - \sum_{j=1}^{n} a_j x^j, \quad \dot{y} = -x,$$

where $D = \mathbb{R}^n = \{(a_1, a_2, \ldots, a_n)|a_j \in \mathbb{R}\}$, $n \geqslant 1$. Han [5] proved $H_D^* = \widehat{H}_D = [\frac{2n-1}{3}]$ for the system

$$\dot{x} = y - \sum_{j=1}^{n} a_j x^j, \quad \dot{y} = -x(1-x), \quad n \geqslant 1.$$

A new proof of this conclusion can be found in [11].

Jiang and Han [6] observed the following theorem from the proof of the above theorem given in [5].

Theorem 1.2. *If there exists* $k \geqslant 1$ *such that for* $j \geqslant k+1$, $B_{2j+1} = O(B_1, B_3, \ldots, B_{2k+1})$ *as* $|B_1|, |B_3|, \ldots, |B_{2k+1}|$ *are sufficiently small, then there exists a neighborhood* U *of the origin such that* (1.1) *has at most* k *limit cycles in* U *for all* $a \in D$.

By using Theorems 1.1 and 1.2, it was proved in [6] that the system

$$\dot{x} = y - \frac{\sum_{i=1}^{n} a_i x^i}{1 + \sum_{i=1}^{m} b_i x^i}, \quad \dot{y} = -g(x),$$

where $g(0) = 0,\ g'(0) > 0,\ g(-x) = -g(x)$, has the cyclicity $[\frac{n+m-1}{2}]$ at the origin. The result implies $H_D^* = \widehat{H}_D = [\frac{n+m-1}{2}]$ with $D = R^{n+m} = \{(a_1, \ldots, a_n, b_1, \ldots, b_m)\}$.

Then, a question arises naturally: Is it true $H_D^* = \widehat{H}_D$ for any system (1.1)? We will give an example in Section 3 to show that the answer is negative, see Proposition 3.1.

Next, consider a polynomial system of the form

$$\dot{x} = y, \quad \dot{y} = -g_m(x) - f_n(x)y, \tag{1.3}$$

where f_n and g_m are polynomials in x of degrees n and m, respectively. Taking all coefficients of f_n and g_m as parameters, one can define two numbers $H_{n,m}^*$ and $\widehat{H}_{n,m}$ for (1.3) as before, see [3]. In other words, $H_{n,m}^*$ is the maximal order of the origin as a focus of (1.3), and $\widehat{H}_{n,m}$ is the cyclicity of (1.3) at the origin for all possible f_n and g_m.

We have $H_{n,1}^* = \widehat{H}_{n,1} = [\frac{n}{2}]$ and $H_{n,2}^* = \widehat{H}_{n,2} = [\frac{2n+1}{3}]$ from the discussion after Theorem 1.1. From the works of [1–4, 7, 9, 10, 12], we can find the values of $H_{n,m}^*$ for many n and m, which suggest $H_{n,m}^* = \widehat{H}_{n,m}$. As suggested in [3, p. 1101] it may hold that $H_{n,m}^* \geqslant \widehat{H}_{n,m}$. However, from Proposition 3.1 and the discussion to (3.7) and (3.8) it must be nontrivial to prove either $H_{n,m}^* \leqslant \widehat{H}_{n,m}$ or $H_{n,m}^* \geqslant \widehat{H}_{n,m}$ for general system (1.3). That is, it is an open problem to prove that any one of the two inequalities is true. From the discussion in [3], one can see that

$$H_{n,3}^* = 2\left[\frac{3n+2}{8}\right] \leqslant \widehat{H}_{n,3} \quad \text{for} \quad 1 \leqslant n \leqslant 50,$$
$$H_{3,m}^* = 2\left[\frac{3m+2}{8}\right] \leqslant \widehat{H}_{3,m} \quad \text{for} \quad 1 \leqslant m \leqslant 50.$$

In Llibre et al. [8], the authors considered a system of the form

$$\dot{x} = y, \quad \dot{y} = -x - \varepsilon g_m(x) - \varepsilon f_n(x)y. \tag{1.4}$$

A number was introduced in [8] for (1.4) which is the maximal number of limit cycles bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ for all possible f_n and g_m. We denote the number by $\widetilde{H}_{n,m}^{(1)}$. By using the averaging theory of order 3, Llibre et al. [8] obtained

$$\widetilde{H}_{n,m}^{(1)} \geqslant \left[\frac{n+m-1}{2}\right].$$

The proof of the above inequality is very technical and complicated.

To our knowledge, we do not find other results for arbitrary m and n. As was shown in [5–7, 11], if the function F or g in (1.1) has a particular form, then one can use Theorems 1.1 and 1.2 to find a sharp estimate of the number of limit cycles in Hopf bifurcations. However, if both of the functions are polynomials of arbitrary degrees, the theorems are very hard to be used to find a sharp estimate. To overcome this difficulty, in this paper, we first introduce a small parameter in (1.1) and establish a general theorem, and then apply it to give a new lower bound of the maximal number of limit cycles for arbitrary m and n. More precisely, we first consider a system of the form with small parameter λ,

$$\dot{x} = y, \quad \dot{y} = -g_0(x) - \lambda g_1(x) - \lambda^2 g_2(x) - [f_0(x) + \lambda f_1(x) + \lambda^2 f_2(x)]y, \tag{1.5}$$

where f_j and g_j are C^∞ functions with

$$g_j(0) = 0, \quad j = 0, 1, 2, \quad g_0'(0) > 0.$$

We will discuss the number of limit cycles in Hopf bifurcation.

Let

$$F(x,\lambda)=F_0(x)+\lambda F_1(x)+\lambda^2F_2(x), \tag{1.3}$$
$$G(x,\lambda)=G_0(x)+\lambda G_1(x)+\lambda^2G_2(x), \tag{1.4}$$

where

$$F_j(x)=\int_0^x f_j(x)dx,\ G_j(x)=\int_0^x g_j(x)dx,\quad j=0,1,2.$$

The following lemma is fundamental.

Lemma 1.3. *Let*

$$\Phi(x,\lambda)=F(\alpha(x,\lambda),\lambda)-F(x,\lambda),$$

where $\alpha(x,\lambda)=-x+O(x^2)$ *satisfies* $G(\alpha(x,\lambda),\lambda)=G(x,\lambda)$ *for* (x,λ) *near* $(0,0)$. *Then*

$$\alpha(x,\lambda)=\alpha_0(x)+\lambda\alpha_1(x)+\lambda^2\alpha_2(x)+\cdots,$$

and

$$\Phi(x,\lambda)=\Phi_0(x)+\lambda\Phi_1(x)+\lambda^2\Phi_2(x)+\cdots,$$

where

$$\begin{aligned}&G_0(\alpha_0(x))=G_0(x),\quad \alpha_1(x)=\frac{G_1(x)-G_1(\alpha_0(x))}{g_0(\alpha_0(x))},\\&\alpha_2(x)=\frac{1}{g_0(\alpha_0(x))}\left[G_2(x)-G_2(\alpha_0(x))-g_1(\alpha_0(x))\alpha_1(x)-\frac{1}{2}g_0'(\alpha_0(x))\alpha_1^2(x)\right],\\&\Phi_0(x)=F_0(\alpha_0(x))-F_0(x),\quad \Phi_1(x)=F_1(\alpha_0(x))-F_1(x)+f_0(\alpha_0(x))\alpha_1(x),\end{aligned}$$

and

$$\Phi_2(x)=F_2(\alpha_0(x))-F_2(x)+f_0(\alpha_0(x))\alpha_2(x)+f_1(\alpha_0(x))\alpha_1(x)+\frac{1}{2}f_0'(\alpha_0(x))\alpha_1^2(x).$$

In particular, if $\Phi_0(x)\equiv 0$, *then*

$$\Phi_1(x)=\frac{1}{g_0(x)}\left\{g_0(x)[F_1(\alpha_0(x))-F_1(x)]-f_0(x)[G_1(\alpha_0(x))-G_1(x)]\right\}. \tag{1.6}$$

Now suppose the functions f_j and g_j in (1.5) depend on a vector parameter $\delta\in\mathbb{R}^m$. Then the function $\Phi=\Phi(x,\lambda,\delta)$ is a function of (x,λ,δ). In this case, the main results can be stated as follows.

Theorem 1.4. *Let for* $k=1$ *or* $k=2$

$$\begin{aligned}&\Phi(x,\lambda,\delta)=\lambda^k\varphi_0(x,\delta)\tilde{\Phi}_k(x,\delta)+O(\lambda^{k+1}),\\&\tilde{\Phi}_k(x,\delta)=\sum_{j\geqslant 1}B_j^*(\delta)x^j,\quad \varphi_0(0,\delta)\neq 0.\end{aligned}$$

Suppose there exists $\delta_0\in\mathbb{R}^m$ *such that*

$$B_{2j+1}^*(\delta_0)=0,\quad j=0,1,\ldots,m-1,$$

and

$$\det\frac{\partial(B_1^*,B_3^*,\ldots,B_{2m-1}^*)}{\partial(\delta_1,\delta_2,\ldots,\delta_m)}(\delta_0)\neq 0.$$

Then

(i) *If* $B_{2m+1}^*(\delta_0)\neq 0$, (1.5) *has at least* m *limit cycles, each having an odd multiplicity, in an arbitrary neighborhood of the origin for some* (λ,δ) *sufficiently closed to* $(0,\delta_0)$.

(ii) *If* $\Phi(x,\lambda,\delta)\equiv 0$ *as* $B_{2j+1}^*=0, j=0,1,\ldots,m-1$, *then there exist a constant* $\varepsilon>0$ *and a neighborhood* U *of the origin such that for all* $|\lambda|<\varepsilon$, $|\delta-\delta_0|<\varepsilon$, (1.5) *has at most* $m-1$ *limit cycles in* U. *Moreover,* $m-1$ *limit cycles can appear in an arbitrary neighborhood of the origin for some* (λ,δ) *near* $(0,\delta_0)$.

We remark that if $\delta \in \mathbb{R}^{m+1}$ and

$$B^*_{2j+1}(\delta_0) = 0, \quad j = 0, 1, \ldots, m,$$
$$\det \frac{\partial(B^*_1, B^*_3, \ldots, B^*_{2m+1})}{\partial(\delta_1, \delta_2, \ldots, \delta_{m+1})}(\delta_0) \neq 0$$

for some δ_0, then by the first conclusion of the above theorem it is obvious that (1.5) has at least m limit cycles in an arbitrary neighborhood of the origin for some (λ, δ) sufficiently closed to $(0, \delta_0)$.

To show that the above theorem is a nice development of Theorem 1.1, we apply it to the polynomial system

$$\dot{x} = y, \quad \dot{y} = -\bar{g}_k(x) - \varepsilon g_m(x) - \varepsilon f_n(x) y, \tag{1.7}$$

where ε is small and $\bar{g}_k$ is a polynomial of degree k satisfying $\bar{g}_k(0) = 0$ and $\bar{g}'_k(0) > 0$. We can obtain the following theorem.

Theorem 1.5. *Let $\widehat{H}^{(k)}_{n,m}$ denote the maximal number of limit cycles near the origin of* (1.7) *for all possible f_n and g_m. Then for $m, n \geqslant 1$*

$$\widehat{H}^{(1)}_{n,m} \geqslant \left[\frac{n+m-1}{2}\right], \quad \widehat{H}^{(2)}_{n,m} \geqslant \max\left\{\left[\frac{m-2}{3}\right] + \left[\frac{2n+1}{3}\right], \left[\frac{n-2}{3}\right] + \left[\frac{2m+1}{3}\right]\right\}.$$

Corollary 1.6. *For* (1.4) *we have* $\widehat{H}_{n,m} \geqslant \max\{[\frac{m-2}{3}] + [\frac{2n+1}{3}], [\frac{n-2}{3}] + [\frac{2m+1}{3}]\}$, *where* $m \geqslant 2$, $n \geqslant 1$.

To our knowledge, the result in the above corollary is the best lower bound for the maximal number of limit cycles in Hopf bifurcations for arbitrary m and n.

It is not hard to prove that

$$h_{m,n} \equiv \max\left\{\left[\frac{m-2}{3}\right] + \left[\frac{2n+1}{3}\right], \left[\frac{n-2}{3}\right] + \left[\frac{2m+1}{3}\right]\right\} \geqslant \left[\frac{n+m-1}{2}\right]. \tag{1.8}$$

See Proposition 3.2. Table 1 gives the values of the number pair $(h_{m,n}, [\frac{n+m-1}{2}])$ for $1 \leqslant n \leqslant 9$ and $1 \leqslant m \leqslant 13$ to show the difference of the two numbers.

In Section 2, we give a proof of Lemma 1.3 and Theorem 1.4. In Section 3, we present an application of Theorem 1.4, proving Theorem 1.5.

Table 1 Some values of the pair $(h_{m,n}, [\frac{n+m-1}{2}])$

$m \diagdown n$	1	2	3	4	5	6	7	8	9
1	(0, 0)	(1, 1)	(1, 1)	(2, 2)	(2, 2)	(3, 3)	(4, 3)	(4, 4)	(5, 4)
2	(1, 1)	(1, 1)	(2, 2)	(3, 2)	(3, 3)	(4, 3)	(5, 4)	(5, 4)	(6, 5)
3	(1, 1)	(2, 2)	(2, 2)	(3, 3)	(3, 3)	(4, 4)	(5, 4)	(5, 5)	(6, 5)
4	(2, 2)	(3, 2)	(3, 3)	(3, 3)	(4, 4)	(4, 4)	(5, 5)	(5, 5)	(6, 6)
5	(2, 2)	(3, 3)	(3, 3)	(4, 4)	(4, 4)	(5, 5)	(6, 5)	(6, 6)	(7, 6)
6	(3, 3)	(4, 3)	(4, 4)	(4, 4)	(5, 5)	(5, 5)	(6, 6)	(6, 6)	(7, 7)
7	(4, 3)	(5, 4)	(5, 4)	(5, 5)	(6, 5)	(6, 6)	(6, 6)	(7, 7)	(7, 7)
8	(4, 4)	(5, 4)	(5, 5)	(5, 5)	(6, 6)	(6, 6)	(7, 7)	(7, 7)	(8, 8)
9	(5, 4)	(6, 5)	(6, 5)	(6, 6)	(7, 6)	(7, 7)	(7, 7)	(8, 8)	(8, 8)
10	(6, 5)	(7, 5)	(7, 6)	(7, 6)	(8, 7)	(8, 7)	(8, 8)	(9, 8)	(9, 9)
11	(6, 5)	(7, 6)	(7, 6)	(7, 7)	(8, 7)	(8, 8)	(8, 8)	(9, 9)	(9, 9)
12	(7, 6)	(8, 6)	(8, 7)	(8, 7)	(9, 8)	(9, 8)	(9, 9)	(10, 9)	(10, 10)
13	(8, 6)	(9, 7)	(9, 7)	(9, 8)	(10, 8)	(10, 9)	(10, 9)	(11, 10)	(11, 10)

2 Proofs of Lemma 1.3 and Theorem 1.4

Consider an equivalent form of (1.5) as follows:

$$\dot{x} = y - F(x,\lambda), \quad \dot{y} = -g_0(x) - \lambda g_1(x) - \lambda^2 g_2(x) \equiv -g(x,\lambda). \tag{2.1}$$

Lemma 2.1. *There exists a unique function $y = \alpha(x,\lambda)$ with $\alpha(0,0) = 0$ and $\frac{\partial\alpha}{\partial x}(0,\lambda) = -1$ such that $G(\alpha(x,\lambda),\lambda) = G(x,\lambda)$. Moreover,*

$$\alpha(x,\lambda) = \alpha_0(x) + \lambda\alpha_1(x) + \lambda^2\alpha_2(x) + \cdots,$$

where

$$G_0(\alpha_0(x)) = G_0(x), \quad \alpha_1(x) = \frac{G_1(x) - G_1(\alpha_0(x))}{g_0(\alpha_0(x))},$$
$$\alpha_2(x) = \frac{1}{g_0(\alpha_0(x))}\left[G_2(x) - G_2(\alpha_0(x)) - g_1(\alpha_0(x))\alpha_1(x) - \frac{1}{2}g_0'(\alpha_0(x))\alpha_1^2(x)\right].$$

Proof. Let

$$H(x,y,\lambda) = \begin{cases} \dfrac{G(y,\lambda) - G(x,\lambda)}{y-x}, & y \neq x, \\ g(x,\lambda), & y = x. \end{cases}$$

Then H is of C^∞ and satisfies

$$H(x,y,\lambda) = \sum_{j=0}^{2} \frac{G_j(y) - G_j(x)}{y-x}\lambda^j.$$

Since $G_j(x) = \frac{1}{2}g_j'(0)x^2 + O(|x|^3)$, we have

$$G_j(y) - G_j(x) = (y-x)\left[\frac{1}{2}g_j'(0)(y+x) + O(|x,y|^2)\right].$$

Hence, the first part follows from the implicit function theorem.

Then twice differentiating both sides of the quality $G(\alpha(x,\lambda),\lambda) = G(x,\lambda)$ in λ yields

$$G_x(\alpha,\lambda)\alpha_\lambda + G_\lambda(\alpha,\lambda) = G_\lambda(x,\lambda),$$

and

$$G_{xx}(\alpha,\lambda)(\alpha_\lambda)^2 + G_x(\alpha,\lambda)\alpha_{\lambda\lambda} + 2G_{\lambda x}(\alpha,\lambda)\alpha_\lambda + G_{\lambda\lambda}(\alpha,\lambda) = G_{\lambda\lambda}(x,\lambda).$$

Thus, taking $\lambda = 0$ in the above two formulas gives the formula of α_1 and α_2. This finishes the proof. □

Lemma 2.2. *Let*

$$\Phi(x,\lambda) = F(\alpha(x,\lambda),\lambda) - F(x,\lambda) = \Phi_0(x) + \lambda\Phi_1(x) + \lambda^2\Phi_2(x) + \cdots.$$

Then

$$\Phi_0(x) = F_0(\alpha_0(x)) - F_0(x), \quad \Phi_1(x) = F_1(\alpha_0(x)) - F_1(x) + f_0(\alpha_0(x))\alpha_1(x), \tag{2.2}$$

$$\Phi_2(x) = F_2(\alpha_0(x)) - F_2(x) + f_0(\alpha_0(x))\alpha_2(x) + f_1(\alpha_0(x))\alpha_1(x) + \frac{1}{2}f_0'(\alpha_0(x))\alpha_1^2(x). \tag{2.3}$$

In particular, if $\Phi_0(x) \equiv 0$, then (1.6) holds.

Proof. We have

$$F(\alpha,\lambda) = F_0(\alpha) + \lambda F_1(\alpha) + \lambda^2 F_2(\alpha),$$

$$F_0(\alpha) = F_0(\alpha_0) + \lambda f_0(\alpha_0)\alpha_1 + \lambda^2\left[f_0(\alpha_0)\alpha_2 + \frac{1}{2}f_0'(\alpha_0)\alpha_1^2\right] + O(\lambda^3),$$

and

$$F_1(\alpha) = F_1(\alpha_0) + \lambda f_1(\alpha_0)\alpha_1 + O(\lambda^2).$$

Then the formula of Φ_0, Φ_1 and Φ_2 follows immediately.

When $\Phi_0(x) \equiv 0$, we have

$$f_0(\alpha_0)\alpha_0' = f_0(x), \quad g_0(\alpha_0)\alpha_0' = g_0(x),$$

which implies

$$\frac{f_0(\alpha_0)}{g_0(\alpha_0)} = \frac{f_0(x)}{g_0(x)}.$$

Hence, (1.6) follows from the formula of α_1 and Φ_1. The proof is completed. □

Let $(x(t), y(t))$ be the solution to (2.1) with the initial value $x(0) = 0,\ y(0) = r_0$. Then there exists a first return time $\tau_0(r_0, \lambda) \in \mathbb{C}^\infty$ such that $x(\tau_0) = 0$. Define $d(r_0, \lambda) = y(\tau_0) - r_0$. It is obvious that the function d is $\mathbb{C}^\infty$ for (r_0, λ) near $(0, 0)$ and system (2.1) has a periodic orbit near the origin if and only if the function has two zeros in r_0 (with one positive and the other negative) near $r_0 = 0$. The function d is called a displacement function or a bifurcation function.

From [5] we have the following lemma.

Lemma 2.3. *Suppose formally*

$$F(\alpha(x,\lambda),\lambda) - F(x,\lambda) = \sum_{i\geqslant 1} B_i(\lambda)x^i, \tag{2.4}$$

and

$$d(r_0,\lambda) = \sum_{i\geqslant 1} d_i(\lambda)r_0^i.$$

Then

$$\begin{aligned}
&d_1 = B_1 N_0^*(B_1),\\
&d_{2j} = O(|B_1, B_3, \dots, B_{2j-1}|),\\
&d_{2j+1} = B_{2j+1}N_j^*(B_1) + O(|B_1, B_3, \dots, B_{2j-1}|),
\end{aligned}$$

with $N_j^* \in C^\infty$ and $N_j^*(0) > 0$ for $j \geqslant 0$. Moreover, we have

$$F(\alpha(x,\lambda),\lambda) - F(x,\lambda) = \sum_{i\geqslant 0} A_{2i+1}(\lambda)u^{2i+1}, \tag{2.5}$$

where $u = (\operatorname{sgn} x)\sqrt{G(x,\lambda)}$,

$$\begin{aligned}
&A_1 = (2/g_x(0,\lambda))^{\frac{1}{2}}\,B_1,\\
&A_{2k+1} = (2/g_x(0,\lambda))^{k+\frac{1}{2}}B_{2k+1} + O(|B_1, B_3, \dots, B_{2k-1}|), \quad k \geqslant 1.
\end{aligned}$$

Using the above lemma it is easy to prove the following lemma.

Lemma 2.4. *Suppose formally*

$$F(\alpha(x,\lambda),\lambda) - F(x,\lambda) = \lambda^k\varphi_0(x)\sum_{i\geqslant 1}\tilde{B}_i(\lambda)x^i, \tag{2.6}$$

where $\varphi_0 \in C^\infty$, $\varphi_0(0) \neq 0$. Then

$$d(r_0,\lambda) = \lambda^k\sum_{i\geqslant 1}\tilde{d}_i(\lambda)r_0^i,$$

where

$$\tilde{d}_1 = \tilde{B}_1(\beta_0 + O(\lambda^k)),$$

$$\tilde{d}_{2j} = O(|\tilde{B}_1, \tilde{B}_3, \ldots, \tilde{B}_{2j-1}|),$$

$$\tilde{d}_{2j+1} = \tilde{B}_{2j+1}(\beta_j + O(\lambda^k)) + O(|\tilde{B}_1, \tilde{B}_3, \ldots, \tilde{B}_{2j-1}|),$$

with $\beta_0, \ \beta_1, \ldots$ *being all nonzero constants.*

Now, we are in a position to prove Lemma 1.3 and Theorem 1.4. It is clear that Lemma 1.3 is direct from Lemmas 2.1 and 2.2. Then let us prove Theorem 1.4. Suppose all conditions of Theorem 1.4 are satisfied. Then

$$\Phi(x, \lambda, \delta) = \lambda^k \varphi_0(x, \delta) \sum_{i \geqslant 1} \tilde{B}_i(\lambda, \delta) x^i,$$

where

$$\tilde{B}_i(\lambda, \delta) = B_i^*(\delta) + O(\lambda), \quad j \geqslant 1.$$

Consider the change of parameters

$$b_j = \tilde{B}_{2j-1}(\lambda, \delta), \quad j = 1, 2, \ldots, m.$$

By our assumptions, we can solve from the above equations

$$\delta = \psi(\lambda, b) = \delta_0 + O(|\lambda, b|),$$

where $b = (b_1, b_2, \ldots, b_m)$. Then by Lemma 2.4 the succession function d has the form

$$d(r_0, \lambda, \delta) = \lambda^k \sum_{i \geqslant 1} \tilde{d}_i(\lambda, \delta) r_0^i \equiv \bar{d}(r_0, \lambda, b),$$

where

$$\tilde{d}_{2j-1} = b_j(\beta_{j-1} + O(|\lambda^k, b|)) + O(|b_1, b_2, \ldots, b_{j-1}|),$$

$$\tilde{d}_{2j} = O(|b_1, b_2, \ldots, b_j|), \quad j = 1, 2, \ldots, m.$$

Therefore, we can rewrite $\bar{d}$ as

$$\bar{d}(r_0, \lambda, b) = \lambda^k \left[\sum_{j=1}^{m} r_0^{2j-1} b_j P_j(r_0, \lambda, b) + r_0^{2m+1} P_{m+1}(r_0, \lambda, b) \right], \tag{2.7}$$

where $P_1, \ldots, P_m$ are polynomials in r_0 of degree at most $2m - 1$, $P_{m+1} \in C^\infty$, $P_j(0, 0, 0) = \beta_{j-1} \neq 0$ for $j = 1, 2, \ldots, m$, and $P_{m+1}(0, 0, 0) = B_{2m+1}^*(\delta_0)\beta_m$. For definiteness, we may suppose $\varphi_0(0, \delta_0) > 0$, which yields $\beta_j > 0$ for all $j \geqslant 1$.

If $B_{2m+1}^*(\delta_0) \neq 0$, we can change $b_m, b_{m-1}, \ldots, b_1$ in turn such that

$$0 \ll |b_j| \ll |b_{j+1}| \ll 1, \quad b_j b_{j+1} < 0, \quad b_m B_{2m+1}^*(\delta_0) < 0.$$

Then by the form (2.7), the function $\bar{d}$ has at least m positive zeros in r_0 near $r_0 = 0$ in this case.

If $\Phi(x, \lambda, \delta) \equiv 0$ as $B_{2j+1}^* = 0, \ j = 0, 1, \ldots, m - 1$, then

$$\tilde{B}_{2j-1}(\lambda, \delta) = B_{2j-1}^*(\delta) + O(\lambda) = O(B_{2j-1}^*(\delta)), \quad j = 1, 2, \ldots, m,$$

which implies

$$B_{2j-1}^*(\delta) = 0, \quad j = 1, 2, \ldots, \ m \Leftrightarrow b = 0.$$

Thus, by Lemma 2.4, in this case we have $\bar{d}(r_0, \lambda, 0) = 0$. Hence, the function P_{m+1} in (2.7) satisfies $P_{m+1}(r_0, \lambda, 0) = 0$. It follows that

$$\bar{d}(r_0, \lambda, b) = \lambda^k \sum_{j=1}^{m} r_0^{2j-1} b_j \bar{P}_j(r_0, \lambda, b),$$

where $\bar{P}_j = P_j + O(r_0^{2(m-j)+2})$. Then using the above, we can prove that $\bar{d}$ has at most $m - 1$ positive zeros in r_0 near $r_0 = 0$ for all small $|b|$, and $m - 1$ positive zeros can appear for some b.

Then the conclusion of Theorem 1.4 follows.

3 Proof of Theorem 1.5 and some propositions

In this section, we consider a system of Liénard type with the form

$$\dot{x} = y, \quad \dot{y} = -g_m(x, \lambda) - f_n(x, \lambda)y, \tag{3.1}$$

where g_m and f_n are polynomials in x of degree m and n, respectively, and have the form

$$g_m(x, \lambda) = g_0(x) + \lambda g_1(x), \quad f_n(x, \lambda) = f_0(x) + \lambda f_1(x). \tag{3.2}$$

First, we take

$$\begin{aligned} g_0(x) &= \sum_{j=0}^{m_1} a_{0j}x^{2j+1}, \quad 2m_1+1 \leqslant m, \quad g_1(x) = \sum_{j=1}^{m} a_{1j}x^j, \\ f_0(x) &= \sum_{j=0}^{n_1} b_{0j}x^{2j+1}, \quad 2n_1+1 \leqslant n, \quad f_1(x) = \sum_{j=0}^{n} b_{1j}x^j, \end{aligned} \tag{3.3}$$

where $a_{00} > 0$.

In this case, we have by (1.6)

$$\Phi_1(x) = \frac{1}{g_0^*(x)}\tilde{\Phi}_1(x),$$

where

$$\begin{aligned} \tilde{\Phi}_1(x) &= g_0^*(x)[F_1(-x) - F_1(x)] - f_0^*(x)[G_1(-x) - G_1(x)], \\ g_0^*(x) &= \frac{g_0(x)}{x} = \sum_{j=0}^{m_1} a_{0j}x^{2j}, \quad f_0^*(x) = \frac{f_0(x)}{x} = \sum_{j=0}^{n_1} b_{0j}x^{2j}, \\ F_1(-x) - F_1(x) &= \sum_{j=0}^{n_2} \bar{b}_j x^{2j+1}, \quad n_2 = [n/2], \quad \bar{b}_j = -\frac{b_{1,2j}}{2j+1}, \end{aligned}$$

and

$$G_1(-x) - G_1(x) = \sum_{j=1}^{m_2} \bar{a}_j x^{2j+1}, \quad m_2 = [m/2], \quad \bar{a}_j = -\frac{a_{1,2j}}{2j+1}.$$

Let $r = x^2$, then it is easy to see that

$$\tilde{\Phi}_1(x) = x\left[\sum_{j=0}^{m_1} a_{0j}r^j \sum_{j=0}^{n_2} \bar{b}_j r^j - \sum_{j=0}^{n_1} b_{0j}r^j \sum_{j=1}^{m_2} \bar{a}_j r^j\right] \equiv xS_{\bar{M}}(r),$$

where

$$\begin{aligned} S_{\bar{M}}(r) &= \sum_{k=0}^{\bar{M}} B_{k+1}r^k, \quad \bar{M} = \max\{m_1+n_2, n_1+m_2\}, \\ B_{k+1} &= \sum_{\substack{i+j=k \\ 0\leqslant i\leqslant m_1 \\ 0\leqslant j\leqslant n_2}} a_{0i}\bar{b}_j - \sum_{\substack{i+j=k \\ 0\leqslant i\leqslant n_1 \\ 1\leqslant j\leqslant m_2}} b_{0i}\bar{a}_j. \end{aligned}$$

Let first m be even. In this case, we take $m_1 = 0$, $n_1 = [\frac{n-1}{2}]$ so that

$$m_1 + n_2 \leqslant \left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right] \leqslant \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right] = \bar{M}.$$

We further take $a_{00} = 1$, $b_{0n_1} \neq 0$, $b_{0i} = 0$ for $0 \leqslant i \leqslant n_1 - 1$, and set

$$\mu = b_{0n_1}, \quad \delta = (\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{n_2}, \bar{a}_{n_2-n_1+1}, \ldots, \bar{a}_{m_2}) \in \mathbb{R}^{\bar{M}+1}.$$

Then

$$B_{k+1} = \begin{cases} \bar{b}_k, & 0 \leqslant k \leqslant n_1, \\ \bar{b}_k - b_{0n_1}\bar{a}_{n_2-n_1}, & k = n_2 \text{ (for } n \text{ even)}, \\ b_{0n_1}\bar{a}_{k-n_1}, & n_2+1 \leqslant k \leqslant \bar{M}. \end{cases}$$

Evidently,

$$B_{k+1} = 0, \quad k = 0, 1, \dots, \ \bar{M} \Leftrightarrow \delta = 0;$$
$$\det \frac{\partial(B_1, \dots, B_{\bar{M}+1})}{\partial \delta}\bigg|_{\delta=0} = (-b_{0n_1})^{\bar{M}-n_2}.$$

Now let m be odd. Then take $n_1 = 0$, $m_1 = [\frac{m-1}{2}]$ so that

$$n_1 + m_2 \leqslant \left[\frac{n-1}{2}\right] + \left[\frac{m}{2}\right] \leqslant \left[\frac{m-1}{2}\right] + \left[\frac{n}{2}\right] = \bar{M}.$$

Further take $a_{00} = 1$, $a_{0m_1} \neq 0$, $b_{00} \neq 0$, $a_{0i} = 0$ for $0 \leqslant i \leqslant m_1 - 1$, and set

$$\mu = (a_{0m_1}, b_{00}), \quad \delta = (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_{m_2}, \bar{b}_0, \dots, \bar{b}_{n_2}) \in \mathbb{R}^{\bar{M}+1}.$$

Then noting $m_1 = m_2$, we have

$$B_{k+1} = \begin{cases} \bar{b}_0, & k = 0, \\ \bar{b}_k - b_{00}\bar{a}_k, & 1 \leqslant k \leqslant m_2 - 1, \\ \bar{b}_k + a_{0m_1}\bar{b}_0 - b_{00}\bar{a}_k, & k = m_2, \\ \bar{b}_k + a_{0m_1}\bar{b}_{k-m_1}, & m_2 + 1 \leqslant k \leqslant \bar{M}. \end{cases}$$

As above

$$B_{k+1} = 0, \quad k = 0, 1, \dots, \ \bar{M} \Leftrightarrow \delta = 0;$$
$$\det \frac{\partial(B_1, \dots, B_{\bar{M}+1})}{\partial \delta}\bigg|_{\delta=0} \neq 0.$$

Then by Theorem 1.4(i) under (3.3) for all $0 < |\lambda| \ll |\mu|$ and some δ near 0, the system (3.1) can have at least $\bar{M}$ limit cycles near the origin.

Note that $\bar{M} = [\frac{m+n-1}{2}]$ and that $|\mu|$ can be taken very small. We have immediately Lemma 3.1.

Lemma 3.1. *For any neighborhood U of the origin there exist $\varepsilon_0 > 0$ and functions g_0, g_1, f_0 and f_1 of the form* (3.3)*, where $a_{00} > 0$, $|a_{0j}| < \varepsilon_0$ for $1 \leqslant j \leqslant m_1$, $|b_{0j}| < \varepsilon_0$ for $0 \leqslant j \leqslant n_1$, $|a_{1j}| < \varepsilon_0$ for $1 \leqslant j \leqslant m$ and $|b_{1j}| < \varepsilon_0$ for $0 \leqslant j \leqslant n$, such that for all $0 < |\lambda| < \varepsilon_0$,* (3.1) *has at least $[\frac{m+n-1}{2}]$ limit cycles in U.*

Next, we take g_0, g_1, f_0 and f_1 in (3.2) to have the form

$$g_0(x) = \bar{g}(x)\sum_{j=0}^{m_1} a_{0j}[\bar{G}(x)]^j, \quad g_1(x) = \sum_{j=1}^{m} a_{1j}x^j,$$
$$f_0(x) = \bar{g}(x)\sum_{j=0}^{n_1} b_{0j}[\bar{G}(x)]^j, \quad f_1(x) = \sum_{j=0}^{n} b_{1j}x^j, \tag{3.4}$$

where

$$a_{00} > 0, \quad \bar{g}(x) = x(1-x), \quad \bar{G}(x) = x^2/2 - x^3/3, \quad 3m_1 + 2 \leqslant m, \quad 3n_1 + 2 \leqslant n.$$

Obviously,

$$G_0(x) = \int_0^x g_0(x)dx = \sum_{j=0}^{m_1} \frac{a_{0j}}{j+1}[\bar{G}(x)]^{j+1},$$

$$F_0(x) = \int_0^x f_0(x)dx = \sum_{j=0}^{n_1} \frac{b_{0j}}{j+1}[\bar{G}(x)]^{j+1},$$

which satisfy

$$G_0(\alpha_0(x)) = G_0(x), \quad F_0(\alpha_0(x)) = F_0(x),$$

where $\alpha_0(x) = -x + O(x^2)$ is defined by $\bar{G}(\alpha_0(x)) = \bar{G}(x)$.

By (1.6), we have

$$\Phi_1(x) = \frac{1}{g_0^*(x)}\tilde{\Phi}_1(x),$$

where

$$\tilde{\Phi}_1(x) = g_0^*(x)[F_1(\alpha_0) - F_1(x)] - f_0^*(x)[G_1(\alpha_0) - G_1(x)],$$

$$g_0^*(x) = \frac{g_0(x)}{\bar{g}(x)} = \sum_{j=0}^{m_1} a_{0j}\bar{G}^j, \quad f_0^*(x) = \frac{f_0(x)}{\bar{g}(x)} = \sum_{j=0}^{n_1} b_{0j}\bar{G}^j.$$

By (2.5), we have

$$F_1(\alpha_0) - F_1(x) = \sum_{j\geqslant 0} \alpha_j u^{2j+1},$$

$$G_1(\alpha_0) - G_1(x) = \sum_{j\geqslant 1} \beta_j u^{2j+1},$$

where $u = (\mathrm{sgn} x)\sqrt{\bar{G}(x)}$. By [5, Theorem 3 and Proof of Theorem 5] and Lemma 2.3 (or [5, Lemma 1]), we know that $\alpha_0, \alpha_1, \ldots, \alpha_{n_3}$ and $\beta_1, \beta_2, \ldots, \beta_{m_3}$ can be taken as free parameters, and

$$\alpha_j = O(|\alpha_0, \alpha_1, \ldots, \alpha_{n_3}|), \quad j \geqslant n_3 + 1,$$

$$\beta_j = O(|\beta_1, \beta_2, \ldots, \beta_{m_3}|), \quad j \geqslant m_3 + 1,$$

where $m_3 = [\frac{2m+1}{3}]$, $n_3 = [\frac{2n+1}{3}]$. Therefore,

$$\tilde{\Phi}_1(x) = u\left[\sum_{j=0}^{m_1} a_{0j}u^{2j}\sum_{j\geqslant 0}\alpha_j u^{2j} - \sum_{j=0}^{n_1} b_{0j}u^{2j}\sum_{j\geqslant 1}\beta_j u^{2j}\right] \equiv uS(u),$$

where

$$S(u) = \sum_{k\geqslant 0} B_{k+1}u^{2k},$$

$$B_{k+1} = \sum_{\substack{i+j=k\\0\leqslant i\leqslant m_1\\0\leqslant j}} a_{0i}\alpha_j - \sum_{\substack{i+j=k\\0\leqslant i\leqslant n_1\\1\leqslant j}} b_{0i}\beta_j$$

Let $M = \max\{m_1 + n_3,\ n_1 + m_3\}$, where $m_1 = [\frac{m-2}{3}]$, $n_1 = [\frac{n-2}{3}]$. Then for $m_1 \geqslant 0$ or $n_1 \geqslant 0$ it is easy to see, as before, that $B_0, B_1, \ldots, B_M$ can be taken as free parameters, which implies the following lemma.

Lemma 3.2. *Let $m, n \geqslant 1$ and $\max\{m, n\} \geqslant 2$. Then for any neighborhood U of the origin there exist $\varepsilon_0 > 0$ and functions g_0, g_1, f_0 and f_1 of the form* (3.4) *such that for all $0 < |\lambda| < \varepsilon_0$,* (3.1) *has at least M limit cycles in U.*

It is clear that Theorem 1.5 follows from Lemmas 3.1 and 3.2.

In the rest, we give two propositions mentioned in the first section.

Proposition 3.3. *For H_D^* and $\widehat{H}_D$ introduced in Section* 1*, each of the cases $H_D^* < \widehat{H}_D$ and $H_D^* > \widehat{H}_D$ can occur in certain examples.*

Proof. We need only to present an example. Let us consider a special system of degree 5 of the form

$$\dot{x} = y - (a_1x + a_2x^3 + a_3x^5), \quad \dot{y} = -x. \tag{3.5}$$

For the system we have

$$B_1 = -2a_1, \quad B_3 = -2a_2, \quad B_5 = -2a_3, \quad B_{2j+1} = 0 \quad \text{for} \quad j \geqslant 3.$$

From [5], the Poincaré map satisfies

$$P(r, a) - r = r[B_1P_1(r, a) + B_3r^2P_2(r, a) + B_5r^4P_3(r, a)], \tag{3.6}$$

where $P_j(r, a) = N^*_{j-1}(B_1) + \mathrm{O}(r)$ with

$$N^*_{j-1}(0) = 2^{j-3/2}\int_0^{2\pi} \cos^{2j}\theta d\theta, \quad j = 1, 2, 3.$$

In fact, we can find

$$N^*_0(0) = \frac{\pi}{\sqrt{2}}, \quad N^*_1(0) = \frac{3\pi}{4\sqrt{2}}, \quad N^*_2(0) = \frac{5\pi}{8\sqrt{2}}.$$

By Theorem 1.1 or using (3.6) one has for (3.5) $H^*_D = \widehat{H}_D = 2$ with $a = (a_1, a_2, a_3) \in D = \mathbb{R}^3$. □

Now we consider two subclasses of (3.5) as follows

$$\dot{x} = y - (a_2x^3 + a_3x^5), \quad \dot{y} = -x \tag{3.7}$$

and

$$\dot{x} = y - (a_1x - a_3^2x^3 + a_3x^5), \quad \dot{y} = -x. \tag{3.8}$$

We claim that

(i) for (3.7) with $a = (a_2, a_3) \in D = \mathbb{R}^2$ we have $H^*_D = 2 > \widehat{H}_D = 1$;

(ii) for (3.8) with $a = (a_1, a_3) \in D = \mathbb{R}^2$ we have $H^*_D = 1 < \widehat{H}_D = 2$.

In fact, for (3.7), (3.6) becomes

$$P(r, a) - r = r^3[B_3P_2(r, a) + B_5r^2P_3(r, a)],$$

which has at most one positive zero in r, giving $\widehat{H}_D = 1$. By Theorem 1.1 above it is clear that $H^*_D = 2$. For (3.8) it is direct by Theorem 1.1 that $H^*_D = 1$. Since (3.8) is a special form of (3.5), it implies that $\widehat{H}_D \leqslant 2$. Now we use (3.6) to prove $\widehat{H}_D \geqslant 2$. First, set $a_1 = 0$. Then noting $B_3 = 2a_3^2$ and $B_5 = -2a_3$ for (3.8) we have from (3.6)

$$P(r, a) - r = r^3[2a_3^2P_2(r, a) - 2a_3r^2P_3(r, a)] = 2a_3r^3[a_3P_2(r, a) - r^2P_3(r, a)],$$

which has a zero of the form

$$r = \sqrt{\frac{N^*_1(0)}{N^*_2(0)}a_3} + \mathrm{O}(a_3) \equiv r_1(a_3)$$

for $0 < a_3 \ll 1$.

In this case, the origin is an unstable focus of order 1 and there exists a stable limit cycle $\Gamma_1(a_3)$ near the origin. Then fix a_3 and change a_1 satisfying $0 < a_1 \ll a_3$ so that a second zero,

$$r = r_2(a_1) = \sqrt{\frac{a_1N^*_0(0)}{a_3^2N^*_1(0)}} + \mathrm{O}(a_1)$$

is produced (the stability of the origin has changed from unstable into stable), while the first zero keeps remain near $r_1(a_3)$. Thus we have that (3.8) has two limit cycles near the origin for $0 < a_1 \ll a_3 \ll 1$, implying $\widehat{H}_D \geqslant 2$. Then the conclusion is shown.

Table 2 The values of $([\frac{m-2}{3}+[\frac{2n+1}{3}, \frac{n-2}{3}+[\frac{2m+1}{3}, [\frac{n+m-1}{2}])$

(i,j)	$([\frac{m-2}{3}]+[\frac{2n+1}{3}], [\frac{n-2}{3}]+[\frac{2m+1}{3}], [\frac{n+m-1}{2}])$
(0,0)	$(l+2k-1, k+2l-1, [\frac{3(k+l)-1}{2}])$
(1,0)	$(l+2k, k+2l-1, [\frac{3(k+l)}{2}])$
(2,0)	$(l+2k, k+2l, [\frac{3(k+l)+1}{2}])$
(0,1)	$(l+2k-1, k+2l, [\frac{3(k+l)}{2}])$
(1,1)	$(l+2k, k+2l, [\frac{3(k+l)+1}{2}])$
(2,1)	$(l+2k, k+2l+1, [\frac{3(k+l)}{2}]+1)$
(0,2)	$(l+2k, k+2l, [\frac{3(k+l)+1}{2}])$
(1,2)	$(l+2k+1, k+2l, [\frac{3(k+l)}{2}]+1)$
(2,2)	$(l+2k+1, k+2l+1, [\frac{3(k+l)+1}{2}]+1)$

Proposition 3.4. *For all $m,n \geqslant 1$, (1.8) holds.*

Proof. Let $n=3k+i$, $m=3l+j$, where $0 \leqslant i,j \leqslant 2$. Then we have Table 2 for the values of the triple $([\frac{m-2}{3}]+[\frac{2n+1}{3}], [\frac{n-2}{3}]+[\frac{2m+1}{3}], [\frac{n+m-1}{2}])$.

From Table 2, it is easy to see that to prove (1.8) we need only to prove the following two inequalities:

$$\max\{l+2k-1, k+2l-1\} \geqslant \left[\frac{3(k+l)-1}{2}\right],$$

and

$$\max\{l+2k, k+2l-1\} \geqslant \left[\frac{3(k+l)}{2}\right].$$

Note that

$$l+2k-1 \geqslant \left[\frac{3(k+l)-1}{2}\right]$$

as $k \geqslant l$,

$$l+2k \geqslant \left[\frac{3(k+l)}{2}\right]$$

as $k \geqslant l-1$, and

$$l+2k-1 \geqslant \left[\frac{3(k+l)}{2}\right]$$

as $k \leqslant l-2$. Then the above two inequalities follow directly.

In particular, for $m=n, n+1$ then (1.8) reduces to

$$h_{m,n} = [(n+m-1)/2],$$

and for $m=n+2, n+3, n+4, n+5, n+6, n+7$, (1.8) reduces to

$$h_{m,n} \geqslant [(n+m-1)/2].$$

For $m=n+8, n+9, n+10$, we have

$$h_{m,n} \geqslant [(n+m-1)/2]+1.$$

And for $m=3n+1, 3n+2$ we have

$$h_{m,n} = \left[\frac{n-2}{3}\right]+2n+1 > 2n = [(n+m-1)/2].$$

□

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