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HOPF BIFURCATIONS FOR NEAR-HAMILTONIAN SYSTEMS*

MAOAN HAN† and JUNMIN YANG
Department of Mathematics, Shanghai Normal University, Shanghai 200234, P. R. China
†*mahan@shnu.edu.cn*

PEI YU
Department of Applied Mathematics, The University of Western Ontario London, Ontario, Canada N6A 5B7

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In this paper, we consider bifurcation of limit cycles in near-Hamiltonian systems. A new method is developed to study the analytical property of the Melnikov function near the origin for such systems. Based on the new method, a computationally efficient algorithm is established to systematically compute the coefficients of Melnikov function. Moreover, we consider the case that the Hamiltonian function of the system depends on parameters, in addition to the coefficients involved in perturbations, which generates more limit cycles in the neighborhood of the origin. The results are applied to a quadratic system with cubic perturbations to show that the system can have five limit cycles in the vicinity of the origin.

Keywords: Hopf cyclicity; Hamiltonian system; parameter; bifurcation; limit cycle; perturbation.

1. Introduction

There have been many studies on the Hopf bifurcations of limit cycles, see [Bautin, 1952; Yu & Han, 2004]. In general, there are two types of such bifurcations leading to limit cycles: either by perturbing a focus point or by perturbing a center. In this paper, we study a C^∞ system of the form

$$\dot{x} = H_y + \varepsilon p(x, y, \delta), \quad \dot{y} = -H_x + \varepsilon q(x, y, \delta), \tag{1}$$

where $H(x, y)$, $p(x, y, \delta)$, $q(x, y, \delta)$ are C^∞ functions, $\varepsilon \geq 0$ is small and $\delta \in D \subset \mathbf{R}^m$ is a vector parameter with D compact.

When $\varepsilon = 0$, system (1) becomes

$$\dot{x} = H_y, \quad \dot{y} = -H_x, \tag{2}$$

which is a Hamiltonian system, and thus (1) is called a near-Hamiltonian system. Now suppose that the Hamiltonian system (2) has an elementary center at the origin, namely the function H satisfies $H_x(0, 0) = H_y(0, 0) = 0$, and

$$\det \frac{\partial(H_y, -H_x)}{\partial(x, y)}(0, 0) > 0.$$

Thus, without loss of generality, we may assume that the expansion of H at the origin can be expanded as

$$H(x, y) = \frac{\omega}{2}(y^2 + x^2) + \sum_{i+j\geq 3} h_{ij} x^i y^j, \quad \omega > 0. \tag{3}$$

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†Author for correspondence

Then, the Hamiltonian system (2) has a family of periodic orbits, given by

$$L_h : H(x, y) = h, \quad h \in (0, \beta)$$

such that L_h approaches the origin as $h \to 0$.

Take $h = h_0 \in (0, \beta)$ and $A(h_0) \in L_{h_0}$. Let l be a cross-section of system (2) passing through $A(h_0)$. Then, for h near h_0 the periodic orbit L_h has a unique intersection point with l, denoted by $A(h)$, i.e. $A(h) = L_h \cap l$. Consider the positive orbit $\gamma(h, \varepsilon, \delta)$ of system (1) starting from $A(h)$. Let $B(h, \varepsilon, \delta)$ denote the first intersection point of the orbit with l. Then, we have

$$\begin{aligned} H(B) - H(A) &= \int_{AB} dH \\ &= \varepsilon[M(h, \delta) + O(\varepsilon)] \\ &= \varepsilon F(h, \varepsilon, \delta), \end{aligned} \tag{4}$$

where

$$\begin{aligned} M(h, \delta) &= \oint_{L_h} (H_y q + H_x p) dt \\ &= \oint_{L_h} (q dx - p dy) \\ &= \iint_{H \le h} (p_x + q_y) dx dy. \end{aligned} \tag{5}$$

The function $F(h, \varepsilon, \delta)$ in (4) is called a bifurcation function of system (1). The resulting map from $A(h)$ to $B(h, \varepsilon, \delta)$ is called a Poincaré map of system (1). Obviously, for small ε system (1) has a limit cycle near the origin if and only if the function $F(h, \varepsilon, \delta)$ has a positive zero in h near $h = 0$.

On the analytical property of the function M and the number of limit cycles near the origin by the function, we have the following theorems (see [Han, 2000] for more details).

Theorem 1. *Let (3) hold. Then $M(h, \delta)$ is C^∞ in $0 \le h \ll 1$ with*

$$M(h, \delta) = h \sum_{l \ge 0} b_l(\delta) h^l \tag{6}$$

formally for $0 \le h \ll 1$. Moreover, if (1) is analytic, so is M.

Theorem 2. *Under the condition of Theorem 1, if there exist $k \ge 1, \delta_0 \in D$ such that $b_k(\delta_0) \ne 0$ and*

$$b_j(\delta_0) = 0, \quad j = 0, 1, \ldots, k-1,$$

$$\det \frac{\partial(b_0, \ldots, b_{k-1})}{\partial(\delta_1, \ldots, \delta_k)}(\delta_0) \ne 0,$$

where $\delta = (\delta_1, \ldots, \delta_m)$, $m \ge k$, then there exist a constant $\varepsilon_0 > 0$ and a neighborhood V of the origin such that for all $0 < |\varepsilon| < \varepsilon_0$ and $|\delta - \delta_0| < \varepsilon_0$ (1) has at most k limit cycles in V. Moreover, for any neighborhood V_1 of the origin there exists (ε, δ) near $(0, \delta_0)$ such that system (1) has k limit cycles in V_1. In other words, system (1) has Hopf cyclicity k for all (ε, δ) near $(0, \delta_0)$.

In light of the above theorems, a key step in studying the Hopf bifurcation problem of system (1) is to find an efficient method to compute the coefficients b_l. The formulas for the first three coefficients $b_j(\delta), j = 0, 1, 2$ were obtained by Hou and Han [2006] by using the double integral in (5). However, such method and computation are hard to be used to establish an algorithm to compute higher-order coefficients b_j. In this paper, we develop a new approach to prove Theorem 1 and then establish a computationally efficient algorithm to systematically compute $b_j(\delta), j = 0, 1, 2, 3, \ldots$. Moreover, we consider the case where the Hamiltonian function H depends on parameters and obtain a generalization of Theorem 2.

This paper is organized as follows. In the next section, we give a new proof for Theorem 1 and further show a generalization of Theorem 2. An efficient algorithm based on the new proof is also developed in this section. Then in Sec. 3 we apply the generalized theorem and the program to discuss the number of limit cycles for a class of cubic systems.

2. Main Results

In this section, we first present a new proof for Theorem 1, and then give a generalization of Theorem 2. Finally, we establish an algorithm based on the new proof and code programs using the computer algebra system — Maple.

2.1. *A new proof of Theorem 1*

To develop a new proof for Theorem 1, we first introduce a change of variables to make the form of the Hamiltonian function simpler. By (3) and using the implicit function theorem, we can show that a unique C^∞ function $\varphi(x)$ exists such that $H_y(x, \varphi(x)) = 0$ for $|x|$ small. Thus, we can write

$$\varphi(x) = \sum_{j \ge 2} e_j x^j. \tag{7}$$

By introducing a new variable $v = y - \varphi(x)$, system (1) can be rewritten as

$$\begin{aligned} \dot{x} &= H_v^*(x, v) + \varepsilon p^*(x, v, \delta), \\ \dot{v} &= -H_x^*(x, v) + \varepsilon q^*(x, v, \delta), \end{aligned} \tag{8}$$

where

$$\begin{aligned} H^*(x, v) &= H(x, v + \varphi(x)), \\ p^*(x, v, \delta) &= p(x, v + \varphi(x), \delta), \\ q^*(x, v, \delta) &= q(x, v + \varphi(x), \delta) \\ &\quad - \varphi'(x) p^*(x, v, \delta). \end{aligned} \tag{9}$$

Further, by (7) and noticing $H_y(x, \varphi(x)) = 0$, we have

$$\begin{aligned} H^*(x, v) &= H_0^*(x) + \sum_{j \geq 1} H_j^*(x) v^{j+1} \\ &\equiv H_0^*(x) + v^2 \tilde{H}(x, v), \end{aligned} \tag{10}$$

where

$$\begin{aligned} H_0^*(x) &= H(x, \varphi(x)) = \sum_{j \geq 2} h_j x^j, \\ H_j^*(x) &= \frac{1}{(j+1)!} \frac{\partial^{j+1} H}{\partial y^{j+1}}(x, \varphi(x)), \quad j \geq 1, \\ \tilde{H}(0, 0) &= H_1^*(0) = \frac{\omega}{2}. \end{aligned} \tag{11}$$

It follows from (3) that there exist a family of periodic orbits surrounding the origin defined by the equation $H(x, y) = h$ or $H^*(x, v) = h$ for $h > 0$ small. Then, it is obvious that

$$\begin{aligned} M(h, \delta) &= \oint_{H^*(x,v)=h} q^* dx - p^* dv \\ &= \iint_{H^* \leq h} (p_x^* + q_v^*) dx dv \\ &= \oint_{H^*(x,v)=h} \bar{q}(x, v, \delta) dx, \end{aligned} \tag{12}$$

where

$$\begin{aligned} \bar{q}(x, v, \delta) &= q^*(x, v, \delta) - q^*(x, 0, \delta) \\ &\quad + \int_0^v p_x^*(x, u, \delta) du \end{aligned} \tag{13}$$

satisfying

$$\bar{q}_v = p_x^* + q_v^* \quad \text{and} \quad \bar{q}(x, 0, \delta) = 0.$$

With (7), (9) and (13), if

$$p_x(x, y, \delta) + q_y(x, y, \delta) = \sum_{i+j \geq 0} c_{ij} x^i y^j, \tag{14}$$

then

$$\bar{q}(x, v, \delta) = v \sum_{i+j \geq 0} \bar{b}_{ij} x^i v^j = \sum_{j \geq 1} q_j(x) v^j, \tag{15}$$

where

$$\begin{aligned} q_{j+1}(x) &= \frac{1}{(j+1)!} \frac{\partial^j}{\partial v^j} (p_x^* + q_v^*)|_{\varepsilon=v=0} \\ &= \frac{1}{(j+1)!} \frac{\partial^j}{\partial y^j} (p_x + q_y)(x, \varphi(x), 0, \delta) \\ &= \sum_{i \geq 0} \bar{b}_{ij} x^i, \quad j \geq 0. \end{aligned} \tag{16}$$

Next, consider the equation $H^*(x, v) = h$. Following the proof of Lemma 2.1 [Han *et al.*, 2009], we have

Lemma 1. *The equation $H^*(x, v) = h$ has exactly two C^∞ solutions $v_1(x, w)$ and $v_2(x, w)$ in v satisfying*

$$\begin{aligned} v_1(x, w) &= \sqrt{2} w (1 + O(|x, w|)), \\ v_2(x, w) &= v_1(x, -w), \end{aligned}$$

where $w = \sqrt{h - H_0^*(x)}$.

In fact, by (10) the equation $H^*(x, v) = h$ is equivalent to

$$w = |v| (\tilde{H}(x, v))^{\frac{1}{2}}. \tag{17}$$

Introduce an equation of the form

$$u = v (\tilde{H}(x, v))^{\frac{1}{2}}.$$

Noticing that $[\tilde{H}(x, v)]^{\frac{1}{2}} \in C^\infty$ for $|x| + |v|$ small the implicit function theorem implies that the above equation has a unique solution in v, denoted by $v^*(x, u)$. Obviously, the solution is C^∞ with $v^*(x, u) = \sqrt{2} u (1 + O(|x, u|))$. Let

$$v_1(x, w) = v^*(x, w), \quad v_2(x, w) = v^*(x, -w).$$

Then, $v_1(x, w)$ and $v_2(x, w)$ are the only solutions of (17) for $v > 0$ and $v < 0$, respectively, with

$$v_1(x, w) = v_2(x, -w) \quad \text{for } |x| + |w| \text{ small.}$$

Further, we write

$$v_1(x, w) = \sum_{j\geq 1} a_j(x) w^j. \tag{18}$$

Then, combining (17) and (10) results in

$$\begin{aligned} w^2 &= (v_1(x, w))^2 \sum_{j\geq 1} H_j^*(x)[v_1(x, w)]^{j-1} \\ &= a_1^2(x)H_1^*(x)w^2 + (2a_1(x)a_2(x)H_1^*(x) \\ &\quad + a_1^3(x)H_2^*(x))w^3 \\ &\quad + [(a_2^2(x) + 2a_1(x)a_3(x))H_1^*(x) \\ &\quad + 3a_1^2(x)a_2(x)H_2^*(x) + a_1^4(x)H_3^*(x)]w^4 + \dots. \end{aligned}$$

Balancing the terms w^j in the above equation, we have

$$\begin{aligned} a_1(x) &= \frac{1}{\sqrt{H_1^*(x)}}, \quad a_2(x) = -\frac{H_2^*(x)}{2(H_1^*(x))^2}, \\ a_3(x) &= -\frac{1}{8(H_1^*)^{\frac{7}{2}}}[4H_1^*H_3^* - 5(H_2^*)^2], \dots. \end{aligned} \tag{19}$$

Let $x_1(h) > 0$ and $x_2(h) < 0$ be the solutions of the equation $H_0^*(x) = h$. Then, it follows from (12) that

$$M(h, \delta) = \int_{x_2(h)}^{x_1(h)} [\bar{q}(x, v_1(x, w)) - \bar{q}(x, v_2(x, w))]dx.$$

By Lemma 1, the function $\bar{q}(x, v_1) - \bar{q}(x, v_2)$ is odd in w. Thus, we can write

$$\bar{q}(x, v_1) - \bar{q}(x, v_2) = \sum_{j\geq 0} \bar{q}_j(x) w^{2j+1}, \tag{20}$$

and hence,

$$M(h, \delta) = \sum_{j\geq 0} \int_{x_2(h)}^{x_1(h)} \bar{q}_j(x) w^{2j+1} dx.$$

In order to compute the above integral, we change the limits of integration. Let $\psi(x) = \operatorname{sgn}(x)[H_0^*(x)]^{\frac{1}{2}}$. Then, by (11) the function ψ is C^∞ for small $|x|$ with $\psi'(0) = h_2^{\frac{1}{2}}$. Therefore, we may introduce the new variable $u = \psi(x)$ to obtain

$$\begin{aligned} M(h, \delta) &= \sum_{j\geq 0} \int_{-h^{\frac{1}{2}}}^{h^{\frac{1}{2}}} \tilde{q}_j(u) w^{2j+1} du \\ &= \sum_{j\geq 0} \int_0^{h^{\frac{1}{2}}} [\tilde{q}_j(u) + \tilde{q}_j(-u)] w^{2j+1} du, \end{aligned}$$

where $w = \sqrt{h - u^2}$ and

$$\tilde{q}_j(u) = \left.\frac{\bar{q}_j(x)}{\psi'(x)}\right|_{x=\psi^{-1}(u)}. \tag{21}$$

It is easy to see that we can assume

$$\tilde{q}_j(u) + \tilde{q}_j(-u) = \sum_{i\geq 0} r_{ij} u^{2i}. \tag{22}$$

Then,

$$M(h, \delta) = \sum_{i+j\geq 0} r_{ij} I_{ij}(h), \tag{23}$$

where

$$\begin{aligned} I_{ij}(h) &= \int_0^{h^{\frac{1}{2}}} u^{2i} w^{2j+1} du \\ &= \int_0^{h^{\frac{1}{2}}} u^{2i}(h - u^2)^j \sqrt{h - u^2} du. \end{aligned}$$

Lemma 2. *Let*

$$\beta_{ij} = \int_0^1 v^{2i}(1 - v^2)^j \sqrt{1 - v^2} dv. \tag{24}$$

Then

$$I_{ij}(h) = \beta_{ij} h^{1+i+j}, \quad h > 0, \quad 0 < \beta_{ij} < 1.$$

Proof. Introducing $v = u/h^{\frac{1}{2}}$, we have

$$I_{ij}(h) = h^{\frac{2i+1}{2}+j+\frac{1}{2}} \int_0^1 v^{2i}(1 - v^2)^j \sqrt{1 - v^2} dv.$$

This ends the proof. ■

Now by (23) and Lemma 2, we have

$$M(h, \delta) = h \sum_{i+j\geq 0} r_{ij}\beta_{ij} h^{i+j} = h \sum_{l\geq 0} b_l(\delta) h^l, \tag{25}$$

where

$$b_l(\delta) = \sum_{i+j=l} r_{ij}\beta_{ij}. \tag{26}$$

Finally based on (20)–(22), we know that if system (1) is analytic, then the series $\sum_{i,j\geq 0} r_{ij} u^{2i} w^{2j+1}$ is convergent for (u, w) near the origin. Then it follows that the series $\sum_{i,j\geq 0} |r_{ij}|\mu^{i+j}$ is convergent for some constant $\mu > 0$, and hence by (25) $M(h, \delta)$ is analytic in h.

This completes the proof of Theorem 1.

2.2. *A generalization of Theorem 2*

In many cases the Hamiltonian system (2) contains some coefficients which can be varied. If we take them as parameters and change them suitably, we may find more limit cycles. More precisely, suppose $H(x, y, a)$ with $a \in \mathbf{R}^n$ satisfies (3), where the coefficients h_{ij} depend on a. Then by Theorem 1, we have

$$M(h, \delta, a) = h \sum_{l \geq 0} b_l(\delta, a) h^l. \quad (27)$$

For simplicity, suppose the functions p and q in system (1) are linear in δ. Then, the coefficients $b_l(\delta, a)$ are linear in δ. Suppose there exist an integer $k > 0$, and vectors $\delta_0 \in \mathbf{R}^m$, $a_0 \in \mathbf{R}^n$ such that

$$\begin{aligned} &b_j(\delta_0, a_0) = 0, \quad j = 0, \ldots, k-1, \\ &\det \frac{\partial(b_0, \ldots, b_{k-1})}{\partial(\delta_1, \ldots, \delta_k)}(a_0) \neq 0. \end{aligned} \quad (28)$$

Then, the linear equations $b_j = 0, j = 0, \ldots, k-1$ of δ have a unique solution in the form of

$$(\delta_1, \ldots, \delta_k) = \xi(\delta_{k+1}, \ldots, \delta_m, a)$$

for a near a_0. Obviously, ξ is linear in $\delta_{k+1}, \ldots, \delta_m$.

Further, let

$$\begin{aligned} &b_{k+j}|_{(\delta_1, \ldots, \delta_k) = \xi(\delta_{k+1}, \ldots, \delta_m, a)} \\ &\quad = L_j(\delta_{k+1}, \ldots, \delta_m)\Delta_j(a), \quad j = 0, \ldots, n. \end{aligned} \quad (29)$$

We have the following theorem.

Theorem 3. *Consider the near-Hamiltonian system (1), where $H(x, y, a)$ with $a \in \mathbf{R}^n$ satisfies (3) and the functions p and q are linear in $\delta \in \mathbf{R}^m$. Suppose there exist integer $k > 0$ and $\delta_0 = (\delta_{10}, \ldots, \delta_{m0}) \in \mathbf{R}^m$, $a_0 \in \mathbf{R}^n$ such that (28) and (29) hold with*

$$\begin{aligned} L_j(\delta_{k+1,0}, \ldots, \delta_{m0}) \neq 0, \quad j = 0, \ldots, n, \\ \Delta_j(a_0) = 0, \quad j = 0, \ldots, n-1, \quad \Delta_n(a_0) \neq 0, \end{aligned} \quad (30)$$

and

$$\det \frac{\partial(\Delta_0, \ldots, \Delta_{n-1})}{\partial(a_1, \ldots, a_n)}(a_0) \neq 0. \quad (31)$$

Then, for all (ε, δ, a) near $(0, \delta_0, a_0)$, system (1) has at most $k + n$ limit cycles near the origin, and for some (ε, δ, a) near $(0, \delta_0, a_0)$ system (1) can have $k + n$ limit cycles near the origin.

Proof. We fix $(\delta_{k+1}, \ldots, \delta_m) = (\delta_{k+1,0}, \ldots, \delta_{m0})$ such that

$$L_j(\delta_{k+1}, \ldots, \delta_m) = L_j(\delta_{k+1,0}, \ldots, \delta_{m0}) \equiv L_{j0} \neq 0.$$

Then noticing that $b_j = 0$ for $j = 0, \ldots, k-1$ as $(\delta_1, \ldots, \delta_k) = \xi(\delta_{k+1}, \ldots, \delta_m, a)$, by (27)–(29), we have

$$\begin{aligned} &M(h, \delta, a)|_{(\delta_1, \ldots, \delta_k) = \xi(\delta_{k+1}, \ldots, \delta_m, a)} \\ &\quad = h^{k+1} \sum_{j \geq 0} L_{j0} \Delta_j(a) h^j \equiv \tilde{M}(h, a). \end{aligned} \quad (32)$$

Due to (31), we can change a near a_0 such that

$$\begin{aligned} L_{i0} L_{i+1,0} \Delta_i \Delta_{i+1} < 0, \quad |\Delta_i| \ll |\Delta_{i+1}|, \\ i = 0, \ldots, n-1 \end{aligned} \quad (33)$$

which implies that the function $\tilde{M}$ in (32) has n positive simple zeros $h_n^* < \cdots < h_1^*$ near $h = 0$. Having obtained a satisfying (33), by (28) we can change $(\delta_1, \ldots, \delta_k)$ near $\xi(\delta_{k+1,0}, \ldots, \delta_{m0}, a)$ such that

$$b_j b_{j+1} < 0, \quad |b_j| \ll |b_{j+1}|, \quad j = 0, \ldots, k-1, \quad (34)$$

which indicates that the function M given by (27) has k simple zeros in the interval $(0, h_n^*)$. Clearly, under (34) the zeros $h_n^*, \ldots, h_1^*$ keep to exist. Thus, under (33) and (34) the function M has $n+k$ positive simple zeros altogether. Hence, by (4) the function F can have $n+k$ positive zeros in h near $h = 0$ which give $n + k$ hyperbolic limit cycles.

Finally, using (28)–(30), we have

$$\begin{aligned} b_j(\delta_0, a_0) = 0, \quad j = 0, \ldots, n+k-1, \\ b_{n+k}(\delta_0, a_0) = L_{n0}\Delta_n(a_0) \neq 0. \end{aligned}$$

Following the proof of Theorem 1 given in [Han, 2000], one can show that system (1) has at most $n+k$ limit cycles near the origin for all (ε, δ, a) near $(0, \delta_0, a_0)$.

This completes the proof. ■

2.3. *Programs for computing* $\{b_j\}$

Based on the formulas given in Sec. 2.1, we have developed Maple programs for computing $\{b_j\}$, as listed below. It contains several subroutines (as shown in the code) for computing e_j, $q_j(x)$, $H_j^*(x)$, $a_j(x)$, $\bar{q}_j(x)$, $\psi(x)$, $\tilde{q}_j$, r_{ij}, β_{ij} and b_j.

```
with(LinearAlgebra): ##########  read the input file  ##########
read input: ##########  compute the  e_j  coefficients  ##########
H:= W/2*(x^2+y^2):
for i from 3 to n+1 do
    for j from 0 to i do
        H := H+h[i-j,j]*x^(i-j)*y^j:
    od:
od:
P := 0:
Q := 0:
for i from 0 to n do
    for j from 0 to i do
        P := P+a[i-j,j]*x^(i-j)*y^j:
        Q := Q+b[i-j,j]*x^(i-j)*y^j:
    od:
od:
phi := 0:
for i from 2 to n do
    phi := phi+e[i]*x^i:
od:
temp := subs(y=phi,diff(H,y)):
temp1[2] := diff(temp,x$2)/2:
for i from 3 to n do
    temp1[i] := diff(temp1[i-1],x)/i:
od:
for i from 2 to n do
    temp1[i] := subs(x=0,temp1[i]):
    e[i]     := solve(temp1[i],e[i]):
od:
##########  compute  q_j(x)  ##########
y     := v+phi:
Pstar := subs(x=x*eps,v=v*eps,P):
Pstar := series(Pstar,eps=0,n+1):
Pstar := convert(Pstar,polynom):
Pstar := subs(eps=1,Pstar):
dphi  := diff(phi,x):
Qstar := subs(x=x*eps,v=v*eps,Q-dphi*Pstar):
Qstar := series(Qstar,eps=0,n+1):
Qstar := convert(Qstar,polynom):
Qstar := subs(eps=1,Qstar):
Qbar  := int(diff(Pstar, x),v,v=0..v):
Qbar  := Qbar+Qstar-subs(v=0,Qstar):
Qbar  := sort(Qbar,v):
for i from 1 to n do
    qjx[i] := series(coeff(Qbar,v,i),x=0,n-i+1):
    qjx[i] := convert(qjx[i],polynom):
od:
##########  compute H*_j(x)  ##########
Hstar  := subs(x=x*eps,v=v*eps,H):
Hstar  := series(Hstar,eps=0,n+2):
Hstar  := convert(Hstar,polynom):
Hstar  := subs(eps=1,Hstar):
```

```
H0star := subs(v=0,Hstar):
v2Htilde := factor(Hstar-H0star-subs(v=0,diff(Hstar,v))*v):
v2Htilde := sort(v2Htilde/v^2,v):
for i from 1 to n do
    Hjxstar[i] := coeff(v2Htilde,v,i-1):
od:
save Hjxstar, `output_Hjxstar`:
Hjxstar := 'Hjxstar':
##########  compute  a_j(x)  ##########
v2Htilde := 0:
for i from 1 to n do
    v2Htilde := v2Htilde+Hjxstar[i]*v^(i+1):
od:
F  := u^2-v2Htilde:
vv := 0:
for i from 1 to n do
    vv := vv+ajx[i]*u^i:
od:
F := subs(v=vv,F):
ajx[1] := 1/Hjxstar[1]^(1/2):
for i from 3 to n+1 do
    ajx[i-1] := solve(subs(u=0,diff(F,u$i)/i!),ajx[i-1]):
od:
read output_Hjxstar:
for i from 1 to n do
    ajx[i] := series(ajx[i],x=0,n+1-i):
    ajx[i] := convert(ajx[i],polynom):
od:
##########  compute  bar_q_j(x)  ##########
v1 := 0:
for i from 1 to n do
    v1 := v1+ajx[i]*w^i:
od:
v2   := subs(w=-w,v1):
qbar := 0:
for i from 1 to n do
    qbar := qbar+qjx[i]*(v1^i-v2^i):
od:
for i from 0 to (n-1)/2 do
    qbarjx[i] := subs(w=0, diff(qbar,w$(2*i+1))/(2*i+1)!):
    qbarjx[i] := series(qbarjx[i],x=0,n-2*i):
    qbarjx[i] := convert(qbarjx[i],polynom):
od:
###  compute  psi(x)  ###
Hstar0      := subs(y=phi,H):
Hstar0_half := series((Hstar0/(W/2)/x^2)^(1/2),x=0,n+1):
Hstar0_half:= x*(W/2)^(1/2)*convert(Hstar0_half,polynom):
psix[0]:=subs(x=0,Hstar0_half):
 for i from 1 to n do
    psix[i] := factor(subs(x=0,diff(Hstar0_half,x$i)/i!)):
```

```
od:
##########  compute  tilde_q_j  ##########
for j from 0 to (n-1)/2 do
    qjutilde[j] := 0:
    for i from 0 to (n-1)-j*2 do
        qjutilde[j] := qjutilde[j]+qjtilde[i,j]*u^i:
    od:
od:
psi := 0:
for i from 1 to n do
    psi := psi+psix[i]*x^i:
od:
dpsi := diff(psi, x):
for j from 0 to (n-1)/2 do
    qjutilde[j] := subs(u=psi,qjutilde[j]):
    qjutilde[j] := series(qjutilde[j],x=0,n-j*2):
    qjutilde[j] := convert(qjutilde[j],polynom):
od:
for j from 0 to (n-1)/2 do
    qjutilde[j] := qjutilde[j]*dpsi:
    qjutilde[j] := series(qjutilde[j],x=0,n-j*2):
    qjutilde[j] := convert(qjutilde[j],polynom):
od:
for j from 0 to (n-1)/2 do
    for i from 0 to (n-1)-j*2 do
        qjtilde[i,j] := solve(coeff(qjutilde[j]-qbarjx[j],x,i),qjtilde[i,j]):
    od:
od:
##########  compute  r_ij  coefficients  ##########
for j from 0 to (n-1)/2 do
    for i from 0 to (n-1)-j*2 by 2 do
        r[i/2,j] := 2*qjtilde[i,j]:
    od:
od:
##########  compute  beta_ij  coefficients  ##########
for i from 0 to (n-1)/2 do
    for j from 0 to i do
        beta[j,i-j] := int(nu^(2*j)*(1-nu^2)^(i-j)*(1-nu^2)^(1/2),nu=0..1):
    od:
od:
##########  compute  b_j  coefficients  ##########
for i from 0 to (n-1)/2 do
    B[i] := 0:
    for j from 0 to i do
        B[i] := B[i]+r[j,i-j]*beta[j,i-j]:
    od:
    B[i] := factor(B[i]):
od:
save B, `output_b`:
quit:
```

Executing the above program yields the expressions, expressed in the original coefficients H_{ij}, a_{ij} and b_{ij}. In the following, we list the final coefficients b_j (the intermediate expressions such as e_j, a_j, etc. are omitted).

$$b_0 = \frac{2\pi}{\omega}(a_{10} + b_{01}).$$

When $b_0 = 0$, i.e. $b_{01} = -a_{10}$,

$$b_1 = \frac{\pi}{\omega^3} \times \{\omega(3a_{30} + b_{21} + a_{12} + 3b_{03}) - (2a_{20} + b_{11})(3h_{30} + h_{12}) - (a_{11} + 2b_{02})(h_{21} + 3h_{03})\},$$

$$\begin{aligned}
b_2 = \frac{\pi}{\omega^6} \times \Bigg\{&\omega^3[(5a_{50} + b_{41}) + (a_{32} + b_{23}) + (a_{14} + 5b_{05})] - \omega^2[(2a_{20} + b_{11})(h_{32} + 5h_{50} + h_{14})\\
&+ (a_{11} + 2b_{02})(h_{41} + 5h_{05} + h_{23}) + (3a_{30} + b_{21})(h_{22} + h_{04} + 5h_{40}) + (2a_{21} + 2b_{12})(h_{31} + h_{13})\\
&+ (a_{12} + 3b_{03})(h_{40} + h_{22} + 5h_{04}) + (4a_{40} + b_{31})(h_{12} + 5h_{30}) + (3a_{31} + 2b_{22})(h_{21} + h_{03})\\
&+ (2a_{22} + 3b_{13})(h_{12} + h_{30}) + (a_{13} + 4b_{04})(h_{21} + 5h_{03})] + \omega\Big[(2a_{20} + b_{11})(3h_{13}h_{21} + 3h_{03}h_{31}\\
&+ 3h_{12}h_{22} + 35h_{30}h_{40} + 5h_{31}h_{21} + 5h_{12}h_{40} + 5h_{30}h_{22} + 5h_{03}h_{13} + 3h_{30}h_{04} + 5h_{04}h_{12})\\
&+ (a_{11} + 2b_{02})(3h_{22}h_{21} + 3h_{12}h_{31} + 5h_{21}h_{40} + 5h_{30}h_{31} + 5h_{21}h_{04} + 35h_{03}h_{04} + 5h_{13}h_{12} + 5h_{03}h_{22}\\
&+ 3h_{03}h_{40} + 3h_{30}h_{13}) + (3a_{30} + b_{21})\left(3h_{03}h_{21} + 5h_{30}h_{12} + \frac{3}{2}h_{12}^2 + \frac{5}{2}h_{03}^2 + \frac{35}{2}h_{30}^2 + \frac{5}{2}h_{21}^2\right)\\
&+ (2a_{21} + 2b_{12})(3h_{12}h_{21} + 5h_{30}h_{21} + 3h_{30}h_{03} + 5h_{03}h_{12}) + (a_{12} + 3b_{03})\Bigg(\frac{5}{2}h_{12}^2 + 3h_{30}h_{12}\\
&+ 5h_{03}h_{21} + \frac{3}{2}h_{21}^2 + \frac{35}{2}h_{03}^2 + \frac{5}{2}h_{30}^2\Bigg)\Big] - \Big[(2a_{20} + b_{11})\Bigg(\frac{105}{2}h_{30}^3 + \frac{35}{2}h_{30}h_{21}^2 + \frac{15}{2}h_{12}h_{21}^2 + \frac{35}{2}h_{12}h_{30}^2\\
&+ 15h_{03}h_{12}h_{21} + 15h_{30}h_{03}h_{21} + \frac{5}{2}h_{12}^3 + \frac{15}{2}h_{30}h_{12}^2 + \frac{35}{2}h_{12}h_{03}^2 + \frac{15}{2}h_{30}h_{03}^2\Bigg)\\
&+ (a_{11} + 2b_{02})\Bigg(\frac{35}{2}h_{21}h_{03}^2 + \frac{15}{2}h_{03}h_{21}^2 + \frac{15}{2}h_{12}^2h_{21} + \frac{35}{2}h_{30}^2h_{21} + 15h_{30}h_{12}h_{21}\\
&+ \frac{5}{2}h_{21}^3 + \frac{105}{2}h_{03}^3 + \frac{35}{2}h_{03}h_{12}^2 + \frac{15}{2}h_{30}^2h_{03} + 15h_{30}h_{03}h_{12}\Bigg)\Big]\Bigg\},
\end{aligned}$$

$$b_3 = \frac{\pi}{\omega^9} \times (\cdots) \text{ (560 lines in outputfile).}$$

3. Application to a Class of Cubic Systems

From [Bautin, 1952; Llibre, 2004; Schlomiuk, 1993], we know that a quadratic Hamiltonian system having an elementary center can be transformed into the following form

$$\begin{aligned}
\dot{x} &= y + bx^2 - 2axy + cy^2,\\
\dot{y} &= -x - ax^2 - 2bxy + ay^2,
\end{aligned}$$

where a, b and c are parameters.

We perturb the above system by cubic polynomials to obtain a near-Hamiltonian system, given by

$$\begin{aligned}
\dot{x} &= y + bx^2 - 2axy + cy^2 + \varepsilon p(x, y),\\
\dot{y} &= -x - ax^2 - 2bxy + ay^2 + \varepsilon q(x, y),
\end{aligned} \tag{35}$$

where

$$p(x, y) = \sum_{1 \le i+j \le 3} a_{ij}x^iy^j, \quad q(x, y) = \sum_{1 \le i+j \le 3} b_{ij}x^iy^j.$$

The unperturbed system $(35)|_{\varepsilon=0}$ is a Hamiltonian system with Hamiltonian:

$$H = \frac{1}{2}(x^2 + y^2) + \frac{a}{3}x^3 + bx^2y - axy^2 + \frac{c}{3}y^3.$$

For system (35), there are two cases: $a = 0$ and $a \neq 0$, which will be discussed separately in the following. First let $a \neq 0$. Without loss of generality, we assume $a = 1$ (otherwise, we can introduce a suitable rescaling of (x, y)). Further, we can assume $b \geq 0$ (otherwise, we need only to change the sign of y and t). Then, system (35) becomes

$$\begin{aligned}\dot{x} &= y + bx^2 - 2xy + cy^2 + \varepsilon p(x, y),\\ \dot{y} &= -x - x^2 - 2bxy + y^2 + \varepsilon q(x, y).\end{aligned} \tag{36}$$

Correspondingly

$$\begin{aligned}H(x, y) &= \frac{1}{2}(x^2 + y^2) + \frac{1}{3}x^3 + bx^2y - xy^2 + \frac{c}{3}y^3\\ &= \frac{1}{2}(x^2 + y^2) + \sum_{i+j=3} h_{ij}x^iy^j, \quad b > 0,\end{aligned} \tag{37}$$

and

$$p_x + q_y = \sum_{0\leq i+j\leq 2} c_{ij}x^iy^j,$$

where

$$\begin{aligned}c_{00} &= a_{10} + b_{01}, \quad c_{10} = 2a_{20} + b_{11},\\ c_{01} &= a_{11} + 2b_{02}, \quad c_{20} = 3a_{30} + b_{21},\\ c_{11} &= 2(a_{21} + b_{12}), \quad c_{02} = a_{12} + 3b_{03}.\end{aligned} \tag{38}$$

Theorem 4. *For system (36), introduce*

$$\delta = (c_{00}, c_{10}, c_{01}, c_{02}, c_{11}, c_{20}), \quad \sigma = (b, c),$$

and

$$\delta_0 = (c^*_{00}, c^*_{10}, c^*_{01}, c^*_{02}, c^*_{11}, c^*_{20}), \quad \sigma_0 = (b^*, c^*),$$

where δ_0 and σ_0 satisfy

$$\begin{gathered}b^* = \frac{2}{13}\sqrt{26}, \quad c^* = 2b^*, \quad c^*_{10} - c^*c^*_{01} + c^*_{02} \neq 0,\\ c^*_{20} = 3b^*c^*_{01} - c^*_{02}, \quad c^*_{00} = 0,\\ c^*_{11} = \frac{15}{4}b^*c^*_{10} - \left(2 + \frac{7}{2}(b^*)^2\right)c^*_{01} + \frac{7}{4}b^*c^*_{02}.\end{gathered}$$

Then, for all $(\varepsilon, \delta, \sigma)$ near $(0, \delta_0, \sigma_0)$ the system (36) has Hopf cyclicity 5 at the origin.

Proof. Executing the Maple program yields

$$b_0 = 2\pi(a_{10} + b_{01}).$$

Letting $b_{01} = -a_{10}$, under which $b_0 = 0$, we obtain the following expressions for b_i's.

$$b_1|_{b_0=0} = \pi[c_{20} + c_{02} - (b + c)c_{01}],$$

$$\begin{aligned}b_2|_{b_0=0} = \pi\Bigg[&\left(\frac{5}{3}b^2 + \frac{5}{3}c^2 + \frac{10}{3}cb\right)c_{10} + \left(-\frac{40}{9}b - \frac{5}{2}cb^2 - \frac{40}{9}c - \frac{35}{18}c^2b - \frac{5}{2}b^3 - \frac{35}{18}c^3\right)c_{01}\\ &+ \left(\frac{5}{18}c^2 + cb + \frac{16}{9} + \frac{5}{2}b^2\right)c_{20} + \left(-\frac{4}{3}b - \frac{4}{3}c\right)c_{11} + \left(\frac{16}{9} + \frac{5}{3}cb + \frac{3}{2}b^2 + \frac{35}{18}c^2\right)c_{02}\Bigg],\end{aligned} \tag{39}$$

$$b_i|_{b_0=0} = \pi(\delta_{i2}c_{10} + \delta_{i3}c_{01} + \delta_{i4}c_{20} + \delta_{i5}c_{11} + \delta_{i6}c_{02}), \quad i = 3, 4, 5,$$

where

$$\delta_{32} = \frac{105}{8}b^4 + \frac{140}{9}b^2 + \frac{140}{9}c^2 + \frac{385}{72}c^4 + \frac{875}{36}c^2b^2 + \frac{280}{9}cb + \frac{175}{6}cb^3 + \frac{245}{18}c^3b,$$

$$\begin{aligned}\delta_{33} = &-\frac{315}{32}b^5 - \frac{1225}{96}b^4c - \frac{175}{16}b^3c^2 - 35b^3 - \frac{385}{48}b^2c^3 - \frac{175}{3}b^2c - \frac{5005}{864}bc^4 - \frac{1435}{27}bc^2 - \frac{560}{27}b\\ &- \frac{5005}{864}c^5 - \frac{805}{27}c^3 - \frac{560}{27}c,\end{aligned}$$

$$\delta_{34} = \frac{385}{864}c^4 + \frac{175}{48}c^2b^2 + \frac{35}{2}b^2 + \frac{175}{24}cb^3 + \frac{35}{24}c^3b + \frac{245}{54}c^2 + \frac{140}{27} + \frac{315}{32}b^4 + \frac{35}{3}cb,$$

$$\delta_{35} = -\frac{35}{4}b^3 - \frac{175}{12}b^2c - \frac{385}{36}bc^2 - \frac{70}{9}b - \frac{175}{36}c^3 - \frac{70}{9}c,$$

$$\delta_{36} = \frac{175}{32}b^4 + \frac{175}{24}b^3c + \frac{105}{16}b^2c^2 + \frac{35}{2}b^2 + \frac{385}{72}bc^3 + \frac{245}{9}bc + \frac{5005}{864}c^4 + \frac{1085}{54}c^2 + \frac{140}{27},$$

$$\delta_{42} = \frac{3080}{27}c^2 + \frac{6160}{27}cb + \frac{3080}{27}b^2 + \frac{29645}{54}c^2b^2 + \frac{5005}{9}cb^3 + \frac{8855}{27}c^3b + \frac{2695}{12}b^4 + \frac{11935}{108}c^4 + \frac{3773}{16}cb^5$$
$$+ \frac{8085}{32}c^2b^4 + \frac{13475}{72}c^3b^3 + \frac{3003}{32}b^6 + \frac{17017}{864}c^6 + \frac{95095}{864}c^4b^2 + \frac{23023}{432}c^5b,$$
$$\delta_{43} = -\frac{3003}{64}b^7 - \frac{4851}{64}b^6c - \frac{4851}{64}b^5c^2 - \frac{2079}{8}b^5 - \frac{35035}{576}b^4c^3 - \frac{13475}{24}b^4c - \frac{25025}{576}b^3c^4 - \frac{2695}{4}b^3c^2$$
$$- \frac{3080}{9}b^3 - \frac{17017}{576}b^2c^5 - \frac{20405}{36}b^2c^3 - \frac{6160}{9}b^2c - \frac{323323}{15552}bc^6 - \frac{235235}{648}bc^4 - \frac{52360}{81}bc^2$$
$$- \frac{24640}{243}b - \frac{323323}{15552}c^7 - \frac{109109}{648}c^5 - \frac{24640}{81}c^3 - \frac{24640}{243}c,$$
$$\delta_{44} = \frac{13475}{648}c^4 + \frac{4480}{243} + \frac{280}{3}cb + \frac{980}{9}b^2 + \frac{3220}{81}c^2 + \frac{1715}{12}c^2b^2 + \frac{1085}{6}cb^3 + \frac{1155}{8}b^4 + \frac{1295}{18}c^3b$$
$$+ \frac{2695}{144}c^3b^3 + \frac{1617}{32}cb^5 + \frac{2205}{64}c^2b^4 + \frac{3003}{64}b^6 + \frac{17017}{15552}c^6 + \frac{5005}{576}c^4b^2 + \frac{1001}{288}c^5b,$$
$$\delta_{45} = -\frac{231}{4}b^5 - \frac{245}{2}b^4c - \frac{245}{2}b^3c^2 - \frac{350}{3}b^3 - \frac{770}{9}b^2c^3 - \frac{2030}{9}b^2c - \frac{5005}{108}bc^4 - \frac{4970}{27}bc^2 - \frac{1120}{27}b$$
$$- \frac{1001}{54}c^5 - \frac{2030}{27}c^3 - \frac{1120}{27}c,$$
$$\delta_{46} = \frac{1617}{64}b^6 + \frac{1323}{32}b^5c + \frac{2695}{64}b^4c^2 + \frac{1085}{8}b^4 + \frac{5005}{144}b^3c^3 + \frac{1715}{6}b^3c + \frac{5005}{192}b^2c^4 + \frac{1295}{4}b^2c^2 + 140b^2$$
$$+ \frac{17017}{864}bc^5 + \frac{13475}{54}bc^3 + \frac{6440}{27}bc + \frac{323323}{15552}c^6 + \frac{85085}{648}c^4 + \frac{12460}{81}c^2 + \frac{4480}{243},$$
$$\delta_{52} = \frac{350350}{243}c^4 + \frac{1121120}{729}cb + \frac{560560}{81}cb^3 + \frac{1821820}{243}c^2b^2 + \frac{1310309}{1944}c^6 + \frac{560560}{729}b^2 + \frac{560560}{729}c^2$$
$$+ \frac{70070}{27}b^4 + \frac{1121120}{243}c^3b + \frac{175175}{72}b^6 + \frac{1296295}{162}c^3b^3 + \frac{665665}{72}c^2b^4 + \frac{763763}{108}b^5c + \frac{9844835}{1944}c^4b^2$$
$$+ \frac{2277275}{972}c^5b + \frac{77077}{36}c^3b^5 + \frac{875875}{576}c^4b^4 + \frac{85085}{128}b^8 + \frac{15015}{8}b^7c + \frac{77077}{32}c^2b^6 + \frac{595595}{648}c^5b^3$$
$$+ \frac{11316305}{23328}c^6b^2 + \frac{323323}{1458}c^7b + \frac{7436429}{93312}c^8,$$
$$\delta_{53} = -\frac{255255}{1024}b^9 - \frac{495495}{1024}b^8c - \frac{143143}{256}b^7c^2 - \frac{185185}{96}b^7 - \frac{385385}{768}b^6c^3 - \frac{161161}{32}b^6c - \frac{595595}{1536}b^5c^4$$
$$- \frac{231231}{32}b^5c^2 - \frac{49049}{12}b^5 - \frac{11316305}{41472}b^4c^5 - \frac{6341335}{864}b^4c^3 - \frac{3538535}{324}b^4c - \frac{11316305}{62208}b^3c^6$$
$$- \frac{5080075}{864}b^3c^4 - \frac{805805}{54}b^3c^2 - \frac{700700}{243}b^3 - \frac{7436429}{62208}b^2c^7 - \frac{10125115}{2592}b^2c^5 - \frac{2137135}{162}b^2c^3$$
$$- \frac{1541540}{243}b^2c - \frac{185910725}{2239488}bc^8 - \frac{151638487}{69984}bc^6 - \frac{23158135}{2916}bc^4 - \frac{13313300}{2187}bc^2 - \frac{1121120}{2187}b$$
$$- \frac{185910725}{2239488}c^9 - \frac{64341277}{69984}c^7 - \frac{8275267}{2916}c^5 - \frac{5745740}{2187}c^3 - \frac{1121120}{2187}c,$$
$$\delta_{54} = \frac{160160}{2187} + \frac{255255}{1024}b^8 + \frac{7436429}{2239488}c^8 + \frac{160160}{243}cb + \frac{160160}{243}b^2 + \frac{640640}{2187}c^2 + \frac{205205}{81}cb^3 + \frac{385385}{162}c^2b^2$$
$$+ \frac{35035}{27}c^3b + \frac{1056055}{2916}c^4 + \frac{55055}{36}b^4 + \frac{175175}{108}c^3b^3 + \frac{35035}{16}c^2b^4 + \frac{49049}{24}b^5c + \frac{385385}{432}c^4b^2$$

$$+\frac{235235}{648}c^5b+\frac{3250247}{34992}c^6+\frac{55055}{48}b^6+\frac{77077}{384}c^3b^5+\frac{175175}{1536}c^4b^4+\frac{77077}{256}c^2b^6+\frac{595595}{10368}c^5b^3 +\frac{1616615}{62208}c^6b^2+\frac{323323}{31104}c^7b+\frac{45045}{128}b^7c,$$

$$\delta_{55}=-\frac{25025}{64}b^7-\frac{63063}{64}b^6c-\frac{77077}{64}b^5c^2-\frac{23023}{18}b^5-\frac{595595}{576}b^4c^3-\frac{175175}{54}b^4c-\frac{1226225}{1728}b^3c^4 -\frac{35035}{9}b^3c^2-\frac{10010}{9}b^3-\frac{2127125}{5184}b^2c^5-\frac{245245}{81}b^2c^3-\frac{190190}{81}b^2c-\frac{9376367}{46656}bc^6-\frac{785785}{486}bc^4 -\frac{490490}{243}bc^2-\frac{160160}{729}b-\frac{3556553}{46656}c^7-\frac{251251}{486}c^5-\frac{190190}{243}c^3-\frac{160160}{729},$$

$$\delta_{56}=\frac{135135}{1024}b^8+\frac{33033}{128}b^7c+\frac{77077}{256}b^6c^2+\frac{49049}{48}b^6+\frac{35035}{128}b^5c^3+\frac{21021}{8}b^5c+\frac{2977975}{13824}b^4c^4+\frac{175175}{48}b^4c^2 +\frac{205205}{108}b^4+\frac{1616615}{10368}b^3c^5+\frac{385385}{108}b^3c^3+\frac{385385}{81}b^3c+\frac{2263261}{20736}b^2c^6+\frac{1176175}{432}b^2c^4 +\frac{35035}{6}b^2c^2+\frac{80080}{81}b^2+\frac{7436429}{93312}bc^7+\frac{3250247}{1944}bc^5+\frac{1056055}{243}bc^3+\frac{1281280}{729}bc+\frac{185910725}{2239488}c^8 +\frac{26835809}{34992}c^6+\frac{5260255}{2916}c^4+\frac{2322320}{2187}c^2+\frac{160160}{2187}.$$

Let

$$\tilde{b}_j=b_j|_{b_0=0},\quad j=1,2,3,4,5. \tag{40}$$

Noticing (39), we can solve c_{20}, c_{11} from $\tilde{b}_1=\tilde{b}_2=0$ as follows:

$$c_{20}=(b+c)c_{01}-c_{02},$$
$$c_{11}=\frac{5}{4}(c+b)c_{10}+\left(-\frac{5}{4}c^2+\frac{3}{4}cb-2\right)c_{01}+\left(\frac{5}{4}c-\frac{3}{4}b\right)c_{02}. \tag{41}$$

Substituting (41) into $\tilde{b}_3, \tilde{b}_4$ and $\tilde{b}_5$ results in

$$\tilde{b}_{3+i}=\Delta_i\pi(c_{10}-c_{01}c+c_{02})\equiv\Delta_iL,\quad i=0,1,2, \tag{42}$$

where $L=\pi(c_{10}-c_{01}c+c_{02})$, and

$$\Delta_0=\frac{35}{16}b^4-\frac{175}{24}b^2c^2+\frac{35}{6}b^2-\frac{35}{6}bc^3+\frac{35}{3}bc-\frac{35}{48}c^4+\frac{35}{6}c^2$$
$$=-\frac{35}{48}(b+c)^2(c+3b-2\sqrt{3b^2+2})\times(c+3b+2\sqrt{3b^2+2}),$$

$$\Delta_1=-\frac{7}{288}(b+c)^2(-891b^4+1350b^3c+396b^2c^2-3240b^2+858bc^3+1200bc+143c^4-680c^2-2560),$$

$$\Delta_2=-\frac{1001}{20736}(b+c)^2(-3645b^6+4050b^5c+2349b^4c^2-17280b^4+4860b^3c^3+5184b^3c+1581b^2c^4+192b^2c^2-24960b^2+1938bc^5+7872bc^3-3840bc+323c^6-576c^4-9600c^2-10240).$$

For c near c^* and (c_{10}, c_{01}, c_{02}) near $(c^*_{10}, c^*_{01}, c^*_{02})$ we have $L\neq 0$. Thus, $\tilde{b}_3=0$ if and only if $\Delta_0=0$. For (b,c) near (b^*,c^*) we have $b+c\neq 0$. Then $\Delta_0=0$ if and only if $c=c_1(b)$ or $c=c_2(b)$, where

$$c_i(b)=-3b-(-1)^i 2\sqrt{3b^2+2},\quad i=1,2.$$

When $c=c_i(b)$, we have

$$\Delta_1=-28(b+(-1)^i\sqrt{3b^2+2})^2(43b^4+(-1)^i b(25b^2+11)\sqrt{3b^2+2}+33b^2+4)\equiv\overline{\Delta}_1(b).$$

It is easy to observe from the fact $b\geq 0$ that $\Delta_1<0$ for $i=2$, and $\Delta_1=0$ if and only if

$b = (2/13)\sqrt{26} \equiv b^*$ for $i = 1$. Thus, it follows that when $\Delta_1 = 0$ we have $c = 2b^*$ and

$$\overline{\Delta}_1'(b^*) \neq 0, \quad \Delta_2 = -\frac{898128}{2197}.$$

Hence, it is clear that the conclusion follows from Theorem 3.

The proof is complete. ■

Now we turn to the case $a = 0$. Similar to the case $a = 1$, we have $b_0 = 2\pi c_{00}$. For simplicity, let $b_0 = 0$, i.e. $c_{00} = a_{10} + b_{01} = 0$. Taking $a = 0$ in the program and using the same procedure as that for the case $a = 1$, we obtain the following expressions for b_j's:

$$b_1|_{b_0=0} = \pi[-(b+c)c_{01} + c_{20} + c_{02}],$$

$$b_2|_{b_0=0} = \pi\left[\left(-\frac{5}{2}cb^2 - \frac{5}{2}b^3 - \frac{35}{18}c^2b - \frac{35}{18}c^3\right)c_{01} + \left(cb + \frac{5}{2}b^2 + \frac{5}{18}c^2\right)c_{20} + \left(\frac{3}{2}b^2 + \frac{5}{3}cb + \frac{35}{18}c^2\right)c_{02}\right],$$

$$b_3|_{b_0=0} = \pi\left[\left(-\frac{1225}{96}b^4c - \frac{315}{32}b^5 - \frac{175}{16}c^2b^3 - \frac{385}{48}c^3b^2 - \frac{5005}{864}c^4b - \frac{5005}{864}c^5\right)c_{01} + \left(\frac{175}{24}b^3c + \frac{315}{32}b^4 + \frac{175}{48}c^2b^2 + \frac{35}{24}c^3b + \frac{385}{864}c^4\right)c_{20} + \left(\frac{175}{32}b^4 + \frac{175}{24}b^3c + \frac{105}{16}c^2b^2 + \frac{385}{72}c^3b + \frac{5005}{864}c^4\right)c_{02}\right].$$

Similar to (40), let

$$\tilde{b}_j = b_j|_{b_0=0}, \quad j = 1, 2, 3.$$

Then, letting $\tilde{b}_1 = 0$ gives

$$-(b+c)c_{01} + c_{20} + c_{02} = 0,$$

or

$$c_{20} = (b+c)c_{01} - c_{02}. \tag{43}$$

Substituting (43) into $\tilde{b}_2$ and $\tilde{b}_3$ yields

$$\tilde{b}_2 = \Delta_0\tilde{L}, \quad \tilde{b}_3 = \Delta_1\tilde{L}, \tag{44}$$

where $\tilde{L} = \pi(-c_{01}c + c_{02})$ and

$$\Delta_0 = \frac{2}{3}cb - b^2 + \frac{5}{3}c^2 = \frac{1}{3}(b+c)(5c-3b),$$

$$\Delta_1 = \frac{35}{72}(b+c)(11c^3 - 3bc^2 + 9b^2c - 9b^3).$$

Further, let (b, c) and (c_{20}, c_{01}) satisfy $b + c \neq 0$ and $\tilde{L} \neq 0$. Then

$$\tilde{b}_2 = 0 \Leftrightarrow \Delta_0 = 0 \Leftrightarrow c = \frac{3}{5}b,$$

for which we have

$$\Delta_1 = -\frac{224}{125}b^4.$$

Summarizing the above results gives the following theorem.

Theorem 5. *For system (35), let $a = 0$ and introduce*

$$\delta = (c_{00}, c_{10}, c_{01}, c_{02}, c_{11}, c_{20}), \quad \sigma = (b, c),$$

and

$$\overline{\delta} = (\bar{c}_{00}, \bar{c}_{10}, \bar{c}_{01}, \bar{c}_{02}, \bar{c}_{11}, \bar{c}_{20}), \quad \overline{\sigma} = (\bar{b}, \bar{c}),$$

where $\overline{\delta}$ and $\overline{\sigma}$ satisfy

$$\bar{b} \neq 0, \quad \bar{c} = \frac{3}{5}\bar{b}, \quad \bar{c}_{00} = 0,$$

$$\bar{c}_{20} = (\bar{b} + \bar{c})\bar{c}_{01} - \bar{c}_{02}, \quad \bar{c}_{02} - \bar{c}\bar{c}_{01} \neq 0.$$

Then, for all $(\varepsilon, \delta, \sigma)$ near $(0, \overline{\delta}, \overline{\sigma})$ system (35) has Hopf cyclicity 3 at the origin.

4. Conclusions

We have studied bifurcation of limit cycles for near-Hamiltonian systems. It is assumed that the Hamiltonian function of the system depends on parameters, in addition to the coefficients involved in perturbations. We have extended the existing theorem to a new theorem for this case, which can generate more limit cycles. Moreover, based on a new method developed in this paper, a computationally efficient algorithm has been established to systematically compute the coefficients of the Melinkov function. As an illustrative example, a quadratic system with cubic perturbations has been considered to show that the system can have five limit cycles, two more than the case when all the coefficients of the Hamiltonian function are fixed. The method and program developed in this paper can be extended to study nonpolynomial systems.

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