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# LIMIT CYCLES IN TWO TYPES OF SYMMETRIC LIÉNARD SYSTEMS

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Liénard systems and their generalized forms are classical and important models of nonlinear oscillators, and have been widely studied by mathematicians and scientists. The main problem considered is the maximal number of limit cycles that the system can have. In this paper, two types of symmetric polynomial Liénard systems are investigated and the maximal number of limit cycles bifurcating from Hopf singularity is obtained. A global result is also presented.

*Keywords*: Liénard equation; limit cycles; normal form.

## 1. Introduction and Main Results

Consider the following system in the Liénard plane:

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{1}$$

where $F(x)$ is of degree $m+1$ and $g(x)$ is of degree $n$, given by

$$F(x) = \sum_{i=1}^{m+1} a_i x^i, \quad g(x) = \sum_{i=1}^{n} b_i x^i \quad (b_1 > 0). \tag{2}$$

Then the origin of system (1) is a singular point with index +1.

This paper is concerned with limit cycles in symmetric Liénard systems and there are potential applications, particularly in engineering, when considering large-amplitude limit cycle bifurcations when modeling wing rock phenomena and surge in jet engines (for example, see [Lynch & Christopher, 1999; Agarwal & Ananthkrishnan, 2000; Owens *et al.*, 2004]). Engineers consider limit cycles as steady state behavior and they are interested in hard (dangerous) and soft (safe) bifurcations.

An interesting problem studied widely in recent years is to find the maximal number of limit cycles in a small neighborhood of the origin of (1), namely, the Hopf cyclicity of (1) at the origin. Recall that system (1) has Hopf cyclicity $k$ at the origin if the

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following two conditions are satisfied:

(i) There exists a neighborhood $U$ of the origin such that for any functions $F$ and $g$ of the form (2), Eq. (1) has at most $k$ limit cycles in $U$.
(ii) For any neighborhood $U_0$ of the origin with $U_0 \subset U$ there exist functions $F$ and $g$ of the form (2) such that system (1) has exactly $k$ limit cycles in $U_0$.

Clearly, the cyclicity $k$ depends only on $m$ and $n$, denoted by $\hat{H}(m,n)$, that is, $k = \hat{H}(m,n)$. In 1984, Blows and Lloyd [1984] proved that $\hat{H}(m,1) = \lfloor m/2 \rfloor$ and in 1988, Lloyd and Lynch [1988] considered a number of classes of systems, and in particular, proved that $\hat{H}(1,n) = \lfloor n/2 \rfloor$, where $\lfloor\,\rfloor$ denotes the integer part. Han [1999] proved that $\hat{H}(m,2) = \lfloor (2m+1)/3 \rfloor$ and Gasull and Torregrosa [1999] provided results for a number of cases for varying degrees of $m$ and $n$. In [Christopher & Lynch, 1999], it is shown that $\hat{H}(m,2) = \hat{H}(2,m) = \lfloor (2m+1)/3 \rfloor$ for all natural numbers $m$, $n$, and $\hat{H}(m,3) = \hat{H}(3,m) = 2\lfloor 3(m+2)/8 \rfloor$, for all $1 < m \le 50$, for generalized Liénard systems

$$\dot{x} = h(y) - F(x), \quad \dot{y} = -g(x),$$

where $h(y)$ is analytic with $h'(y) > 0$. Other results for larger values of $m$ and $n$ are also listed in this paper. Recently, Yu and Han considered small limit cycles bifurcating from symmetric Hopf critical points [Yu & Han, 2006].

Christopher and Lloyd [1996] have proven that $\hat{H}(m,n) = \hat{H}(n,m)$ but only in the restricted cases where the quadratic coefficient in $F(x)$ is nonzero.

In this paper, we consider two types of symmetric Liénard systems of the form

$$\dot{x} = y - \sum_{i=0}^{m} a_i x^{2i+1}, \quad \dot{y} = -\sum_{i=0}^{n} b_i x^{2i+1}, \tag{3}$$

and

$$\dot{x} = y - \sum_{i=0}^{m} a_i x^{2i}, \quad \dot{y} = -\sum_{i=0}^{n} b_i x^{2i+1}, \tag{4}$$

where

$$\sum_{i=0}^{n} b_i = 0, \quad \sum_{i=0}^{n} (2i+1) b_i > 0. \tag{5}$$

Condition (5) implies that system (3) or system (4) has two singular points of index +1 away from the origin. One is at $A(1, y_0)$, and the other at $B(-1, -y_0)$ for system (3) or $B(-1, y_0)$ for system (4), where

$$y_0 = \sum_{i=0}^{m} a_i.$$

Denote by $\hat{H}_0(m,n)$ (resp. $\hat{H}_e(m,n)$) the Hopf cyclicity of system (3) (resp. system (4)) at the point $A$. Then the maximal number of small-amplitude limit cycles of system (3) (resp. system (4)) is at least $2\hat{H}_0(m,n)$ (resp. $2\hat{H}_e(m,n)$). Yu and Han have showed [2006] that $\hat{H}_0(m,1) = m$, for $1 \le m \le 10$, and conjectured that $\hat{H}_0(m,1) = m$, for all $m \ge 1$. By applying a previously developed theorem obtained by Han [1999] we have obtained the following main results.

**Theorem 1.** *Suppose condition (5) is satisfied. Then for system (3) $\hat{H}_0(m,1) = m$. Hence, the maximal number of small-amplitude limit cycles of system (3) is $2m$ for $n = 1$.*

**Theorem 2.** *Suppose condition (5) holds. Then for system (4) $\hat{H}_e(m,n) = \hat{H}(m-1,n)$. In particular, the maximal number of small-amplitude limit cycles of system (4) is $2\lfloor (m-1)/2 \rfloor$ for $n = 1$ and $2\lfloor (2m-1)/3 \rfloor$ for $n = 2$.*

For a global result we have

**Theorem 3.** *When $m \le 3$, $n \le 2$, system (4) has at most two limit cycles on the plane.*

## 2. Proof of the Main Results

In order to prove Theorem 1, here we will apply a theorem given by Han [1999]. To state the theorem, consider a Liénard system of the form

$$\dot{x} = p(y) - F(x,a), \quad \dot{y} = -g(x), \tag{6}$$

where $F, g$ and $p$ are $C^\infty$ functions near the origin with

$$g(0) = 0, \quad g'(0) > 0, \quad p(0) = 0, \quad p'(0) > 0,$$
$$F(0,a) = 0, \quad a \in \mathrm{R}^n.$$

Let $\alpha(x) = -x + O(x^2)$ satisfying $G(\alpha(x)) = G(x)$ for $|x| \ll 1$, where $G(x) = \int_0^x g(x)dx$.

**Theorem 4.** *Suppose $F$ is linear in $a$ and*

$$F(\alpha(x), a) - F(x,a) = \sum_{i\ge 1} B_i(a) x^i,$$
$$a = (a_1, \ldots, a_m)$$

*formally for $|x| \ll 1$.*

(1) *If*

$$B_{2j+1}(a) = 0, \quad j = 0, \ldots, k-1, \quad B_{2k+1}(a) \neq 0$$

*for a fixed* $a \in \mathrm{R}^m$, *then for this value of* $a$ *system (6) has a focus of order* $k$ *at the origin.*

(2) *If there exists* $1 \leq k \leq m-1$ *such that*

(i) $rank[(\partial(B_1, B_3, \ldots, B_{2k+1}))/\partial(a_1, a_2, \ldots, a_m)] = k+1$;

(ii) $F(\alpha(x), a) - F(x, a) = 0$ *when* $B_{2j+1} = 0$, $j = 0, 1, \ldots, k$.

*Then system (6) has Hopf cyclicity* $k$ *at the origin.*

*Proof of Theorem 1.* Consider system (3) with $n = 1$. By (5) we have

$$b_0 x + b_1 x^3 = -b_0 x(x^2 - 1), \quad b_0 < 0.$$

Rescaling system (3) by $y \to \sqrt{-b_0}y$, $t \to t/\sqrt{-b_0}$, we can suppose $b_0 = -1$. Hence (3) becomes

$$\dot{x} = y - \sum_{i=0}^{m} a_i x^{2i+1}, \quad \dot{y} = -x(x^2 - 1). \tag{7}$$

Let $u = x^2 - 1$, $v = y - y_0$, where $y_0 = \sum_{i=0}^m a_i$. Then on the half plane $x > 0$, system (7) can be further written as

$$\dot{u} = 2\sqrt{1+u}\left[v - \sqrt{1+u}\sum_{i=0}^{m} a_i(u+1)^i + y_0\right],$$
$$\dot{v} = -u\sqrt{1+u},$$

which is equivalent to

$$\begin{aligned} \dot{u} &= v - \left[\sqrt{1+u}\sum_{i=0}^{m} y_i u^i - y_0\right], \\ \dot{v} &= -\frac{1}{2}u \end{aligned} \tag{8}$$

on $u > -1$, where

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ & 1 & 2 & \cdots & m \\ & & 1 & \cdots & \frac{m(m-1)}{2} \\ & & & \ddots & \vdots \\ & & & & 1 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{pmatrix}. \tag{9}$$

Let

$$\begin{aligned} F(u, \beta) &= \sqrt{1+u}\sum_{i=0}^{m} y_i u^i - y_0, \\ \beta &= (y_0, y_1, \ldots, y_m) \in \mathrm{R}^{m+1}. \end{aligned} \tag{10}$$

The function $F$ is analytic for $|u| < 1$. Hence we can write

$$F(u, \beta) = \sum_{i=1}^{\infty} c_i u^i \quad \text{for } |u| < 1. \tag{11}$$

We first have

**Lemma 1.** *The function* $F$ *is even in* $u$ *for* $|u| < 1$ *if and only if* $\beta = 0$.

In fact, if $\beta = 0$ then $F = 0$ and hence it is even. Conversely, let $F$ be even, we want to prove $\beta = 0$. Otherwise, $y_k \neq 0$ and $y_{k+1} = \cdots = y_m = 0$ for some $k \leq m$. It follows from (10) that

$$\begin{aligned} (F + y_0)^2 &= (1+u)(y_0 + y_1 u + \cdots + y_k u^k)^2 \\ &= y_0^2 + y_0(y_0 + 2y_1)u + \cdots + y_k^2 u^{2k+1}. \end{aligned}$$

Since $F$ is even, then $(F + y_0)^2$ is even, implying $y_k = 0$, a contradiction. Therefore $\beta = 0$.

Further we prove

**Lemma 2.** *Let* $c_i$ *be given in (11). Then*

(i) $\beta = 0$ *if and only if* $c_{2j+1} = 0, j = 0, 1, \ldots, m$.

(ii) $\det[\partial(c_1, c_3, \ldots, c_{2m+1})/\partial(y_0, y_1, \ldots, y_m)] \neq 0$.

In fact, if $\beta = 0$, then by (10) we have $F = 0$. Hence, $c_{2j+1} = 0$ for all $j \geq 0$ by (11). Conversely, let $c_{2j+1} = 0, j = 0, 1, \ldots, m$. Then by (11) we can write

$$F(u, \beta) + y_0 = F_0(u) + u^{(2m+3)}F_1(u), \tag{12}$$

where

$$F_0(u) = \sum_{i=0}^{\infty} c_{2j}u^{2j}, \quad c_0 = y_0,$$

$$F_1(u) = \sum_{j=m}^{\infty} c_{2j+3}u^{2(j-m)}.$$

There are two cases to consider.

***Case 1.*** $F_0 = 0$. In this case, by (12) we have

$$(F + y_0)^2 = u^{2(2m+3)}F_1^2.$$

Note that $(F + y_0)^2$ is a polynomial having degree at most $2m + 1$. It implies $F_1 = 0$. Thus we have $F = 0$ in this case. Then $\beta = 0$ by Lemma 1.

***Case 2.*** $F_0 \neq 0$. In this case we can write

$$F_0 = u^{2k} \sum_{j=k}^{\infty} c_{2j} u^{2(j-k)}, \quad c_{2k} \neq 0, \quad k \geq 0. \quad (13)$$

By (12) we have

$$(F + y_0)^2 = F_0^2 + 2u^{2m+3} F_0 F_1 + u^{2(2m+3)} F_1^2. \quad (14)$$

If $F_1 \neq 0$, then

$$F_1 = c_{2n+3} u^{2(n-m)} + O(u^{2(n-m)+2}), \quad c_{2n+3} \neq 0, n \geq m. \quad (15)$$

It follows from (13)–(15) that

$$(F + y_0)^2 = (F_0^2 + u^{2(2m+3)} F_1^2) + 2c_{2k} c_{2n+3} u^{2l+3} + O(u^{2l+5}),$$

where $l = n + k \geq m$. Note that the function $F_0^2 + u^{2(2m+3)} F_1^2$ is even in $u$. The above equality implies that a nontrivial term of degree $2l + 3$ appears in the polynomial $(F + y_0)^2$. This contradicts that $(F + y_0)^2$ is of degree at most $2m + 1$. Hence, it must have $F_1 = 0$, and then by (12) $F + y_0 = F_0$ is even. Therefore, $\beta = 0$ by Lemma 1. Thus, the conclusion (i) follows.

By (10) and (11) each $c_j$ is linear in $\beta$. Hence

$$(c_1, c_3, \ldots, c_{2m+1})^T = Q(y_0, \ldots, y_m)^T,$$

where $Q$ is a constant matrix of order $m+1$. By conclusion (i) we have $\det Q \neq 0$. Hence, the conclusion (ii) follows.

Now we continue our proof for Theorem 1. For (8) we have $\alpha(u) = -u$. By (11) we have

$$F(\alpha(u), \beta) - F(u, \beta) = -\sum_{i=0}^{\infty} c_{2i+1} u^{2i+1}.$$

Hence, by Lemma 2 and Theorem 4 we know that system (8) has Hopf cyclicity $n$ at the origin. The proof of Theorem 1 is thus completed. ■

*Proof of Theorem 2.* Let

$$u = x^2 - 1, \quad v = y - \sum_{i=0}^{m} a_i = y - y_0.$$

Then similar to (8) we obtain from (4)

$$\dot{u} = v - \sum_{i=1}^{m} y_i u^i, \qquad \dot{v} = -\frac{1}{2} \sum_{i=1}^{n} z_i u^i \quad (16)$$

on $u > -1$, where

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{bmatrix} 1 & 2 & \cdots & m \\ & 1 & \cdots & \frac{m(m-1)}{2} \\ & & \ddots & \vdots \\ & & 1 & m \\ & & & 1 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix},$$

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix} = \begin{bmatrix} 1 & 2 & \cdots & n \\ & 1 & \cdots & \frac{n(n-1)}{2} \\ & & \ddots & \vdots \\ & & 1 & n \\ & & & 1 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

By (5) we have

$$z_1 = b_1 + 2b_2 + \cdots + nb_n > 0.$$

Noting (2) and comparing (16) with (1) we know that the Hopf cyclicity of (16) at the origin is $\hat{H}(m-1, n)$. Hence, the Hopf cyclicity of (4) at the point $A(1, y_0)$ is $\hat{H}(m-1, n)$. That is, $\hat{H}_e(m, n) = \hat{H}(m-1, n)$.

By the result of Lloyd and Lynch [1988] or Theorem 4, $\hat{H}(m-1, 1) = \lfloor (m-1)/2 \rfloor$. This follows that the maximal number of small-amplitude limit cycles of system (4) is $2\lfloor (m-1)/2 \rfloor$ for $n = 1$. Since $\hat{H}(m-1, 2) = \lfloor (2m-1)/3 \rfloor$ by Han [1999], the conclusion follows for the case of $n = 2$. This completes the proof of Theorem 2. ■

For the case of $(m, n) = (3, 1)$, system (4) becomes

$$\dot{x} = y - (a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6), \qquad \dot{y} = -b_1 x(x^2 - 1), \quad b_1 > 0. \quad (17)$$

Further, without loss of generality suppose $b_1 = 1, a_0 = 0$. The system (17) becomes

$$\dot{x} = y - (a_1 x^2 + a_2 x^4 + a_3 x^6), \qquad \dot{y} = -x(x^2 - 1). \quad (18)$$

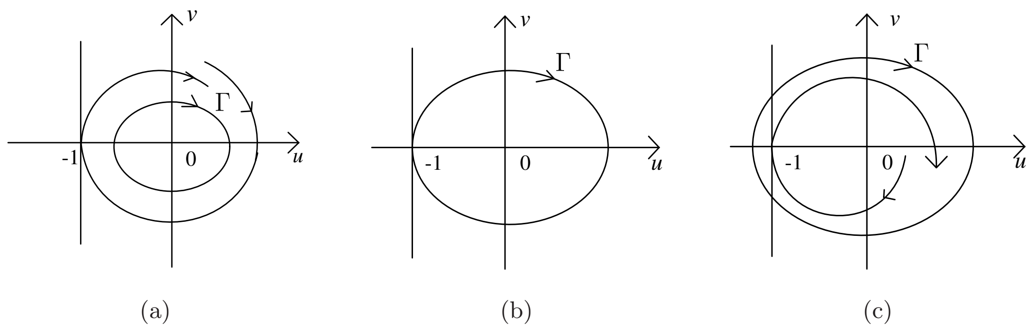


Fig. 1. Existence of the limit cycle $\Gamma(a_{11})$ for system (19) as $a_1$ is varied: (a) $a_{11} < a_1 < a_{10}$, (b) $a_1 = a_{11}$, and (c) $a_1 < a_{11}$.

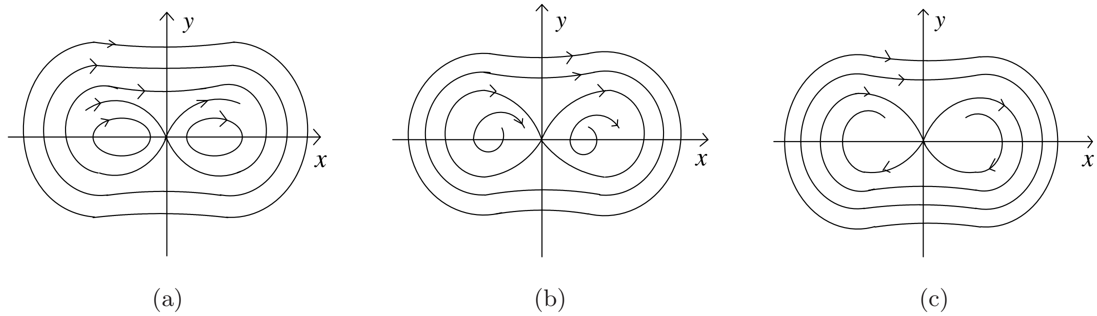


Fig. 2. Existence of the limit cycle $\Gamma(a_{11})$ for system (18) as $a_1$ is varied: (a) $a_{11} < a_1 < a_{10}$, (b) $a_1 = a_{11}$, and (c) $a_1 < a_{11}$.

Then the corresponding Eq. (16) has the form

$$\begin{aligned} \dot{u} &= v - u(a_1 + 2a_2 + 3a_3 + (a_2 + 3a_3)u + a_3u^2), \\ \dot{v} &= -\frac{1}{2}u. \end{aligned} \tag{19}$$

By the results of Lins–deMelo–Pugh [Lins *et al.*, 1977], system (19) has at most one limit cycle and it exists if and only if $a_3(a_1 + 2a_2 + 3a_3) < 0$. For definiteness, let $a_3 > 0$. Note that system (19) forms a rotated vector field with respect to $a_1$ (see [Han, 1999]). Hence, for $a_1 + 2a_2 + 3a_3 < 0$ and $a_1$ near $a_{10} = -2a_2 - 3a_3$, say, system (19) has a stable limit cycle, denoted by $\Gamma(a_1)$. The limit cycle expands as $a_1$ decreases. Thus, there is a unique value $a_{11}$, with $a_{11} < a_{10}$, such that the limit cycle $\Gamma(a_{11})$ is tangent to the line $u = -1$, and for $a_{11} < a_1 < a_{10}$ the limit cycle $\Gamma(a_1)$ is on the region $u > -1$. The above discussion is demonstrated in Fig. 1.

Let $v_0 = -(a_1 + a_2 + a_3)$. Denote by $\gamma$ the orbit of system (19) passing through the point $(-1, v_0)$. Then $\gamma = \Gamma(a_{11})$ and $\Lambda^+(\gamma) = \Gamma(a_1)$ for $a_{11} < a_1 < a_{10}$, or $a_1 < a_{11}$. Also, $\gamma$ is inside $\Gamma(a_1)$ if and only if $a_1 < a_{11}$. This corresponds to Fig. 2 for system (18).

Hence, we have proved the following result.

**Proposition 1.** *Let $a_3 > 0$ in (18). Then there exist $a_{10} = -2a_2 - 3a_3$ and $a_{11} < a_{10}$ such that (18) has exactly two limit cycles on the plane if and only if $a_{11} < a_1 < a_{10}$.*

Similarly, by Theorem 3.3.1 of Luo–Wang–Zhu–Han [Lu *et al.*, 1997] we have

**Proposition 2.** *The system (4) with $(m, n) = (3, 2)$ has at most two limit cycles on the plane.*

Then Theorem 3 follows.

## 3. Conclusion

In this paper, we have investigated two types of symmetric Liénard systems. It has been shown that the maximal number of small-amplitude limit cycles for one type of system is $2m$, for $n = 1$, while that for another type of system is $2\lfloor (m-1)/2 \rfloor$, for $n = 1$ and $2\lfloor (2m-1)/3 \rfloor$, for $n = 2$.

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