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Delay-Induced Triple-Zero Bifurcation in a Delayed Leslie-Type Predator–Prey Model with Additive Allee Effect

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In this paper, a Leslie-type predator–prey model with ratio-dependent functional response and Allee effect on prey is considered. We first study the existence of the multiple positive equilibria and their stability. Then we investigate the effect of delay on the distribution of the roots of characteristic equation and obtain the conditions for the occurrence of simple-zero, double-zero and triple-zero singularities. The formulations for calculating the normal form of the triple-zero bifurcation of the delay differential equations are derived. We show that, under certain conditions on the parameters, the system exhibits homoclinic orbit, heteroclinic orbit and periodic orbit.

Keywords: Predator–prey model; Allee effect; time delay; triple-zero bifurcation; stability.

1. Introduction

Study of the interaction between predators and their prey is one of the most popular problems in both ecology and mathematical ecology. While investigating these biological phenomena, there are many factors which affect the dynamical properties of biological and mathematical models. One of the familiar nonlinear factors is functional response. After the pioneer work of Lotka and Volterra [Lotka, 1925, 1956; Volterra, 1926], Holling [1965] suggested three different kinds of functional response for different types of species in modeling the phenomenon of predation, which made the standard Lotka–Volterra system more realistic. The Holling type functional response is also called prey-dependent functional response which only depends on the prey species. The predator–prey models with prey-dependent functional response have been widely studied [Freedman, 1980; Murray, 1989]. Recently, based upon experimental evidences and the analysis of collected field data, Arditi and Ginzburg [1989] have proposed the so-called ratio-dependent functional response, which can be roughly stated as that the per capita predator growth rate should be a function of the ratio of prey to predator abundance. It has been shown that the ratio-dependent functional response is especially applicable to the situation when predators have to search

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for food and therefore have to share or compete for food. Thus, the predator–prey models having ratio-dependent functional response can exhibit more realistic, complex dynamics than that of the traditional prey-dependent predator–prey models [Hsu *et al.*, 2001a, 2001b; Jost *et al.*, 1999; Kuang & Beretta, 1998; Kuang, 1999]. Recently, Turing instability, spatiotemporal dynamics and chaos for diffusive predator–prey models with ratio-dependent functional response have also been investigated in [Banerjee & Banerjee, 2012; Song & Zou, 2014a, 2014b].

Another important contribution to the study of the predator–prey models is the outstanding work by Leslie [1948] for improving the predator equation. He introduced a predator–prey model under the assumption that the reduction in a predator population has a reciprocal relationship with per capita availability of its preferred food, and the carrying capacity of the predator environment is proportional to the number of prey. This interesting formulation for the predator dynamics has also been extensively discussed by many researchers [Aguirre *et al.*, 2009a, 2009b; Chen *et al.*, 2009, 2012; Gupta & Chandra, 2013; Leslie & Gower, 1960].

In this paper, we mainly focus on a delayed Leslie-type predator–prey model with ratio-dependent functional response and additive Allee effect in the growth of the prey population, given in the form of

$$\begin{aligned}\dot{x}(t) &= \left[r\left(1-\frac{x(t)}{K}\right)-\frac{m}{x(t)+b}\right]x(t) \\ &\quad -\frac{\alpha x(t)y(t)}{ny(t)+x(t)}, \\ \dot{y}(t) &= s\left(1-\beta\frac{y(t-\tau)}{x(t-\tau)}\right)y(t),\end{aligned} \tag{1}$$

where the time delay $\tau \geq 0$ is introduced due to the fact that the reproduction of predator after consuming the prey is not instantaneous, but is mediated by some time lag required for gestation. Without the predator species, the natural growth equation for the prey is described by

$$\dot{x}(t) = \left[r\left(1-\frac{x(t)}{K}\right)-\frac{m}{x(t)+b}\right]x(t),$$

which is also identified as having an Additive Allee effect. This natural growth equation with an Additive Allee effect was first deduced in [Dennis, 1989; Stephens & Sutherland, 1999] and has been studied by Aguirre *et al.* [Aguirre *et al.*, 2009a, 2009b]. For more information on the parameters of model (1), see [Aguirre *et al.*, 2009a, 2009b; Banerjee & Banerjee, 2012] and references therein. Although there have been many published research works on the dynamics of the predator–prey models, to the best of our knowledge, there is little work on the dynamics of system (1). In this paper, the existence and stability of positive equilibria and the bifurcation phenomenon will be investigated. Especially, we focus on the triple-zero bifurcation. For the triple-zero bifurcation of delay differential equations, Campbell and Yuan [2008] and Qiao *et al.* [2010] have derived the corresponding normal form and determined how the coefficients of the normal form depend on the original parameters of the system. The method developed in [Campbell & Yuan, 2008; Qiao *et al.*, 2010] is based on the basic assumption that the equilibria of the system exist for any positive values of the parameters. In this case, the triple-zero bifurcation comes from the pitch-fork or transcritical bifurcation. However, in this paper, the triple-zero bifurcation comes from a saddle-node bifurcation and the basic assumption in [Campbell & Yuan, 2008] is not applicable. Motivated by our results obtained recently on the bifurcation related to the saddle-node type in delay differential equations [Jiang & Song, 2014; Jiang *et al.*, 2016], we derive the formulas for calculating the normal form of triple-zero bifurcation related to the saddle-node type for the delay differential equations.

The rest of the paper is organized as follows. In Sec. 2, the existence and stability of system (1) without delay are studied. In Sec. 3, the conditions are obtained on delay-induced Bogdanov–Takens bifurcation and triple-zero bifurcation. In Sec. 4, the normal form associated with triple-zero bifurcation is derived. In Sec. 5, some numerical simulations are presented to illustrate and extend the theoretical results. Finally, the paper ends by a conclusion section.

2. Linear Stability and Bifurcations for the Case of $\tau = 0$

For the biological model to be physically meaningful, we are only interested in the dynamics of system (1) near an interior equilibrium in the first quadrant of the x–y plane. By a simple calculation, we have the following lemma on the positive equilibrium of system (1).

Lemma 1. *Assume that* $r(n+\beta)-\alpha>0$, *and let*

$$m_* = b\left(r - \frac{\alpha}{n+\beta}\right),\qquad m^* = \frac{K}{4r}\left(r - \frac{\alpha}{n+\beta} + \frac{br}{K}\right)^2 \tag{2}$$

and

$$x^*_\pm = \frac{K}{2r}\left[\left(r - \frac{\alpha}{n+\beta} - \frac{br}{K}\right) \pm \sqrt{\triangle}\right],\quad y^*_\pm = \frac{x^*_\pm}{\beta},$$

with $\triangle = (r-\alpha/(n+\beta)-br/K)^2-(4r/K)(m-br+\alpha b/(n+\beta))$. *Then we have the following results.*

(i) *When either* $m<m_*$, *or* $m=m_*$ *when* $K>rb(n+\beta)/(r(n+\beta)-\alpha)$, *system (1) has a unique positive equilibrium* $E^*_+=(x^*_+,y^*_+)$;
(ii) *When* $m_*<m<m^*$ *and* $K>rb(n+\beta)/(r(n+\beta)-\alpha)$, *system (1) has two positive equilibria* $E^*_+=(x^*_+,y^*_+)$ *and* $E^*_-=(x^*_-,y^*_-)$;
(iii) *When* $m=m^*$ *and* $K>rb(n+\beta)/(r(n+\beta)-\alpha)$, *system (1) has a unique positive equilibrium* $E^*=(x^*,y^*)$ *with*

$$x^* = \frac{K}{2r}\left(r - \frac{\alpha}{n+\beta} - \frac{br}{K}\right),\quad y^* = \frac{x^*_+}{\beta}.$$

By a direct calculation, we obtain the characteristic equation for these equilibria as follows:

$$\lambda^2 + T\lambda + D = 0,$$

where

$$T = \begin{cases} \dfrac{x^*_+\sqrt{\Delta}}{x^*_++b} - \dfrac{\alpha\beta}{(n+\beta)^2} + s, & \text{for } E^*_+,\\[2ex] -\dfrac{\alpha\beta}{(n+\beta)^2} + s, & \text{for } E^*,\\[2ex] -\dfrac{x^*_-\sqrt{\Delta}}{x^*_-+b} - \dfrac{\alpha\beta}{(n+\beta)^2} + s, & \text{for } E^*_-,\end{cases}$$

$$D = \begin{cases} \dfrac{sx^*_+\sqrt{\Delta}}{x^*_++b} > 0, & \text{for } E^*_+,\\[2ex] 0, & \text{for } E^*,\\[2ex] -\dfrac{sx^*_-\sqrt{\Delta}}{x^*_-+b} < 0, & \text{for } E^*_-.\end{cases}$$

Then based on the signs of T and D, we can easily obtain the following result.

Lemma 2. *Assume that* $r(n+\beta)-\alpha>0$ *and let*

$$s^* = \frac{\alpha\beta}{(n+\beta)^2},\quad s_* = s^* - \frac{x^*_+\sqrt{\Delta}}{x^*_++b}. \tag{3}$$

Then for system (1) with $\tau=0$, *the following holds.*

(i) *The positive equilibrium* E^*_- *is a saddle.*
(ii) *The positive equilibrium* E^*_+ *is stable for* $s>s_*$ *and unstable for* $s<s_*$. *System (1) with* $\tau=0$ *undergoes a Hopf bifurcation at* $s=s_*$ *near the positive equilibrium* E^*_+.
(iii) *When* $K>rb(n+\beta)/(r(n+\beta)-\alpha)$ *and* $s\neq s^*$, *system (1) with* $\tau=0$ *undergoes a saddle-node bifurcation at* $m=m^*$ *near the positive equilibrium* E^*.

3. Delay-Induced Triple-Zero Bifurcation

In this section, we investigate the effect of the delay on the dynamics of system (1). Especially, we are interested in the delay-induced triple-zero bifurcation near the positive equilibrium E^*. Linearizing system (1) at the equilibrium E^* yields the following linear system,

$$\dot{x}(t) = \frac{\alpha\beta}{(n+\beta)^2}x(t) - \frac{\alpha\beta^2}{(n+\beta)^2}y(t),$$
$$\dot{y}(t) = \frac{s}{\beta}x(t-\tau) - sy(t-\tau),$$

for which the corresponding characteristic equation is given by

$$P(\lambda,\tau) = \lambda^2 - \left(\frac{\alpha\beta}{(n+\beta)^2} - se^{-\lambda\tau}\right)\lambda = 0. \tag{4}$$

Lemma 3. *If* $m=m^*$, $K>rb(n+\beta)/(r(n+\beta)-\alpha)$ *and* $\tau^*=1/s^*$, *then for the positive equilibrium* E^*, *we have*

(i) *when* $s\neq s^*$, $\lambda=0$ *is a simple-zero eigenvalue of Eq. (4);*
(ii) *when* $s=s^*$ *and* $\tau\neq\tau^*$, $\lambda=0$ *is a double-zero eigenvalue of Eq. (4);*
(iii) *when* $s=s^*$ *and* $\tau=\tau^*$, $\lambda=0$ *is a triple-zero eigenvalue of Eq. (4).*

Proof. For the characteristic polynomial $P(\lambda,\tau)$, taking the partial derivative with respective to λ

yields

$$\frac{\partial P(\lambda,\tau)}{\partial\lambda} = 2\lambda - \frac{\alpha\beta}{(n+\beta)^2} + se^{-\lambda\tau}(1-\lambda\tau).$$

It is easy to see that $\partial P(0,\tau)/\partial\lambda = 0$ if and only if $s = s^*$. Moreover, we also have

$$\frac{\partial^2 P(\lambda,\tau)}{\partial^2\lambda} = 2 + se^{-\lambda\tau}(\lambda\tau^2 - 2\tau)$$

and

$$\frac{\partial^3 P(\lambda,\tau)}{\partial^3\lambda} = se^{-\lambda\tau}(3\tau^2 - \lambda\tau^3),$$

which implies that $\partial^2 P(0,\tau)/\partial^2\lambda = 0$ if and only if $s = s^*$, $\tau = 1/s^*$, but $\partial^3 P(0,\tau)/\partial^3\lambda \neq 0$.

This completes the proof. ■

It is easy to show that when $s = s^*$ and $\tau = \tau^*$, except for the triple-zero eigenvalue, all the other eigenvalues of Eq. (4) have strictly negative real parts. Thus, at $s = s^*$ and $\tau = \tau^*$, system (1) undergoes a triple-zero bifurcation from the equilibrium E^*. In the next section, we derive a versal unfolding for system (1) at the triple-zero singularity.

4. Normal Form of Triple-Zero Bifurcation

In this section, we employ the similar method used by Faria and Magalhães [1995a, 1995b] to derive the normal forms on the center manifold, which can be used to study the dynamics near the triple-zero singularity. Note that E^* is the unique interior equilibrium under the conditions, $m = m^*$ and $K > rb(n+\beta)/(r(n+\beta)-\alpha)$. For the case of simple-zero eigenvalue, we can conclude that system (1) undergoes a saddle-node bifurcation at $m = m^*$. Thus, the existence of equilibrium of system (1) near E^* is determined by the parameter m, so we cannot directly apply the technique developed in [Campbell & Yuan, 2008; Qiao *et al.*, 2010], where the basic assumption is that a unique equilibrium always exists for all parameter values.

For simplicity, we take the time scaling $\bar{t} \mapsto t/\tau$ to normalize the delay, and introduce three new bifurcation parameters μ_1, μ_2, μ_3 by setting $\tau = \tau^* + \mu_1$, $s = s^* + \mu_2$ and $m = m^* + \mu_3$, such that system (1) exhibits a triple-zero bifurcation at $(\mu_1, \mu_2, \mu_3) = (0, 0, 0)$ in the neighborhood of E^*. Then, dropping the bar on $\bar{t}$ we obtain

$$\begin{aligned}\dot{x}(t) &= (\tau^* + \mu_1)\left[r\left(1 - \frac{x(t)}{K}\right) - \frac{m^* + \mu_3}{x(t) + b} \right.\\ &\qquad \left. - \frac{\alpha y(t)}{ny(t) + x(t)}\right]x(t),\\ \dot{y}(t) &= (\tau^* + \mu_1)(s^* + \mu_2)\left(1 - \beta\frac{y(t-1)}{x(t-1)}\right)y(t).\end{aligned} \tag{5}$$

Note that the parameter μ_3 affects the number of equilibria of system (5). To apply the method developed by Faria and Magalhães [1995a, 1995b], based on our recent results on the bifurcation related to the saddle-node type in delay differential equations [Jiang & Song, 2014; Jiang *et al.*, 2016], we, as commonly used in center manifold reduction, introduce the equation $\dot{\mu}_3 = 0$ into (5). Further, let $z_1(t) = x(t) - x^*$ and $z_2(t) = y(t) - y^*$. Then system (5) becomes

$$\begin{aligned}\dot{z}_1(t) &= (\tau^* + \mu_1)\left[\frac{\alpha\beta}{(n+\beta)^2}z_1(t)\right.\\ &\qquad - \frac{\alpha\beta^2}{(n+\beta)^2}z_2(t) - \frac{x^*}{x^* + b}\mu_3\\ &\qquad \left. + \sum_{i+j+k\geq 2}\frac{1}{i!j!k!}f^{(1)}_{ijk}z_1^i(t)z_2^j(t)\mu_3^k\right],\\ \dot{z}_2(t) &= (\tau^* + \mu_1)(s^* + \mu_2)\left[\frac{1}{\beta}z_1(t-1) - z_2(t-1)\right.\\ &\qquad \left. + \sum_{i+j+k\geq 2}\frac{1}{i!j!k!}f^{(2)}_{ijk}z_1^i(t-1)z_2^j(t-1)z_2^k(t)\right],\end{aligned}$$

where

$$\begin{aligned}f^{(1)} &= \left[r\left(1 - \frac{x}{K}\right) - \frac{m^* + \mu_3}{x + b} - \frac{\alpha y}{ny + x}\right]x,\\ f^{(2)} &= \left(1 - \beta\frac{y_1}{x_1}\right)y,\\ f^{(1)}_{ijk} &= \left.\frac{\partial^{i+j+k}f^{(1)}}{\partial x^i \partial y^j \partial\mu_3^k}\right|_{(x^*,y^*,0)},\\ f^{(2)}_{ijk} &= \left.\frac{\partial^{i+j+k}f^{(2)}}{\partial x_1^i \partial y_1^j \partial y^k}\right|_{(x^*,y^*,0)}.\end{aligned}$$

Now, let $z(t) = (z_1(t), z_2(t), \mu_3)^T \in \mathbb{R}^3$, $\mu = (\mu_1, \mu_2)$ and $C_3 = C([-1, 0]; \mathbb{R}^3)$, which is the Banach space of continuous mappings from $[-1, 0]$ to $\mathbb{R}^3$ with supremum norm. Consider the following retarded functional differential equations with parameters in the phase space C_3,

$$\dot{z}(t) = L_0(z_t) + L_1(\mu)z_t + F(z_t, \mu), \tag{6}$$

where for $\varphi = (\varphi_1, \varphi_2, \mu_3)^T \in C_3$ and $\gamma = (\tau^* x^*)/(x^* + b)$

$$L_0(\varphi) = \begin{pmatrix} \varphi_1(0) - \beta\varphi_2(0) - \gamma\mu_3 \\ \frac{1}{\beta}\varphi_1(-1) - \varphi_2(-1) \\ 0 \end{pmatrix}, \tag{7}$$

$$L_1(\mu)\varphi = \begin{pmatrix} \mu_1 s^*(\varphi_1(0) - \beta\varphi_2(0) - \gamma\mu_3) \\ (\mu_1 s^* + \mu_2\tau^*)\left(\frac{1}{\beta}\varphi_1(-1) - \varphi_2(-1)\right) \\ 0 \end{pmatrix} \tag{8}$$

and

$$F(\varphi, \mu) = \sum_{l\geq 2} \frac{1}{l!} F_l(\varphi, \mu)$$

$$= \mu_1\mu_2 \begin{pmatrix} 0 \\ \frac{1}{\beta}\varphi_1(-1) - \varphi_2(-1) \\ 0 \end{pmatrix} + (\tau^* + \mu_1) \begin{pmatrix} \sum_{i+j+k\geq 2} \frac{1}{i!j!k!} f_{ijk}^{(1)} \varphi_1^i(0)\varphi_2^j(0)\mu_3^k \\ (s^* + \mu_2)\left(\sum_{i+j+k\geq 2} \frac{1}{i!j!k!} f_{ijk}^{(2)} \varphi_1^i(-1)\varphi_2^j(-1)\varphi_2^k(0)\right) \\ 0 \end{pmatrix}. \tag{9}$$

Therefore, the equilibrium $(x^*, y^*, 0)$ is translated to the origin and the zero equilibrium always exists for μ_1, μ_2 and any other parameters. From (7), the linearized system of (6) has the characteristic matrix

$$\Gamma(\lambda) = \lambda I - L(e^{\lambda \cdot} I) = \begin{pmatrix} \lambda - 1 & \beta & \gamma \\ -\frac{1}{\beta}e^{-\lambda} & \lambda + e^{-\lambda} & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

which in turn yields the characteristic equation:

$$\det \Gamma(\lambda) = \lambda^2(\lambda + e^{-\lambda} - 1) = 0. \tag{10}$$

From the analysis given in Sec. 3, it is obvious that $\lambda = 0$ is the root of multiplicity 4 of the characteristic equation (10). Assuming A_0 is the infinitesimal generator of $\dot{z}(t) = L_0(z_t)$, consider $\Lambda = \{0\}$ and denote by P the invariant space of A_0 associated with the eigenvalue $\lambda = 0$ and P^* the space adjoint with P. Using the formal adjoint theory for RFDEs in [Hale & Verduyn Lunel, 1993; Xu & Huang, 2008], we can decompose C_3 by Λ as $C_3 = P \oplus Q$, where $Q = \{\varphi \in C_3 : \langle\psi, \varphi\rangle = 0, \forall\, \psi \in P^*\}$. Let $\Phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta), \phi_4(\theta))$ be the bases for P, where

$$\phi_1(\theta) = \phi_1^0,$$
$$\phi_2(\theta) = \phi_2^0 + \theta\phi_1^0,$$
$$\phi_3(\theta) = \phi_3^0 + \theta\phi_2^0 + \frac{\theta^2}{2!}\phi_1^0,$$
$$\phi_4(\theta) = \phi_4^0 + \theta\phi_3^0 + \frac{\theta^2}{2!}\phi_2^0 + \frac{\theta^3}{3!}\phi_1^0, \quad -1 \leq \theta \leq 0,$$

and ϕ_1^0, ϕ_2^0, ϕ_3^0, $\phi_4^0 \in \mathbb{R}^3$ satisfy the following linear equations:

$$\begin{aligned}
&\text{(1a)}\ \ \Gamma(0)\phi_1^0 = 0,\\
&\text{(1b)}\ \ \Gamma(0)\phi_2^0 + \Gamma'(0)\phi_1^0 = 0,\\
&\text{(1c)}\ \ \Gamma(0)\phi_3^0 + \Gamma'(0)\phi_2^0 + \frac{1}{2}\Gamma''(0)\phi_1^0 = 0,\\
&\text{(1d)}\ \ \Gamma(0)\phi_4^0 + \Gamma'(0)\phi_3^0 + \frac{1}{2!}\Gamma''(0)\phi_2^0\\
&\qquad + \frac{1}{3!}\Gamma'''(0)\phi_1^0 = 0,
\end{aligned}$$

which yield

$$\phi_1^0 = \begin{pmatrix}\beta\\1\\0\end{pmatrix},\quad \phi_2^0 = \begin{pmatrix}\beta\\0\\0\end{pmatrix},$$

$$\phi_3^0 = \begin{pmatrix}0\\-1\\0\end{pmatrix},\quad \phi_4^0 = \begin{pmatrix}0\\ \frac{1}{2}\\ -\frac{\beta}{2\gamma}\end{pmatrix}.$$

Denote $C_3^* = C([0,1], \mathbb{R}^{3*})$, where $\mathbb{R}^{3*}$ is the three-dimensional vector space of row vectors. The bases $\Psi(s) = \mathrm{col}(\psi_1(s), \psi_2(s), \psi_3(s), \psi_4(s))$ for the dual space P^* in C_3^* can be chosen as

$$\begin{aligned}
&\psi_4(s) = \psi_4^0,\\
&\psi_3(s) = \psi_3^0 - s\psi_4^0,\\
&\psi_2(s) = \psi_2^0 - s\psi_3^0 + \frac{s^2}{2!}\psi_4^0,\\
&\psi_1(s) = \psi_1^0 - s\psi_2^0 + \frac{s^2}{2!}\psi_3^0 - \frac{s^3}{3!}\psi_4^0, \quad 0 \le s \le 1
\end{aligned}$$

and ψ_1^0, ψ_2^0, ψ_3^0, $\psi_4^0 \in C^*$ satisfying the following linear equations:

$$\begin{aligned}
&\text{(2a)}\ \ \psi_4^0\Gamma(0) = 0,\\
&\text{(2b)}\ \ \psi_3^0\Gamma(0) + \psi_4^0\Gamma'(0) = 0,\\
&\text{(2c)}\ \ \psi_2^0\Gamma(0) + \psi_3^0\Gamma'(0) + \frac{1}{2!}\psi_4^0\Gamma''(0) = 0,\\
&\text{(2d)}\ \ \psi_1^0\Gamma(0) + \psi_2^0\Gamma'(0) + \frac{1}{2!}\psi_3^0\Gamma''(0)\\
&\qquad + \frac{1}{3!}\psi_4^0\Gamma'''(0) = 0,
\end{aligned}$$

such that $\langle\Psi(s), \Phi(\theta)\rangle = I_4$, which yield

$$\begin{aligned}
\psi_4^0 &= \left(0, 0, -\frac{2\gamma}{\beta}\right),\\
\psi_3^0 &= \left(\frac{2}{\beta}, -2, \frac{4\gamma}{3\beta}\right),\\
\psi_2^0 &= \left(-\frac{4}{3\beta}, \frac{4}{3}, -\frac{43\gamma}{18\beta}\right),\\
\psi_1^0 &= \left(\frac{43}{18\beta}, -\frac{25}{18}, \frac{323\gamma}{135\beta}\right).
\end{aligned}$$

It is known that $\dot{\Phi} = \Phi B$ and $-\dot{\Psi} = B\Psi$, where B is given by

$$B = \begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{pmatrix}.$$

To apply the method of Faria and Magalhães [1995a, 1995b], it is necessary to enlarge the phase space C_3 by considering the space BC_3 of functions from $[-1, 0]$ into $\mathbb{R}^3$ which are uniformly continuous on $[-1, 0]$ and with a jump discontinuity at 0. Then the projection of C_3 upon P, associated with the decomposition $C_3 = P \oplus Q$, is now replaced by $\pi : BC_3 \to P$, which leads to the decomposition $BC_3 = P \oplus \mathrm{Ker}\,\pi$. Let

$$\tilde{x} = \begin{pmatrix}x\\ \tilde{\mu}\end{pmatrix},\quad x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},$$

$$\tilde{y} = \begin{pmatrix}y\\0\end{pmatrix},\quad y = \begin{pmatrix}y_1\\y_2\end{pmatrix}$$

with $\tilde{\mu} = -(2\gamma/\beta)\mu_3$. We now decompose $z_t = (z_1(t), z_2(t), \mu_3)^T = \Phi\tilde{x} + \tilde{y}$ with $\tilde{x} \in \mathbb{R}^4$, $\tilde{y} \in C_3^1 \cap \mathrm{Ker}\,\pi = C_3^1 \cap Q \equiv Q^1$. Hence, system (6) is equivalent to the system,

$$\begin{aligned}
\dot{\tilde{x}} &= B\tilde{x} + \Psi(0)[L_1(\mu)(\Phi\tilde{x} + \tilde{y}) + F(\Phi\tilde{x} + \tilde{y}, \mu)],\\
\dot{\tilde{y}} &= A_{Q^1}\tilde{y} + (I - \pi)X_0[L_1(\mu)(\Phi\tilde{x} + \tilde{y})\\
&\quad + F(\Phi\tilde{x} + \tilde{y}, \mu)],
\end{aligned} \tag{11}$$

where $X_0 = X_0(\theta)$ is defined by

$$X_0(\theta) = \begin{cases} I_3, & \theta = 0,\\ 0, & -1 \le \theta < 0.\end{cases}$$

We write the formal Taylor expansion

$$\begin{aligned}&\Psi(0)[L_1(\mu)(\Phi\tilde{x}+\tilde{y})+F(\Phi\tilde{x}+\tilde{y},\mu)]\\&\quad=\sum_{j\ge 2}\frac{1}{j!}f_j^1(\tilde{x},\tilde{y},\mu),\\&(I-\pi)X_0[L_1(\mu)(\Phi\tilde{x}+\tilde{y})+F(\Phi\tilde{x}+\tilde{y},\mu)]\\&\quad=\sum_{j\ge 2}\frac{1}{j!}f_j^2(\tilde{x},\tilde{y},\mu),\end{aligned}\tag{12}$$

where $f_j^i(\tilde{x},\tilde{y},\mu)$ $(i=1,2)$ are homogeneous polynomials in variables $(\tilde{x},\tilde{y},\mu)$ of degree j with coefficients in the Banach space $\mathbb{R}^4\times\operatorname{Ker}\pi$. Let $V_j^6(\mathbb{R}^4)$ denote the linear space of the homogeneous polynomials of degree j in six real variables $(x_1,x_2,x_3,\tilde{\mu},\mu_1,\mu_2)$. For $j\ge 2$, M_j^1 denote the operator associated with these changes of variables acted in $V_j^6(\mathbb{R}^4)$ as

$$(M_j^1p)(\tilde{x},\mu)=D_{\tilde{x}}p(\tilde{x},\mu)B\tilde{x}-Bp(\tilde{x},\mu),\quad p(\tilde{x},\mu)\in V_j^6(\mathbb{R}^4).\tag{13}$$

Following the procedure described in [Faria and Magalhães, 1995a, 1995b], after the successive transformations of variables, system (11) is transformed into the normal form on the center manifold of the origin as

$$\dot{\tilde{x}}=B\tilde{x}+\frac{1}{2!}g_2^1(\tilde{x},0,\mu)+\frac{1}{3!}g_3^1(\tilde{x},0,\mu)+\text{h.o.t.},\tag{14}$$

where $g_j^1=(I-P_{I,j}^1)\tilde{f}_j^1(\tilde{x},0,\mu)\in\operatorname{Im}(M_j^1)^c$, $\tilde{f}_j^1$ denotes the terms of order j in $(\tilde{x},\mu)$ after the previous transformations of variables, and $P_{I,j}^1\tilde{f}_j^1$ denotes the projection of $\tilde{f}_j^1$ on the image space $\operatorname{Im}(M_j^1)$ of the operator M_j^1. For $p(\tilde{x},\mu)=(p_1(\tilde{x},\mu),p_2(\tilde{x},\mu),p_3(\tilde{x},\mu),p_4(\tilde{x},\mu))^T\in V_2^6(\mathbb{R}^4)$, it follows from (13) that

$$M_2^1\begin{pmatrix}p_1\\p_2\\p_3\\p_4\end{pmatrix}=\begin{pmatrix}\frac{\partial p_1}{\partial x_1}x_2+\frac{\partial p_1}{\partial x_2}x_3+\frac{\partial p_1}{\partial x_3}\tilde{\mu}-p_2\\ \frac{\partial p_2}{\partial x_1}x_2+\frac{\partial p_2}{\partial x_2}x_3+\frac{\partial p_2}{\partial x_3}\tilde{\mu}-p_3\\ \frac{\partial p_3}{\partial x_1}x_2+\frac{\partial p_3}{\partial x_2}x_3+\frac{\partial p_3}{\partial x_3}\tilde{\mu}-p_4\\ \frac{\partial p_4}{\partial x_1}x_2+\frac{\partial p_4}{\partial x_2}x_3+\frac{\partial p_4\xi}{\partial x_3}\tilde{\mu}\end{pmatrix},\quad\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}\in V_2^4(\mathbb{R}^3).\tag{15}$$

Note that $\Psi(0)=\operatorname{col}(\psi_1^0,\psi_2^0,\psi_3^0,\psi_4^0)$, $\psi_4^0=(0,0,-2\gamma/\beta)$. From (8) and (9), the equation for $\tilde{\mu}$ in (11) is $\dot{\tilde{\mu}}=0$. Moreover, note that the parameter $\tilde{\mu}$ is not a function of time, so we can consider the space $V_2^6\binom{\mathbb{R}^3}{0}$ instead of $V_2^6(\mathbb{R}^4)$. Then from (15), we decompose $V_2^6\binom{\mathbb{R}^3}{0}=\operatorname{Im}(M_2^1)\oplus\operatorname{Im}(M_2^1)^c$, with a possible choice for the basis of $\operatorname{Im}(M_2^1)$, given by

$$\begin{pmatrix}x_1x_2\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}x_2x_3\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}x_3\tilde{\mu}\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}x_2^2+x_1x_3\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}x_2\tilde{\mu}\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}x_2\mu_i\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}\tilde{\mu}^2\\0\\0\\0\end{pmatrix},$$

$$\begin{pmatrix}x_3\mu_i\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}\tilde{\mu}\mu_i\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_1^2\\2x_1x_2\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_2^2\\2x_2x_3\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_3^2\\2x_3\tilde{\mu}\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_1x_2\\x_2^2+x_1x_3\\0\\0\end{pmatrix},$$

$$\begin{pmatrix}\mu_i^2\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_1x_3\\x_2x_3+x_1\tilde{\mu}\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_1\tilde{\mu}\\x_2\tilde{\mu}\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_1\mu_i\\x_2\mu_i\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_2\tilde{\mu}\\x_3\tilde{\mu}\\0\\0\end{pmatrix},\ \begin{pmatrix}-x_2\mu_i\\x_3\mu_i\\0\\0\end{pmatrix},$$

$$\begin{pmatrix} -x_2x_3 \\ x_3^2+x_2\tilde{\mu} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_3\tilde{\mu} \\ \tilde{\mu}^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -x_3\mu_i \\ \tilde{\mu}\mu_i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \mu_1\mu_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_1^2 \\ 2x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_2^2 \\ 2x_2x_3 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ -x_3^2 \\ 2x_3\tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_i^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_1x_2 \\ x_2^2+x_1x_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_1x_3 \\ x_2x_3+x_1\tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_1\tilde{\mu} \\ x_2\tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_1\mu_i \\ x_2\mu_i \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ -x_2x_3 \\ x_3^2+x_2\tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_2\tilde{\mu} \\ x_3\tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_2\mu_i \\ x_3\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_3\tilde{\mu} \\ \tilde{\mu}^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -x_3\mu_i \\ \tilde{\mu}\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \mu_1\mu_2 \\ 0 \\ 0 \end{pmatrix}$$

and that for the complementary space of $\mathrm{Im}(M_2^1)$ in $V_2^6\binom{\mathbb{R}^3}{0}$, given by

$$\mathrm{Im}(M_2^1)^c = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 0 \\ x_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_2^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mu_i^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1x_3 \\ 0 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 0 \\ 0 \\ x_1\tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_2\tilde{\mu} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_1\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_2\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ x_3\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \mu_1\mu_2 \\ 0 \end{pmatrix}\right\},$$

with $i = 1, 2$. According to (8), (9) and (12), we have

$$\begin{aligned}
\frac{1}{2!}f_2^1(\tilde{x},0,\mu) &= \Psi(0)\left[L_1(\mu)(\Phi\tilde{x}) + \frac{1}{2!}F_2(\Phi\tilde{x},0)\right] \\
&= \Psi(0)\begin{pmatrix} \mu_1 s^*(\varphi_1(0) - \beta\varphi_2(0) - \gamma\mu_3) \\ (\mu_1 s^* + \mu_2\tau^*)\left(\frac{1}{\beta}\varphi_1(-1) - \varphi_2(-1)\right) \\ 0 \end{pmatrix} \\
&\quad + \tau^*\Psi(0)\begin{pmatrix} \frac{1}{2}f_{200}^{(1)}\varphi_1^2(0) + f_{110}^{(1)}\varphi_1(0)\varphi_2(0) + f_{101}^{(1)}\varphi_1(0)\mu_3 + \frac{1}{2}f_{020}^{(1)}\varphi_2^2(0) \\ s^*\left(\frac{1}{2}f_{200}^{(2)}\varphi_1^2(-1) + f_{110}^{(2)}\varphi_1(-1)\varphi_2(-1)\right. \\ \left. + f_{101}^{(2)}\varphi_1(-1)\varphi_2(0) + f_{011}^{(2)}\varphi_2(-1)\varphi_2(0)\right) \\ 0 \end{pmatrix},
\end{aligned}$$

where

$$\varphi_1(0) = \beta(x_1 + x_2), \quad \varphi_2(0) = x_1 - x_3 + \frac{1}{2}\tilde{\mu},$$

$$\mu_3 = -\frac{\beta}{2\gamma}\tilde{\mu}, \quad \varphi_1(-1) = \beta\left(x_1 - \frac{1}{2}x_3 + \frac{1}{3}\tilde{\mu}\right),$$

$$\varphi_2(-1) = x_1 - x_2 - \frac{1}{2}x_3 + \frac{4}{3}\tilde{\mu}.$$

Then by (14) and removing the auxiliary equation introduced for handing the dependent parameter μ_3, the normal form of (5) for the positive equilibrium $E(x^*, y^*)$ becomes

$$\begin{aligned}
\dot{x}_1 &= x_2 + \text{h.o.t.},\\
\dot{x}_2 &= x_3 + \text{h.o.t.},\\
\dot{x}_3 &= \kappa_0 + \kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3 + \tilde{A}_1 x_1 x_3\\
&\quad + \tilde{A}_2 x_1 x_2 + \tilde{A}_3 x_1^2 + \tilde{A}_4 x_2^2 + \text{h.o.t.},
\end{aligned} \tag{16}$$

where

$$\kappa_0 = \tilde{\mu} = -\frac{2\gamma}{\beta}\mu_3,$$

$$\kappa_1 = \left(-\frac{8r\beta\gamma}{9K} - \frac{4n\beta}{9(n+\beta)x^*} + \frac{b\beta}{x^*(x^*+b)} - \frac{\beta}{3x^*}\right)\tilde{\mu},$$

$$\kappa_2 = \left(\frac{43r\beta\gamma}{27K} - \frac{145n\beta}{27(n+\beta)x^*} + \frac{b\beta}{3x^*(x^*+b)} - \frac{\beta}{9x^*}\right)\tilde{\mu} - 2\tau^*\mu_2,$$

$$\kappa_3 = 2s^*\mu_1 + \frac{4}{3}\tau^*\mu_2,$$

$$\tilde{A}_1 = -\frac{19r\beta\gamma}{9K}, \quad \tilde{A}_2 = -\frac{4r\beta\gamma}{3K}, \quad \tilde{A}_3 = -\frac{2r\beta\gamma}{K},$$

$$\tilde{A}_4 = -\frac{37r\beta\gamma}{9K} + \frac{2n\beta}{(n+\beta)x^*}.$$

Now we make a shift of the coordinates,

$$x_1 \to x_1 - \frac{\kappa_1}{2\tilde{A}_3}, \quad x_2 \to x_2, \quad x_3 \to x_3,$$

to obtain the normal form of system (16):

$$\begin{aligned}
\dot{x}_1 &= x_2 + \text{h.o.t.},\\
\dot{x}_2 &= x_3 + \text{h.o.t.},\\
\dot{x}_3 &= \varepsilon_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + \tilde{A}_1 x_1 x_3\\
&\quad + \tilde{A}_2 x_1 x_2 + \tilde{A}_3 x_1^2 + \tilde{A}_4 x_2^2 + \text{h.o.t.},
\end{aligned}$$

where $\varepsilon_1 = \kappa_0 - \kappa_1^2/(4\tilde{A}_3)$, $\varepsilon_2 = \kappa_2 - (\tilde{A}_2/(2\tilde{A}_3))\kappa_1$, $\varepsilon_3 = \kappa_3 - (\tilde{A}_1/(2\tilde{A}_3))\kappa_1$. Using the method developed in [Gamero *et al.*, 1999] and [Freire *et al.*, 2002], we obtain the following truncated hypernormal form up to second order

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_3,\\
\dot{x}_3 &= \varepsilon_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3 + A_1 x_1 x_3\\
&\quad + A_2 x_1 x_2 - \frac{1}{2}x_1^2,
\end{aligned} \tag{17}$$

where $A_1 = -\tilde{A}_1/(2\tilde{A}_3)$, $A_2 = -\tilde{A}_2/(2\tilde{A}_3)$. It is easy to show that system (17) undergoes three codimension-one bifurcations: a saddle-node bifurcation from the origin at the critical point $\varepsilon_1 = 0$, and two Hopf bifurcations from the two nontrivial equilibria $(x_\pm, 0, 0)$ at the critical point, defined by $(\varepsilon_3 + A_1 x_\pm)(\varepsilon_2 + A_2 x_\pm) = x_\pm$, $\varepsilon_2 + A_2 x_\pm < 0$, $\varepsilon_1 > 0$, where $x_\pm = \pm\sqrt{2\varepsilon_1}$. Also, system (17) exhibits two codimension-two bifurcations from the origin: a saddle-node Hopf bifurcation at $\varepsilon_1 = \varepsilon_3 = 0$, $\varepsilon_2 < 0$ and a Bogdanov–Takens bifurcation at $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 \neq 0$. Freire *et al.* [2002] have derived the complete set of bifurcations in the three-parameter family (17) for the Bogdanov–Takens and saddle-node Hopf singularities. The details can be found in [Freire *et al.*, 2002; Qiao *et al.*, 2010], and we omit them here for brevity.

5. Dynamical Classification Near the Triple-Zero Bifurcation Point Based on the Normal Form and Numerical Simulations

By the center manifold theory [Carr, 1981] and the method of the normal form for FDEs [Faria & Magalhães, 1995b; Hale & Verduyn Lunel, 1993], the dynamics of system (1) near a bifurcation point is topologically equivalent to that of normal form (17) in the sufficiently small neighborhood of the critical point $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 0, 0)$. Hence, when the perturbation parameters (μ_1, μ_2, μ_3) vary in a small neighborhood of the origin, the local representations of the bifurcation curves can be determined by the normal form (17). Then the dynamical classification of the original system (1) near the positive equilibrium E^* is obtained as the parameters (τ, s, m) vary in a small neighborhood of the triple-zero point (τ^*, s^*, m^*). The local representations of the bifurcation curves related to the perturbation parameters μ_1, μ_2 and μ_3 can be determined as follows:

(i) a saddle-node bifurcation of the equilibrium (x^*, y^*) at $\mu_3 = 0$;

(ii) a saddle-node Hopf bifurcation of the equilibrium (x^*, y^*) at

$$\mu_3 = 0, \quad \mu_2 = -\left(\frac{\tilde{\kappa}_{31}}{\tilde{k}_{32}}\right)\mu_1, \quad \mu_1 < 0;$$

(iii) a Bogdanov–Takens bifurcation of the equilibrium (x^*, y^*) when $\mu_2 = \mu_3 = 0$, $\mu_1 \neq 0$.

Taking $\alpha = 1.5$, $\beta = 0.8$, $n = 0.2$, $r = 1.8$, $K = 6$, $b = 0.5$, then it follows from (2) and (3) that $m_* = 0.15$, $m^* = 0.1688$, $s^* = 1.2$, $\tau^* = 1/s^* = 0.8333$. By the procedure described in the previous section, we have

$$\varepsilon_1 = -0.6944444442 * \mu_3,$$

$$\varepsilon_2 = 2.233539095 * \mu_3 - 1.666666667 * \mu_2,$$

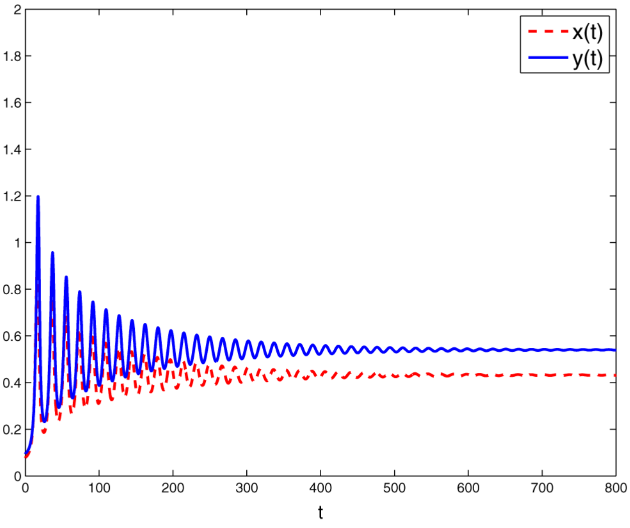
$$\varepsilon_3 = 2.400000000 * \mu_1 + 1.111111111 * \mu_2 + 0.2649748513 * \mu_3,$$

$$A1 = -0.5277777780, \quad A2 = -0.3333333333.$$

We investigate the dynamics near the triple-zero bifurcation point (m^*, s^*, τ^*) by using the results in [Freire *et al.*, 2002]. For fixed $\mu_1 = -0.3833$, i.e. $\tau = 0.45$, a Bogdanov–Takens bifurcation point is given by $(s, m) = (1.2, 0.1688)$, and a saddle-node Hopf bifurcation point is given by $(s, m) = (2.7622, 0.1688)$ in the parameter (s, m) plane. For $\mu_3 = -0.01$ (corresponding to $m = 0.1588$), system (1) has two equilibria $E^*_+(0.4321, 0.5401)$ and $E^*_-(0.0679, 0.0849)$. The variation of the parameter s does not change the two equilibria, but changes their stability, giving rise to bifurcations. In the following, we study how the dynamics of system (1) changes with respect to the variation of s.

5.1. *Bogdanov–Takens bifurcation*

First, we restrict s to the neighborhood of the Bogdanov–Takens bifurcation point, $s = s^* = 0.1688$, with the perturbation $s = s^* + \mu_2$. There exists a critical value s_H for which system (1) undergoes a supercritical Hopf bifurcation from the equilibrium E^*_+, i.e. when $s > s_H$, the equilibrium E^*_+ is asymptotically stable, and there exists a heteroclinic orbit connecting the two equilibria. The case taking $\mu_2 = -0.001$ is shown in Fig. 1. Here, a heteroclinic orbit (sometimes called a heteroclinic connection) is identified as a path in phase space which joins two different equilibrium points. When $s < s_H$, the equilibrium E^*_+ is unstable and there



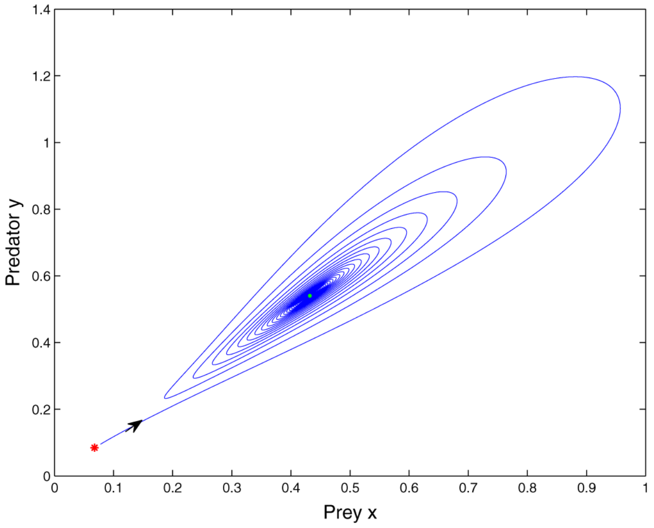


Fig. 1. The dynamics of system (1) near the Bogdanov–Takens bifurcation point for $\tau = 0.45$, $\mu_2 = -0.001$, $\mu_3 = -0.01$. The positive equilibrium $E^*_+(0.4321, 0.5401)$ is asymptotically stable and there exists a heteroclinic orbit connecting the two equilibria $E^*_+(0.4321, 0.5401)$ and $E^*_-(0.0679, 0.0849)$. Initial value: $(x(0), y(0)) = (x^*_- + 0.01, y^*_- + 0.01)$.

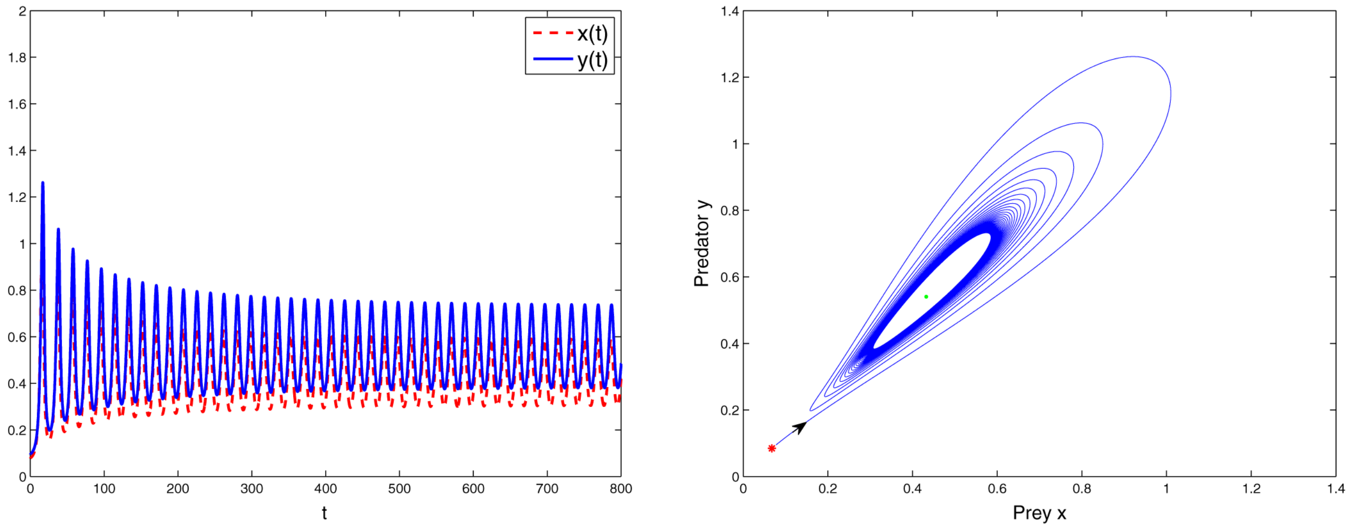


Fig. 2. The dynamics of system (1) near the Bogdanov–Takens bifurcation point for $\tau = 0.45$, $\mu_2 = -0.01$, $\mu_3 = -0.01$. The positive equilibrium $E_+^*(0.4321, 0.5401)$ is unstable and there exists a stable periodic orbit surrounding this positive equilibrium. Initial value: $(x(0), y(0)) = (x_-^* + 0.01, y_-^* + 0.01)$.

exists a stable periodic orbit. For example, the case choosing $\mu_2 = -0.01$ is depicted in Fig. 2. When μ_2 decreases further to some critical value, it follows from the result in [Freire *et al.*, 2002] that there exists an attractive homoclinic connection of the positive equilibrium $E_-^*(0.0679, 0.0849)$. Figure 3 shows the existence of the attractive homoclinic connection of the equilibrium E_-^* for μ_2 being decreased to -0.02746.

5.2. *Saddle-node Hopf bifurcation*

Next, we investigate the dynamics with the parameter s taking the values in the neighborhood of the saddle-node Hopf bifurcation point, $s = 2.7622$ for $m = m^* - 0.01$ and $\tau = 0.45$. For $s = 2.7$, the positive equilibria $E_+^*(0.4321, 0.5401)$ is asymptotically stable and the other positive equilibrium $E_-^*(0.0679, 0.0849)$ is unstable, as shown in Fig. 4.

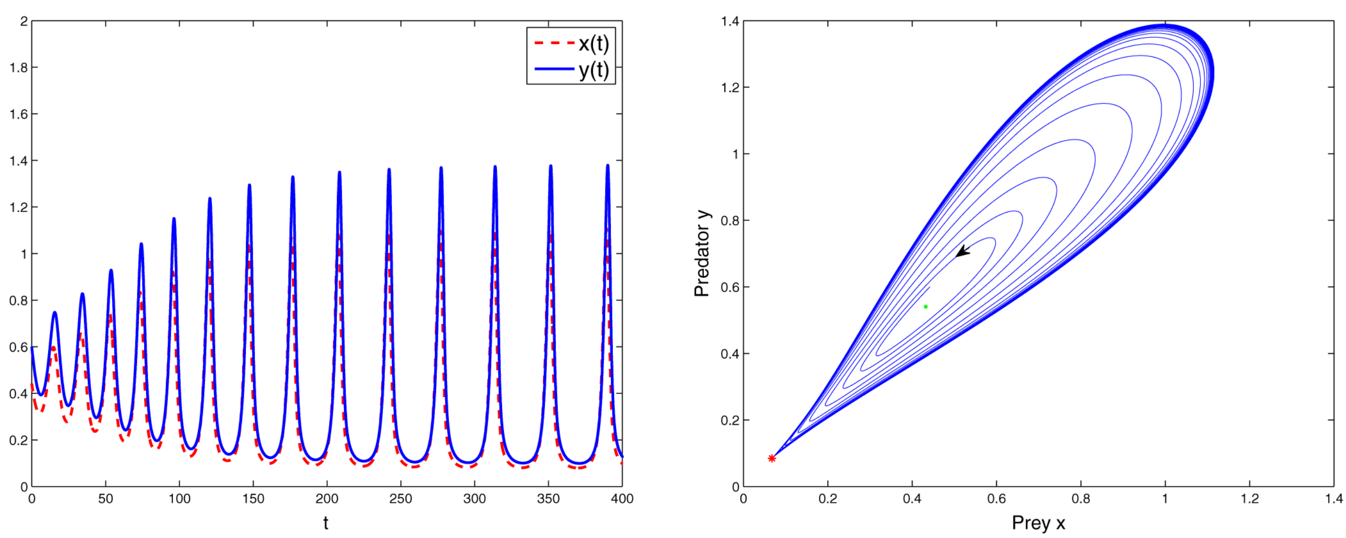


Fig. 3. The dynamics of system (1) near the Bogdanov–Takens bifurcation point for $\tau = 0.45$, $\mu_2 = -0.02746$, $\mu_3 = -0.01$. The positive equilibrium $E_+^*(0.4321, 0.5401)$ is unstable and there exists a homoclinic orbit connecting the equilibrium $E_-^*(0.0679, 0.0849)$. Initial value: $(x(0), y(0)) = (x_+^* - 0.01, y_+^* + 0.06)$.

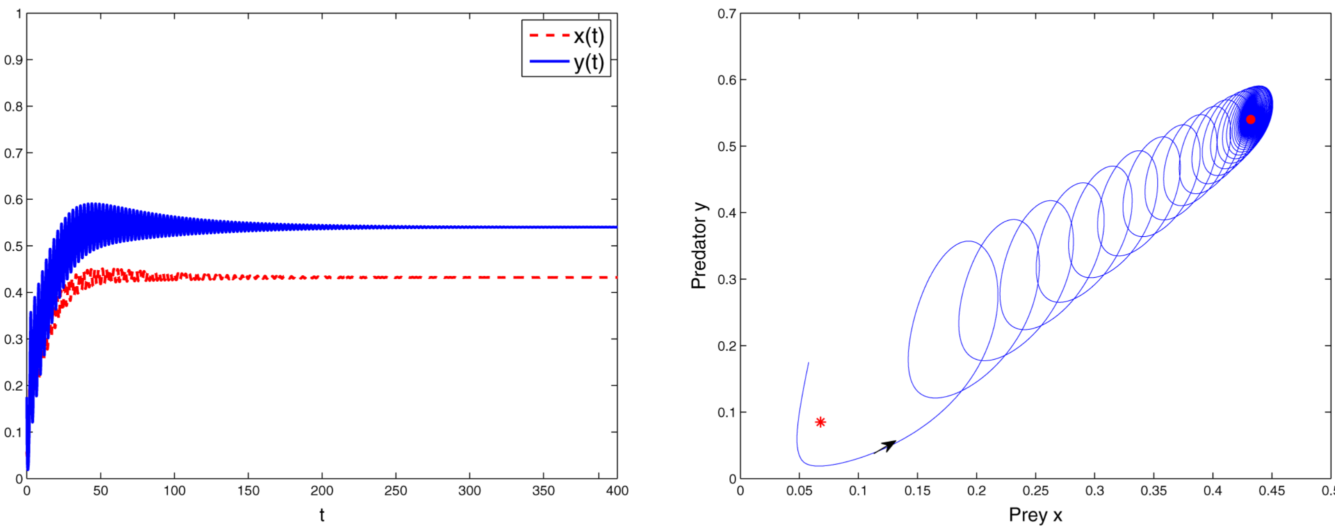


Fig. 4. The dynamics of system (1) near the saddle-node Hopf bifurcation point for $\tau = 0.45$, $s = 2.7$, $m = 0.1588$. The positive equilibrium $E_+^*(0.4321, 0.5401)$ is asymptotically stable. Initial value: $(x(0), y(0)) = (x_-^* - 0.01, y_-^* + 0.09)$.

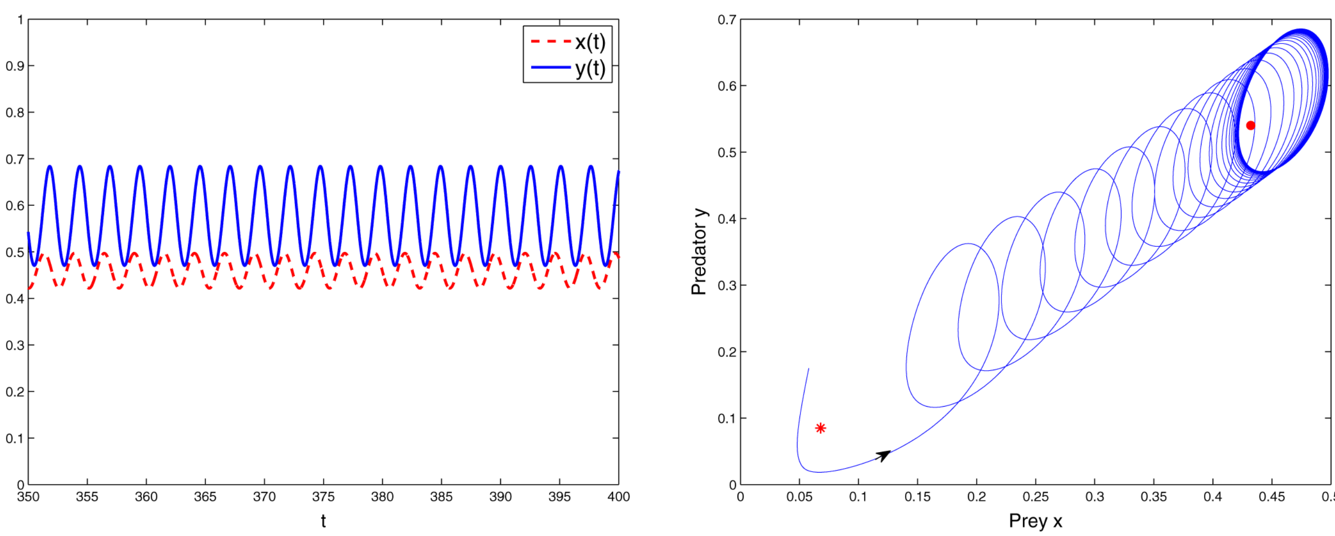


Fig. 5. The dynamics of system (1) near the saddle-node Hopf bifurcation point for $\tau = 0.45$, $s = 2.732$, $m = 0.1588$. There exists a small amplitude periodic orbit near the positive equilibrium $E_+^*(0.4321, 0.5401)$ arising from a supercritical Hopf bifurcation. Initial value: $(x(0), y(0)) = (x_-^* - 0.01, y_-^* + 0.09)$.

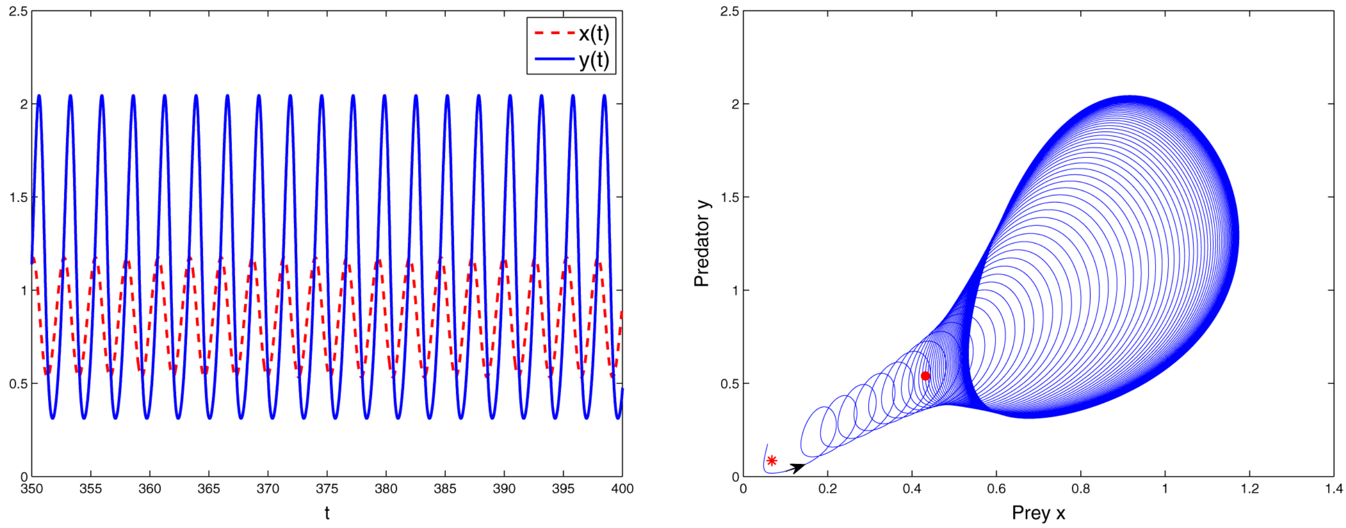


Fig. 6. The dynamics of system (1) near the saddle-node Hopf bifurcation point for $\tau = 0.45$, $s = 2.8$, $m = 0.1588$. There exists a large amplitude periodic orbit. Initial value: $(x(0), y(0)) = (x^*_- - 0.01, y^*_- + 0.02)$.

When s increases, it follows from the result in [Freire *et al.*, 2002] that the positive equilibrium $E^*_+(0.4321, 0.5401)$ loses its stability and bifurcates into a periodic orbit, arising from a supercritical Hopf bifurcation, and the other positive equilibrium $E^*_-(0.0679, 0.0849)$ undergoes a subcritical Hopf bifurcation. Figure 5 illustrates this result with $s = 2.732$.

When s is changed far away from the Hopf bifurcation point, the stable periodic orbit still exists and its amplitude increases with the increasing of s. For example, taking $s = 2.8$ yields a large amplitude periodic orbit as shown in Fig. 6.

6. Conclusions

We have presented a detailed study on a Leslie-type predator–prey model with ratio-dependent functional response and Allee effect on prey. Without delay, the existence of multiple positive equilibria and their stability are explicitly determined. The effect of delay on the dynamics is investigated by analyzing the triple-zero bifurcation. Motivated by our recent results on the bifurcation related to the saddle-node type, we employ the normal form theory of delay differential equations developed by Faria and Magalhães [1995a, 1995b] to compute the normal form of the triple-zero bifurcation, arising from a saddle-node bifurcation of delay differential equations. The coefficients of the normal form are expressed in terms of the coefficients of the Taylor series of the right-hand side of the system. Although the algorithm is developed for system (1), it can be easily generalized to any delay models which have the same type of triple-zero bifurcation as studied in this paper. We have shown that under certain parameter values of system (1) heteroclinic orbits can occur which connect one of the positive equilibria to another; and that homoclinic orbits connecting one of the positive equilibria and stable periodic orbit surrounding the positive equilibrium inside. We have also shown the existence of large amplitude periodic orbits near the saddle-node Hopf bifurcation.

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