Qualitative Theory of Dynamical Systems (2024) 23:8
https://doi.org/10.1007/s12346-023-00863-3

A Complete Classification on the Center-Focus Problem of a Generalized Cubic Kukles System with a Nilpotent Singular Point

Feng Li[1] · Ting Chen[2] · Yuanyuan Liu[1] · Pei Yu[3]

Received: 25 March 2023 / Accepted: 28 August 2023

Abstract
In this paper, we study the center-focus problem for a generalized cubic Kukles system with a nilpotent singular point, which consists of a cubic system with an extra 4th-order term. A complete classification is given on the center conditions which are explicitly expressed in term of the system parameters. A total of 15 cases are obtained, among them 4 for the generalized cubic Kukles system and 12 for the cubic Kukles system, with one common for both. One of the center conditions is analytic. Moreover, it is shown that 8 small-amplitude limit cycles can bifurcate in the neighborhood of the singular point for the generalized cubic Kukles system, while only 7 small-amplitude limit cycles can exist around the singular point for the cubic Kukles system. The center-focus problem for the generalized cubic Kukles system with a nilpotent origin is thoroughly solved.

Keywords Generalized cubic Kukles system · Center-focus problem · Nilpotent singular point · Center · Analytic center

✉ Pei Yu
pyu@uwo.ca

Feng Li
lf0539@126.com

Ting Chen
chenting0715@126.com

Yuanyuan Liu
liuyuanyuan@lyu.edu.cn

1 School of Mathematics and Statistics, Linyi University, Linyi 276005, Shandong, China

2 School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou 510320, Guangdong, China

3 Department of Mathematics, Western University, London, ON N6A 5B7, Canada

Published online: 22 September 2023

1 Introduction

Classical problems on the Kukles systems with an elementary singular point have been investigated intensively for many years. The center conditions and bifurcation of limit cycles for the Kukles systems with an elementary singular point have been studied in [9, 28, 29, 34]. In particular, the solution to the center problem of the cubic Kukles system was obtained independently by Lloyd and Pearson [29], and Sadovskii [34] using different methods. A predator–prey model which can be reduced to the Kukles system was considered by Pitchford and Brindley [31], and Lloyd and Pearson [18], in which bifurcation of limit cycles was analyzed in detail. The center problem for a linear center perturbed by homogeneous polynomials (with odd or even degree), called Kukles homogeneous systems, was proposed by Giné [11], and investigated by Giné et al. [14, 15]. Recently, the Kukles system was reconsidered [30] by using the advanced technique in symbolic computation to obtain the first seven focal values, which were used to classify the center and isochronous center conditions of the Kukles system. The extended Kukles systems were also considered recently by many researchers, with attention paid to some classical problems such as center problem and isochronous center problem, for example see [13, 16, 17, 33] and references therein. A class of generalizations of the Kukles systems with an elementary singular point, described by

$$\begin{aligned} \frac{dx}{dt} &= y\,(1 + kx + lx^2),\\ \frac{dy}{dt} &= -x + a_1x^2 + 3a_2xy + a_3y^2 + a_4x^3 + 3a_5x^2y + a_6xy^2 + a_7y^3, \end{aligned} \tag{1.1}$$

was proposed by Bondar and Sadovskii [4], and further considered by Sadovskii and Shcheglova [35], in which 25 center conditions were classified and all of them were proved to be sufficient [4] and necessary [35]. Around the same time, Kushner and Sadovskii [19] proposed a more generalized Kukles system with an elementary singular point, which can be reduced to the following system,

$$\begin{aligned} \frac{dx}{dt} &= y(1 + kx + lx^2 + mx^3),\\ \frac{dy}{dt} &= -x + a_1x^2 + 3a_2xy + a_3y^2 + a_4x^3 + 3a_5x^2y\\ &\quad + a_6xy^2 + a_7y^3 + a_8x^4 + 3a_9x^3y + a_{10}x^2y^2 + a_{11}y^4. \end{aligned} \tag{1.2}$$

Center conditions were carefully studied in [36] for some even more complex generalized Kukles systems. Averaging theory has been applied [32] to consider bifurcation of limit cycles for a family of perturbed Kukles differential systems.

However, because of the computational difficulty, very little attention has been paid to the Kukles systems with a nilpotent singular point. The following system,

$$
\begin{aligned}
\frac{dx}{dt} &= y, \\
\frac{dy}{dt} &= a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3,
\end{aligned}
\tag{1.3}
$$

was studied by many authors, see for example [1, 2, 5, 11, 13–15] and references therein. Alvarez and Gasull [2], who proved that 3 limit cycles can bifurcate from the nilpotent singular point. Later, the result was improved by Liu and Li [25], who showed that there can exist 4 limit cycles in a perturbed system of (1.3). Recently, bifurcation of limit circles in a class of Z_2-equivalent cubic planar differential systems with two nilpotent singular points was studied by Li et al. to prove the existence of $6 \times 2 = 12$ limit cycles. In another article [20], a class of generalized Kukles systems with a nilpotent singular point, written as

$$
\begin{aligned}
\frac{dx}{dt} &= y(1 + a_{11}x), \\
\frac{dy}{dt} &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3,
\end{aligned}
\tag{1.4}
$$

was studied to derive the center conditions and investigate bifurcation of limit cycles, and to prove the existence of 6 limit cycles around the nilpotent singular point.

In this paper, as a continuous work, we will consider the following system,

$$
\begin{aligned}
\frac{dx}{dt} &= y(1 + a_{11}x + a_{21}x^2 + a_{31}x^3), \\
\frac{dy}{dt} &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3,
\end{aligned}
\tag{1.5}
$$

which has a nilpotent singular point at the origin. Compared to the Kukles system (1.3) and the generalized Kukles system (1.4), our proposed new system (1.5) contains three more terms in the first equation $\frac{dx}{dt}$, in particular, including the 4th-order term $a_{31}x^3y$. This work is motivated by the series results obtained from the study of Kukles system and the generalized Kukles system, which promoted the research in this direction, especially for the bifurcation of limit cycles in such polynomial systems. Our aim is to demonstrate that the center problem becomes much more complex even for just adding one 4th-order term, and to try developing new techniques in solving center problem of more complex dynamical systems.

We will present a complete classification on the center conditions of the system (1.5), showing that a total of 15 cases are classified, among them 4 for the case $a_{31} \neq 0$ and 12 for the case $a_{31} = 0$, with one common for both. Moreover, we will show that 8 small-amplitude limit cycles can bifurcate in the neighborhood of the singular point for the case $a_{31} \neq 0$, while only 7 small-amplitude limit cycles can exist around the singular point for $a_{31} = 0$.

The rest of the paper is organized as follows. In the next section, the classification of the nilpotent origin will be given. In Section 3, the first eight Lyapunov constants will be computed for the nilpotent foci, which are used to obtain the necessary center conditions, and they are further proved to be sufficient. An analytic center condition is also classified. Finally, a conclusion is drawn in Section 4.

2 Classification of the Singular Point of System (1.5)

In this section, we present the classification on the nilpotent origin of system (1.5). For the planar polynomial differential systems, described by

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{2.1}$$

if its linear part has a double-zero eigenvalue and the matrix of the linearized system at the origin is not identically null, then the origin of the system is called a nilpotent singular (or critical) point. In [38], it is shown that there exist many different kinds of topological phase constructions around a nilpotent singular point. Some early results on this topic can be found in Sections 17–19 of [3].

Planar autonomous analytic systems with a nilpotent singular point can always be transformed into the following form,

$$\begin{aligned} \frac{dx}{dt} &= \Phi(x, y) = y + \sum_{k+j=2}^{\infty} a_{kj} x^k y^j, \\ \frac{dy}{dt} &= \Psi(x, y) = \sum_{k+j=2}^{\infty} b_{kj} x^k y^j, \end{aligned} \tag{2.2}$$

by a proper linear transformation, where $\Phi(x, y), \Psi(x, y)$ are analytic in the neighborhood of the origin.

Suppose the equation,

$$\Phi(x, f(x)) = 0, \quad f(0) = 0, \tag{2.3}$$

has an unique solution $y = f(x)$, satisfying that

$$\begin{aligned} &\Psi(x, f(x)) = \alpha_k x^k + o(x^k), \quad a_k \neq 0, \\ &\left[\frac{\partial \Phi(x, y)}{\partial x} + \frac{\partial \Psi(x, y)}{\partial y}\right]_{y=f(x)} = \beta_n x^n + o(x^n). \end{aligned} \tag{2.4}$$

By using Theorems 7.2 and 7.3 in [38], we have the following result.

Theorem 2.1 *For system (2.4), the following holds:*

$$\textit{The origin is a} \begin{cases} \textit{degenerate point}, & \textit{if } b_{20} \neq 0; \\ \textit{saddle point}, & \textit{if } b_{20} = 0,\ b_{30} > 0; \\ \textit{degenerate point}, & \textit{if } b_{20} = 0,\ b_{30} < 0,\ b_{11}^2 + 8b_{30} \geq 0; \\ \textit{center or a focus}, & \textit{if } b_{20} = 0,\ b_{30} < 0,\ b_{11}^2 + 8b_{30} < 0. \end{cases}$$

The origin is a nilpotent center or focus when $b_{20}=0$, $b_{30}<0$ and $b_{11}^2+8b_{30}<0$. So we can always suppose that $b_{11}^2+8b_{30}=-16$ in system (1.5). Otherwise, we can change system (1.5) to

$$\begin{aligned}\frac{du}{d\tau}&=v(1+a_{11}r^2u+a_{21}r^3u+a_{31}r^4u^4),\\ \frac{dv}{d\tau}&=r^2(rb_{11}xy+b_{02}y^2+b_{30}r^4x^3+b_{21}r^3x^2y+b_{12}r^2xy^2+b_{03}ry^3),\end{aligned}$$

by using the transformation, $x=r^2u$, $y=rv$, and the time rescaling $dt=rd\tau$. Then, it can be shown [22] that the nilpotent origin is a center or focus if and only if $b_{30}<0$, and $r^6(b_{11}^2+8b_{30})=-16<0$, which leads to $b_{11}=4\mu$, $b_{30}=-2-2\mu^2$, under which system (1.5) becomes

$$\begin{aligned}\frac{dx}{dt}&=y(1+a_{11}x+a_{21}x^2+a_{31}x^3),\\ \frac{dy}{dt}&=4\mu xy+b_{02}y^2-2(1+\mu^2)x^3+b_{21}x^2y+b_{12}xy^2+b_{03}y^3.\end{aligned}\tag{2.5}$$

In the following, we will use (2.5) to study the original system (1.5).

3 Center Conditions of System (2.5)

In order to study center conditions, the only way is to compute the Lyapunov constants of the system under consideration. There are many methods for computing the Lyapunov constants, such as the inverse integrating method [25], the method of normal forms [37], etc. In this paper, both the inverse integrating method and the method of normal forms will be adopted, so that their results can be cross checked to guarantee the correctness.

We first consider the case $a_{31}\neq 0$ for which (2.5) is a generalized cubic Kukles system, and then the case $a_{31}=0$ for which (2.5) is a cubic Kukles system.

3.1 Center Conditions of System (2.5) for $a_{31}\neq 0$

In order to simplify the analysis in finding the center conditions and the maximal number of bifurcating limit cycles, we first suppose $a_{11}\neq 0$ and use it to make a scaling. Then, we consider the case $a_{11}=0$ which will certainly not generate maximal number of bifurcating limit cycles, but may have center conditions. With $a_{11}\neq 0$, we introduce the following scaling:

$$\begin{aligned}&x=\frac{X}{a_{11}},\qquad y=\frac{Y}{a_{11}},\qquad \tau=\frac{t}{a_{11}},\\ &a_{21}=A_{21}a_{11}^2,\ a_{31}=A_{31}a_{11}^3,\ b_{02}=B_{02}a_{11},\ b_{21}=B_{21}a_{11},\\ &b_{12}=B_{12}a_{11}^2,\ b_{03}=B_{03}a_{11}^3,\end{aligned}\tag{3.1}$$

into (2.5) to obtain the following scaled system,

$$\begin{aligned}\frac{dx}{d\tau} &= Y(1 + X + A_{21}X^2 + A_{31}X^3),\\ \frac{dy}{d\tau} &= 4\mu XY + B_{02}Y^2 - 2(1+\mu^2)X^3 + B_{21}X^2Y + B_{12}XY^2 + B_{03}Y^3.\end{aligned} \tag{3.2}$$

Then, once we obtain the center conditions from the system (3.2), we can easily use (3.1) to get the center conditions for the original system (2.5).

In order to consider the center-focus problem of system (3.2), a powerful method it to compute the so-called the generalized Lyapunov constants. For dynamical systems associated with an elementary center, three methods are mainly used for computing Lyapunov constants: the method of normal forms, the method of Poincaré return map or focus value method, and the method of Lyapunov function. More details can be found, for example, in [37] and references therein. The above mentioned three methods have been used to study the center-focus problem associated with nilpotent critical points, see for example [1, 5]. But the method of normal forms was only recently applied to compute the so-called generalized Lyapunov constants in determining the lower bound of cyclicity [2, 37]. The method developed in [37] is a generalization of the approach for computing the focus values associated with the singular point of elementary center to the case associated with the singular point of nilpotent point. All computations are purely algebraic for solving linear polynomial equations in an iterative procedure, and thus the computation is significantly efficient. The program developed using the computer algebra system – Maple can be easily executed by an end-user to obtain the normal form or the generalized Lyapunov constants. Therefore, in this paper, we apply the method of normal forms [37] and its associated Maple program to compute the Lyapunov constants of the system (3.2), given in the following theorem.

Theorem 3.1 *When $a_{31}a_{11} \neq 0$, the first 8 Lyapunov constants at the origin of system (3.2) are given as follows:*

$$\begin{aligned}
L_1 =\ & \frac{1}{5}\big[5B_{21} + 4\mu(B_{02} - 2)\big],\\
L_2 =\ & \frac{1}{35}\big[B_{21}(65A_{21} - 30B_{12} - B_{02}^2 + 9B_{02} - 4) + 10(21 + 5\mu^2)B_{03}\\
& - 40\mu(2A_{21} + A_{31} - B_{12})\big],\\
L_3 =\ & \frac{1}{27000}\big\{44125B_{03}B_{21}^2 + \big[40A_{31}(1081B_{02} + 3868) - 256500\mu B_{03}\\
& - 400(454A_{21}^2 + 123B_{12}^2) + 189800A_{21}B_{12} + 16A_{21}(1112B_{02}^2 + 5667B_{02} - 8757)\\
& - 8B_{12}(833B_{02}^2 + 5063B_{02} - 7008) + 8(2196B_{02}^2 - 6604B_{02} + 2624)\big]B_{21}\\
& - 48\big[1200(9A_{21} - 4B_{12}) - 791B_{02}^2 - 7591B_{02} + 23146\big]B_{03}\\
& + 1600\mu\big[6(11A_{21} - 6B_{12} - 4)A_{31} + 142A_{21}^2 + 41B_{12}^2\\
& - 153A_{21}B_{12} + 72A_{21} - 36B_{12}\big]\big\},\\
L_4 =\ & \frac{-1}{160224243656400000}\big\{125B_{03}(11950520375044135A_{21} - 13675341705133535B_{12}\\
& + 1132574198104706B_{02}^2 - 13314054421345314B_{02} - 1461890469098436)B_{21}^2\\
& + \big[112104949701150000A_{31}^2 + 100A_{31}\big(21501738022823625\mu B_{03}
\end{aligned}$$

$$
\begin{aligned}
&+500A_{21}(63923003350555B_{02}+521300229167026)\\
&-8330B_{12}(2714562059597B_{02}+17929731667646)+1685974367750722B_{02}^3\\
&-24025434900710222B_{02}^2+10582263135571991 4B_{02}+883278821847120084\big)\\
&-156062574990000B_{03}^2(40317+14573\mu^2)-7500\mu B_{03}(2491547435642281A_{21}\\
&-1658506710437731B_{12}+14722849389811216)-18229032669417900000A_{21}^3\\
&+2000A_{21}^2(14653641096336150B_{12}-67966702067004B_{02}^2\\
&+11907433842385217B_{02}-50677420717253798)\\
&-200A_{21}\big(10B_{12}(7884191910763950B_{12}+12222351922188939B_{02}\\
&-51671506143086548)-107052696707698B_{02}^4+1746566514541716B_{02}^3\\
&-6526202245432629B_{02}^2-30574925482301258 5B_{02}+359341684006642976\big)\\
&+2000B_{12}\big(1415543485778100B_{12}^2+40B_{12}(77754173334187B_{02}\\
&-325982227731669)-1302425791607320 3B_{02}+13713279937386821\big)\\
&+8(167973039509422B_{02}^6-3604144943410354B_{02}^5+38196182144361745B_{02}^4\\
&-314027474456124170B_{02}^3+2016088131205650555B_{02}^2\\
&-4230716226799244974B_{02}+15768163876398464 88)\big]B_{21}\\
&+208083433320000\mu(40317+14573\mu)B_{03}^2-\big[52020858330000A_{31}(1188-5643B_{02}\\
&+57278\mu^2)+7440000A_{21}^2(6975604460847-59534982311\mu^2)\\
&+74970000B_{12}^2(171292589973-3860104507\mu^2)-30000A_{21}B_{12}(1724684386382607\\
&-29770381203185\mu^2)+30000A_{21}(72427326580007B_{02}^2-2922070221604217B_{02}\\
&+14086990920322148+140200837275646\mu^2)+6000B_{12}(37332698609101B_{02}^2\\
&+7028813384491871B_{02}-33240523945545311-187486063469005\mu^2)\\
&+2400(42503280435413B_{02}^4-840097072912349B_{02}^3+11844823687901162B_{02}^2\\
&-137423809804840476B_{02}+278177525762443488+619048214127000\mu^2)\big]B_{03}\\
&-40000\mu\big[4213689524730A_{31}^2-(264481516712790A_{21}^2-292061524988571A_{21}B_{12}\\
&+81875773202853B_{12}^2+1066400289515844A_{21}-553967364982377B_{12}\\
&-390970428808920)A_{31}-(2A_{21}-B_{12})(285236198374521A_{21}^2\\
&-325309999165989A_{21}B_{12}+94369565718540B_{12}^2+1415022304995647A_{21}\\
&-802740822864698B_{12}+660741428908620)\big]\big\},
\end{aligned}
$$

where $L_k,\ k=2,3,4$ *have been simplified by using the Groebner basis approach, and the lengthy* $L_k,\ k=5,6,7,8$*, are not listed here for brevity.*

Based on the Lyapunov constants given in Theorem 3.1, we obtain the following result.

Proposition 3.1 *When* $a_{31}a_{11}\neq 0$*, the first* 8 *Lyapunov constants at the origin of system (2.5) are zero if and only if one of the following conditions holds:*

(a) $b_{03} = b_{21} = \mu = 0$;

(b) $b_{03} = b_{21} = a_{21} + 24a_{11}^2 = a_{31} - 36a_{11}^3 = b_{02} - 2a_{11} = b_{12} + 12a_{11}^2 = 0$;

(c) $b_{03} = b_{02} = b_{12} = 25a_{21} - 8a_{11}^2 = 125a_{31} - 4a_{11}^3 = 5b_{21} - 8\mu a_{11} = 0$;

(d) $b_{03} = 529a_{21} - 174a_{11}^2 = 12167a_{31} - 432a_{11}^3 = 23b_{02} - a_{11}$
$= 23b_{21} - 36\,\mu a_{11} = 529b_{12} - 6a_{11}^2 = 0$.

Proof We first apply the Lyapunov constants at the origin of system (3.2) to obtain the conditions and then use (3.1) to get the conditions for system (2.5), as listed in the proposition. In general, the center condition candidates found by setting $L_k = 0,\ k = 1, 2, \ldots$, are possible center conditions, since some center conditions might be missed. Unless one can show that all such conditions are obtained, then after the sufficiency of these conditions is proved, one can claim that the obtained center condition candidates are necessary and sufficient. In the following, we will use the Lyapunov constants given in Theorem 3.1 to show the details that the conditions (a), (b), (c) and (d) are all possible center condition candidates which can be obtained from the Lyapunov constants.

It is easy to see that the equation $L_1 = 0$ yields two cases: (I) $B_{21} = 0$ and (II) $B_{21} \neq 0$.

(I) For $B_{21} = 0$, there are two sub-cases: (I-a) $\mu = 0$ (B_{02} is free), and (I-b) $B_{02} = 2$ ($\mu \neq 0$).

(I-a) For this case, we have $B_{21} = \mu = 0$, which leads to $L_2 = 6B_{03}$. Setting $L_2 = 0$ gives $B_{03} = 0$, yielding $L_3 = L_4 = \cdots = L_8 = 0$. Hence, we obtain the first condition for the system (3.2) as $B_{21} = B_{03} = \mu = 0$. Then using the transformation (3.1) directly yields the condition (a) for system (2.5).

(I-b) For this case, we have

$$L_2 = \frac{2}{7}\big[(21 + 5\mu^2)B_{03} - 4(A_{31} + 2A_{21} - B_{12}\mu\big].$$

Setting $L_2 = 0$ yields $B_{03} = \frac{4(A_{31}+2A_{21}-B12)\mu}{5\mu^2+21}$ under which

$$L_3 = \frac{8\mu}{27(5\mu^2 + 21)}\big\{6M_1A_{31} + 2(2A_{21} - B_{12})\big[(39 + 71\mu^2)A_{21} + (57 + 41\mu^2)B_{12} + 36(1 + \mu^2)\big]\big\},$$

where

$$M_1 = (3 + 11\mu^2)A_{21} - 6(1 + \mu^2)B_{12} - 4(9 + \mu^2).$$

There are two possibilities: (I-b-i) $M_1 = 0$ or (I-b-ii) $M_1 \neq 0$.

(I-b-i) When $M_1 = 0$, we have $A_{21} = \frac{6(1+\mu^2)B_{12}+4(9+\mu^2)}{3+11\mu^2}$, for which L_3 becomes

$$L_3 = -\frac{8\mu}{27(5\mu^2 + 21)}(B_{12} - 8)\big[(5\mu^2 - 3)B_{12} - (136\mu^2 + 72)\big].$$

Note that if $A_{21}=0$, i.e., $B_{12}=-\frac{2(9+\mu^2)}{3(1+\mu^2)}$, leading to $L_3 \neq 0$. Thus, $L_3=0$ gives either the solution $B_{12}=8$, or the solution $B_{12}=\frac{8(17\mu^2+9)}{5\mu^2-3}$ if $B_{12}\neq 8$, noticing that $\mu^2=\frac{3}{5}$ yields $L_3\neq 0$ under the condition $B_{12}\neq 8$. When $B_{12}=8$, we have $A_{31}=-\frac{17(5\mu^2+21)}{41\mu^2+9}$, and then $L_3=L_4=L_5=0$, but $L_6\neq 0$. When $B_{12}=\frac{8(17\mu^2+9)}{5\mu^2-3}$, $(5\mu^2-3\neq 0)$, $A_{21}=\frac{4(19\mu^2+27)}{5\mu^2-3}$. Then, we obtain

$$\begin{aligned}
L_4 &= \frac{128\mu(\mu^2+9)}{55(5\mu^2+21)^2(5\mu-3)^3}\,\overline{L}_4,\\
L_5 &= -\frac{64\mu(\mu^2+9)}{1365(5\mu^2+21)^2(5\mu-3)^3}\,\overline{L}_5,\\
L_6 &= -\frac{16\mu(\mu^2+9)}{51975(5\mu^2+21)^2(5\mu-3)^3}\,\overline{L}_6,
\end{aligned}$$

where $\overline{L}_k,\ k=4,5,6$ are polynomials in A_{31} and μ. With the help of Maple, eliminating A_{31} and μ from these three polynomials results in two resultants, one of which is positive, showing that there do not exist solutions for A_{31} and μ such that $\overline{L}_4=\overline{L}_5=\overline{L}_6=0$ simultaneously, which indicates that the case (I-b-i) does not have possible solutions such that $L_k=0,\ k=1,2,\ldots,8$.

(I-b-ii) For this case, we have

$$\begin{aligned}
A_{31} = \frac{1}{3M_1}&(2A_{21}-B_{12})\big[(39+71\mu^2)A_{21}\\
&+(57+41\mu^2)B_{12}+36(1+\mu^2)\big], \quad (B_{12}\neq 2A_{21}),
\end{aligned} \tag{3.3}$$

under which $L_3=0$. Then, $L_k,\ k=4,5,6$ are obtained as

$$\begin{aligned}
L_4 &= -\frac{4\mu(2A_{21}-B_{12})}{495M_1^2}\,\widetilde{L}_4,\\
L_5 &= \frac{2\mu(2A_{21}-B_{12})}{12285M_1^3}\,\widetilde{L}_5,\\
L_6 &= \frac{\mu(2A_{21}-B_{12})}{5613300M_1^3}\,\widetilde{L}_6,
\end{aligned}$$

where $\widetilde{L}_k,\ k=4,5,6$ are polynomials in A_{21}, B_{12} and μ. With the help of Maple, eliminating A_{21} from these three polynomials yields solutions $A_{21}=A_{21}(B_{12},\mu)$ and two resultants:

$$\mathrm{R}_{45}=C_1\,\tilde{\mathrm{R}}_{45}, \quad \mathrm{R}_{46}=C_1\,\tilde{\mathrm{R}}_{46},$$

where the common factor C_1 is given by

$$\begin{aligned}
C_1 = B_{12}\,&(\mu^2+1)(\mu^2+9)(B_{12}+12)(B_{12}+8)\\
&\times\big[(5\mu^2-3)B_{12}-8(17\mu^2+9)\big],
\end{aligned}$$

and $\tilde{R}_{45}$ and $\tilde{R}_{46}$ are polynomials in B_{12} and μ. The common factor C_1 has three possible solutions for B_{12}: $B_{12}=-12$ generates the condition (b); while the other two solutions $B_{12} = -8$ and $B_{21} = \frac{8(17\mu^2+9)}{5\mu^2-3)}$ yield $M_1 = 0$, as discussed above. The resultant of $\tilde{R}_{45}$ and $\tilde{R}_{46}$ with respect to B_{12} has a factor μ^2-3 for which $L_k,\ k = 4, 5, 6, 7, 8$ have a common factor:

$$\begin{aligned}M_2 = {} & 826208126B_{12}^9 - 51264404745B_{12}^8 - 1999127381612B_{12}^7 \\ & -58287364950288B_{12}^6 - 1350601624681984B_{12}^5 \\ & -19328008337820160B_{12}^4 - 171358789352458240B_{12}^3 \\ & -822123861520384000B_{12}^2 - 1025400150491136000B_{12} \\ & +340332138987520000 0,\end{aligned}$$

for which it can be shown that $A_{31}=B_{03}=0$. The above discussions indicate that the case (I-b-ii) also does not yield additional conditions such that $L_k = 0,\ k = 1, 2, \ldots, 8$.

(II) Now, we consider the case $B_{21} \neq 0$. $L_1 = 0$ gives

$$B_{21} = -\frac{4}{5}\,\mu\,(B_{02} - 2) \neq 0.$$

Then, setting $L_2=0$ we have the solution,

$$\begin{aligned}B_{03} = {} & \frac{4\mu}{50(5\mu^2 + 21)}\big[50A_{31} + 5(13B_{02} - 6)A_{21} - 10(3B_{02} - 1)B_{12} \\ & -(B_{02} - 2)(B_{02}^2 - 9B_{02} + 4)\big].\end{aligned}$$

Next, solving $L_3 = 0$ we obtain the solution $A_{31} = \frac{A_{31n}}{A_{31d}}$, where

$$\begin{aligned}A_{31n} = {} & -2500\big[(227\mu^2 + 111)B_{02} - 9(11\mu^2 + 3)\big]A_{21}^2 \\ & +125\big\{5\big[(949\mu^2 + 933)B_{02} - 16(23\mu^2 + 21)\big]B_{12} \\ & -(473\mu^2 - 375)B_{02}^3 - 9(133\mu^2 + 293)B_{02}^2 \\ & +(3014\mu^2 + 2118)B_{02} - 96(11\mu^2 + 3)\big\}A_{21} \\ & -1250(3B_{02} - 1)(41\mu^2 + 57)B_{12}^2 + 125\big[(257\mu^2 - 15)B_{02}^3 \\ & +(563\mu^2 + 1155)B_{02}^2 - 8(187\mu^2 + 195)B_{02} \\ & +4(121\mu^2 + 105)\big]B_{12} + (B_{02} - 2)\big[(1765\mu^2 + 2373)B_{02}^4 \\ & -8(1265\mu^2 - 177)B_{02}^3 - (8515\mu^2 + 34323)B_{02}^2 \\ & +2(12635\mu^2 + 11307)B_{02} - 24(365\mu^2 + 93)\big], \\ A_{31d} = {} & 375\,\big[100(11\mu^2+3)A_{21} - 600(\mu^2+1)B_{12} - (125\mu^2+1197)B_{02}^2 \\ & +8(25\mu^2 + 81)B_{02} - 12(25\mu^2 + 9)\big].\end{aligned}$$

If $A_{31\mathrm{d}}=0$, then L_3 does not contain A_{31}, and A_{21} can be solved from $A_{31\mathrm{d}}=0$, given by

$$A_{21}=\frac{600(\mu^2+1)B_{12}+(125\mu^2+1197)B_{02}^2-8(25\mu^2+81)B_{02}+12(25\mu^2+9)}{100(11\mu^2+3)}.$$

Then, eliminating B_{12} from $L_3=L_4=L_5=0$ yields the solution for B_{12}, and two resultants from which the solution $\widetilde{A}_{31}$ for A_{31} is obtained by using the Maple command "eliminate", with a resultant which contains the following three factors,

$$(B_{02}-2)(3B_{02}-1)\big[(97\mu^2+1137)B_{02}^2+12(\mu^2-39)B_{02}-12(\mu^2+1)\big].$$

But these three factors happen to be involved in the numerator of the solution $\widetilde{A}_{31}$, implying that letting $A_{31\mathrm{d}}=0$ does not yield conditions satisfying $L_k=0,\ k=1,2,\ldots,8$.
Now, assuming $A_{31\mathrm{d}}\neq 0$, we have the solution $A_{31}=\frac{A_{31\mathrm{n}}}{A_{31\mathrm{d}}}$ such that $L_3=0$. Then, $L_k,\ k=4,5,6,7$, become

$$\begin{aligned}
L_4&=-\frac{4\mu}{116015625A_{31\mathrm{d}}^2}\,L_{4\mathrm{b}},\\
L_5&=\frac{2\mu}{14396484375A_{31\mathrm{d}}^3}\,L_{5\mathrm{b}},\\
L_6&=\frac{\mu}{274086914062500A_{31\mathrm{d}}^3}\,L_{6\mathrm{b}},\\
L_7&=-\frac{\mu}{5451588720703125000A_{31\mathrm{d}}^4}\,L_{7\mathrm{b}},
\end{aligned}$$

where $L_{\mathrm{kb}},\ k=4,5,6,7$, are lengthy polynomials in μ^2, B_{02}, A_{21} and B_{12}. Eliminating μ^2 from these four polynomials gives a solution $\tilde{\mu}^2=\tilde{\mu}^2(A_{21},B_{12},B_{02})$, and three resultants R_{45}, R_{46} and R_{47}, which have two common factors F_1 and F_2:

$$F_1=50A_{21}-25B_{12}-B_{02}^2+9B_{02}-14,$$

and

$$\begin{aligned}
F_2=\;&12500B_{02}A_{21}^2-25\big[50(17B_{02}-4)B_{12}-2053B_{02}^3+1068B_{02}^2+64B_{02}+24\big]\\
&A_{21}+2500(3B_{02}-1)B_{12}^2-100(7B_{02}-4)(37B_{02}^2+2B_{02}-2)B_{12}\\
&-(B_{02}-2)(1081B_{02}^4-8023B_{02}^3+4394B_{02}^2-392B_{02}+96).
\end{aligned}$$

$F_1=0$ yields $\mu^2=-9<0$. To verify whether $F_2=0$ can produce the conditions such that $L_k=0,\ k=1,2,\cdots,8$, we use the Maple command *eliminate*$(\{F_2,L_{4\mathrm{b}},L_{5\mathrm{b}},L_{6\mathrm{b}}\},\{A_{21},\mu^2\})$ to find that no resultants can be obtained from this operation, implying that the factor F_2 does not yield the required conditions.

Finally, in order to find if there exist the required conditions from setting the resultants $R_{45}=R_{46}=R_{47}=0$, we use the Maple built-in command *resultant* to obtain the two resultants:

$$\begin{aligned} R_{456} &= resultant(R_{45}, R_{46}, A_{21}) = C_2\, \widetilde{R}_{456}, \\ R_{457} &= resultant(R_{45}, R_{47}, A_{21}) = C_2\, \widetilde{R}_{457}, \end{aligned} \tag{3.4}$$

where the common factor C_2 contains four factors, given by

$$\begin{aligned} C_2 = &\left[5B_{12} + B_{02}(B_{02} - 2)\right](25B_{12} + 11B_{02}^2 + B_{02} + 4) \\ &\times(25B_{12}+17B_{02}^2-43B_{02}+18)(25B_{12}+11B_{02}^2-19B_{02}-6), \end{aligned}$$

and $\widetilde{R}_{456}$ and $\widetilde{R}_{457}$ are lengthy polynomials in B_{12} and B_{02}. Each of the four factors in C_2 has a linear solution for B_{12}, which is then used to determine $A_{31} = \frac{A_{31n}}{A_{31d}}$. It is found that the first three factors yield $\tilde{\mu}^2 = 3$ and $A_{31n} = 0$ ($A_{31d} \neq 0$), while the last factor gives $\tilde{\mu}^2 = -1$. Hence, these four factors do not produce the required conditions.
Next, the lengthy polynomial factors $\widetilde{R}_{456}$ and $\widetilde{R}_{457}$ in (3.4), with the command *resultant*, yield the following factors:

$$\begin{aligned} C_3 = &(B_{02} - 2)^{160} B_{02}^5 (23B_{02} - 1)(3B_{02} - 1)^{35}(2B_{02} + 1)^{13} \\ &\times(3B_{02} + 4)^2(23B_{02} - 6)^2(11B_{02} - 2)(11B_{02} + 3)\, F_3\, F_4, \end{aligned}$$

where

$$\begin{aligned} F_3 &= 334B_{02}^3 - 339B_{02}^2 + 98B_{02} - 12, \\ F_4 &= 70127B_{02}^5 - 107890B_{02}^4 + 58365B_{02}^3 - 13790B_{02}^2 + 100B_{02} + 216. \end{aligned}$$

There are a total of 11 factors in C_3. Since $B_{02} - 2 \neq 0$, we only need to consider the remaining 10 factors. For each of the 10 factors, we use L_{4b}, L_{5b} and L_{6b} to eliminate A_{21} to get a solution $\widetilde{A}_{21}$ for A_{21}, and two resultants. Verifying the common factors of the two resultants we obtain the following results. First, for the 6 factors, $3B_{02}-1$, $2B_{02}+1$, $3B_{02}+4$, $23B_{02}-6$, $11B_{02}-2$ and $11B_{02}+3$, we can show that these factors yield either $A_{31n} = 0$, or $A_{31n} = A_{31d} = 0$. For example, for the solution $B_{02} = \frac{1}{3}$ from the factor $3B_{02}-1$, we obtain the following common factors from the two resultants:

$$B_{12}(\mu^2 - 3)\left[216(\mu^2 + 1)B_{12} + \mu^2 - 15\right]\left[9(\mu^2 + 9)B_{12} - 4(4\mu^2 + 3)\right].$$

Then, we can use the formulas A_{31n} and A_{31d}, as well as the solution $\widetilde{A}_{21}$ obtained above to verify that $A_{31n} = 0$ for the first two factors; and $A_{31n} = A_{31d} = 0$ for the other two factors. Similarly, we can prove that the other five factors yield either $A_{31n} = 0$ or $A_{31n} = A_{31d} = 0$.
Next, consider the factor B_{02}. Using $B_{02} = 0$ we obtain the common factors:

$$B_{12}(25B_{12} + 2)\left[(625B_{12}^2 + 650B_{12} + 78)\mu^2 + 3\,(1875B_{12}^2 - 50B_{12} - 6)\right],$$

which gives four roots:

$$B_{12} = \frac{-13\mu^2 + 3 \pm \sqrt{91\mu^4 - 762\mu^2 + 171}}{25(\mu^2+9)}, \quad -\frac{2}{25}, \quad 0.$$

The first two roots yield $A_{31\mathrm{n}} = A_{31\mathrm{d}} = 0$, and the third root gives $A_{31\mathrm{n}} = 0$. For the last root, we obtain the solution:

$$B_{03} = B_{02} = B_{12} = 0, \quad A_{21} = \frac{8}{25}, \quad A_{31} = \frac{4}{125}, \quad B_{21} = \frac{8}{5}\mu,$$

which, with the transformation (3.1), leads to the condition (c).
From the factor $23B_{02} - 1$, we have the solution $B_{02} = \frac{1}{23}$, and then obtain the common factors from the two resultants, yielding the following roots:

$$B_{12} = \frac{-156273\mu^4 + 64326\mu^2 + 28215 \pm \sqrt{\Delta}}{8464(47\mu^4 + 438\mu^2 + 135)}, \quad -\frac{36}{529}, \quad \frac{6}{529},$$

where

$$\Delta = 10997677537\mu^8 - 151771646796\mu^6 - 105265288170\mu^4 \\ - 22658655852\mu^2 - 1489591215.$$

Similarly, we can show that the first two roots yield $A_{31\mathrm{n}} = A_{31\mathrm{d}} = 0$, and the third root gives $A_{31\mathrm{n}} = 0$. For the last root, we obtain the solution:

$$B_{03} = 0, \quad B_{12} = \frac{6}{529}, \quad A_{21} = \frac{174}{529}, \quad A_{31} = \frac{432}{12167}, \quad B_{21} = \frac{36}{23}\mu,$$

under which $L_k = 0,\ k = 1, 2, \ldots, 8$. Then, using the transformation (3.1) to the above parameter solutions, we obtain the condition (d) in Proposition 3.1.
For the factors F_3 and F_4, it is easy to show that $F_3 = 0$ has one real solution, and $F_4 = 0$ has three real solutions. But it can be proved that none of these four real solutions can yield a common factor in $\widetilde{\mathrm{R}}_{456}$ and $\widetilde{\mathrm{R}}_{457}$.

The proof of Proposition 3.1 is complete. □

Next, we consider the case $a_{31} \neq 0$, $a_{11} = 0$. For this case, we apply the following scaling

$$\begin{aligned} x &= \frac{X}{\sqrt[3]{a_{31}}}, \qquad y = \frac{Y}{\sqrt[3]{a_{31}^2}}, \qquad \tau = \frac{t}{\sqrt[3]{a_{31}}}, \\ a_{11} &= A_{11}\sqrt[3]{a_{31}},\ a_{21} = A_{21}\sqrt[3]{a_{31}^2},\ b_{02} = B_{02}\sqrt[3]{a_{31}}, \\ b_{21} &= B_{21}\sqrt[3]{a_{31}},\ b_{12} = B_{12}\sqrt[3]{a_{11}^2},\ b_{03} = B_{03}a_{31}, \end{aligned} \tag{3.5}$$

into (2.5) yields the scaled system,

$$\frac{dx}{d\tau} = Y(1 + A_{21}X^2 + X^3),$$

$$\frac{dy}{d\tau} = 4\mu XY + B_{02}Y^2 - 2(1+\mu^2)X^3 + B_{21}X^2Y + B_{12}XY^2 + B_{03}Y^3. \quad (3.6)$$

Similarly, we apply the method of normal forms [37] to compute the Lyapunov constants of the system (3.6), which are given in the following theorem.

Theorem 3.2 *When $a_{31} \neq 0$, $a_{11} = 0$, the first 8 Lyapunov constants at the origin of system (3.6) are*

$$\begin{aligned}
L_1 =& B_{21} + \frac{4}{5}\mu B_{02},\\
L_2 =& \frac{1}{35}\big[10B_{03}(5\mu^2+21) + B_{21}(65A_{21} - 30B_{12} - B_{02}^2) - 40\mu\big],\\
L_3 =& -\frac{1}{27000}\big\{B_{03}\big[57600(9A_{21} - 4B_{12}) - (44125B_{21}^2 + 37968B_{02}^2)\big]\\
&+ 40B_{21}(4540A_{21}^2 + 1230B_{12}^2 - 1081B_{02}) - 8A_{21}(2224B_{02}^2B_{21}\\
&+ 23725B_{12}B_{21} + 13200\mu) + 8B_{12}(833B_{02}^2B_{21} + 7200\mu)\big\},\\
L_4 =& \frac{1}{160224243656400000}\big\{156062574990000B_{03}^2B_{21}(14573\mu^2 + 40317)\\
&-25B_{03}\big[297600A_{21}^2(59534982311\mu^2 - 6975604460847)\\
&+2998800B_{12}^2(3860104507\mu^2 - 171292589973)\\
&-1200A_{21}B_{12}(29770381203185\mu^2 - 1724684386382607)\\
&+4165B_{21}^2(1359632890882B_{02}^2 - 16416976836895B_{12})\\
&-25A_{21}(3476511675840336B_{02}^2 - 2390104075008827B_{21}^2)\\
&-349860(25609808684B_{02}^2B_{12} - 245832481825B_{21}\mu)\\
&-233146704B_{02}(17501062B_{02}^3 - 50363775)\big]\\
&+300000B_{21}(60763442231393A_{21}^3 - 9436956571854B_{12}^3)\\
&+12000A_{21}^2(11327783677834B_{02}^2B_{21} - 2442273516056025B_{12}B_{21}\\
&-881605055709300\mu) + 60000B_{12}^2(262806397025465A_{21}B_{21}\\
&-54583848801902\mu) + 116824609995428400000A_{21}B_{12}\mu\\
&-8B_{21}\big[125B_{02}(3196150167527750A_{21} - 2261230195644301B_{12})\\
&+2676317417692450A_{21}B_{02}^4 + 693889(242074798B_{02}^6\\
&+30371831225B_{02}^3 + 20195043750)\big]\big\},\\
&\vdots
\end{aligned}$$

where L_k, $k = 2, 3, 4$ have been simplified by using the Groebner basis approach, and the lengthy polynomials L_5 and L_6 are omitted here for brevity.

Based on the Layponov constants given in Theorem 3.2, we have the following result.

Proposition 3.2 *When $a_{31} \neq 0,\ a_{11} = 0$, except for the condition (a) in Proposition 3.1, there do not exist conditions such that the first 6 Lyapunov constants at the origin of system (2.5) are zero.*

Proof Since when $a_{31} \neq 0,\ a_{11} = 0$, the original system (2.5) is equivalent to the system (3.6), we consider the Lypunov constants at the origin of the system (3.6), given in Theorem 3.2.

Similar to proving Proposition 3.1, we have two cases: (I) $B_{21}=0$ and (II) $B_{21}\neq 0$.

(I) For this case, there are two sub-cases: (I-a) $\mu=0$ (B_{02} is free), and (I-b) $B_{02}=0$ ($\mu\neq 0$).

(I-a) When $\mu=0$, it is easy to get $B_{03}=0$ from $L_2=0$, yielding the condition (a) in Proposition 3.1.

(I-b) When $B_{02}=0$ ($\mu\neq 0$), we have $B_{03} = \frac{4\mu}{5\mu^2+21}\neq 0$ under which $L_2=0$. Then, L_3 is given by

$$L_3 = \frac{16\mu[(11\mu^2+3)A_{21} - 6(\mu^2+1)B_{12}]}{9(5\mu^2+21)}.$$

$L_3=0$ yields two solutions: either $A_{21}=B_{12}=0$, or $A_{21} = \frac{6(\mu^2+1)}{11\mu^2+3}B_{12}\neq 0$. When $A_{21}=B_{12}=0$, we have $L_3=L_4=L_6=L_7=0$, but

$$L_5 = \frac{-64\mu(\mu^2+9)(395\mu^4+118\mu^2-21)}{91(5\mu^2+21)^3},$$
$$L_8 = \frac{-256\mu(\mu^2+9)(6911555\mu^8+6470470\mu^6+147264\mu^4-422982\mu^2-11907)}{25925(5\mu^2+21)^5}.$$

It is obvious that $L_8\neq 0$ when $L_5=0$. Similarly, we can prove that when $A_{21}=\frac{6(\mu^2+1)}{11\mu^2+3}B_{12}$, $L_2=L_3=0$, while $L_4=0$ yields $L_5\neq 0$.

(II) For this case, $B_{21}\neq 0$. $L_1=0$ gives $B_{21}=-\frac{4}{5}\mu B_{02}\neq 0$. Then,

$$L_2 = \frac{2}{175}\left\{25(5\mu^2+21)B_{03} + 2\left[B_{02}(B_{02}^2 - 65A_{21} + 30B_{12}) - 50\right]\mu\right\}.$$

Considering $L_2=0$, there are two cases: (II-a) $B_{03}=0$ and (II-b) $B_{03}\neq 0$.

(II-a) For $B_{03}=0$, $L_2=0$ gives $B_{12}=\frac{1}{30B_{02}}[50+B_{02}(65A_{21}-B_{02}^2)]$. Then, $L_1=L_2=0$, and we can use L_3, L_4 and L_5 to show that $L_5\neq 0$ when $L_3=L_4=0$ since there are only two free parameters A_{21} and B_{02} are involved in these Lyapunov constants.

(II-b) When $B_{03}\neq 0$, we obtain

$$B_{03} = \frac{2\mu[50 - B_{02}(B_{02}^2 - 65A_{21} + 30B_{12})]}{25(5\mu^2+21)}$$

from $L_2=0$. Then, L_3 is a linear function in μ^2. Solving μ^2 from $L_3=0$ yields $\bar{\mu}^2 = \frac{3\bar{\mu}_n^2}{5\bar{\mu}_d^2}$, where

$$\begin{aligned}
\bar{\mu}_n^2 &= -625B_{02}(148A_{21}^2-311A_{21}B_{12}+114B_{12}^2)+625B_{02}^3(25A_{21}-B_{12})\\
&\quad -37500(A_{21}-2B_{12})+7B_{02}^2(113B_{02}^3+21375),\\
\bar{\mu}_d^2 &= 125B_{02}(908A_{21}^2-949A_{21}B_{12}+246B_{12}^2)+25B_{02}^3(473A_{21}-257B_{12})\\
&\quad +7500(11A_{21}-6B_{12})-B_{02}^2(353B_{02}^3+9375).
\end{aligned}$$

If $\bar{\mu}_d^2=0$, then μ is free, and $L_3=0$ implies $\bar{\mu}_n^2=0$. Then, eliminating A_{21} from the equations: $\bar{\mu}_d^2=\bar{\mu}_n^2=L_4=0$ yields two resultants which do not have common factors. This implies that $\bar{\mu}_d^2=0$ does not yield conditions such that $L_3=L_4=\cdots=0$. So assuming $\bar{\mu}_d^2\neq 0$, we have

$$\begin{aligned}
L_4 &= -\frac{9216\mu[5(13A_{21}-6B_{12})B_{02}-B_{02}^3+50]^2}{4296875(5\mu^2+21)^2(\bar{\mu}_d^2)^2}L_{4c},\\
L_5 &= -\frac{221184\mu[5(13A_{21}-6B_{12})B_{02}-B_{02}^3+50]^3}{35546875(5\mu^2+21)^3(\bar{\mu}_d^2)^3}L_{5c},\\
L_6 &= \frac{1536\mu[5(13A_{21}-6B_{12})B_{02}-B_{02}^3+50]^3}{93994140625(5\mu^2+21)^3(\bar{\mu}_d^2)^3}L_{6c},\\
L_7 &= -\frac{12288\mu[5(13A_{21}-6B_{12})B_{02}-B_{02}^3+50]^4}{20772705078125(5\mu^2+21)^4(\bar{\mu}_d^2)^4}L_{7c},
\end{aligned}$$

where L_{4c}, L_{5c}, L_{6c} and L_{7c} are polynomials in A_{21}, B_{12} and B_{02}. Eliminating A_{21} from L_{4c}, L_{5c}, L_{6c} and L_{7c} results in a solution for A_{21}, and three resultants, $\mathrm{R}_{45c}=G_1\mathrm{R}_{45d}$, $\mathrm{R}_{46c}=G_1\mathrm{R}_{46d}$, and $\mathrm{R}_{47c}=G_1\mathrm{R}_{47d}$, which have a common factor,

$$G_1 = 36250B_{02}B_{12}^2+1250(103B_{02}^3-540)B_{12}+B_{02}^2(44872B_{02}^3-1798875).$$

Then, eliminating B_{12} from G_1, L_{4c}, L_{5c} and L_{6c} yields a solution for B_{12}, and three resultants which have a common factor,

$$G_2=18125B_{02}A21^2+250(137B_{02}^3-600)A_{21}+3B_{02}^2(3107B_{02}^3-67000).$$

Now, eliminating A_{21} and B_{12} from G_1, G_2, L_{4c}, L_{5c} and L_{6c} shows that no resultants can be obtained, implying that the common factors G_1 and G_2 do not yield the required conditions.
Finally, we consider the possibility from R_{45d}, R_{46d} and R_{47d} which are lengthy polynomials in B_{12} and B_{02}. Using the Maple built-in command *resultant* we obtain two resultants:

$$\begin{aligned}
\mathrm{R}_{4546d} &= resultant(\mathrm{R}_{45d},\mathrm{R}_{46d},B_{12}),\\
\mathrm{R}_{4547d} &= resultant(\mathrm{R}_{45d},\mathrm{R}_{47d},B_{12}),
\end{aligned}$$

which do not have common factors, implying that the polynomials R_{45d}, R_{46d} and R_{47d} do not have possible solutions such that $L_k = 0,\ k = 1, 2, \ldots, 8$.

Summarizing the above results we have shown that when $a_{31} \neq 0$, $a_{11} = 0$, except for the case (a) in Proposition 3.1, there are no more solutions such that the first 8 Lyapunov constants vanish. In other words, the case $a_{11} = 0$ is included in the condition (a) in Proposition 3.1.

This completes the proof of Proposition 3.2. □

Now, we prove that the four conditions given in Proposition 3.1 are necessary and sufficient for the origin of system (2.5) to be a center.

Theorem 3.3 *When $a_{31} \neq 0$, the nilpotent origin of system* (2.5) *is a center if and only if one of the* 4 *conditions in Proposition 3.1 holds.*

Proof The necessity has been shown in the proof of Proposition 3.1, since only those four conditions satisfy $L_k = 0,\ k = 1, 2, \ldots, 8$. In the following, the sufficiency will be proved one by one. The main idea of proving the sufficiency is to transform the system (2.5) under each of the four conditions to a Liénard system so that two primitives are formed for proving the sufficiency. More details on the theory and methodology of this development can be found in [6–8, 10, 12]. More precisely, Cherkas established the method [6, 7], which was further improved and generalized in [8, 10, 12], from which we particularly apply Corollary 6 in [12] to prove the sufficiency of a center in Liénard systems. The detailed steps can be seen in the following proof.

When the condition (a) in Proposition 3.1 holds, system (2.5) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y(1 + a_{11}x + a_{21}x^2 + a_{31}x^3),\\ \frac{dy}{dt} &= -2x^3 + b_{02}y^2 + b_{12}xy^2,\end{aligned}$$

which is obviously a revertible system (symmetric with respect to the x-axis).

When the condition (b) holds, system (2.5) becomes

$$\begin{aligned}\frac{dx}{dt} &= (1 - 2a_{11}x)(1 - 3a_{11}x)(1 + 6a_{11}x)y,\\ \frac{dy}{dt} &= -2(x^3 - a_{11}y^2 + 6a_{11}^2xy^2 - 2xy\mu + x^3\mu^2),\end{aligned}$$

which can be changed into a Liénard system in the form of

$$\begin{aligned}\frac{dx}{d\tau} &= y,\\ \frac{dy}{d\tau} &= \frac{-2(x^3 - a_{11}y^2 + 6a_{11}^2xy^2 - 2xy\mu + x^3\mu^2)}{(1 - 2a_{11}x)(1 - 3a_{11}x)(1 + 6a_{11}x)}\\ &= p_0(x) + p_1(x)y + p_2(x)y^2,\end{aligned} \tag{3.7}$$

by $d\tau = (1-2a_{11}x)(1-3a_{11}x)(1+6a_{11}x)\,dt$. Now, we construct

$$W_1(x) = \frac{(p_0(x)p_1(x)p_2(x) - p_1(x)p_0'(x) + p_0(x)p_1'(x))}{p_1(x)^3},$$
$$W_2(x) = \frac{W_1'[x]p_0(x)}{p_1(x)^2}.$$

Then, the system (3.7) has a center if and only if $W_1(x) - W_1(y)$ and $W_2(x) - W_2(y)$ have a common factor with the form $x + y +$ h.o.t. It is easy to verify that

$$\begin{aligned} W_1(x) - W_1(y) =& \frac{9a_{11}^2(x-y)(1+\mu^2)}{2\mu^2}\,h_1(x,y), \\ W_2(x) - W_2(y) =& \frac{9a_{11}^2(x-y)}{2\mu^2}\,h_1(x,y)(1-3a_{11}x-3a_{11}y)(1+\mu^2)^2 \\ & \times (1+3a_{11}x-18a_{11}^2x^2+3a_{11}y+18a_{11}^2xy-18a_{11}^2y^2), \end{aligned}$$

where

$$h_1(x,y) = -x + 2a_{11}x^2 - y + 2a_{11}xy + 2a_{11}y^2,$$

is a common factor, implying that the origin of the system is a nilpotent center.

When the condition (c) is satisfied, system (2.5) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} =& \frac{1}{125}\,(5+a_{11}x)(5+2a_{11}x)^2 y, \\ \frac{dy}{dt} =& -\frac{2}{5}\,x(5x^2 - 10y\mu - 4a_{11}xy\mu + 5x^2\mu^2), \end{aligned}$$

which can be changed into a Liénard system,

$$\begin{aligned} \frac{dx}{d\tau} &= y, \\ \frac{dy}{d\tau} &= -\frac{50x((5x^2+5x^2\mu^2) - y(10\mu + 4a_{11}x\mu))}{(5+a_{11}x)(5+2a_{11}x)^2} \\ &= p_0(x) + p_1(x)y + p_2(x)y^2, \end{aligned} \tag{3.8}$$

by $d\tau = \frac{(5+a_{11}x)(5+2a_{11}x)^2}{125}\,dt$. Similar to the proof for the case (b), let

$$W_1(x) = \frac{p_0(x)p_1(x)p_2(x) - p_1(x)p_0'(x) + p_0(x)p_1'(x)}{p_1(x)^3},$$
$$W_2(x) = \frac{W_1'[x]p_0(x)}{p_1(x)^2}.$$

Then, the system (3.8) has a center if and only if $W_1(x) - W_1(y)$ and $W_2(x) - W_2(y)$ have a common factor in the form of $x + y +$ h.o.t. It is easy to verify that

$$\begin{aligned} W_1(x) - W_1(y) =& \frac{a_{11}^2(x-y)(1+\mu^2)}{20(5+2a_{11}x)(5+2a_{11}y)\mu^2} h_2(x,y), \\ W_2(x) - W_2(y) =& -\frac{a_{11}^2(x-y)(1+\mu^2)^2(5+a_{11}x+a_{11}y)}{400(5+2a_{11}x)^2(5+2a_{11}y)^2\mu^4} \\ & \times (25+5a_{11}x+5a_{11}y+2a_{11}^2xy)h_2(x,y), \end{aligned}$$

where

$$h_2(x,y) = 5x + 5y + 2a_{11}xy,$$

is a common factor, showing that the origin of the system is a nilpotent center.

When the condition (d) is satisfied, system (2.5) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} =& (1+6b_{02}x)(1+8b_{02}x)(1+9b_{02}x)y, \\ \frac{dy}{dt} =& -2x^3 + b_{02}y^2 + 6b_{02}^2xy^2 + 4xy\mu + 36b_{02}x^2y\mu - 2x^3\mu^2, \end{aligned}$$

which can be changed into a Liénard system,

$$\begin{aligned} \frac{dx}{d\tau} &= y, \\ \frac{dy}{d\tau} &= \frac{-2x^3 + b_{02}y^2 + 6b_{02}^2xy^2 + 4xy\mu + 36b_{02}x^2y\mu - 2x^3\mu^2}{(1+6b_{02}x)(1+8b_{02}x)(1+9b_{02}x)}, \end{aligned} \tag{3.9}$$

by $d\tau = (1+6b_{02}x)(1+8b_{02}x)(1+9b_{02}x)dt$. Similar to the proof for the case (b), let

$$\begin{aligned} W_1(x) &= \frac{p_0(x)p_1(x)p_2(x) - p_1(x)p_0'(x) + p_0(x)p_1'(x)}{p_1(x)^3}, \\ W_2(x) &= \frac{W_1'[x]p_0(x)}{p_1(x)^2}. \end{aligned}$$

Then, the system (3.9) has a center if and only if $W_1(x) - W_1(y)$ and $W_2(x) - W_2(y)$ have a common factor in the form of $x + y +$ h.o.t. It is easy to verify that

$$\begin{aligned} W_1(x) - W_1(y) =& \frac{27b_{02}^2(x-y)(1+\mu^2)}{4(1+9b_{02}x)^2(1+9b_{02}y)^2\mu^2} h_3(x,y), \\ W_2(x) - W_2(y) =& -\frac{27b_{02}^2(x-y)(1+\mu^2)^2}{16(1+9b_{02}x)^4(1+9b_{02}y)^4\mu^4} h_3(x,y) \\ & \times (1 + 18b_{02}x + 108b_{02}^2x^2 + 216b_{02}^3x^3 + 18b_{02}y \\ & + 324b_{02}^2xy + 1944b_{02}^3x^2y + 3888b_{02}^4x^3y + 108b_{02}^2y^2 \\ & + 1944b_{02}^3xy^2 + 10935b_{02}^4x^2y^2 + 17496b_{02}^5x^3y^2 \end{aligned}$$

$$+216b_{02}^3y^3 + 3888b_{02}^4xy^3 + 17496b_{02}^5x^2y^3),$$

where

$$h_3(x, y) = x + 8b_{02}x^2 + y + 26b_{02}xy + 144b_{02}^2x^2y + 8b_{02}y^2 + 144b_{02}^2xy^2 + 648b_{02}^3x^2y^2,$$

is a common factor, showing that the origin of the system is a nilpotent center. □

Furthermore, we want to prove that under the condition (a) in Proposition 3.1, the origin of system (2.5) is an analytic center. To prove this, we apply the definitions and relative theorems given in [5, 27], which are listed below for convenience. It should be noted that Theorems 6 and 12 in [27] are only applicable for cubic-order systems.

Lemma 3.1 [5, 27] *The origin of system* (2.5) *is an analytic center if and only if the origin of system* (2.5) *is a center and for any natural number* k, $L_k = 0$.

Lemma 3.2 [5, 27] *If system* (2.5) *is symmetric with respect to the* x*-axis, then the origin of system* (2.5) *is an analytic center.*

For system (2.5), we have the following result.

Theorem 3.4 *The origin of system* (2.5) *is an analytic center if and only if the condition* (a) *in Proposition* 3.1 *is satisfied, i.e.,*

$$b_{03} = b_{21} = \mu = 0.$$

Proof When the condition holds, system (2.5) can be brought into the form,

$$\frac{dx}{dt} = y(1 + a_{11}x + a_{21}x^2 + a_{31}x^3),$$
$$\frac{dy}{dt} = -2x^3 + b_{02}y^2 + b_{12}xy^2,$$

which is symmetric with respect to the x-axis.

By the change of state variables, $u = x,\ v = y^2$ and the time rescaling $\tau = yt$, the above system can be changed to the new system,

$$\frac{du}{d\tau} = 1 + a_{11}u + a_{21}u^2 + a_{31}u^3,$$
$$\frac{dv}{d\tau} = -2u^3 + b_{02}v^2 + b_{12}uv^2,$$

whose origin is a regular point, and by Lemmas 3.1 and 3.2 it is analytical integrability. □

In the following, we present one of our main results in this paper.

Theorem 3.5 *When* $a_{31} \neq 0$, *system (2.5) can have at least* 8 *small-amplitude limit cycles around the origin by small parameter perturbation.*

Proof To obtain maximal number of the limit cycles around the origin of the system (2.5), we need to find the conditions such that $L_j = 0,\ j = 1, 2, \ldots, k-1$, but $L_k \neq 0$ for as large k as possible. For $a_{11} \neq 0$, the system (2.5) is equivalent to the system (3.2) under the transformation (3.1). The system with $a_{11} = 0$ is obviously to have less limit cycles than that of the system with $a_{11} \neq 0$. Thus, we consider system (3.1) which has 7 independent parameters. In general, with a linear perturbation, the system may have 8 limit cycles bifurcating from the origin. It has been clearly shown in the proof of Proposition 3.1 that bifurcation of maximal limit cycles can only come from the case (II) $B_{21} \neq 0$ when $B_{03} \neq 0$. If $B_{03} = 0$, it can be shown that the maximal number of limit cycles bifurcating from the origin is 6. If $B_{03} \neq 0$, it is seen from (3.4) that the two polynomials $\widetilde{\mathrm{R}}_{456}$ and $\widetilde{\mathrm{R}}_{457}$ still contain two parameters B_{12} and B_{02}. Thus, it may be possible to have solutions for B_{12} and B_{02} such that $\widetilde{\mathrm{R}}_{456} = \widetilde{\mathrm{R}}_{457} = 0$, which may lead to $L_4 = L_5 = L_6 = L_7 = 0$, but $L_8 \neq 0$. To achieve this, eliminating B_{12} from $\widetilde{\mathrm{R}}_{456}$ and $\widetilde{\mathrm{R}}_{457}$ yields a resultant polynomial in B_{02},

$$\begin{aligned}\widetilde{\mathrm{R}}_{4567} = {} & (334B_{02}^3 - 339B_{02}^2 + 98B_{02} - 12)(70127B_{02}^5 - 107890B_{02}^4 \\ & +58365B_{02}^3 - 13790B_{02}^2 + 100B_{02} + 216)\, F_{667}(B_{02}),\end{aligned}$$

where F_{667} represent a 667th-degree polynomial in B_{02}. The first two factors have 4 real solutions, but none of them yields solution B_{12} to satisfy $\widetilde{\mathrm{R}}_{456} = \widetilde{\mathrm{R}}_{457} = 0$. The polynomial F_{667} yields 73 real solutions:

$$-2.6540242544\cdots,\ -2.0351509596\cdots,\quad \cdots,\quad 4.4358229258\cdots,$$

which have corresponding 73 solutions for B_{12} such that $\widetilde{\mathrm{R}}_{456} = \widetilde{\mathrm{R}}_{457} = 0$. Then, using the polynomials R_{45}, R_{46} and R_{47} we obtain corresponding 73 solutions for A_{21} such that $\mathrm{R}_{45} = \mathrm{R}_{46} = \mathrm{R}_{47} = 0$. Finally, we use the solutions $\tilde{\mu}^2(A_{21}, B_{12}, B_{02})$ to check if $\tilde{\mu}^2 > 0$ for the set of 73 solutions $(A_{21}, B_{12}, B_{02})_k,\ k = 1, 2, \ldots, 73$. It is found that 31 of the 73 solutions satisfy $\tilde{\mu}^2 > 0$, 7 of them yield $L_8 < 0$ and 24 of them give $L_8 > 0$. To guarantee the existence of 8 limit cycles, we compute the determinant of the Jacobian to obtain

$$\det_7 = \det\left[\frac{\partial(L_1, L_2, L_3, L_4, L_5, L_6, L_7)}{\partial(B_{21}, B_{03}, A_{31}, A_{21}, B_{12}, B_{02})}\right]_{\mathrm{C}},$$

where the subscript C denotes a critical point using one solution of the above 31 solutions. In the following, we list two solutions, with one for $v_8 < 0$ and one for $v_8 > 0$:

$$v_8 = -3.79957976\cdots \begin{cases} B_{21} = 0.35951158\cdots, & B_{03} = -0.72206432\cdots, \\ A_{31} = 0.12284334\cdots, & A_{21} = -6.71259965\cdots, \\ B_{12} - 7.71405139\cdots, & B_{02} = 1.63917500\cdots \\ \det_7 = -181397.86\cdots; & \end{cases}$$

$$v_8 = 1.02997402\cdots \begin{cases} B_{21} = 2.51956740\cdots, & B_{03} = 0.39729450\cdots, \\ A_{31} = -0.32360952\cdots, & A_{21} = -9.03972383\cdots, \\ B_{12} - 12.26798825\cdots, & B_{02} = 0.15355900\cdots \\ \det_7 = 495.225881\cdots. \end{cases}$$

Note that under the time rescaling given in (3.1), the Lyapunov constants for the original system (2.5) are $a_{11}L_k$, and thus the nonzero free parameter a_{11} can be used to adjust the order of the Lyapunov constants. This shows, with a linear perturbation, the existence of 8 small-amplitude limit cycles bifurcating from the origin of the system (2.5).

This finishes the proof of Theorem 3.5. □

We should point out that for the generalized Kukles system (2.5) one may apply the so-called "double bifurcation" analysis [21] to obtain one more limit cycle. That is,

Theorem 3.6 *When $a_{31} \neq 0$, at least 9 small-amplitude limit cycles can bifurcate from the origin of system (2.5) by using the double bifurcation analysis.*

3.2 Center Conditions of System (2.5) for $a_{31} = 0$

We first compute the Lyapunov constants of system (2.5) at the origin to obtain the following result.

Theorem 3.7 *The first 7 Lyapunov constants at the origin of system (2.5) with $a_{31} = 0$ are given as follows:*

$$\begin{aligned}
u_1 &= \frac{1}{15}(1+\mu^2)[5b_{21} + 4(b_{02} - 2a_{11})\mu],\\
u_2 &= \frac{2}{875}(1+\mu^2)(525b_{03} - 16a_{11}^3\mu + 44a_{11}^2 b_{02}\mu - 22a_{11}b_{02}^2\mu + 2b_{02}^3\mu\\
&\qquad + 125b_{03}\mu^2 - 20a_{11}\mu b_{12} + 60b_{02}\mu b_{12} + 60a_{11}a_{21}\mu - 130a_{21}b_{02}\mu),\\
u_3 &= -\frac{4\mu(1+\mu^2)}{590625(21+5\mu^2)} f_1,\\
u_4 &= -\frac{199766310912\mu(1+\mu^2)(14a_{11}^2 - 9a_{11}b_{02} + b_{02}^2 + 25b_{12} - 50a_{21})^4}{34375(9+\mu^2)(21+5\mu^2)^2} f_2^2 f_3^4 f_4^2,\\
u_5 &= -\frac{75144747810816\mu(1+\mu^2)(14a_{11}^2 - 9a_{11}b_{02} + b_{02}^2 + 25b_{12} - 50a_{21})^4}{1001(4+\mu^2)(9+\mu^2)(21+5\mu^2)^3} f_3^6 f_5^2 f_6^2,\\
u_6 &= -\frac{33241631799575052288\mu(1+\mu^2)(14a_{11}^2 - 9a_{11}b_{02} + b_{02}^2 + 25b_{12} - 50a_{21})^4}{3072265625(4+\mu^2)(9+\mu^2)(21+5\mu^2)^3(1+9\mu^2)(25+9\mu^2)} f_3^6 f_7^2 f_8^2 f_9^2,\\
u_7 &= -\frac{245083902931906945508966 4\mu(1+\mu^2)(14a_{11}^2 - 9a_{11}b_{02} + b_{02}^2 + 25b_{12} - 50a_{21})^4}{5611328125(4+\mu^2)(9+\mu^2)(1+4\mu^2)(9+4\mu^2)(21+5\mu^2)^3(1+9\mu^2)(25+9\mu^2)} f_3^8 f_{10}^2 f_{11}^2 f_{12}^2,
\end{aligned}$$

where

$$
\begin{aligned}
f_1 = {} & 4464a_{11}^5 - 36000a_{11}^3a_{21} + 67500a_{11}a_{21}^2 - 47460a_{11}^4b_{02} + 264750a_{11}^2a_{21}b_{02} \\
& - 277500a_{21}^2b_{02} + 91260a_{11}^3b_{02}^2 - 329625a_{11}a_{21}b_{02}^2 - 37155a_{11}^2b_{02}^3 + 46875a_{21}b_{02}^3 \\
& - 3330a_{11}b_{02}^4 + 2373b_{02}^5 + 17520a_{11}^5\mu^2 - 132000a_{11}^3a_{21}\mu^2 + 247500a_{11}a_{21}^2\mu^2 \\
& - 59300a_{11}^4b_{02}\mu^2 + 376750a_{11}^2a_{21}b_{02}\mu^2 - 567500a_{21}^2b_{02}\mu^2 + 42300a_{11}^3b_{02}^2\mu^2 \\
& - 149625a_{11}a_{21}b_{02}^2\mu^2 + 11725a_{11}^2b_{02}^3\mu^2 - 59125a_{21}b_{02}^3\mu^2 - 13650a_{11}b_{02}^4\mu^2 \\
& + 1765b_{02}^5\mu^2 + 52500a_{11}^3b_{12} - 210000a_{11}a_{21}b_{12} - 195000a_{11}^2b_{02}b_{12} \\
& + 583125a_{21}b_{02}b_{12} + 144375a_{11}b_{02}^2b_{12} - 1875b_{02}^3b_{12} + 60500a_{11}^3\mu^2b_{12} \\
& - 230000a_{11}a_{21}\mu^2b_{12} - 187000a_{11}^2b_{02}\mu^2b_{12} + 593125a_{21}b_{02}\mu^2b_{12} \\
& + 70375a_{11}b_{02}^2\mu^2b_{12} + 32125b_{02}^3\mu^2b_{12} + 71250a_{11}b_{12}^2 - 213750b_{02}b_{12}^2 \\
& + 51250a_{11}\mu^2b_{12}^2 - 153750b_{02}\mu^2b_{12}^2, \\
f_2 = {} & 456a_{11}^3 - 1800a_{11}a_{21} - 1154a_{11}^2b_{02} + 4025a_{21}b_{02} + 87a_{11}b_{02}^2 + 563b_{02}^3 \\
& + 650a_{11}b_{12} - 1950b_{02}b_{12}, \\
f_3 = {} & 8a_{11}^3 - 30a_{11}a_{21} - 22a_{11}^2b_{02} + 65a_{21}b_{02} + 11a_{11}b_{02}^2 - b_{02}^3 + 10a_{11}b_{12} - 30b_{02}b_{12}, \\
f_5 = {} & 65616a_{11}^5 - 492000a_{11}^3a_{21} + 922500a_{11}a_{21}^2 - 189740a_{11}^4b_{02} + 1242250a_{11}^2a_{21}b_{02} \\
& - 1992500a_{21}^2b_{02} + 77940a_{11}^3b_{02}^2 - 268875a_{11}a_{21}b_{02}^2 + 84055a_{11}^2b_{02}^3 - 283375a_{21}b_{02}^3 \\
& - 51270a_{11}b_{02}^4 + 4687b_{02}^5 + 189500a_{11}^3b_{12} - 710000a_{11}a_{21}b_{12} - 553000a_{11}^2b_{02}b_{12} \\
& + 1789375a_{21}b_{02}b_{12} + 137125a_{11}b_{02}^2b_{12} + 130375b_{02}^3b_{12} + 133750a_{11}b_{12}^2 \\
& - 401250b_{02}b_{12}^2, \\
f_7 = {} & 2832a_{11}^5 - 24000a_{11}^3a_{21} + 45000a_{11}a_{21}^2 - 45980a_{11}^4b_{02} + 250750a_{11}^2a_{21}b_{02} \\
& - 241250a_{21}^2b_{02} + 97380a_{11}^3b_{02}^2 - 352125a_{11}a_{21}b_{02}^2 - 43265a_{11}^2b_{02}^3 \\
& + 60125a_{21}b_{02}^3 - 2040a_{11}b_{02}^4 + 2449b_{02}^5 + 51500a_{11}^3b_{12} - 207500a_{11}a_{21}b_{12} \\
& - 196000a_{11}^2b_{02}b_{12} + 581875a_{21}b_{02}b_{12} + 153625a_{11}b_{02}^2b_{12} - 6125b_{02}^3b_{12} \\
& + 73750a_{11}b_{12}^2 - 221250b_{02}b_{12}^2, \\
f_8 = {} & 24864a_{11}^5 - 186000a_{11}^3a_{21} + 348750a_{11}a_{21}^2 - 65960a_{11}^4b_{02} + 439750a_{11}^2a_{21}b_{02} \\
& - 730625a_{21}^2b_{02} + 14760a_{11}^3b_{02}^2 - 48375a_{11}a_{21}b_{02}^2 + 39220a_{11}^2b_{02}^3 - 118750a_{21}b_{02}^3 \\
& - 19455a_{11}b_{02}^4 + 1423b_{02}^5 + 65000a_{11}^3b_{12} - 241250a_{11}a_{21}b_{12} - 182500a_{11}^2b_{02}b_{12} \\
& + 598750a_{21}b_{02}b_{12} + 28750a_{11}b_{02}^2b_{12} + 51250b_{02}^3b_{12} + 40000a_{11}b_{12}^2 - 120000b_{02}b_{12}^2, \\
f_{10} = {} & 336a_{11}^5 - 12000a_{11}^3a_{21} + 22500a_{11}a_{21}^2 - 130540a_{11}^4b_{02} + 682250a_{11}^2a_{21}b_{02} \\
& - 542500a_{21}^2b_{02} + 322740a_{11}^3b_{02}^2 - 1168875a_{11}a_{21}b_{02}^2 - 160345a_{11}^2b_{02}^3 \\
& + 246625a_{21}b_{02}^3 + 330a_{11}b_{02}^4 + 7727b_{02}^5 + 149500a_{11}^3b_{12} - 610000a_{11}a_{21}b_{12} \\
& - 593000a_{11}^2b_{02}b_{12} + 1739375a_{21}b_{02}b_{12} + 507125a_{11}b_{02}^2b_{12} - 39625b_{02}^3b_{12} \\
& + 233750a_{11}b_{12}^2 - 701250b_{02}b_{12}^2, \\
f_{11} = {} & 46608a_{11}^5 - 348000a_{11}^3a_{21} + 652500a_{11}a_{21}^2 - 114620a_{11}^4b_{02} + 777250a_{11}^2a_{21}b_{02} \\
& - 1332500a_{21}^2b_{02} + 5220a_{11}^3b_{02}^2 - 9375a_{11}a_{21}b_{02}^2 + 84715a_{11}^2b_{02}^3 - 239875a_{21}b_{02}^3 \\
& - 36510a_{11}b_{02}^4 + 2131b_{02}^5 + 111500a_{11}^3b_{12} - 410000a_{11}a_{21}b_{12} - 301000a_{11}^2b_{02}b_{12}
\end{aligned}
$$

$$+\,1001875a_{21}b_{02}b_{12}+18625a_{11}b_{02}^2b_{12}+98875b_{02}^3b_{12}+58750a_{11}b_{12}^2$$
$$-\,176250b_{02}b_{12}^2,$$

and the lengthy f_4*,* f_6*,* f_9 *and* f_{12} *are not listed here for brevity.*

Based on the Lyapunov constants given in Theorem 3.7, we obtain the following result.

Proposition 3.3 *The first 7 Lyapunov constants at the origin of system (2.5) with* $a_{31}=0$ *are zero if and only if one of the following conditions holds:*

(1) $b_{03}=b_{21}=\mu=0;$

(2) $b_{03}=b_{21}=a_{11}=b_{02}=0;$

(3) $b_{03}=b_{21}=b_{02}-2a_{11}=b_{12}-2a_{21}=0;$

(4) $b_{03}=3b_{21}-4a_{11}\mu=3b_{02}-a_{11}=9a_{21}-2a_{11}^2=0;$

(5) $b_{03}=5b_{21}-4(2a_{11}-b_{02})\mu=25a_{21}-(2a_{11}-b_{02})(3a_{11}+b_{02})$
$=25b_{12}+(2a_{11}-b_{02})(a_{11}-3b_{02})=0;$

(6) $b_{03}=5b_{21}-12b_{02}\mu=25b_{12}+3b_{02}^2=25a_{21}-9b_{02}^2=a_{11}-2b_{02}=0;$

(7) $b_{03}=b_{21}-6b_{02}\mu=2b_{12}-3b_{02}^2=2a_{21}-9b_{02}^2=4a_{11}-17b_{02}=0;$

(8) $b_{03}=b_{21}-2b_{02}\mu=b_{12}+b_{02}^2=2a_{21}+b_{02}^2=4a_{11}-7b_{02}=0;$

(9) $\mu^2-3=9000b_{03}-\mu(2a_{11}-b_{02})^2(4a_{11}-17b_{02})=5b_{21}-4\mu(2a_{11}-b_{02})$
$=5b_{12}-b_{02}(2a_{11}-b_{02})=100a_{21}-3(2a_{11}-b_{02})(4a_{11}+3b_{02})=0;$

(10) $\mu^2-3=225b_{03}-\mu(2a_{11}-b_{02})^2(a_{11}-2b_{02})=5b_{21}-4\mu(2a_{11}-b_{02})$
$=25b_{12}+(2a_{11}-b_{02})(9a_{11}-17b_{02})=25a_{21}+3(2a_{11}-b_{02})(a_{11}-3b_{02})=0;$

(11) $\mu^2-3=1125b_{03}-\mu(a_{11}+2b_{02})^2(4a_{11}-7b_{02})=5b_{21}-4\mu(2a_{11}-b_{02})$
$=25b_{12}+4a_{11}^2+b_{02}a_{11}+11b_{02}^2=25a_{21}-(4a_{11}+3b_{02})(a_{11}-3b_{02})=0;$

(12) $\mu^2-3=72b_{03}-\mu\,b_{02}(2b_{02}^2-3b_{12})=b_{21}-4\mu\,b_{02}=4a_{21}-3(2b_{02}^2+b_{12})$
$=3b_{02}-a_{11}=0.$

Remark 3.1 Linh and Sadovskii considered the Kukles system in the following format [23]:

$$\begin{aligned}\frac{dx}{dt}&=y\,(1+Dx+Px^2),\\ \frac{dy}{dt}&=-\,x^3+Axy+By^2+Kx^2y+Lxy^2+My^3,\end{aligned}\tag{3.10}$$

which can be transformed into (2.5) with $a_{31}=0$, under the following transformation:

$$\begin{aligned}&\xi=\sqrt{2(1+\mu^2)},\quad x\to x,\quad y\to\xi\,y,\quad t\to\xi\,t,\\ &D=a_{11},\quad P=a_{21},\quad A=\frac{4\mu}{\xi},\quad B=b_{02},\quad K=\frac{b_{21}}{\xi},\quad L=b_{12},\ M=\xi\,b_{03}.\end{aligned}$$

(i) 14 center conditions were obtained in [23], in which the first 12 conditions are the same as the 12 conditions (1)–(12) given in Proposition 3.3, while the last two conditions (13) and (14) given in [23] are actually not center conditions. These two cases are characterized by $A^2 = 9$, which corresponds to our system (2.5) with $a_{31} = 0$ for $\mu^2 = -9$ (yielding $b_{30} = 16 > 0$), showing that these two cases lead to a nilpotent saddle point, not a nilpotent center (see Theorem 2.1).

(ii) In [23], the authors claimed that the origin of the Kukles system (3.10) is a focus of 8th order. This is not true since our analysis given above clearly shows that the 7 parameters/coefficients are not independent, and a simple scaling reduces the number of parameters from 7 to 6. Hence, the origin of the system is a focus of 7th order.

Proof The proof is similar to that for proving Proposition 3.1. Again, it is seen from $u_1 = 0$ that

$$b_{21} = \frac{4}{5}\,\mu\,(2a_{11} - b_{02}). \tag{3.11}$$

There are two cases: (I) $b_{21} = 0$ and (II) $b_{21} \neq 0$.

(I) This case has two sub-cases: (I-a) $\mu = 0$ and (I-b) $b_{02} - 2a_{11} = 0$ ($\mu \neq 0$).

(I-a) For this case, $u_2 = 0$ gives $b_{03} = 0$ and so $u_3 = u_4 = \cdots = 0$, yielding the condition (1).

(I-b) Under this condition, u_2 becomes

$$u_2 = \frac{2}{35}(1+\mu^2)\big[(5\mu^2+21)b_{03} + 4\mu a_{11}(b_{12} - 2a_{21})\big],$$

which gives the solution $b_{03} = -\frac{4\mu a_{11}(b_{12}-2a_{21})}{5\mu^2+21}$ by setting $u_2 = 0$. Then, $u_3, u_4, \cdots$ have common factors $a_{11}(b_{12} - 2a_{21})$. Setting $a_{11} = 0$ yields the condition (2), while letting $b_{12} - 2a_{21} = 0$ leads to the condition (3).

(II) For this case, $b_{21} \neq 0$, i.e., $\mu\,(2a_{11} - b_{02}) \neq 0$. Then, $u_2 = 0$ yields

$$\begin{aligned} b_{03} = -\frac{2\mu}{25(5\mu^2+21)}\big[&5(6a_{11} - 13b_{02})a_{21} - 10(a_{11} - 3b_{02})b_{12} \\ &- (2a_{11} - b_{02})(4a_{11}^2 - 9a_{11}b_{02} + b_{02}^2)\big]. \end{aligned} \tag{3.12}$$

With the solutions (3.11) and (3.12), we have $u_1 = u_2 = 0$, and

$$\begin{aligned} u_3 &= \frac{-4\mu(1+\mu^2)}{590625(21+5\mu^2)}\,\overline{u}_3, \\ u_4 &= \frac{-199766310912\mu(1+\mu^2)}{34375(9+\mu^2)(21+5\mu^2)^2}\,\overline{u}_4, \\ u_5 &= \frac{-75144747810816\mu(1+\mu^2)}{1001(4+\mu^2)(9+\mu^2)(21+5\mu^2)^3}\,\overline{u}_5, \\ u_6 &= \frac{-33241631799575052288\mu(1+\mu^2)}{3072265625(4+\mu^2)(9+\mu^2)(21+5\mu^2)^3(1+9\mu^2)(25+9\mu^2)}\,\overline{u}_6, \end{aligned} \tag{3.13}$$

where $\overline{u}_3, \overline{u}_4, \overline{u}_5$ and $\overline{u}_6$ are polynomials in $a_{11}, a_{21}, b_{11}, b_{02}$ and b_{12}. Eliminating b_{12} from the equations $\overline{u}_3 = \overline{u}_4 = \overline{u}_5 = \overline{u}_6 = 0$ we obtain a solution $\widetilde{b}_{12} = \widetilde{b}_{12}(a_{11}, a_{21}, b_{11}, b_{02})$ and three resultants:

$$\mathrm{R}_{34} = \overline{C}_1\, \widetilde{\mathrm{R}}_{34}, \quad \mathrm{R}_{35} = \overline{C}_1\, \widetilde{\mathrm{R}}_{35}, \quad \mathrm{R}_{36} = \overline{C}_1\, \widetilde{\mathrm{R}}_{36}, \tag{3.14}$$

where the common factor $\overline{C}_1$ is

$$\overline{C}_1 = (\mu^2+9)(5\mu^2+21)(2a_{11}-b_{02})\big[(2a_{11}-b_{02})(3a_{11}+b_{02})-25a_{21}\big].$$

The factor in the square bracket gives the condition (5). Further, eliminating a_{21} from $(\widetilde{\mathrm{R}}_{34}, \widetilde{\mathrm{R}}_{35}, \widetilde{\mathrm{R}}_{36})$, we obtain the solution $\widetilde{a}_{21} = \widetilde{a}_{21}(a_{11}, b_{11}, b_{02})$ and two resultants which have the following common factors:

$$\begin{aligned}\overline{C}_2 = &(a_{11}-3b_{02})(a_{11}-2b_{02})(4a_{11}-17b_{02})(4a_{11}-7b_{02})\\ &\times(2a_{11}-b_{02})(a_{11}+2b_{02})(\mu^2-3).\end{aligned}$$

It is easy to verify that the first 4 factors yield the conditions (4), (6), (7) and (8), respectively, all of which actually generate $b_{03}=0$. Note that the factor $2a_{11}–b_{02} \neq 0$. For the factor $a_{11}+2b_{02}$, we have $b_{02}=-\frac{1}{2}a_{11}$, and then use the above obtained solution $\widetilde{a}_{21}$ to get $a_{21}=\frac{1}{4}a_{11}^2$, leading to that the last factor in $\overline{C}_1$ is satisfied. This implies that the factor $a_{11}+2b_{02}$ gives a special case of the condition 5.
The remaining factor is μ^2-3 which gives $\mu^2=3$. Under this condition, the polynomials in (3.13) become the functions in a_{11}, a_{21}, b_{12} and b_{02}. Eliminating a_{21} from these polynomials results in three resultants which have the common factors:

$$\begin{aligned}\overline{C}_3 = &(2a_{11}-b_{02})\big[(2a_{11}-b_{02})(a_{11}-3b_{02})+25b_{12}\big](a_{11}-3b_{02})\\ &\times\big[b_{02}(2a_{11}-b_{02})-5b_{12}\big]\big[(2a_{11}-b_{02})(9a_{11}-17b_{02})+25b_{12}\big]\\ &\times\big[4a_{11}^2+a_{11}b_{02}+11b_{02}^2+25b_{12}\big].\end{aligned}$$

All these factors satisfy $u_1=u_2=\cdots=u_7=0$. Note that the first two factors have appeared in the factor $\overline{C}_1$. The next four factors yield the conditions (9), (10), (11) and (12), respectively. Note that although the third factor $a_{11}-3b_{02}$ has appeared in the condition (4) which has $b_{03}=0$, but the condition (9) contains $b_{03}\neq 0$.

This finishes the proof of Proposition 3.3. □

The following theorem directly follows Proposition 3.3.

Theorem 3.8 *When $a_{31}=0$, the nilpotent origin of system (2.5) is a center if and only if one of the conditions in Proposition 3.3 holds.*

Proof The necessity is directly given by Proposition 3.3 since all the 7 Lyapunov constants vanish under these conditions. In the following, we prove that these conditions are also sufficient.

Since the condition (1) is identical to the condition (a) in Proposition 3.1 for the case $a_{31} \neq 0$, we start the proof from the condition (2) under which system (2.5) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y(1 + a_{21}x^2),\\ \frac{dy}{dt} &= x(-2x^2 + b_{12}y^2 + 4\mu y - 2\mu^2 x^2),\end{aligned}$$

which is symmetric with respect to the y-axis.

When condition (3) holds, system (2.5) is reduced to

$$\begin{aligned}\frac{dx}{dt} &= y(1 + a_{11}x + a_{21}x^2),\\ \frac{dy}{dt} &= 4\mu xy + 2a_{11}y^2 - 2(1 + \mu^2)x^3 + 2a_{21}xy^2,\end{aligned}$$

which has two invariant algebraic curves:

$$f_{31} = 1 + a_{11}x + a_{21}x^2 \quad \text{and} \quad f_{32} = y^2 - 2\mu x^2 y + x^4(1 + \mu^2),$$

and an inverse integrating factor $I_3 = f_{31} f_{32}$.

When the condition (4) holds, system (2.5) can be brought into

$$\begin{aligned}\frac{dx}{dt} &= y\Big(1 + a_{11}x + \frac{2}{9}a_{11}^2 x^2\Big),\\ \frac{dy}{dt} &= \frac{1}{3}a_{11}y^2 + b_{12}xy^2 + 4\mu xy + \frac{4}{3}a_{11}\mu x^2 y - 2x^3(1 + \mu^2).\end{aligned}$$

By the transformation,

$$\begin{aligned}u &= \frac{x}{1 + \frac{a_{11}x}{3}},\\ v &= \Big(1 + \frac{a_{11}x}{3}\Big)^{1 - \frac{9b_{12}}{a_{11}^2}} \Big(1 + \frac{2a_{11}x}{3}\Big)^{-1 + \frac{9b_{12}}{a_{11}^2}} y,\end{aligned}$$

and the time scaling,

$$\tau = \Big(1 + \frac{a_{11}u}{3}\Big)^{-2}\Big(1 - \frac{a_{11}^2 u^2}{9}\Big)^{\frac{9b_{12}}{a_{11}^2}} t,$$

the system can be changed to

$$\begin{aligned}\frac{du}{d\tau} &= v,\\ \frac{dy}{d\tau} &= -2u^3\Big(1 - \frac{a_{11}^2 u^2}{9}\Big)^{-3 + \frac{9b_{12}}{a_{11}^2}} + 4\mu u\Big(1 - \frac{a_{11}^2 u^2}{9}\Big)^{-2 + \frac{9b_{12}}{a_{11}^2}} v,\end{aligned}$$

which is symmetric with respect to the v-axis.

When the condition (5) holds, system (2.5) can be rewritten as

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{25} y(5 + 2a_{11}x - b_{02}x)(5 + 3a_{11}x + b_{02}x),\\
\frac{dy}{dt} &= \frac{1}{25}(-50x^3 + 25b_{02}y^2 - 2a_{11}^2 xy^2 + 7a_{11}b_{02}xy^2 - 3b_{02}^2 xy^2 \\
&\quad + 100\mu xy + 40\mu a_{11}x^2 y - 20b_{02}\mu x^2 y - 50\mu^2 x^3),
\end{aligned}
$$

which has three invariant algebraic curves:

$$
\begin{aligned}
f_{51} &= 5 + 3a_{11}x + b_{02}x,\\
f_{52} &= 5 + 2a_{11}x - b_{02}x,\\
f_{53} &= 25x^4 + 25y^2 + (20a_{11} + 10b_{02})xy^2 + (4a_{11}^2 - 4a_{11}b_{02+b_{02}^2} x^2 y^2 \\
&\quad - 50\mu x^2 y - 20a_{11}x^3 y\mu + 10b_{02}x^3 y\mu + 25x^4\mu^2,
\end{aligned}
$$

and admit an inverse integrating factor $I_5 = f_{51} f_{52}^{-1} f_{53}$.

When the condition (6) is satisfied, system (2.5) becomes

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{4} y(4 - b_{02}x)(1 + 2b_{02}x),\\
\frac{dy}{dt} &= -2x^3 + b_{02}y^2 - b_{02}^2 xy^2 + 4\mu xy + 2\mu b_{02}x^2 y - 2\mu^2 x^3,
\end{aligned}
$$

which can be further changed to a Liénard system,

$$
\begin{aligned}
\frac{dx}{d\tau} &= y,\\
\frac{dy}{d\tau} &= \frac{(-2x^3 + b_{02}y^2 - b_{02}^2 xy^2 + 4\mu xy + 2\mu b_{02}x^2 y - 2\mu^2 x^3)}{(4 - b_{02}x)(1 + 2b_{02}x)}\\
&= p_0(x) + p_1(x)y + p_2(x)y^2,
\end{aligned}
$$

by a time rescaling $\tau = \frac{1}{4} y(4 - b_{02}x)(1 + 2b_{02}x)\, t$. Let

$$
\begin{aligned}
W_1(x) &= \frac{(p_0(x)p_1(x)p_2(x) - p_1(x)p_0'(x) + p_0(x)p_1'(x))}{p_1(x)^3},\\
W_2(x) &= \frac{W_1'[x]p_0(x)}{p_1(x)^2}.
\end{aligned}
$$

Then, the system 3.15 has a center if and only if $W_1(x) - W_1(y)$ and $W_2(x) - W_2(y)$ has a common factor with the form $x + y + \text{h.o.t.}$ It can be shown that

$$
W_1(x) - W_1(y) = \frac{-9(1 + \mu^2)b_{02}^2(x - y)}{8\mu^2(2 + b_{02}x)^3(2 + b_{02}y)^3}\, h(x, y),
$$

$$W_2(x) - W_2(y) = \frac{9(1+\mu^2)^2 b_{02}^2 (x-y)(4 - b_{02}x - b_{02}y - 2b_{02}^2 xy)}{64\mu^2(2+b_{02}x)^6(2+b_{02}y)^6} \\ \times h(x,y)(32 + 56b_{02}x - 16b_{02}^2x^2 + 56b_{02}y + 116b_{02}^2xy \\ + 8b_{02}^3x^2y - 16b_{02}^2y^2 + 8b_{02}^3xy^2 - b_{02}^4x^2y^2),$$

where the common factor $h(x, y)$ is given by

$$h(x, y) = 8x + 8y + 12b_{02}xy - b_{02}^3x^2y^2.$$

So the origin is a nilpotent center.

When the condition (7) holds, system (2.5) becomes

$$\frac{dx}{dt} = \frac{1}{25}\, y\,(5 - b_{02}x)(5 + b_{02}x),$$
$$\frac{dy}{dt} = \frac{1}{25}(-50x^3 + 25b_{02}y^2 - 3b_{02}^2xy^2 + 100\mu xy - 20\mu b_{02}x^2y - 50\mu^2x^3),$$

which has three invariant algebraic curves:

$$f_{81} = 5 - b_{02}x,$$
$$f_{82} = 5 + b_{02}x,$$
$$f_{83} = 25x^4 + 25y^2 - 10b_{02}xy^2 + b_{02}^2x^2y^2 - 50\mu x^2y + 10b_{02}\mu x^3y + 25\mu^2x^4,$$

and admits an inverse integrating factor $I_8 = f_{81}^{-1} f_{82} f_{83}$.

When the condition (8) holds, system (2.5) can be rewritten as

$$\frac{dx}{dt} = \frac{1}{4}y(1 + 2b_{02}x)(4 + 9b_{02}x),$$
$$\frac{dy}{dt} = \frac{1}{2}(-4x^3 + 2b_{02}y^2 + 3b_{02}^2xy^2 + 8\mu xy + 12\mu b_{02}x^2y - 4\mu^2x^3).$$

When $\mu = 0$, the system has a first integral,

$$H_1 = \frac{1}{(9b_{02}x + 4)^8}(324b_{02}^6x^2y^2 + 324b_{02}^5xy^2 + 216b_{02}^4x^4 \\ + 81b_{02}^4y^2 - 720b_{02}^3x^3 - 1440b_{02}^2x^2 - 768b_{02}x - 128)^3.$$

When $b_{02} = 0$, the system has a first integral,

$$H_2 = (\mu^2x^4 - 2\mu x^2y + x^4 + y^2)e^{-2\mu \arctan \frac{\mu x^2 - y}{x^2}}.$$

When $\mu b_{02} \neq 0$, the system can be transformed into

$$\begin{aligned}\frac{du}{d\tau} &= v,\\ \frac{dv}{d\tau} &= \frac{-u^3c+4(6u+1)v(u+v)}{(8u+1)(9u+1)},\end{aligned}$$

by $u = \frac{4}{b}x$, $v = \frac{16}{ab^2}y$, $\tau = \frac{4a}{b}(72u^2+17u+1)t$, where $c = \sqrt{\frac{2}{\mu-2}}$. Furthermore, with $z = \frac{8u+1}{(9u+1)^{\frac{4}{3}}}v$, $\tau = \frac{8u+1}{(9u+1)^{\frac{4}{3}}}T$, the above system can be changed to

$$\begin{aligned}\frac{du}{dT} &= z,\\ \frac{dz}{dT} &= -\frac{cu^3(8u+1)}{(9u+1)^{\frac{11}{3}}}+\frac{4u(6u+1)z}{(9u+1)^{\frac{7}{3}}} = -f(u)-g(u)z.\end{aligned} \tag{3.15}$$

Now, let

$$\begin{aligned}F(x) &= \int_0^x f(u) = \frac{(9x+1)^{\frac{4}{3}}-36x^2-12x-1}{9(9x+1)^{\frac{4}{3}}},\\ G(x) &= \int_0^x g(u) = \frac{c\,(x+\frac{1}{9})^2}{8(9x+1)^2}+\frac{c\,(432x^4-360x^3-180x^2-24x-1)}{648(9x+1)^{\frac{8}{3}}}.\end{aligned}$$

Then, the system (3.15) has a center if and only if $G(x)$ is a function of $F(x)$. It is easy to verify that

$$\begin{aligned}G(x) = {} & \left[\frac{1}{24}F^2(x)-\frac{1}{108}F(x)+\frac{1}{486}-\frac{1}{486}\sqrt{1-9F(x)}\right]c\\ & +\frac{c}{16}F^2(x)+\frac{3c}{32}F^3(x)+o(F^4(x)),\end{aligned}$$

which satisfies

$$\begin{aligned}& F^2(x)\big[27F^2(x)-12F(x)+4\big]c^2\\ & -16G(x)\big[81F^2(x)-18F(x)+4\big]c+15552G^2(x)=0.\end{aligned}$$

So the origin is a nilpotent center.

For the remaining four cases (9)–(12), we will only consider $\mu = \sqrt{3}$, since for $\mu = -\sqrt{3}$, we can use the transformation $u = x$, $v = -y$, $T = -t$ to obtain the same system as that obtained for $\mu = \sqrt{3}$.

When the condition (9) holds for $\mu=\sqrt{3}$, system (2.5) can be written as

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{100}y(10+6a_{11}x-3b_{02}x)(10+4a_{11}x+3b_{02}x),\\
\frac{dy}{dt} &= -8x^3+4\sqrt{3}xy+\frac{4}{5}\sqrt{3}(2a_{11}-b_{02})x^2y+b_{02}y^2\\
&\quad+\frac{1}{5}(2a_{11}-b_{02})b_{02}xy^2+\frac{(4a_{11}-17b_{02})(2a_{11}-b_{02})^2}{3000\sqrt{3}}y^3,
\end{aligned}
$$

which has three invariant algebraic curves:

$$
\begin{aligned}
f_{101} &= 10+6a_{11}x-3b_{02}x,\\
f_{102} &= 10+4a_{11}x+3b_{02}x,\\
f_{103} &= -12000\sqrt{3}x^4+18000x^2y+7200a_{11}x^3y-3600b_{02}x^3y-3000\sqrt{3}y^2\\
&\quad-2400\sqrt{3}a_{11}xy^2+1200\sqrt{3}b_{02}xy^2-480\sqrt{3}a_{11}^2x^2y^2+480\sqrt{3}a_{11}b_{02}x^2y^2\\
&\quad-120\sqrt{3}b_{02}^2x^2y^2+60a_{11}^2y^3-60a_{11}b_{02}y^3+15b_{02}^2y^3\\
&\quad+32a_{11}^3xy^3-48a_{11}^2b_{02}xy^3+24a_{11}b_{02}^2xy^3-4b_{02}^3xy^3,
\end{aligned}
$$

and admits an inverse integrating factor $I_{10}=f_{101}^{-\frac{1}{3}}f_{102}f_{103}$.

When the condition (10) holds for $\mu=\sqrt{3}$, system (2.5) can be rewritten as

$$
\begin{aligned}
\frac{dx}{dt} &= -\frac{1}{25}y(-5+a_{11}x-3b_{02}x)(5+6a_{11}x-3b_{02}x),\\
\frac{dy}{dt} &= -8x^3+4\sqrt{3}xy+\frac{4}{5}\sqrt{3}(2a_{11}-b_{02})x^2y+b_{02}y^2\\
&\quad+\frac{1}{25}(9a_{11}-17b_{02})(2a_{11}-b_{02})xy^2+\frac{(a_{11}-2b_{02})(2a_{11}-b_{02})^2}{75\sqrt{3}}y^3,
\end{aligned}
$$

which has three invariant algebraic curves:

$$
\begin{aligned}
f_{111} &= -5+a_{11}x-3b_{02}x,\\
f_{112} &= 5+6a_{11}x-3b_{02}x,\\
f_{113} &= -3000\sqrt{3}x^4+4500x^2y+1800a_{11}x^3y-900b_{02}x^3y-750\sqrt{3}y^2\\
&\quad-600\sqrt{3}a_{11}xy^2+300\sqrt{3}b_{02}xy^2-120\sqrt{3}a_{11}^2x^2y^2+120\sqrt{3}a_{11}b_{02}x^2y^2\\
&\quad-30\sqrt{3}b_{02}^2x^2y^2+60a_{11}^2y^3-60a_{11}b_{02}y^3+15b_{02}^2y^3\\
&\quad+8a_{11}^3xy^3-12a_{11}^2b_{02}xy^3+6a_{11}b_{02}^2xy^3-b_{02}^3xy^3,
\end{aligned}
$$

and admits an inverse integrating factor $I_{11}=f_{111}f_{112}^{\frac{1}{3}}f_{113}$.

When the condition (11) holds for $\mu = \sqrt{3}$, system (2.5) can be similarly brought into

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{25} y(5 + a_{11}x - 3b_{02}x)(5 + 4a_{11}x + 3b_{02}x),\\
\frac{dy}{dt} &= -8x^3 + 4\sqrt{3}xy + \frac{4}{5}\sqrt{3}(2a_{11} - b_{02})x^2 y + b_{02}y^2\\
&\quad + \frac{1}{25}(-4a_{11}^2 - a_{11}b_{02} - 11b_{02}^2)xy^2 + \frac{(4a_{11} - 7b_{02})(a_{11} + 2b_{02})^2}{375\sqrt{3}} y^3,
\end{aligned}
$$

which has three invariant algebraic curves:

$$
\begin{aligned}
f_{121} &= 5 + a_{11}x - 3b_{02}x,\\
f_{122} &= 5 + 4a_{11}x + 3b_{02}x,\\
f_{123} &= 3000\sqrt{3}x^4 - 4500x^2 y - 1800a_{11}x^3 y + 900b_{02}x^3 y + 750\sqrt{3}y^2\\
&\quad + 600\sqrt{3}a_{11}xy^2 - 300\sqrt{3}b_{02}xy^2 + 120\sqrt{3}a_{11}^2 x^2 y^2 - 120\sqrt{3}a_{11}b_{02}x^2 y^2\\
&\quad + 30\sqrt{3}b_{02}^2 x^2 y^2 - 15a_{11}^2 y^3 - 60a_{11}b_{02}y^3 - 60b_{02}^2 y^3\\
&\quad - 4a_{11}^3 xy^3 - 9a_{11}^2 b_{02}xy^3 + 12a_{11}b_{02}^2 xy^3 + 28b_{02}^3 xy^3,
\end{aligned}
$$

and admits an inverse integrating factor $I_{12} = f_{121}^{-\frac{1}{3}} f_{122}^{\frac{1}{3}} f_{123}$.

Finally, when the condition (12) for $\mu = \sqrt{3}$ holds, system (2.5) becomes

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{1}{4} y(4 + 12b_{02}x + 6b_{02}^2 x^2 + 3b_{12}x^2),\\
\frac{dy}{dt} &= \frac{1}{72}(-576x^3 + 288\sqrt{3}xy + 288\sqrt{3}b_{02}x^2 y + 72b_{02}y^2\\
&\quad + 72b_{12}xy^2 + 2\sqrt{3}b_{02}^3 y^3 - 3\sqrt{3}b_{02}b_{12}y^3),
\end{aligned}
$$

which has two invariant algebraic curves:

$$
\begin{aligned}
f_{91} &= 4 + 12b_{02}x + 6b_{02}^2 x^2 + 3b_{12}x^2,\\
f_{92} &= -288x^4 + 144\sqrt{3}x^2 y + 144\sqrt{3}b_{02}x^3 y - 72y^2 - 144b_{02}xy^2\\
&\quad - 72b_{02}^2 x^2 y^2 + 6\sqrt{3}b_{02}^2 y^3 - 3\sqrt{3}b_{12}y^3 + 4\sqrt{3}b_{02}^3 xy^3,
\end{aligned}
$$

and admits an inverse integrating factor $I_9 = f_{91}^{\frac{1}{3}} f_{92}$.

The proof of Theorem 3.8 is complete. □

Similar to Theorem 3.5, we have a theorem on the number of limit cycles for the system (2.5) when $a_{31} = 0$. Since the proof is similar to that for Theorem 3.5, we state the results without proof.

Theorem 3.9 *When $a_{31}=0$, system (2.5) can have at least 7 small-amplitude limit cycles around the origin by small parameter perturbation, and 8 small-amplitude limit cycles around the origin by the double bifurcation analysis.*

To end this section, as a summary, we list the 15 center conditions obtained for the generalized Kukles system (2.5), given in Theorems 3.3 and 3.8, in the following theorem.

Theorem 3.10 *The nilpotent origin of system (2.5) is a center if and only if one of the following 15 conditions holds.*

(1) $b_{03}=b_{21}=\mu=0;$

(2) $b_{03}=b_{21}=a_{21}+24a_{11}^2=a_{31}-36a_{11}^3=b_{02}-2a_{11}=b_{12}+12a_{11}^2=0;$

(3) $b_{03}=b_{02}=b_{12}=25a_{21}-8a_{11}^2=125a_{31}-4a_{11}^3=5b_{21}-8\mu a_{11}=0;$

(4) $b_{03}=529a_{21}-174a_{11}^2=12167a_{31}-432a_{11}^3=23b_{02}-a_{11}$
$=23b_{21}-36\,\mu a_{11}=529b_{12}-6a_{11}^2=0;$

(5) $a_{31}=b_{03}=b_{21}=a_{11}=b_{02}=0;$

(6) $a_{31}=b_{03}=b_{21}=b_{02}-2a_{11}=b_{12}-2a_{21}=0;$

(7) $a_{31}=b_{03}=3b_{21}-4a_{11}\mu=3b_{02}-a_{11}=9a_{21}-2a_{11}^2=0;$

(8) $a_{31}=b_{03}=5b_{21}-4(2a_{11}-b_{02})\mu=25a_{21}-(2a_{11}-b_{02})(3a_{11}+b_{02})$
$=25b_{12}+(2a_{11}-b_{02})(a_{11}-3b_{02})=0;$

(9) $a_{31}=b_{03}=5b_{21}-12b_{02}\mu=25b_{12}+3b_{02}^2=25a_{21}-9b_{02}^2=a_{11}-2b_{02}=0;$

(10) $a_{31}=b_{03}=b_{21}-6b_{02}\mu=2b_{12}-3b_{02}^2=2a_{21}-9b_{02}^2=4a_{11}-17b_{02}=0;$

(11) $a_{31}=b_{03}=b_{21}-2b_{02}\mu=b_{12}+b_{02}^2=2a_{21}+b_{02}^2=4a_{11}-7b_{02}=0;$

(12) $a_{31}=\mu^2-3=9000b_{03}-\mu(2a_{11}-b_{02})^2(4a_{11}-17b_{02})=5b_{21}-4\mu(2a_{11}-b_{02})$
$=5b_{12}-b_{02}(2a_{11}-b_{02})=100a_{21}-3(2a_{11}-b_{02})(4a_{11}+3b_{02})=0;$

(13) $a_{31}=\mu^2-3=225b_{03}-\mu(2a_{11}-b_{02})^2(a_{11}-2b_{02})=5b_{21}-4\mu(2a_{11}-b_{02})$
$=25b_{12}+(2a_{11}-b_{02})(9a_{11}-17b_{02})=25a_{21}+3(2a_{11}-b_{02})(a_{11}-3b_{02})=0;$

(14) $a_{31}=\mu^2-3=1125b_{03}-\mu(a_{11}+2b_{02})^2(4a_{11}-7b_{02})=5b_{21}-4\mu(2a_{11}-b_{02})$
$=25b_{12}+4a_{11}^2+b_{02}a_{11}+11b_{02}^2=25a_{21}-(4a_{11}+3b_{02})(a_{11}-3b_{02})=0;$

(15) $a_{31}=\mu^2-3=72b_{03}-\mu\,b_{02}(2b_{02}^2-3b_{12})=b_{21}-4\mu b_{02}=4a_{21}-3(2b_{02}^2+b_{12})$
$=3b_{02}-a_{11}=0.$

4 Conclusion

In this paper, the cubic Kukles systems with an extra 4th-order term, associated with the nilpotent origin is studied. It is shown that such systems can have only 4 center conditions, while 12 center conditions are obtained without this extra term. One of the center conditions is an analytic center condition. Moreover, for the Kukles systems with the extra term, at least 8 limit cycles can bifurcate from the nilpotent singular point, one more than that of the cubic Kukles systems.

Acknowledgements This research was partially supported by National Natural Science Foundation of China, No. 12071198 (F. Li), and the Natural Sciences and Engineering Research Council of Canada, No. R2686A02 (P. Yu).

Author Contributions Feng Li: Conceptualization, methodology, computation. Ting Chen: Program verification. Yuanyuan Liu: Check some calculations and proofreading. Pei Yu: Supervision, computation, writing, reviewing and editing.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. Alvarez, M.J., Gasull, A.: Monodromy and stability for nilpotent critical points. Int. J. Bifurc. Chaos **15**, 1253–1265 (2005)
2. Alvarez, M.J., Gasull, A.: Generating limits cycles from a nilpotent critical point via normal forms. J. Math. Anal. Appl. **318**, 271–287 (2006)
3. Amelbkin, V.V., Lukasevnky, N.A., Sadovski, A.P.: Nonlinear Oscillations in Second Order Systems. BGY Lenin. B.I. Press, Minsk (1992). (**in Russian**)
4. Bondar, Y.L., Sadovskii, A.P.: Variety of the center and limit cycles of a cubic system, which is reduced to Liénard form. Bul. Acad. Ştiinţe Repub. Mold. Mat. **3**(46), 71–99 (2004)
5. Chavarriga, J., Giacomin, H., Giné, J.: Local analytic integrability for nilpotent centers. Ergodic Theory Dyn. Syst. **23**, 417–428 (2003)
6. Cherkas, L.A.: Center conditions for some equations of the form $yy' = P(x) + Q(x)y + R(x)y^2$. Differ. Uravn. **8**(8), 1435–1439 (1972)
7. Cherkas, L.A.: On the conditions for a center for certain equations of the form $yy' = P(x) + Q(x)y + R(x)y^2$. Differ. Equ. **8**, 1104–1107 (1972)
8. Christopher, C.J., Lloyd, N.G., Pearson, J.M.: On Cherkas's method for centre conditions. Nonlinear World **2**(4), 459–469 (1995)
9. Christopher, C.J., Lloyd, N.G.: On the paper of Jin and Wang concerning the conditions for a centre in certain cubic systems. Bull. London Math. Soc. **22**, 5–12 (1990)
10. Colin, C.: An algebraic approach to the classification of centres in polynomial Liénard systems. J. Math. Anal. Appl. **229**, 319–329 (1999)
11. Giné, J.: Conditions for the existence of a center for the Kukles homogeneous systems. Comput. Math. Appl. **43**(10–11), 1261–1269 (2002)
12. Giné, J.: Center conditions for polynomial Liénard systems. Qual. Theory Dyn. Syst. **16**(1), 119–126 (2017)
13. Giné, J.: Center conditions for generalized polynomial Kukles systems. Commun. Pure Appl. Anal. **16**(2), 417–425 (2017)
14. Giné, J., Llibre, J., Valls, C.: Centers for the Kukles homogeneous systems with odd degree. Bull. London Math. Soc. **47**(2), 315–324 (2015)
15. Giné, J., Llibre, J., Valls, C.: Centers for the Kukles homogeneous systems with even degree. J. Appl. Anal. Comp. **7**(4), 1534–1548 (2017)
16. Hill, J.M., Lloyd, N.G., Pearson, J.M.: Algorithmic derivation of isochronicity conditions. Nonlinear Anal. **67**, 52–69 (2007)
17. Hill, J.M., Lloyd, N.G., Pearson, J.M.: Centres and limit cycles for an extended Kukles system. Electron. J. Differ. Equ. **119**, 1–23 (2007)
18. Hill, J.M., Lloyd, N.G., Pearson, J.M.: Limit cycles of a predator-prey model with intratrophic predation. J. Math. Anal. Appl. **349**, 544–555 (2009)

19. Kushner, A.A., Sadovskii, A.P.: Center conditions for Liénard-type systems of degree four. Vestn. Beloruss. Gos. Univ. Ser. 1 Fiz. Mat. Inform. **2**, 119–122 (2011) (**in Russian**)
20. Li, F., Li, S.: Integrability and limit cycles in cubic Kukles systems with a nilpotent singular point. Nonlinear Dyn. **96**(1), 553–563 (2019)
21. Li, F., Li, H.W., Liu, Y.Y.: New double bifurcation of nilpotent focus. Int. J. Bifurc. Chaos **31**(4), 2150053 (2021)
22. Li, F., Liu, Y., Liu, Y., Yu, P.: Bi-center problem and bifurcation of limit cycles from nilpotent singular points in Z_2-equivariant cubic vector fields. J. Differ. Equ. **265**(10), 4965–4992 (2018)
23. Linh, L.V., Sadovskii, A.P.: The centre-focus problem for analytical systems of Liénard form in degenerate case. Bul. Acad. Stiinte Repub. Mold. Mat. **2**, 37–50 (2003)
24. Liu, Y., Li, F.: Double bifurcation of nilpotent foci. Int. J. Bifurc. Chaos **25**(3), 1550036 (2015)
25. Liu, Y., Li, J.: New study on the center problem and bifurcations of limit cycles for the Lyapunov system II. Int. J. Bifurc. Chaos **19**(09), 3087–3099 (2009)
26. Liu, T., Liu, Y., Li, F.: A kind of bifurcation of limit cycle from a nilpotent critical point. J. Appl. Anal. Comp. **8**, 10–18 (2018)
27. Liu, T., Wu, L., Li, F.: Analytic center of nilpotent critical points. Int. J. Bifurc. Chaos **22**(08), 1250198 (2012)
28. Lloyd, N.G., Pearson, J.M.: Conditions for a centre and the bifurcation of limit cycles. In: Francoise, J.P., Roussarie, R. (eds) Bifurcations of Planar Vector Fields, Lecture Notes Math. 1455, pp. 230–242, Springer, New York (1990)
29. Lloyd, N.G., Pearson, J.M.: Computing centre conditions for certain cubic systems. J. Comp. Appl. Math. **40**, 323–336 (1992)
30. Pearson, J.M., Lloyd, N.G.: Kukles revisited: Advances in computing techniques. Comp. Math. Appl. **60**(10), 2797–2805 (2010)
31. Pitchford, J.W., Brindley, J.: Intratrophic predation in simple predator-prey models. Bull. Math. Biol. **60**, 937–953 (1998)
32. Rebollo-Perdomo, S., Vidal, C.: Bifurcation of limit cycles for a family of perturbed Kukles differential systems. Discrete Contin. Dyn. Syst. Ser. A **38**(8), 4189–4202 (2018)
33. Romanovski, V.G., Shafer, D.S.: The Center and Cyclicity Problems: A Computational Algebra Approach. Birkhauser, Boston (2009)
34. Sadovskii, A.P.: Solution of the center and foci problem for a cubic system of nonlinear oscillations. Differ. Uravn. **33**(2), 236–244 (1997). (**in Russian**)
35. Sadovskii, A.P., Shcheglova, T.V.: Solution of the center-foci problem for a cubic system with nine parameters. Differ. Uravn. **47**(2), 209–224 (2011) (**in Russian**); Differ. Equ. **47**(2), 208–223 (2011)
36. Sadovskii, A.P., Shcheglova, T.V.: Center conditions for a polynomial differential system. Differ. Uravn. **49**(2),151164 (2013) (**in Russian**); Differ. Equ. **49**(2), 151–165 (2013)
37. Yu, P., Li, F.: Bifurcation of limit cycles in a cubic-order planar system around a nilpotent critical point. J. Math. Anal. Appl. **453**(2), 645–667 (2017)
38. Zhang, Z., Ding, T., Huang, W., Dong, Z.: Qualitative Theory of Differential Equations. Science Press, Beijing (1997)