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Robust absolute stability of Lurie interval control systems

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SUMMARY

This paper considers robust absolute stability of Lurie control systems. Particular attention is given to the systems with parameters having uncertain, but bounded values. Such so-called Lurie interval control systems have wide applications in practice. In this paper, a number of sufficient and necessary conditions are derived by using the theories of Hurwitz matrix, M matrix and partial variable absolute stability. Moreover, several algebraic sufficient and necessary conditions are provided for the robust absolute stability of Lurie interval control systems. These algebraic conditions are easy to be verified and convenient to be used in applications. Three mathematical examples and a practical engineering problem are presented to show the applicability of theoretical results. Numerical simulation results are also given to verify the analytical predictions. Copyright © 2007 John Wiley & Sons, Ltd.

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KEY WORDS: Lurie control system; robust absolute stability; interval control; Lyapunov function; M matrix

1. INTRODUCTION

In modelling a physical or an engineering problem, strictly speaking, a mathematical model is only an approximate description of a real system since the available information of the system coefficients are usually the upper and lower bounds, not the exact values [1, 2]. In the past two decades, the study on the stability of linear control systems with parameters varied in finite closed intervals has received considerable attention in control society, and some results are

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obtained [3–6]. However, not many results have been given to consider the stability of nonlinear control systems with varying parameters in intervals. In this paper, we will introduce robust stability of control systems with parameters varied in intervals. In fact, such idea and methodology can be generalized to consider other Lurie control systems. Due to the importance in both theoretical development and applications, the research of the absolute stability of Lurie control systems [7, 8] is expected to be continuously active in future [9–14].

In [15–17], we have used a linear transform to change a general Lurie control system (including direct, indirect and critical controls) into two types of nonlinear control systems with separable variables in which feedback states become state variables. Therefore, without loss of generality, here we assume that the system is given in a standard form after the transformation.

Consider the following Lurie interval control system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A_I x + h_I f(x_n) \tag{1}$$

and a simpler system:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = B_I y + r_I f(y_n) \tag{2}$$

where x, y and f are n-dimensional vectors, and

$$f(\cdot) \in F := \{x_n | 0 < x_n f(x_n) \leqslant +\infty \text{ when } x_n \neq 0,\ f(0) = 0,\ f(x_n) \in C(-\infty, +\infty)\}$$

$$A_I := \{A(a_{ij})_{n\times n} :\ \underline{A} \leqslant A \leqslant \bar{A},\ \text{i.e. } \underline{a_{ij}} \leqslant a_{ij} \leqslant \overline{a_{ij}},\ i,j = 1,2,\ldots,n\}$$

$$h_I := \{h :\ \underline{h} \leqslant h \leqslant \bar{h},\ \text{i.e. } \underline{h_i} \leqslant h_i \leqslant \overline{h_i},\ i = 1,2,\ldots,n\}$$

$$B_I := \{B(b_{ij})_{n\times n} :\ \underline{B} \leqslant B \leqslant \bar{B},\ \text{i.e. } \underline{b_{ij}} \leqslant b_{ij} \leqslant \overline{b_{ij}},\ i,j = 1,2,\ldots,n\}$$

$$r_I := \{r :\ \underline{r} \leqslant r \leqslant \bar{r},\ \text{i.e. } \underline{r_i} \leqslant r_i \leqslant \overline{r_i},\ i = 1,\ldots,n;\ \underline{r}_i = \bar{r}_i = 0,\ i = 1,\ldots,n-1\}$$

in which $\underline{A}, \bar{A}, \underline{B}, \bar{B}$ are known $n \times n$ matrices, $\underline{h}, \bar{h}, \underline{r}, \bar{r}$ are known n-dimensional vectors, while A, B, h and r are not precisely known. It is easy to see that system (2) is a special case of system (1). Since many practical problems are described by the form of (2), we list this system explicitly for convenience. However, we do not need to discuss this system in detail since all the results for system (2) can be directly deduced from that of system (1) (given as corollaries).

Then, $\forall A \in A_I,\ \forall h \in h_I\ (\forall B \in B_I,\ \forall r \in r_I)$, the corresponding Lurie systems of (1) and (2) are given, respectively, by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax + hf(x_n) \tag{1$'$}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = By + rf(y_n) \tag{2$'$}$$

Definition 1

If $\forall A \in A_I,\ \forall h \in h_I\ (\forall B \in B_I,\ \forall r \in r_I)$, the zero solutions of the corresponding systems $(1)'$ and $(2)'$ are absolutely stable, i.e. $\forall f(\cdot) \in F$, the zero solutions of the systems $(1)'$ and $(2)'$ are globally

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asymptotically stable, then it is said that the zero solutions of the Lurie interval control systems (1) and (2) are absolutely robust stable.

Definition 2
If $\forall A \in A_I,\ \forall h \in h_I\ (\forall B \in B_I,\ \forall r \in r_I)$, the zero solutions of the corresponding systems $(1)'$ and $(2)'$ are absolutely stable with respect to the partial variables $x_{j+1}, x_{j+2}, \ldots, x_n$ $(y_{j+1}, y_{j+2}, \ldots, y_n)$, i.e. $\forall f(\cdot) \in F$, the zero solutions of the systems $(1)'$ and $(2)'$ are globally asymptotically stable with respect to the partial variables $x_{j+1}, x_{j+2}, \ldots, x_n$ $(y_{j+1}, y_{j+2}, \ldots, y_n)$, then it is said that the zero solutions of the Lurie interval control systems (1) and (2) are absolutely robust stable with respect to the partial variables $x_{j+1}, x_{j+2}, \ldots, x_n$ $(y_{j+1}, y_{j+2}, \ldots, y_n)$.

Definition 3
If $\forall A \in A_I$, $(\forall B \in B_I)$, $A(B)$ is an Hurwitz matrix, then $A_I(B_I)$ is called an interval Hurwitz matrix [18].

2. SUFFICIENT AND NECESSARY CONDITIONS FOR ROBUST ABSOLUTE STABILITY OF LURIE INTERVAL CONTROL SYSTEMS

Since system (2) is a special case of system (1), we only discuss system (1), and the results for system (2) can be directly obtained from the results of system (1), listed as the corollaries of the theorems obtained from system (1).

Theorem 1
The sufficient and necessary conditions for the robust absolute stability of the zero solution of the Lurie interval control system (1) are the following:

(1) $A_I + (O_{n\times(n-1)}, h_I\theta)$ is an interval Hurwitz matrix, where $\theta = 0$ or 1, and $O_{n\times(n-1)}$ is an $n \times (n-1)$ zero matrix, i.e.

$$(O_{n\times(n-1)}, h_I\theta) = \begin{bmatrix} 0 & \cdots & 0 & h_1\theta \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & h_n\theta \end{bmatrix}_{n\times n}, \quad h_i \in [\,\underline{h_i}, \overline{h_i}\,], \quad i = 1, 2, \ldots, n$$

(2) the zero solution of system (1) is absolutely robust stable with respect to the partial variable x_n.

Proof (Necessity)
When the Lurie interval control system (1) is a direct control system, i.e. A_I is an Hurwitz matrix, we take $\theta = 0$; otherwise, choose $f(x_n) = x_n$, $\theta = 1$. Thus, $A_I + (O_{n\times(n-1)}, h_I\theta)$ is an Hurwitz matrix, implying that condition (1) holds. Condition (2) is obvious since the robust absolute stability of the zero solution of system (1) implies that the zero solution is absolutely robust stable with respect to (w.r.t.) the partial variable x_n.

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Sufficiency. Let $W = A + (O_{n\times(n-1)}, h\theta)$. Then, the zero solution of $(1)'$ can be expressed as

$$x(t,t_0;x_0) = \mathrm{e}^{W(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{W(t-\tau)}h[f(x_n(\tau)) - \theta x_n(\tau)]\,\mathrm{d}\tau \tag{3}$$

Since W is an Hurwitz matrix, there exists $M \geqslant 1$ and $\alpha > 0$ such that

$$||\mathrm{e}^{W(t-t_0)}|| \leqslant M\mathrm{e}^{-\alpha(t-t_0)} \quad \text{for all } t \geqslant t_0$$

In addition, since $f(x_n(t)) \to 0$ as $t \to +\infty$, $x_n(t)$ continuously depends on x_0, and $f(x_n(t))$ is a continuous function of x_0, $\forall \varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ and $t_1 > t_0$ such that

$$\int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)}[||hf(x_n(\tau))|| + ||h\theta x_n(\tau)||]\,\mathrm{d}\tau < \frac{\varepsilon}{3}$$

$$\int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)}[||hf(x_n(\tau))|| + ||h\theta x_n(\tau)||]\,\mathrm{d}\tau < \frac{\varepsilon}{3} \quad \text{for all } t \geqslant t_1$$

$$||\mathrm{e}^{W(t-t_0)}|| \leqslant M\mathrm{e}^{-\alpha(t-t_0)} < \frac{\varepsilon}{3\delta_1(\varepsilon)} \quad \text{for all } t \geqslant t_0$$

provided $||x_0|| < \delta_1(\varepsilon)$.

Thus, it follows that

$$\begin{aligned}
||x(t)|| &\leqslant ||\mathrm{e}^{W(t-t_0)}x_0|| + \int_{t_0}^{t} \mathrm{e}^{W(t-\tau)}[||hf(x_n(\tau))|| + ||h\theta x_n(\tau)||]\,\mathrm{d}\tau \\
&\leqslant M\mathrm{e}^{-\alpha(t-t_0)}||x_0|| + \int_{t_0}^{t_1} M\mathrm{e}^{-\alpha(t-\tau)}[||hf(x_n(\tau))|| + ||h\theta x_n(\tau)||]\,\mathrm{d}\tau \\
&\quad + \int_{t_1}^{t} M\mathrm{e}^{-\alpha(t-\tau)}[||hf(x_n(\tau))|| + ||h\theta x_n(\tau)||]\,\mathrm{d}\tau \\
&< \frac{\varepsilon}{3\delta_1(\varepsilon)}\delta_1(\varepsilon) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{for all } t \geqslant t_0
\end{aligned}$$

for any $x_0 \in R^n$. Then applying the L'Hospital rule to the above inequality yields

$$\begin{aligned}
0 &\leqslant \lim_{t\to+\infty} ||x(t)|| \\
&\leqslant \lim_{t\to+\infty} M\mathrm{e}^{-\alpha(t-t_0)}||x_0|| + \lim_{t\to+\infty} \int_{t_0}^{t} M\mathrm{e}^{-\alpha(t-\tau)}[||hf(x_n(\tau))|| + ||h\theta x_n(\tau)||]\,\mathrm{d}\tau \\
&= 0 + M \lim_{t\to+\infty} \frac{\int_{t_0}^{t} \mathrm{e}^{\alpha\tau}[||hf(x_n(\tau))|| + ||h\theta x_n(\tau)||]\,\mathrm{d}\tau}{\mathrm{e}^{\alpha t}} \\
&= M \lim_{t\to+\infty} \frac{\mathrm{e}^{\alpha t}}{\alpha \mathrm{e}^{\alpha t}}[||hf(x_n(t))|| + ||h\theta x_n(t)||] \quad (\text{by the L'Hospital rule}) \\
&= \frac{M}{\alpha} \lim_{t\to+\infty} [||hf(x_n(t))|| + ||h\theta x_n(t)||] \\
&= 0
\end{aligned}$$

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which clearly shows that $\lim_{t\to+\infty} ||x(t)|| = 0$. Therefore, the zero solution of system $(1)'$ is globally asymptotically stable. So the zero solution of system (1) is absolutely robust stable. This completes the proof. □

Corollary 1

The sufficient and necessary conditions for the robust absolute stability of the zero solution of the Lurie interval control system (2) are given by the following:

(1) $B_I + (O_{n\times(n-1)}, r_I\theta)$ is an interval Hurwitz matrix, where $\theta = 0$ or 1;
(2) the zero solution of system (2) is absolutely robust stable with respect to the partial variable y_n.

Since in system (2), $\underline{r}_i = \bar{r}_i = 0$ $(i = 1, 2, \ldots, n-1)$, (2) is a special case of system (1).

Theorem 2

The sufficient and necessary conditions for the robust absolute stability of the zero solution of the Lurie interval control system (1) are the following:

(1) there exists an n-dimensional interval vector η_I such that $A_I + (O_{n\times(n-1)}, \eta_I)$ is an interval Hurwitz matrix;
(2) the zero solution of system (1) is absolutely robust stable with respect to the partial variable x_n.

Proof (Necessity)

The existence of condition (1) is obvious. When A_I is an Hurwitz matrix, we can choose $\eta_I = (0, 0, \ldots, 0)^{\mathrm{T}}$; otherwise, take $\eta_I = h_i$, $f(x_n) = x_n$. It is easy to check under these choices that condition (1) holds. Condition (2) is obviously true.

Sufficiency. $\forall A \in A_I$, let $\tilde{W} = A + (O_{n\times(n-1)}, \eta)$. Then, rewrite $(1)'$ as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \tilde{W}x + hf(x_n) - \eta x_n \tag{4}$$

Now for system (4), applying the method of constant variation yields

$$x(t, t_0; x_0) = \mathrm{e}^{\tilde{W}(t-t_0)}x(t_0) + \int_{t_0}^{t} \mathrm{e}^{\tilde{W}(t-\tau)}[hf(x_n(t)) - \eta x_n(\tau)]\,\mathrm{d}\tau.$$

The remaining part of the proof can follow Theorem 2. This completes the proof. □

Corollary 2

The zero solution of the Lurie interval control system (2) is absolutely robust stable if and only if the following conditions are satisfied:

(1) there exists an n-dimensional interval vector η_I such that $B_I + (O_{n\times(n-1)}, \eta_I)$ is an interval Hurwitz matrix;
(2) the zero solution of system (2) is absolutely robust stable with respect to the partial variable y_n.

Remark

Compared to the constructive conditions in Theorem 1 and Corollary 1, the existence conditions in Theorem 2 and Corollary 2 are not so convenient in applications. However, if they are chosen properly, sometimes the verification of these conditions can be simplified.

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Similar to Theorems 1 and 2, we can prove the following results.

Theorem 3
The zero solution of the Lurie interval control system (1) is absolutely robust stable if and only if

(1) condition (1) in Theorem 1 or condition (1) in Theorem 2 holds;
(2) the zero solution of system (1) is absolutely robust stable w.r.t. the partial variables $x_{j+1}, x_{j+2}, \ldots, x_n$.

Corollary 3
The zero solution of the Lurie interval control system (2) is absolutely robust stable if and only if

(1) condition (1) in Corollary 1 or condition (1) in Corollary 2 holds;
(2) the zero solution of system (2) is absolutely robust stable w.r.t. the partial variables $y_{j+1}, y_{j+2}, \ldots, y_n$.

3. SUFFICIENT CONDITIONS FOR ROBUST ABSOLUTE STABILITY OF LURIE INTERVAL CONTROL SYSTEMS

In the previous section, we have used the absolute stability with respect to the partial variable x_n and the Hurwitz stability of the linearized part of the system to obtain the sufficient and necessary conditions of the robust absolute stability of the whole system. In this section, we strength the condition on the partial variable x_n to partial variables $x_j, \ldots, x_n$, but weaken the condition of Hurwitz stability of linearized system with respect to $x_1, x_2, \ldots, x_n$ to partial variables $x_1, \ldots, x_j$ while the terms associated with the remaining variables are non-homogeneous terms. We will obtain absolute stability criteria which are different from that given in Section 2. This increases possibility of our results in applications.

First, we introduce the following notations:

$$A_I^{(j_0)} = \begin{bmatrix} a_{11} & \cdots & a_{1j_0} \\ \vdots & & \vdots \\ a_{j_0 1} & \cdots & a_{j_0 j_0} \end{bmatrix}, \quad A_I^{(j_0)C} = \begin{bmatrix} a_{1(j_0+1)} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{j_0(j_0+1)} & \cdots & a_{j_0 n} \end{bmatrix}, \quad 1 \leqslant j_0 < n$$

Similarly we define $B_I^{(j_0)}$ and $B_I^{(j_0)C}$.

Theorem 4
If the following conditions:

(1) $A_I^{(j_0)}$ is an interval Hurwitz matrix;
(2) the zero solution of system (1) is absolutely robust stable w.r.t. partial variables x_{j_0+1}, $x_{j_0+2}, \ldots, x_n$,

are satisfied, then the zero solution of system (1) is absolutely robust stable w.r.t. all the state variables.

Proof
$\forall A \in A_I$, $h \in h_I$, let $x^{(j_0)}(t) := x^{(j_0)}(t, t_0; x_0) = (x_1(t, t_0; x_0), \ldots, x_{j_0}(t, t_0; x_0))^{\mathrm{T}}$, $x^{(j_0)C}(t) := (x_{j_0+1}(t, t_0; x_0), \ldots, x_n(t, t_0; x_0))^{\mathrm{T}}$, $h^{(j_0)}(t) := (h_1, \ldots, h_{j_0})^{\mathrm{T}}$. Thus, the first j_0 solutions, $x^{(j_0)}(t)$, of system $(1)'$ can be expressed as

$$x^{(j_0)}(t) = \mathrm{e}^{A^{(j_0)}(t-t_0)} x^{(j_0)}(t_0) + \int_{t_0}^{t} \mathrm{e}^{A^{(j_0)}(t-\tau)} A^{(j_0)C} x^{(j_0)C}(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t} \mathrm{e}^{A^{(j_0)}(t-\tau)} h^{(j_0)} f(x_n(\tau))\,\mathrm{d}\tau$$

Since $||x^{(j_0)}(t_0)|| \leqslant ||x(t_0)||$, the zero solution of $(1)'$ is absolutely robust stable w.r.t. $x^{(j_0)}$. In fact, we may follow the proof for the sufficiency of Theorem 1 to show that $\forall \varepsilon > 0$, there exists $\delta(\varepsilon)$ such that when $||x^{(j_0)}(t_0)|| < ||x(t_0)|| < \delta$, we have $||x^{(j_0)}(t)|| < \varepsilon$, and $\forall x_0 \in R^n$, $\lim_{t\to+\infty} x^{(j_0)}(t) = 0$. Thus, the zero solution of system (1) is absolutely robust stable w.r.t. $x^{(j_0)}(t)$, and thus also absolutely robust stable w.r.t. all the state variables. □

Corollary 4
If the following conditions:

(1) $B_I^{(j_0)}$ is an interval Hurwitz matrix;
(2) the zero solution of system (2) is absolutely robust stable w.r.t. partial variables y_{j_0+1}, $y_{j_0+2}, \ldots, y_n$,

hold, then the zero solution of system (2) is absolutely robust stable w.r.t. all the state variables.

Theorem 5
If there exists constants $c_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$-c_j\overline{a_{jj}} > \sum_{i=1, i\neq j}^{n} c_i a_{ij}^{(m)}, \quad j = 1, 2, \ldots, n-1$$

$$-c_n\overline{a_{nn}} \geqslant \sum_{i=1}^{n-1} c_i a_{in}^{(m)}$$

$$-c_n\overline{h_n} \geqslant \sum_{i=1}^{n-1} c_i h_i^{(m)} \tag{5}$$

and at least one of the last two inequalities in (5) is a strict inequality. Then, the zero solution of system (1) is absolutely robust stable. Here, $a_{ij}^{(m)} := \max_{i,j=1,2,\ldots,n} \{|\underline{a_{ij}}|, |\overline{a_{ij}}|\}$ and $h_i^{(m)} := \max_{i=1,2,\ldots,n-1} \{|\underline{h_i}|, |\overline{h_i}|\}$.

Proof
$\forall A \in A_I$, $h \in h_I$, construct the positive definite and radially unbounded Lyapunov function:

$$V = \sum_{i=1}^{n} c_i |x_i|$$

Note here that V is not a smooth function which does not have conventional derivative. However, one can find Dini derivative function which is a standard formulation for non-smooth functions (e.g. see [17,19,20]). Using the definition of Dini derivative, differentiating V with respect to time t along the trajectory of $(1)'$ yields

$$
\begin{aligned}
D^+V|_{(1)'} &= \sum_{i=1}^{n} c_i \dot{x}_i \,\mathrm{sign}(x_i(t)) \\
&= \sum_{i=1}^{n} c_i \left[\sum_{j=1}^{n} a_{ij}x_j(t) + h_i f(x_n(t))\right] \mathrm{sign}(x_i(t)) \\
&= \sum_{i=1}^{n} c_i a_{ii}|x_i(t)| + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_{ij}x_j \,\mathrm{sign}(x_i(t)) + \sum_{i=1}^{n} h_i f(x_n(t))\,\mathrm{sign}(x_i(t)) \\
&\leqslant \sum_{j=1}^{n} c_j a_{jj}|x_j(t)| \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} |a_{ij}|\,|x_j(t)| + \sum_{j=1}^{n-1} c_i|h_i|\,|f(x_n(t))| + c_n h_n |f(x_n(t))| \\
&\leqslant \sum_{j=1}^{n} \left[c_j a_{jj} + \sum_{i=1, i\neq j}^{n} c_i |a_{ij}|\right] |x_j(t)| + \left[c_n h_n + \sum_{j=1}^{n-1} c_i |h_i|\right] |f(x_n(t))| \\
&\leqslant \sum_{j=1}^{n} \left[c_j \overline{a_{jj}} + \sum_{i=1, i\neq j} c_i a_{ij}^{(m)}\right] |x_j(t)| + \left[c_n \overline{h_n} + \sum_{j=1}^{n-1} c_i h_i^{(m)}\right] |f(x_n(t))| \\
&< 0 \quad \text{when } x \neq 0
\end{aligned}
$$

due to arbitrary of $A \in A_I$.

Therefore, the zero solution of system (1) is absolutely robust stable. □

Corollary 5

When $\overline{r_n} < 0$, if there exists constants $c_i > 0$ $(i = 1, 2, \ldots, n)$ such that

$$
\begin{aligned}
-c_j \overline{b_{jj}} &> \sum_{i=1, i\neq j}^{n} c_i b_{ij}^{(m)}, \quad j = 1, 2, \ldots, n-1 \\
-c_n \overline{b_{nn}} &\geqslant \sum_{i=1}^{n-1} c_i b_{in}^{(m)}
\end{aligned}
\tag{6}
$$

while when $\overline{r_n} \leqslant 0$, the last inequality in (6) is a strict inequality, then the zero solution of system (2) is absolutely robust stable.

Theorem 6

If the following conditions are satisfied:

(1) $A_I^{(j_0)}$ is an interval Hurwitz matrix;

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(2) if there exists constants $c_i \geqslant 0$ $(i = 1, 2, \ldots, j_0)$, $c_j > 0$, $j = j_0 + 1, \ldots, n$ and $\varepsilon > 0$ such that

$$\begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{a_{11}} & m_{12} & \cdots & m_{1n} & m_{1(n+1)} \\ m_{21} & 2c_2\overline{a_{22}} & \cdots & m_{2n} & m_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & 2c_n a_{nn} & m_{n(n+1)} \\ m_{(n+1)1} & m_{(n+1)2} & \cdots & m_{(n+1)n} & 2\,\overline{h_n} \end{bmatrix} \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix}$$

$$\leqslant \begin{cases} -\varepsilon \sum_{i=j_0+1}^{n} x_i^2 \quad \text{or} \\ -\varepsilon \sum_{i=j_0+1}^{n-1} x_i^2 - \varepsilon f^2(x_n) \quad \text{or} \\ -\varepsilon \sum_{i=j_0+1}^{n-1} x_i^2 - \varepsilon x_n f(x_n) \end{cases}$$

where

$$m_{ij} = m_{ji} = \max_{\underline{a_{ij}} \leqslant a_{ij} \leqslant \overline{a_{ij}}} [|c_i\, a_{ij} + c_j\, a_{ji}|], \quad i \neq j, \quad 1 \leqslant i, j \leqslant n$$

$$m_{(n+1)i} = m_{i(n+1)} = \max_{\substack{\underline{a_{ni}} \leqslant a_{ni} \leqslant \overline{a_{ni}} \\ \underline{h_i} \leqslant h_i \leqslant \overline{h_i}}} [|c_i h_i + a_{ni}|], \quad 1 \leqslant i \leqslant n$$

then the zero solution of the Lurie interval control system (1) is absolutely robust stable.

Proof

For the variables $x_{j_0+1}, \ldots, x_n$, construct the positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n} c_i x_i^2 + 2 \int_0^{x_n} f(x_n)\,\mathrm{d}x_n$$

Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)'}$$

$$= \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1 a_{11} & c_1 a_{12} + c_2 a_{21} & \cdots & c_1 a_{1n} + c_2 a_{n1} & c_1 b_1 + a_{n1} \\ c_1 a_{21} + c_2 a_{12} & 2c_2 a_{22} & \cdots & c_1 a_{2n} + c_2 a_{n2} & c_1 b_2 + a_{n2} \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ c_1 a_{n1} + c_2 a_{1n} & c_1 a_{n2} + c_2 a_{2n} & \cdots & 2c_n a_{nn} & c_1 b_n + a_{n2} \\ c_1 b_1 + a_{n1} & c_1 b_2 + a_{n2} & \cdots & c_1 b_n + a_{n2} & 2h_n \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ f(x_n) \end{pmatrix}$$

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$$
\leqslant \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{a_{11}} & m_{12} & \cdots & m_{1n} & m_{1(n+1)} \\ m_{21} & 2c_2\overline{a_{22}} & \cdots & m_{2n} & m_{2(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & 2c_n a_{nn} & m_{n(n+1)} \\ m_{(n+1)1} & m_{(n+1)2} & \cdots & m_{(n+1)n} & 2\overline{h_n} \end{bmatrix} \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \\ |f(x_n)| \end{pmatrix}
$$

$$
\leqslant \begin{cases} -\varepsilon \sum_{i=j_0+1}^{n} x_i^2 \quad \text{or} \\ -\varepsilon \sum_{i=j_0+1}^{n-1} x_i^2 - \varepsilon f^2(x_n) \quad \text{or} \\ -\varepsilon \sum_{i=j_0+1}^{n-1} x_i^2 - \varepsilon x_n f(x_n) \end{cases} \tag{7}
$$

Therefore, the zero solution of $(1)'$ is globally asymptotically stable w.r.t. the partial variables $x_{j_0+1}, \ldots, x_n$. So the zero solution of (1) is absolutely robust stable w.r.t. the partial variables $x_{j_0+1}, \ldots, x_n$. Further due to condition (1) in Theorem 6, we know by Theorem 4 that the conclusion of Theorem 6 is true. □

Theorem 7

If the following conditions are satisfied: for $\overline{r_n} \leqslant 0$,

(1) $B_I^{(j_0)}$ is an interval Hurwitz matrix;
(2) if there exists constants $c_i \geqslant 0,\ i = 1, 2, \ldots, j_0$, $c_j > 0,\ j = j_0 + 1, \ldots, n$, and $\varepsilon > 0$ such that

$$
\begin{pmatrix} |y_1| \\ \vdots \\ |y_n| \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} 2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n} \\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & 2c_n\overline{b_{nn}} \end{bmatrix} \begin{pmatrix} |y_1| \\ \vdots \\ |y_n| \end{pmatrix} \leq -\varepsilon \sum_{i=j_0+1}^{n} y_i^2
$$

and for $\overline{r_n} < 0$,

(1) B_I is an interval Hurwitz matrix;
(2) if there exists constants $c_i \geqslant 0,\ i = 1, 2, \ldots, n-1$, and $c_n > 0$ such that

$$
\begin{bmatrix} 2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n} \\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & 2c_n\overline{b_{nn}} \end{bmatrix} \leqslant 0
$$

then the zero solution of system (2) is absolutely robust stable. Here, $m_{ij} = m_{ji} = \max_{\underline{b_{ij}} \leqslant b_{ij} \leqslant \overline{b_{ij}}} [|c_i b_{ij} + c_j b_{ji}|],\ i \neq j,\ 1 \leqslant i, j \leqslant n$.

Proof
When $\overline{r_n}\leqslant 0$, for the variables $y_{j_0+1},\ldots,y_n$, construct the positive definite and radially unbounded Lyapunov function:

$$V(y)=\sum_{i=1}^{n} c_i y_i^2$$

Then

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2)'}\leqslant\begin{pmatrix}|y_1|\\ \vdots\\ |y_n|\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n}\\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ m_{n1} & m_{n2} & \cdots & 2c_n\overline{b_{nn}}\end{bmatrix}\begin{pmatrix}|y_1|\\ \vdots\\ |y_n|\end{pmatrix}\leqslant-\varepsilon\sum_{i=j_0+1}^{n} y_i^2 \tag{8}$$

Equation (8) indicates that the zero solution of $(2)'$ is globally asymptotically stable w.r.t. the partial variables $y_{j_0+1},\ldots,y_n$. Further following the proof of Theorem 4 for the robust absolute stability w.r.t. $y^{(j_0)}$, we can show that the zero solution of $(2)'$ is also absolutely stable w.r.t. the partial variables $y_1,\ldots,y_{j_0}$.

Next, consider $\overline{r_n}<0$. For variable y_n, construct the positive definite and radially unbounded Lyapunov function:

$$V(y)=\sum_{i=1}^{n} c_i y_i^2$$

Then, differentiating V w.r.t. time t along the trajectory of system $(2)'$ yields

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(2)'}&\leqslant\begin{pmatrix}|y_1|\\ \vdots\\ |y_n|\end{pmatrix}^{\mathrm{T}}\begin{bmatrix}2c_1\overline{b_{11}} & m_{12} & \cdots & m_{1n}\\ m_{21} & 2c_2\overline{b_{22}} & \cdots & m_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ m_{n1} & m_{n2} & \cdots & 2c_n\overline{b_{nn}}\end{bmatrix}\begin{pmatrix}|y_1|\\ \vdots\\ |y_n|\end{pmatrix}+2c_n\overline{r_n}y_nf(y_n)\\ &\leqslant 2c_n\overline{r_n}y_nf(y_n)<0 \quad \text{when } y_n\neq 0\end{aligned} \tag{9}$$

Further, use the method of constant variation to express $y(t)$ in form of

$$y(t,t_0;y_0)=\mathrm{e}^{B(t-t_0)}y_0+\int_{t_0}^{t}\mathrm{e}^{B(t-\tau)}r_nf(y_n(\tau))\,\mathrm{d}\tau$$

and follow Theorem 1 to finish the proof. □

4. ALGEBRAIC SUFFICIENT AND NECESSARY CONDITIONS FOR ROBUST ABSOLUTE STABILITY OF SPECIAL LURIE INTERVAL CONTROL SYSTEMS

For an interval matrix, it is difficult to verify if it is an Hurwitz matrix. Although we have applied finite cover theorem to show that the Hurwitz stability of an infinite number of interval

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matrices can be found from the Hurwitz stability of a finite number of interval matrices [21], determining these finite number of matrices are very difficult.

In this section, we will consider some special Lurie interval control systems and derive several very simple algebraic sufficient and necessary conditions for the robust absolute stability of these special systems.

Again consider system (1), but now assume that $-\underline{a_{ij}} = \overline{a_{ij}},\ i \neq j,\ i,j = 1,2,\ldots,n,\ \overline{a_{ii}} < 0,$ $i = 1,2,\ldots,n;\quad \overline{h_n} < 0,\quad -\underline{h_i} = \overline{h_i},\ i = 1,2,\ldots,n-1,$ and $\lambda\overline{h_i} = \overline{a_{in}},\ i = 1,2,\ldots,n,\quad \lambda > 0.$ Then, we have the following theorem.

Theorem 8

The sufficient and necessary condition for the zero solution of the Lurie interval control system (1) being absolutely robust stable is that $-\bar{A}$ is an M matrix.

(The definition of M matrix can be found, say, in [22] or [17, pp. 7–8].)

Proof (Necessity)

Take $\bar{A} \in A_I$, $\bar{h} \in h_I$, $f(x_n) = x_n$. Substituting these expressions into system (1) yields

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \bar{A}x + \bar{h}x_n = [\bar{A} + (O_{n\times(n-1)}, \bar{h})]x \tag{10}$$

Since (10) is a linear system with constant coefficient, its coefficient matrix, $[\bar{A} + (O_{n\times(n-1)}, \bar{h})]$, must be an Hurwitz matrix. The diagonal elements of this matrix are negative, and non-diagonal elements are non-negative. Thus, the matrix $-[\bar{A} + (O_{n\times(n-1)}, \bar{h})]$ is an M matrix. Hence, there exists constants $c_i > 0,\ i = 1,2,\ldots,n$ such that

$$-c_j\overline{a_{jj}} > \sum_{i=1, i\neq j}^{n} c_i\overline{a_{ij}}, \quad j = 1,2,\ldots,n-1 \tag{11}$$

and

$$-c_n(\overline{a_{nn}} + \overline{h_n}) > \sum_{i=1}^{n-1} c_i(\overline{a_{in}} + \overline{h_i}) \tag{12}$$

Equation (12) can be rewritten as $-c_n(1+\frac{1}{\lambda})\overline{a_{nn}} > \sum_{i=1}^{n-1} c_i(1+1/\lambda)\overline{a_{in}}$, i.e.

$$-c_n\overline{a_{nn}} > \sum_{i=1}^{n-1} c_i\overline{a_{in}} \tag{13}$$

Equations (12) and (13) imply that $-\bar{A}$ is an M matrix.

Sufficiency. $\forall A \in A_I,\ h \in h_I$, for system (1)$'$, choose the positive definite and radially unbounded Lyapunov function:

$$V(x) = \sum_{i=1}^{n} c_i|x_i|$$

where c_i's are determined by Equations (11) and (12). It follows from

$$-c_n\overline{a_{nn}} > \sum_{i=1}^{n-1} c_i\overline{a_{in}} \quad \text{and} \quad \lambda\overline{h_i} = \overline{a_{in}}\ \ (i = 1,2,\ldots,n)$$

that

$$-c_n\overline{h_n} > \sum_{i=1}^{n-1} c_i\overline{h_i}$$

Now, we compute the Dini derivative of V along the trajectory of $(1)'$ to obtain

$$\begin{aligned} D^+V(x)|_{(1)'} &= \sum_{i=1}^{n} c_i\dot{x}_i \operatorname{sign}(x_i(t)) \\ &= \sum_{i=1}^{n} c_i\left[\sum_{j=1}^{n} a_{ij}x_j(t) + h_i f(x_n(t))\right]\operatorname{sign}(x_i(t)) \\ &\leqslant \sum_{j=1}^{n} c_j a_{jj}|x_j(t)| + \sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n} |a_{ij}|\,|x_j(t)| + \left[c_n h_n + \sum_{i=1}^{n-1} c_i|h_i|\right]|f(x_n(t))| \\ &\leqslant \sum_{j=1}^{n}\left[c_j\overline{a_{jj}} + \sum_{i=1,i\neq j}^{n} c_i\overline{a_{ij}}\right]|x_j| + \left[c_n\overline{h_n} + \sum_{i=1}^{n-1} c_i\overline{h_i}\right]|f(x_n)| \\ &< 0 \quad \text{when } x\neq 0 \end{aligned} \tag{14}$$

Because of arbitrary of $A \in A_I$ and $h \in h_I$, the zero solution of system (1) is absolutely robust stable. The proof is complete. □

Next, consider system (2). Assume that $\overline{b_{ii}}<0$, $i = 1,2,\ldots,n$, $-\underline{b_{ij}} = \overline{b_{ij}}$, $i\neq j$, $i,j = 1,2,\ldots,n$; $-\underline{r_i} = \overline{r_i}$, $i = 1,2,\ldots,n-1$ and $\overline{r_n}<0$.

Theorem 9
The sufficient and necessary conditions for the zero solution of the Lurie interval control system (2) being absolutely robust stable are the following:

(1) the zero solution of system (2) is absolutely robust stable w.r.t. the partial variable y_n;
(2)

$$-\overline{B_{(n-1)}} := \begin{bmatrix} \overline{b_{11}} & \cdots & \overline{b_{1(n-1)}} \\ \vdots & \cdots & \vdots \\ \overline{b_{(n-1)1}} & \cdots & \overline{b_{(n-1)(n-1)}} \end{bmatrix}$$

is an M matrix.

Proof (Necessity)
(1) The robust absolute stability w.r.t. y_n is obvious. For (2), substituting $f(y_n) = y_n$ into (2) results in an interval system:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = [B_I + (O_{n\times(n-1)}, r_I)]y$$

So $[B_I + (O_{n\times(n-1)}, r_I)]$ is an interval Hurwitz matrix. Thus, $[\bar{B} + (O_{n\times(n-1)}, \bar{r})]$ is an Hurwitz matrix, indicating that $-[\bar{B} + (O_{n\times(n-1)}, \bar{r})]$ is an M matrix. In particular, $-\bar{B}_{(n-1)}$ is an M matrix.

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Sufficiency. Let

$$B^{(n-1)} := \begin{bmatrix} b_{11} & \cdots & b_{1(n-1)} \\ \vdots & & \vdots \\ a_{(n-1)1} & \cdots & a_{(n-1)(n-1)} \end{bmatrix}$$

$$B^{(n-1)C} := (b_{1n}, b_{2n}, \dots, b_{(n-1)n})^{\mathrm{T}}$$

$$r^{(n-1)} := (r_1, r_2, \dots, r_{n-1})^{\mathrm{T}} = (0, \dots, 0)^{\mathrm{T}}_{(n-1)\times 1}$$

and $y^{(n-1)}(t) := (y_1(t), y_2(t), \dots, y_{n-1}(t))^{\mathrm{T}}$. Then, the first $n-1$ solutions of system $(2)'$ can be expressed as

$$y^{(n-1)}(t) = \mathrm{e}^{B^{(n-1)}(t-t_0)} y^{(n-1)}(t_0) + \int_{t_0}^{t} \mathrm{e}^{B^{(n-1)}(t-\tau)} B^{(n-1)C} y_n(\tau)\,\mathrm{d}\tau + \int_{t_0}^{t} \mathrm{e}^{B^{(n-1)}(t-\tau)} r^{(n-1)} f(y_n(\tau))\,\mathrm{d}\tau$$

$$\mathrm{Re}\,\lambda_{\max}(B^{(n-1)}) \leqslant \mathrm{Re}\,\lambda_{\max}(\bar{B}^{(n-1)}) \tag{15}$$

where $\mathrm{Re}\,\lambda_{\max}$ denotes the largest real part of the eigenvalues of the corresponding matrix. There exists constants $M \geqslant 1$ and $\alpha > 0$ such that

$$||\mathrm{e}^{B^{(n-1)}(t-t_0)}|| \leqslant ||\mathrm{e}^{\bar{B}^{(n-1)}(t-t_0)}|| \leqslant M \mathrm{e}^{\mathrm{Re}\,\lambda_{\max}(\bar{B}^{(n-1)})(t-t_0)} \leqslant M \mathrm{e}^{-\alpha(t-t_0)}$$

Hence,

$$||y^{(n-1)}(t)|| \leqslant M \mathrm{e}^{\mathrm{Re}\,\lambda_{\max}(\bar{B}^{(n-1)})(t-t_0)} ||y^{(n-1)}(t_0)|| + \int_{t_0}^{t} M \mathrm{e}^{\mathrm{Re}\,\lambda_{\max}(\bar{B}^{(n-1)})(t-\tau)} ||\bar{B}^{(n-1)C}||\, ||y_n(\tau)||\,\mathrm{d}\tau$$

$$\leqslant M \mathrm{e}^{-\alpha(t-t_0)} ||y^{(n-1)}(t_0)|| + \int_{t_0}^{t} M \mathrm{e}^{-\alpha(t-\tau)} ||\bar{B}^{(n-1)C}||\, ||y_n(\tau)||\,\mathrm{d}\tau$$

Due to condition (1), $y_n(t) \to 0$ as $t \to +\infty$, we can follow the proof of Theorem 1 to show that the zero solution of (2) is absolutely robust stable w.r.t. the partial variable $y^{(n-1)}(t)$. □

Remark

Although the conditions given in Theorems 8 and 9 are obtained in special cases, they are quite useful in realizing robust absolute stability *via* feedback controls.

5. APPLICATIONS

In this section, we present several examples to demonstrate the applicability of theorems given in the previous two sections. The first three systems are mathematical examples with numerical simulation results to verify the analytical predictions. The last example is a practical engineering system for which we employ the theorems presented in this paper to obtain stronger conclusions than the existing results.

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Example 1
The first example is to consider the robust absolute stability of the zero solution of the following Lurie interval control system:

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} \end{pmatrix} = \begin{bmatrix} [-1.2,-1] & [-0.5,1.5] \\ [-2,2] & [-5,-4] \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} [-2,2] \\ [-4.5,-4.2] \end{pmatrix} f(x_2) \tag{16}$$

where $f(\cdot) \in F$.

It is easy to verify that the conditions in Theorem 5 are satisfied: $\overline{a_{11}} = -1$, $\overline{a_{22}} = -4$, $a_{12}^{(m)} = \frac{3}{2}$, $a_{21}^{(m)} = 2$, $h_1^{(m)} = 2$, $\overline{h_2} = -4.2$. Take $c_1 = 2.1$ and $c_2 = 1$. Then, we construct the positive definite and radially unbounded Lyapunov function:

$$V = c_1|x_1| + c_2|x_2|$$

and find that

$$\begin{aligned} D^+V|_{(16)} &\leqslant (c_1\overline{a_{11}} + c_2 a_{21}^{(m)})|x_1| + (c_2\overline{a_{22}} + c_1 a_{12}^{(m)})|x_2| + (c_2\overline{b_2} + c_1 b_1^{(m)})|f(x_2)| \\ &= (-2.1+2)|x_1| + (-4 + 2.1 \times \tfrac{3}{2})|x_2| + (-4.2+4.2)|f(x_2)| \\ &\leqslant -0.1|x_1| - 0.85|x_2| \\ &< 0 \quad \text{when } |x_1| + |x_2| \neq 0 \end{aligned}$$

Thus, all the conditions in Theorem 5 are satisfied. Hence, the zero solution of system (16) is absolutely robust stable.

To simulate this example, we take the upper bounds of the system coefficients to obtain

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -x_1 + 1.5x_2 + 2f(x_2) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 - 4x_2 - 4.2f(x_2) \end{aligned} \tag{17}$$

and, for definiteness, take $f(x_2) = x_2^3$. The simulation results are depicted in Figure 1, where two different initial conditions are chosen, given by

$$(x_1, x_2) = (0.5, -2.0) \quad \text{and} \quad (x_1, x_2) = (-28.0, 20.0) \tag{18}$$

It is seen from Figures 1(a) and (b) that the two trajectories starting from different initial points converge to the same equilibrium point—the origin.

Example 2
Analyse the stability of the zero solution of the following Lurie interval control system:

$$\begin{pmatrix} \frac{\mathrm{d}x_1}{\mathrm{d}t} \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} \end{pmatrix} = \begin{bmatrix} [-5,-4] & [-3,3] \\ [-2,2] & [-4,-3] \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} [-2,2] \\ [-3,-2] \end{pmatrix} f(x_2) \tag{19}$$

where $f(x_2) \in F$.

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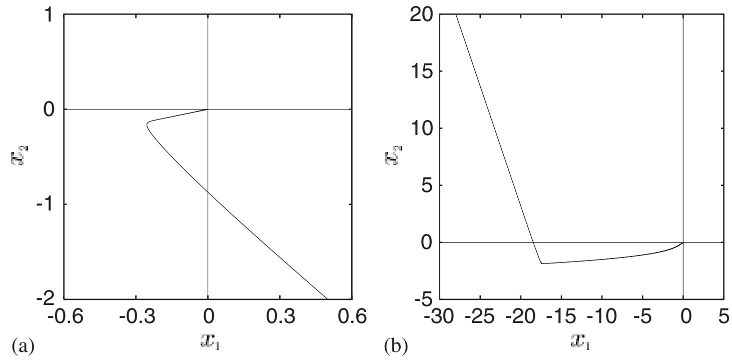

Figure 1. Simulated trajectories of system (17) for Example 1 converge to the origin, with the initial points: (a) $(x_1, x_2) = (0.5, -2.0)$ and (b) $(x_1, x_2) = (-28, 20)$.

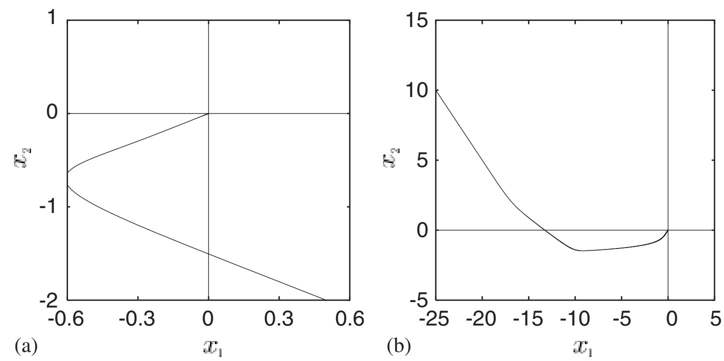

Figure 2. Simulated trajectories of system (20) for Example 2 converge to the origin, with the initial points: (a) $(x_1, x_2) = (0.5, -2.0)$ and (b) $(x_1, x_2) = (-25, 10)$.

Since

$$\bar{A} = \begin{bmatrix} -4 & 3 \\ 2 & -3 \end{bmatrix}$$

it is obvious to see that $-\bar{A}$ is an M matrix. Further, from $\overline{h_1} = 2$, $\overline{h_2} = -2$, $\overline{a_{12}} = 3$, $\overline{a_{22}} = -3$, and taking $\lambda = \frac{3}{2}$ yields $\lambda\overline{h_i} = \overline{a_{i2}}$, $i = 1, 2$. So all the conditions in Theorem 8 are satisfied. Thus, the zero solution of system (19) is absolutely robust stable.

For simulation, we consider

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= -4x_1 + 3x_2 + 2f(x_2) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= 2x_1 - 3x_2 - 2f(x_2) \end{aligned} \tag{20}$$

and choose $f(x_2) = x_2^5$. The simulation results are given in Figure 2 which again shows that trajectories from difference initial points converge to the origin. The two initial points are

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chosen as

$$(x_1, x_2) = (0.5, -2.0) \quad \text{and} \quad (x_1, x_2) = (-25.0, 10.0) \tag{21}$$

respectively.

Example 3
Consider the stability of the zero solution of a 3-D simplified Lurie interval control system, described by

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} = \begin{bmatrix} [-3,-2] & [1,2] & [0,1] \\ [-2,-1] & [-4,-1] & [-1,1] \\ [-1,0] & [-1,1] & [-3,-2] \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ [-1,-\frac{1}{2}] \end{pmatrix} f(y_3) \tag{22}$$

where $f(\sigma) \in F$.

Construct the positive definite and radially unbounded Lyapunov function:

$$V = \tfrac{1}{2}(y_1^2 + y_2^2 + y_3^2)$$

Then, we obtain

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(22)} &= [-3,-2]y_1^2 + [-4,-1]y_2^2 + [-3,-2]y_3^2 + [1,2]y_1y_2 + [-2,-1]y_2y_1 \\ &\quad + [0,1]y_1y_3 + [-1,0]y_3y_1 + [-1,1]y_2y_3 + [-1,1]y_3y_2 + [-1,-\tfrac{1}{2}]y_3f(y_3) \\ &\leq -2y_1^2 - y_2^2 - 2y_3^2 + |y_1y_2| + |y_1y_3| + 2|y_2y_3| - \tfrac{1}{2}y_3f(y_3) \\ &= \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & 1 \\ \frac{1}{2} & 1 & -2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} - \tfrac{1}{2}y_3f(y_3) < 0 \quad \text{when } |y| \neq 0 \end{aligned}$$

Thus, the zero solution of system (22) is absolutely robust stable.

For simulation, we consider two particular systems associated with system (22), given below:

$$\begin{pmatrix} \frac{dy_1}{dt} \\ \frac{dy_2}{dt} \\ \frac{dy_3}{dt} \end{pmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} f(y_3) \tag{23}$$

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and

$$\begin{pmatrix} \dfrac{dy_1}{dt} \\ \dfrac{dy_2}{dt} \\ \dfrac{dy_3}{dt} \end{pmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -2 & -1 & -1 \\ -1 & -1 & -2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix} f(y_3) \tag{24}$$

where $f(y_3)$ is taken as $f(y_3) = y_3^3$ for system (23) and $f(y_3) = y_3^5$ for system (24), respectively, in simulation. Two initial points are chosen as

$$(y_1, y_2, y_3) = (0.5, -2.0, 1.5) \quad \text{and} \quad (y_1, y_2, y_3) = (-25.0, 10.0, 20) \tag{25}$$

The simulation results for system (23) and (24) are shown, respectively, in Figures 3 and 4. These two figures again show that all the trajectories converge to the origin, as expected.

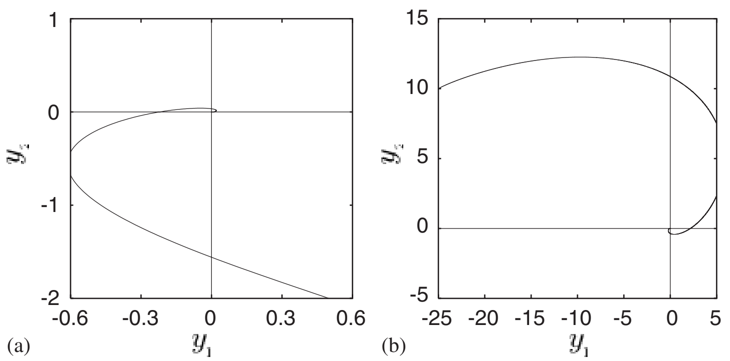

Figure 3. Simulated trajectories of system (23) for Example 3, projected on the y_1–y_2 plane, converge to the origin, with the initial points: (a) $(y_1, y_2, y_3) = (0.5, -2.0, 1.5)$ and (b) $(y_1, y_2, y_3) = (-28, 20, 20)$.

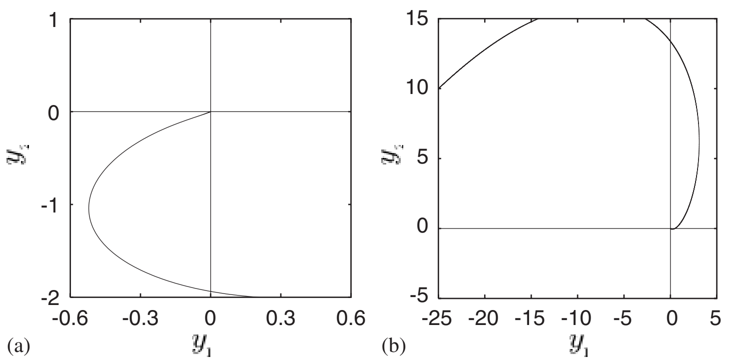

Figure 4. Simulated trajectories of system (24) for Example 3, projected on the y_1–y_2 plane, converge to the origin, with the initial points: (a) $(y_1, y_2, y_3) = (0.5, -2.0, 1.5)$ and (b) $(y_1, y_2, y_3) = (-28, 20, 20)$.

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Example 4

Finally, we consider the following system which describes the motion of an aircraft in the longitudinal direction [17, 20, 23, 24]:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\rho_i x_i + \sigma$$
$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum_{i=1}^{4} \beta_i x_i - rp - f(\sigma) \tag{26}$$

where $\rho_i \in [\underline{\rho_i}, \overline{\rho_i}]$, $\beta_i \in [\underline{\beta_i}, \overline{\beta_i}]$, $rp \in [\underline{rp}, \overline{rp}]$, and $\overline{\rho_i} > \underline{\rho_i} > 0$, $-\infty < \underline{\beta_i} \leqslant 0 \leqslant \overline{\beta_i} < +\infty$, $\overline{rp} > \underline{rp} > 0$.

Note that if $\underline{\rho_i} = \overline{\rho_i}$, $\underline{\beta_i} = \overline{\beta_i}$, $\underline{rp} = \overline{rp}$, then the absolute stability of system (26) has been studied by many authors (e.g. see [17, 23, 25]). However, strictly speaking, it is more practical to allow the coefficients to take values in intervals.

In the following, we shall show that for the system's coefficient taking interval values, the zero solution of system (26) is absolutely robust stable if

$$\underline{rp} \geqslant \sum_{i=1}^{4} \frac{1 + \operatorname{sign} \overline{\beta_i}}{2} \frac{\overline{\beta_i}}{\underline{\rho_i}} \tag{27}$$

Proof

For any values of the system coefficients $\rho_i \in [\underline{\rho_i}, \overline{\rho_i}]$, $\beta_i \in [\underline{\beta_i}, \overline{\beta_i}]$ and $rp \in [\underline{rp}, \overline{rp}]$, we can construct the positive definite and radially unbounded Lyapunov function:

$$V(x, \sigma) = \sum_{i=1}^{4} c_i x_i^2 + \sigma^2 \tag{28}$$

where c_i is defined as

$$c_i = \begin{cases} -\beta_i & \text{when } \beta_i < 0 \\ \epsilon_i & \text{when } \beta_i = 0 \ (0 < \epsilon_i \ll 1) \\ \beta_i & \text{when } \beta_i > 0 \end{cases} \tag{29}$$

Differentiating V with respect to t along the trajectory of (26) and simplifying the result yields

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(26)} = 2\sum_{i=1}^{4} c_i x_i \dot{x}_i + 2\sigma\dot{\sigma}$$
$$= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \sigma \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2c_1\rho_1 & 0 & 0 & 0 & c_1 + \beta_1 \\ 0 & -2c_2\rho_2 & 0 & 0 & c_2 + \beta_2 \\ 0 & 0 & -2c_3\rho_3 & 0 & c_3 + \beta_3 \\ 0 & 0 & 0 & -2c_4\rho_4 & c_4 + \beta_4 \\ c_1 + \beta_1 & c_2 + \beta_2 & c_3 + \beta_3 & c_4 + \beta_4 & -2rp \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \sigma \end{pmatrix} - 2r\sigma f(\sigma)$$
$$\equiv (x_1 \ x_2 \ x_3 \ x_4 \ \sigma) D (x_1 \ x_2 \ x_3 \ x_4 \ \sigma)^{\mathrm{T}} - 2r\sigma f(\sigma) \tag{30}$$

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Thus, if we can prove that the first term of (30) is negative semi-definite, then (30) is negative definite about the variable σ. Therefore, we only need to prove that $D \leqslant 0$.

It is easy to see that $D \leqslant 0$ if and only if

$$2rp - \sum_{i=1}^{4} \frac{(c_i + \beta_i)^2}{2c_i\rho_i} \geqslant 0 \tag{31}$$

Further, we can show that the condition

$$rp \geqslant \sum_{i=1}^{4} \frac{1 + \text{sign}\,\beta_i}{2} \frac{\beta_i}{\rho_i} \tag{32}$$

implies that (31) holds. We only need to compare $(c_i + \beta_i)^2/2c_i\rho_i$ with $(1 + \text{sign}\,\beta_i)\beta_i/\rho_i$. By the definition of c_i given in (29), we see that when $c_i = \beta_i$, $0 = (c_i + \beta_i)^2/2c_i\rho_i = (1 + \text{sign}\,\beta_i)\beta_i/\rho_i = 0$; when $c_i = \beta_i > 0$, $(c_i + \beta_i)^2/2c_i\rho_i = 2\beta_i\rho_i = (1 + \text{sign}\,\beta_i)\beta_i/\rho_i$. For $\beta_i = 0$, we have $(c_i + \beta_i)^2/2c_i\rho_i = \epsilon_i/2\rho_i$. Since $rp > 0$ is a constant, rp can be chosen such that $rp \geqslant \epsilon_i/2\rho_i$ due to ϵ_i being arbitrarily small.

On the other hand, it follows from condition (27) that

$$rp \geqslant \underline{rp} \geqslant \sum_{i=1}^{4} \frac{1 + \text{sign}\,\overline{\beta}_i}{2} \frac{\overline{\beta}_i}{\underline{\rho}_i} \geqslant \sum_{i=1}^{4} \frac{1 + \text{sign}\,\beta_i}{2} \frac{\beta_i}{\rho_i}$$

which indicates that (27) is a sufficient condition for (32) to hold. This shows that the first term of (30) is negative semi-definite, and so $\mathrm{d}V/\mathrm{d}t|_{(26)}$ is negative definite about σ.

Let $f(\sigma) = \sigma$. Then (26) becomes a linear system with interval coefficients:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -\rho_i x_i + \sigma$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \sum_{i=1}^{4} \beta_i x_i - (rp + 1)\sigma \tag{33}$$

Again using the Lyanpunov function (28), we can show that the zero solution of system (33) is globally asymptotically stable. Thus, the coefficient matrix of system (33) is an Hurwitz matrix. Hence, by Theorem 1 we know that the zero solution of system (26) is absolutely robust stable.

In particular, when $\underline{\rho_i} = \overline{\rho_i}$, $\underline{\beta_i} = \overline{\beta_i}$, $\underline{\gamma\rho} = \overline{\gamma\rho}$, the results given above recover the existing results obtained in [17, 24], and also recover the results of [20, 23] as special cases. This shows that the theorems given in this paper generalize the existing results in the literature.

6. CONCLUSION

In this paper, we have considered robust absolute stability of Lurie interval control systems. We presented a number of sufficient and necessary conditions under which a Lurie system with interval feedback controls can be robustly absolutely stabilized. Besides, we have also provided several algebraic sufficient and necessary conditions for the robust absolute stability of Lurie interval control systems. These algebraic conditions can be easily verified and are thus convenient to be used in applications. Four examples including a practical engineering problem, with numerical simulations, are presented to verify analytical predictions.

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