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Globally exponentially attractive sets of the family of Lorenz systems

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In this paper, the concept of globally exponentially attractive set is proposed and used to consider the ultimate bounds of the family of Lorenz systems with varying parameters. Explicit estimations of the ultimate bounds are derived. The results presented in this paper contain all the existing results as special cases. In particular, the critical cases, $b \to 1^+$ and $a \to 0^+$, for which the previous methods failed, have been solved using a unified formula.

the family of Lorenz systems, globally exponentially attractive set, Lagrange stability, generalized Lyapunov function

1 Introduction

Since Lorenz discovered the Lorenz chaotic attractor in 1963, extensive studies have been to given to the well-known Lorenz system (see refs. [1—5]):

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = cx - y - xz, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where a, b, and c are parameters. The typical parameter values for system (1) to exhibit a chaotic attractor are as follows: $a = 10$, $b = 8/3$, $c = 28$. The Lorenz system has played a fundamental role in the area of nonlinear science and chaotic dynamics. Although everyone believes the existence of the Lorenz attractor, no rigorous mathematical proof has been given so far. This

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problem has been listed as one of the fundamental mathematical problems, proposed by Smale, for the 21st century. In ref. [6], this problem is extensively discussed with the aid of numerical computation. It points out that it is extremely difficult to obtain the information of the chaotic attractor directly from the differential eq. (1). Most of the results in the literature are based on computer simulations. Even by us calculating the Lyapunov exponents of the system, one needs to assume the system being bounded in order to conclude that the system is chaotic. Therefore, the study of the globally attractive set of the Lorenz system is not only theoretically significant, but also practically important.

Russian scholar Leonov[7] is the first one to present a globally attractive set of eq. (1) with respect to y and z, given as follows:

$$y^2+(z-c)^2 \leqslant \frac{b^2c^2}{4(b-1)}. \tag{2}$$

No formal estimation on the variable x is given in this paper, but rather showing a few numerical estimations

$$|x|<21,\quad |x|<28.92,\quad |x|<39.246,\quad |x|<21.412, \text{ and } \quad |x|<22.821. \tag{3}$$

These five varying numbers do not provide any clue for the trend of the variable x with respect to the system parameters a, b, and c. Thus, another formula consisting of all the three variables x, y, z is given in the same paper[7]:

$$x^2+y^2+(z-a-c)^2 \leqslant \frac{b^2(a+c)^2}{4(b-1)}. \tag{4}$$

Obviously, the estimation given by inequality (4) on y and z is more conservative than that given in inequality (2).

Late, in ref. [8], a conjecture was presented to estimate the variable x

$$x^2 \leqslant \frac{b^2c^2}{4(b-1)}, \tag{5}$$

but no proof was given.

In our papers[9,10], a new ellipsoid estimation for the globally attractive set of the system is given, which improves and generalizes the results in refs. [7, 8]. Moreover, in these two papers, we simplified Leonov's proofs and rigorously proved the estimation (5). Recently, in ref. [11] the Lagrange product and optimal methods are used to estimate the globally attractive set of the family of Lorenz systems, where the critical case $b \to 1^+$ is solved. However, the formulas presented in ref. [11] contain all the three variables x, y, and z, and thus are still conservative as that of ref. [7]. Besides, the method developed in ref. [11] fails for the critical case when $a \to 0^+$.

Up to now, the concept of globally exponentially attractive set has not been formally proposed in the literature for studying the bounds of chaotic attractors. Thus, the convergent speed of trajectories from outside of the globally attractive set to the boundary of the set is unknown. In this paper, we define the globally exponentially attractive set and apply it to obtain the exponential estimation of such set. Our results contain the various existing results on the globally exponentially attractive set as special cases. Furthermore, the critical cases, $b \to 1^+$ and $a \to 0^+$, which have not been solved in the existing literature, are uniformly solved by using our proposed method[12,13].

2 Globally attractive set of finite interval Lorenz family

Chen and Lü[5] proposed the following Lorenz family:

$$\begin{cases} \dot{x} = (25\alpha + 10)(y - x), \\ \dot{y} = (28 - 35\alpha)x - xz + (29\alpha - 1)y, \\ \dot{z} = xy - \dfrac{\alpha + 8}{3} z, \end{cases} \tag{6}$$

where $\alpha \in [0,1]$. In ref. [12] the globally attractive set of system (6) is obtained for $\alpha \in [0,1/29)$. What we wish to investigate in this paper is to consider the globally exponentially attractive set of system (6) for $\alpha \in [0,1/29)$. For distinction, we call this family of Lorenz systems, described by eq. (6), the finite interval Lorenz family.

For simplicity, let

$$a_\alpha = 25\alpha + 10,\ b_\alpha = \frac{\alpha + 8}{3},\ c_\alpha = 28 - 35\alpha,\ d_\alpha = 1 - 29\alpha,$$

where $\alpha \in [0,1/29)$, $a_\alpha \in [10, 10 + 35/29)$, $b_\alpha \in [8/3,\ (8 + 1/29)/3)$, $c_\alpha \in (28 - 35/29, 28]$, and $d_\alpha \in (0,1]$. Then, system (6) can be rewritten as

$$\begin{cases} \dot{x} = a_\alpha (y - x), \\ \dot{y} = c_\alpha x - xz - d_\alpha y, \\ \dot{z} = xy - b_\alpha z. \end{cases} \tag{7}$$

In ref. [9], the function, given by

$$V_\lambda(X) = \frac{1}{2}\left[\lambda x^2 + y^2 + (z - \lambda a_\alpha - c_\alpha)^2\right] \qquad (\lambda \geqslant 0), \tag{8}$$

is a generalized positive definite and radically unbounded Lyapunov function for system (7), where $X = (x, y, z)$.

Definition 1. If there exists a positive number $L_\lambda > 0$ such that for $V_\lambda(X_0) > L_\lambda$ it holds $V_\lambda(X(t)) = V_\lambda(X(t, t_\alpha, X_0)) \to L_\lambda$, as $t \to +\infty$. Then, the set $\Omega_\lambda = \{X \mid V_\lambda(X) \leqslant L_\lambda\}$ is called a globally attractive set of eq. (7). If $\forall X_0 \in \Omega_\lambda,\ X(t, t_0, X_0) \subseteq \Omega_\lambda$, then the set Ω_λ is called positive invariant set.

If, moreover, there exists a positive number $r_\lambda > 0$ such that $\forall X_0 \in R^3$, we have the following estimation

$$V_\lambda\left(X(t)\right) - L_\lambda \leqslant \left[V_\lambda(X_0) - L_\lambda\right] \mathrm{e}^{-r_\lambda (t - t_0)},$$

when $V_\lambda(X(t)) > L_\lambda,\ t \geqslant t_0$. Then, the set Ω_λ is a globally exponentially attractive set of eq. (7). Ω_λ is also called a positive invariant set.

Theorem 1. Define $L_\lambda = \dfrac{b_\alpha^2 (\lambda a_\alpha + c_\alpha)^2}{8(b_\alpha - d_\alpha) d_\alpha}$. Then, we have an estimation of the globally exponentially attractive set of system (6), given by

$$V_\lambda\left(X(t)\right) - L_\lambda \leqslant \left[V_\lambda(X_0) - L_\lambda\right] \mathrm{e}^{-2 d_\alpha (t - t_0)}. \tag{9}$$

Especially, the set

$$\Omega_\lambda = \{X \mid V_\lambda(X) \leqslant L_\lambda\} = \left\{ X \,\middle|\, \lambda x^2 + y^2 + (z - \lambda a_\alpha - c_\alpha)^2 \leqslant \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)d_\alpha} \right\}$$

is the globally attractive and positive invariant set of eq. (7).

Proof. Let $f(z) = -(b_\alpha - d_\alpha)z^2 + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)z$. Then $f'(z) = -2(b_\alpha - d_\alpha)z + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)$.

Following the approach used in ref. [10], setting $f'(z) = 0$ yields

$$z_0 = \frac{(b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)}{2(b_\alpha - d_\alpha)}.$$

Since $b_\alpha > 2 > d_\alpha$, $0 < d_\alpha \leqslant 1$, it follows that $z_0 > 0$ and $f''(z_0) = -2(b_\alpha - d_\alpha) < 0$. Thus,

$$\sup_{z \in R} f(z) = f(z_0) = \frac{(b_\alpha - 2d_\alpha)^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)}.$$

Using the facts that $a_\alpha > 1$ and $0 < d_\alpha \leqslant 1$, we obtain

$$\begin{aligned}
\left.\frac{\mathrm{d}V_\lambda}{\mathrm{d}t}\right|_{(7)} &= \lambda x\dot{x} + y\dot{y} + (z - \lambda a_\alpha - c_\alpha)\dot{z} = -\lambda a_\alpha x^2 - d_\alpha y^2 - b_\alpha z^2 + b_\alpha(\lambda a_\alpha + c_\alpha)z \\
&= -\lambda a_\alpha x^2 - d_\alpha y^2 - d_\alpha z^2 + 2d_\alpha(\lambda a_\alpha + c_\alpha)z - (b_\alpha - d_\alpha)z^2 + (b_\alpha - 2d_\alpha)(\lambda a_\alpha + c_\alpha)z \\
&\leqslant -\lambda a_\alpha x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + d_\alpha(\lambda a_\alpha + c_\alpha)^2 + f(z) \\
&\leqslant -\lambda d_\alpha x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + d_\alpha(\lambda a_\alpha + c_\alpha)^2 + f(z_0) \\
&= -\lambda d_\alpha x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)} \\
&\leqslant -\lambda d_\alpha x^2 - d_\alpha y^2 - d_\alpha(z - \lambda a_\alpha - c_\alpha)^2 + 2d_\alpha L_\lambda \\
&\leqslant -2d_\alpha V_\lambda + 2d_\alpha L_\lambda \leqslant 0 \quad \text{when } V_\lambda \geqslant L_\lambda.
\end{aligned} \tag{10}$$

By comparison theorem and integrating both sides of formula (10) yields

$$V_\lambda(X(t)) \leqslant V_\lambda(X_0)\mathrm{e}^{-2d_\alpha(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{-2d_\alpha(t-\tau)} 2d_\alpha L_\lambda d\tau = V_\lambda(X_0)\mathrm{e}^{-2d_\alpha(t-t_0)} + L_\lambda(1 - \mathrm{e}^{-2d_\alpha(t-t_0)}).$$

Thus, if $V_\lambda(X(t)) > L_\lambda$, $t \geqslant t_0$, we have the following estimation for the globally exponentially attractive set

$$V_\lambda(X(t)) - L_\lambda \leqslant [V_\lambda(X_0) - L_\lambda]\mathrm{e}^{-2d_\alpha(t-t_0)}.$$

By the definition, taking limit on both sides of the above inequality as $t \to +\infty$ results in

$$\overline{\lim_{t \to +\infty}} V_\lambda(X(t)) \leqslant L_\lambda,$$

namely, the set

$$\Omega_\lambda = \{X \mid V_\lambda(X(t)) \leqslant L_\lambda\} = \left\{ X \,\middle|\, \lambda x^2 + y^2 + (z - \lambda a_\alpha - c_\alpha)^2 \leqslant \frac{b_\alpha^2(\lambda a_\alpha + c_\alpha)^2}{4(b_\alpha - d_\alpha)d_\alpha} \right\}$$

is the globally exponentially attractive and positive invariant set of eq. (7).

Remark 1. 1) Taking $\alpha = 0$, $\lambda \geqslant 0$, Theorem 1 is a generalization of Theorem 1 of ref. [9].

2) Taking $\alpha = 0,\ \lambda = 0$, Theorem 1 is a generalization of the Leonov's estimation (2).

3) Taking $\alpha = 0,\ \lambda = 1$, Theorem 1 is a generalization of the Leonov's estimation (4).

4) Taking $\alpha \in [0, 1/29),\ \lambda = 1$, Theorem 1 is a generalization of Theorem 1 of ref. [11].

Here, the generalization means that the globally attractive set[7,9−11] is extended to the globally exponentially attractive set.

Theorem 2. Let $V_0 = \frac{1}{2}\left[y^2 + (z - c_\alpha)^2\right]$ and $L_0 = \frac{b_\alpha^2 c_\alpha^2}{8(b_\alpha - d_\alpha)d_\alpha}$.

Then, the estimation of the globally exponentially set of the interval Lorenz family eq. (7) is

$$\begin{cases} V_0\left(X(t)\right) - L_0 \leqslant [V_0(X_0) - L_0]\mathrm{e}^{-2d_\alpha(t-t_0)} \leqslant [V_0(X_0) - L_0]\mathrm{e}^{-\min(2d_\alpha, a_\alpha)(t-t_0)}, \\ x^2(t) - 2L_0 \leqslant \left(x_0^2 - 2L_0\right)\mathrm{e}^{-a_\alpha(t-t_0)} \leqslant \left(x_0^2 - 2L_0\right)\mathrm{e}^{-\min(2d_\alpha, a_\alpha)(t-t_0)}. \end{cases}$$

Especially, the set

$$\Omega_0 = \left\{ X \left| \begin{array}{c} V_0(X) \leqslant L_0 \\ x^2 \leqslant 2L_0 \end{array} \right. \right\} = \left\{ X \left| \begin{array}{l} y^2 + (z-c)^2 \leqslant \frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} \\ x^2 \leqslant \frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} \end{array} \right. \right\} \tag{11}$$

is the globally attractive and positive invariant set of eq. (7), where $X = (y, z)$.

Proof. Setting $\lambda = 0$ in Theorem 1, we analogously obtain an estimation for the globally exponentially attractive set with respect to the variables y and z,

$$V_0\left(X(t)\right) - L_0 \leqslant [V_0(X_0) - L_0]\ \mathrm{e}^{-2d_\alpha(t-t_0)}. \tag{12}$$

Then, taking limit on both sides of inequality (12) leads to

$$\overline{\lim_{t\to+\infty}}\ V_0\left(X(t)\right) \leqslant L_0,$$

i.e.,

$$\overline{\lim_{t\to+\infty}}\ \left(y^2(t) + (z(t) - c_\alpha)^2\right) \leqslant \frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha - d_\alpha)d_\alpha} = 2L_0.$$

Thus, the estimation of the ultimate bound for y is

$$y^2 \leqslant 2L_0.$$

Next, for the first equation of system (7), we construct a radically unbounded and positive definite Lyapunov function

$$V = \frac{1}{2}x^2.$$

Then,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(7)} = -a_\alpha x^2 + a_\alpha xy \leqslant -a_\alpha x^2 + a_\alpha |x||y|$$
$$= -a_\alpha x^2 + \frac{1}{2}a_\alpha x^2 + a_\alpha L_0 = -a_\alpha V + a_\alpha L_0.$$

Hence,

$$V\left(X(t)\right) - L_0 \leqslant [V(X_0) - L_0]\mathrm{e}^{-a_\alpha(t-t_0)}, \tag{13}$$

i.e., $$x^2(t)-\frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha-d_\alpha)d_\alpha}\leqslant\left[x_0^2-\frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha-d_\alpha)d_\alpha}\right]\mathrm{e}^{-a_\alpha(t-t_0)}.$$

Therefore, the ultimate bound is given by the limit

$$\overline{\lim_{t\to\infty}}\, x^2(t)\leqslant\frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha-d_\alpha)d_\alpha}=2L_0.$$

This implies that Ω_0 is a globally attractive and positive invariant set.

Remark 2. The globally attractive set Ω_0 given in Theorem 2 improves and extends Theorem 1 of ref. [11], from which we have

$$R_1^2:=\frac{(19-5\alpha)^2(8+\alpha)^2}{(15+264\alpha)(1-29\alpha)}=\frac{b_\alpha^2(c_\alpha+a_\alpha)^2}{4(b_\alpha-d_\alpha)d_\alpha}.$$

Since $\frac{2L_0}{R_1^2}=\frac{c_\alpha^2}{(c_\alpha^2+a_\alpha^2)}<1$, our estimation is sharper than that given in ref. [11]. Further, it is easy to show that

$$x^2(t)\leqslant y^2(t)\leqslant\frac{b_\alpha^2 c_\alpha^2}{4(b_\alpha-d_\alpha)d_\alpha}<\frac{b_\alpha^2(c_\alpha+a_\alpha)^2}{4(b_\alpha-d_\alpha)d_\alpha},$$

and

$$c_\alpha+a_\alpha-\frac{b_\alpha(c_\alpha+a_\alpha)}{2\sqrt{(b_\alpha-d_\alpha)d_\alpha}}<c_\alpha-\frac{b_\alpha c_\alpha}{2\sqrt{(b_\alpha-d_\alpha)d_\alpha}}<z$$
$$\leqslant c_\alpha+\frac{b_\alpha c_\alpha}{2\sqrt{(b_\alpha-d_\alpha)d_\alpha}}\leqslant c_\alpha+a_\alpha+\frac{b_\alpha(a_\alpha+c_\alpha)}{2\sqrt{(b_\alpha-d_\alpha)d_\alpha}}.$$

This clearly indicates that the estimation obtained in this paper is between the lower and upper bounds given in ref. [11].

3 Globally exponentially attractive set of infinity interval Lorenz systems family

In this section, we return to system (1), but now assume that the system parameters are defined as $a\in(0,+\infty),\ b\in(1,+\infty), c\in(0,+\infty)$. For convenience, we call such system (1) the infinite interval Lorenz family. Comparing with eq. (7), though system (1) has one less parameter (d_α), its parameter values are unbounded and analysis is different from that of system (7). For certain values of the parameters, system (1) may be not chaotic. Here, we consider the globally exponentially attractive and positive invariant sets of system (1), regardless whether it is chaotic or not.

Theorem 3. Let

$$\overline{V}_\lambda=\frac{1}{2}\left[\lambda x^2+y^2+(z-\lambda a-c)^2\right],\ \overline{L}_\lambda^{(1)}=\frac{(\lambda a+c)^2 b^2}{8(b-1)},\quad \overline{L}_\lambda^{(2)}=\frac{(\lambda a+c)^2}{2},\quad \overline{L}_\lambda^{(3)}=\frac{(\lambda a+c)^2 b^2}{8a(b-a)}.$$

Then, the globally exponentially attractive and positive invariant sets of the infinity interval Lorenz system (1) are given by

$$\begin{cases} \bar{V}_\lambda\left(X(t)\right)-L_\lambda^{(1)} \leqslant \left[\bar{V}_\lambda(X_0)-L_\lambda^{(1)}\right]\mathrm{e}^{-2(t-t_0)} & \text{when } a \geqslant 1,\ b \geqslant 2, \\ \bar{V}_\lambda\left(X(t)\right)-L_\lambda^{(2)} \leqslant \left[\bar{V}_\lambda(X_0)-L_\lambda^{(2)}\right]\mathrm{e}^{-b(t-t_0)} & \text{when } a > \frac{b}{2},\ b < 2, \\ \bar{V}_\lambda\left(X(t)\right)-L_\lambda^{(3)} \leqslant \left[\bar{V}_\lambda(X_0)-L_\lambda^{(3)}\right]\mathrm{e}^{-2a(t-t_0)} & \text{when } 0 < a < 1,\ b \geqslant 2a. \end{cases} \tag{14}$$

Especially, the sets

$$\Omega_\lambda^{(i)} = \left\{X \,\middle|\, \bar{V}_\lambda(X) \leqslant L_\lambda^{(i)}\right\} = \left\{X \,\middle|\, \lambda x^2 + y^2 + (z-\lambda a - c)^2 \leqslant 2L_\lambda^{(i)}\right\}, \quad i = 1,2,3,$$

are the estimations of the globally exponentially attractive and positive invariant sets of system (1).

Proof. Take $\bar{V}_\lambda = \frac{1}{2}\left[\lambda x^2 + y^2 + (z-\lambda a - c)^2\right]$.

1) When $a \geqslant 1,\ b \geqslant 2$, analogous to the proof of formula (10), we have

$$\left.\frac{\mathrm{d}\bar{V}_\lambda}{\mathrm{d}t}\right|_{(1)} \leqslant -\lambda x^2 - y^2 - (z-\lambda a - c)^2 + 2L_\lambda^{(1)} = -2\bar{V}_\lambda + 2L_\lambda^{(1)},$$

and thus obtain

$$\bar{V}_\lambda\left(X(t)\right)-L_\lambda^{(1)} \leqslant \left[\bar{V}_\lambda(X_0)-L_\lambda^{(1)}\right]\mathrm{e}^{-2(t-t_0)}. \tag{15}$$

2) When $a > \frac{b}{2}, b < 2$, it follows that

$$\begin{aligned} \left.\frac{\mathrm{d}\bar{V}_\lambda}{\mathrm{dt}}\right|_{(1)} &\leqslant -\lambda a x^2 - y^2 - bz^2 + b(\lambda a + c)z \\ &\leqslant -\lambda\frac{b}{2}x^2 - \frac{b}{2}y^2 - \frac{b}{2}z^2 + \frac{b}{2}(2(\lambda a + c))z \\ &\leqslant -\lambda\frac{b}{2}x^2 - \frac{b}{2}y^2 - \frac{b}{2}(z-\lambda a - c)^2 + \frac{b}{2}(\lambda a + c)^2 \\ &\leqslant -\frac{b}{2}\left(2\bar{V}_\lambda - 2L_\lambda^{(2)}\right) \\ &= -b\left(\bar{V}_\lambda - L_\lambda^{(2)}\right). \end{aligned}$$

Thus,

$$\bar{V}_\lambda\left(X(t)\right)-L_\lambda^{(2)} \leqslant \left[\bar{V}_\lambda(X_0)-L_\lambda^{(2)}\right]\mathrm{e}^{-b(t-t_0)}. \tag{16}$$

(3) When $a < 1, b \geqslant 2a$, we have

$$\begin{aligned} \left.\frac{\mathrm{d}\bar{V}_\lambda}{\mathrm{d}t}\right|_{(1)} &= -\lambda a x^2 - y^2 - bz^2 + b(\lambda a + c)z \\ &\leqslant -\lambda a x^2 - a y^2 - a z^2 + 2a(\lambda a + c)z - a(\lambda a + c)^2 \\ &\quad + (a-b)z^2 + (b-2a)(\lambda a + c)z + a(\lambda a + c)^2 \\ &\leqslant -a\left[\lambda x^2 + y^2 + (z-\lambda a - c)^2\right] + (a-b)z^2 + (b-2a)(\lambda a + c)z + a(\lambda a + c)^2 \\ &\leqslant -a\left(2\bar{V}_\lambda - 2L_\lambda^{(3)}\right). \end{aligned}$$

Hence,

$$\overline{V}_\lambda\left(X(t)\right)-L_\lambda^{(3)}\leqslant\left[\overline{V}_\lambda(X(t_0))-L_\lambda^{(3)}\right]\mathrm{e}^{-2a(t-t_0)}. \quad (17)$$

Further, taking limit on both sides of inequalities (15), (16), and (17) yields

$$\overline{\lim_{t\to+\infty}}\left(\lambda x^2(t)+y^2(t)+(z-\lambda a-c)^2\right)\leqslant\begin{cases}\dfrac{(\lambda a+c)^2b^2}{4(b-1)} & \text{when } a\geqslant 1,\ b\geq 2,\\ (\lambda a+c)^2 & \text{when } a>\dfrac{b}{2},\ b<2,\\ \dfrac{(\lambda a+c)^2b^2}{4a(b-a)} & \text{when } a<1,\ b\geqslant 2a.\end{cases} \quad (18)$$

Remark 3. When $\lambda=1$, formula (18) becomes the estimation (12) in ref. [11]. Our Theorem 3 generalizes $\Omega_\lambda^{(i)}$ to globally exponentially attractive and positive invariant sets, and provides explicit exponential estimations for the convergent rate of trajectories. Obviously, the proof of Theorem 3 is simpler than those by using other methods.

However, the third estimation in formula (18) depends on the parameter a. When $a\to 0^+$, this estimation becomes trivial and does not provide any information. Thus, in the following, we give an improved estimation, which is independent of the a.

Theorem 4. Let $\overline{V}_0=\frac{1}{2}\left[y^2+(z-c)^2\right]$ and $\overline{L}_0=\frac{b^2c^2}{8(b-1)}$. Then, the estimation of the globally exponentially attractive and positive invariant set of the infinitive Lorenz system (1) is

$$\begin{aligned}&\overline{V}_0(X(t))-L_0^{(1)}\leqslant\left[\overline{V}_0(X_0)-L_0^{(1)}\right]\mathrm{e}^{-b(t-t_0)}\leqslant\left[\overline{V}_0(X_0)-\overline{L}_0\right]\mathrm{e}^{-\min(b,a)(t-t_0)},\\ &x^2(t)-L_0\leqslant\left(x_0^2-\overline{L}_0\right)\mathrm{e}^{-a(t-t_0)}\leqslant\left(x_0^2-\overline{L}_0\right)\mathrm{e}^{-\min(b,a)(t-t_0)}.\end{aligned} \quad (19)$$

Especially, the set

$$\overline{\Omega}_0=\left\{X\ \middle|\ \begin{aligned}&y^2+(z-c)^2\leqslant\frac{b^2c^2}{4(b-1)}\\ &x^2\leqslant\frac{b^2c^2}{4(b-1)}\end{aligned}\right\} \quad (20)$$

is the globally exponentially attractive set of system (1).

Proof. Similar to the proofs for formulae (13) and (16), differentiating $\overline{V}_0$ with respect to time t and using the second and third equations of system (1) lead to the conclusion. The details are omitted here for brevity.

Remark 4. The estimations given in formulae (19) and (20) hold uniformly for $a\in(0,\infty)$. When $a\to 0^+$, the estimations are also valid. Also, note that formulae (19) and (20) are more accurate than inequalities (17) and (18), respectively.

4 Applications

Equilibrium points, periodic and almost-periodic solutions are all positive invariant sets. Therefore, as a direct application of the results obtained in the previous sections, we have the following theorem.

Theorem 5. Outside the globally attractive sets of the interval Lorenz systems (1) and (7), there are no bounded positive invariant sets that do not intersect the globally attractive sets.

Proof. By contradiction, suppose Ω is the globally attractive set of system (1) and there is a bounded positive invariant set Q outside the set Ω, and $\Omega \cap Q = \Phi$ (empty set). Thus, we have

$$\inf_{\substack{X \in \Omega \\ \bar{X} \in Q}} \left\| X - \bar{X} \right\| > 0.$$

By the definition of positive invariant set, we have $X(t, t_0, X_0) \in Q$ for $X_0 \in Q$ and $t \geqslant t_0$. Hence,

$$\inf_{\substack{X \in \Omega \\ X(t, t_0, X_0) \in Q \\ t \geqslant t_0}} \left\| X - X(t, t_0, X_0) \right\| > 0.$$

On the other hand, since Ω is the globally attractive set, we have $X(t, t_0, X_0) \to \Omega$ for any $X_0 \in R^3$ as $t \to +\infty$

This implies that

$$\inf_{\substack{X \in \Omega \\ X(t, t_0, X_0) \in Q \\ t \geqslant t_0}} \left\| X - X(t, t_0, X_0) \right\| = 0,$$

leading to a contradiction.

To end the paper, we present another application of the established results in this paper to show that the origin (0, 0, 0) of systems (1) and (7) are globally exponentially stable when $c \leqslant 0$. It is well known that the origin of the Lorenz system (1) is globally asymptotically stable for $0 < c < 1$. Here, however, based on Theorems 1—4, we can easily prove that the origin of system (1) or (7) is globally exponentially stable when $c \leqslant 0$.

Theorem 6. If $c \leqslant 0$ in the Lorenz system (1), and if $c_\alpha \leqslant 0$ in the Lorenz family (7), then the origin (0, 0, 0) of the two systems is also globally exponentially stable.

Proof. When taking $c_\alpha < 0$ in system (7), choose $\lambda = -c_\alpha / a_\alpha$ in Theorem 1. Then, we have $L_{-c_\lambda / a_\lambda} = 0$. Thus, it follows from estimation (9) that

$$-\frac{c_\alpha}{a_\alpha} x^2(t) + y^2(t) + z^2(t) \leqslant \left[-\frac{c_\alpha}{a_\alpha} x^2(t_0) + y^2(t_0) + z^2(t_0) \right] \mathrm{e}^{-2d_\alpha (t - t_0)}.$$

When taking $c_\alpha = 0$ in system (7), we have $L_0 = 0$ in Theorem 2. Thus,

$$y^2(t) + z^2(t) \leqslant \left[y^2(t_0) + z^2(t_0) \right] \mathrm{e}^{-2d_\alpha (t - t_0)} \quad \text{and} \quad x^2(t) \leqslant y^2(t) \leqslant y^2(t_0) \mathrm{e}^{-2d(t - t_0)}.$$

When taking $c < 0$ in system (1), we may choose $\lambda = -c / a$, and thus have $\overline{L}_\lambda^{(1)} = \overline{L}_\lambda^{(2)} = \overline{L}_\lambda^{(3)} = 0$. Therefore,

$$\begin{cases} \overline{V}_\lambda\left(X(t)\right) \leqslant \overline{V}_\lambda(X_0)\ \mathrm{e}^{-2(t - t_0)} & \text{when} \quad a \geq 1,\ b \geq 2, \\ \overline{V}_\lambda\left(X(t)\right) \leqslant \overline{V}_\lambda(X_0)\ \mathrm{e}^{-b(t - t_0)} & \text{when} \quad a > \dfrac{b}{2},\ b < 2, \\ \overline{V}_\lambda\left(X(t)\right) \leqslant \overline{V}_\lambda(X_0)\ \mathrm{e}^{-2a(t - t_0)} & \text{when} \quad 0 < a < 1,\ b \geqslant 2a, \end{cases}$$

where $\overline{V_\lambda}(X(t)) = -(c/a)x^2 + y^2 + z^2$. If taking $c=0$ in system (1), we have $\overline{L_0}=0$ for Theorem 4, which yields

$$y^2(t)+z^2(t) \leqslant \left[y^2(t_0)+z^2(t_0) \right] \mathrm{e}^{-2b(t-t_0)} \quad \text{and} \quad x^2(t) \leqslant y^2(t) \leqslant y^2(t_0)\, \mathrm{e}^{-2b(t-t_0)}.$$

This completes the proof.

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