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Integrability and linearizability of cubic Z_2 systems with non-resonant singular points

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Abstract

In this paper, complete integrability and linearizability of cubic Z_2 systems with two non-resonant and elementary singular points are investigated. First of all, four simple normal forms are obtained based on the coefficients and eigenvalues of cubic Z_2 systems. Then, the integrable and linearizable conditions are classified according to the four different cases respectively, and the problem is solved thoroughly for cubic Z_2 systems with two non-resonant singular points.

Keywords: Complex center; Complex isochronous center; Integrability; Linearizability; Invariant algebraic curve

1. Introduction

Hilbert's 16th problem has been proposed for more than one century, but it is still not completely solved even for the simplest quadratic polynomial systems. As far as the number of limit cycles is concerned, the best results for this problem are $H(2) \geq 4$ and $H(3) \geq 13$, where $H(n)$ denotes the number of limit cycles that a planar differential system with degree n can have, see [1–5]. Bifurcation of limit cycles and symmetry are closely connected and symmetry plays an

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important role in the study of Hilbert's 16th problem. Generally speaking, the basic idea, giving rise to an efficient method, is to perturb the symmetric systems with maximal number of centers.

Consider the following planar system,

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),$$

which is called Z_q-equivariant if it is invariant under a planar counterclockwise rotation of angle $\frac{2\pi}{q}$. As a class of special symmetric systems, Z_q-equivariant systems have been investigated intensively since they were proposed, and better results on the number of limit cycles for polynomial differential systems were often obtained from Z_q-equivariant vector fields, for more detail, see [6–10]. In fact, it is difficult to compute higher-order focal values for an isolated focus. Thus, it is hard to obtain more limit cycles in the neighborhood of an isolated focus based on the calculation of focal values, and very few results have been achieved for higher-order polynomial differential systems. Bifurcation of limit cycles at nilpotent critical points in a class of quintic polynomial differential systems are investigated in [11]. In 2012, some new results were obtained for Z_q-equivariant planar polynomial vector fields [12]. A planar system is Z_2-equivariant if the following conditions hold:

$$X(-x, -y) = -X(x, y), \quad Y(-x, -y) = -Y(x, y).$$

In other words, a Z_2-equivariant planar system can always be written as

$$\frac{dx}{dt} = \sum_{k=0}^{\infty} X_{2k+1}(x, y), \quad \frac{dy}{dt} = \sum_{k=0}^{\infty} Y_{2k+1}(x, y).$$

If a Z_2-equivariant planar cubic system has two isolated elementary foci at $(1, 0)$ and $(-1, 0)$, an example for this system with at least 12 small-amplitude limit cycles was first constructed by Yu and Han [13–15]. Liu and Huang [16] confirmed their conclusion and gave some shortened expressions of the Lyapunov constants for this system. Furthermore, in [17], a class of Z_2 cubic-degree systems,

$$\begin{aligned} \frac{dx}{dt} &= -(a_1 + 1)y + a_1 x^2 y + a_2 x y^2 + a_3 y^3, \\ \frac{dy}{dt} &= -\frac{1}{2}x - a_4 y + \frac{1}{2}x^3 + a_4 x^2 y + a_5 x y^2 + a_6 y^3, \end{aligned} \tag{1.1}$$

was studied, and the first six focus values at $(\pm 1, 0)$ of this system were obtained. Then, eleven center conditions were derived, and a complete study on bi-center problem was carried out. The necessary and sufficient conditions for the existence of bi-center were obtained. Further, study on this system showed $H(3) \geq 13$, see [3–5]. Bi-center problem for some Z_2-equivariant quintic systems was studied in [18], and simultaneous existence of centers for two families of planar Z_q-equivariant systems was investigated in [19]. In 2017, the bi-isochronous center problem for a cubic Z_2-equivariant vector field with real coefficients was considered in [20], and two real isochronous center conditions were obtained. For the complex isochronous center problem at $(\pm 1, 0)$ of system (1.1), there are two difficulties encountered in solving this problem: the first one is the computation of periodic constants, and the second one is to find all linearization transformations. Recently, 54 complex linearization centers for system (1.1) were obtained in [21].

In [22], bi-center problem and bifurcation of limit cycles from nilpotent singular points in Z_2-equivariant cubic vector fields were studied, and sufficient and necessary conditions were obtained for two nilpotent singular points of the system to be centers. Moreover, a new perturbation scheme was present in [22] to prove the existence of 12 small-amplitude limit cycles in cubic Z_2-equivariant vector fields, which bifurcate from two nilpotent singular points. The center problem for Z_2-symmetric nilpotent vector fields was also studied in [23].

In this paper, integrability and linearizability of cubic Z_2-equivariant systems with non-resonant and elementary singular points will be investigated. The rest of the paper is organized as follows. In the next section, we present some preliminary results which will be used in the following sections. In Section 3, four simple normal forms are classified based on the coefficients and eigenvalues of cubic Z_2 systems. Section 4 is devoted to study the integrability and linearizability for the four cases. At last, conclusion is drawn in Section 5.

2. Preliminary results

In this section, we present some preliminary results taken from [24], which will be used in next section. Consider the following system,

$$\begin{aligned}\frac{dz}{dT} &= z + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} z^{\alpha} w^{\beta} = Z(z,w),\\ \frac{dw}{dT} &= -w - \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} w^{\alpha} z^{\beta} = -W(z,w),\end{aligned} \tag{2.1}$$

which can be changed into

$$\begin{aligned}\frac{dx}{dt} &= -y + \sum_{\alpha+\beta=2}^{\infty} A_{\alpha\beta} x^{\alpha} y^{\beta} = X(x,y),\\ \frac{dy}{dt} &= x + \sum_{\alpha+\beta=2}^{\infty} B_{\alpha\beta} x^{\alpha} y^{\beta} = Y(x,y),\end{aligned} \tag{2.2}$$

by the following complex transformation,

$$z = x + iy,\quad w = x - iy,\quad T = it,\quad i = \sqrt{-1}.$$

Systems (2.1) and (2.2) are called adjoint systems. The origin of system (2.1) is called a weak saddle in complex domain, and the origin of system (2.2) is called a complex focus (center).

Moreover, with the transformation,

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

system (2.1) can be transformed into

$$\frac{dr}{dt} = i\frac{wZ - zW}{2r} = R(r,\theta), \quad \frac{d\theta}{dt} = \frac{wZ + zW}{2zw} = \Theta(r,\theta), \tag{2.3}$$

where

$$R(r,\theta) = \frac{ir}{2}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k+1}^{\infty}\left[a_{\alpha\beta}e^{i(\alpha-\beta-1)\theta} - b_{\alpha\beta}e^{-i(\alpha-\beta-1)\theta}\right]r^k,$$

$$\Theta(r,\theta) = 1 + \frac{1}{2}\sum_{k=1}^{\infty}\sum_{\alpha+\beta=k+1}^{\infty}\left[a_{\alpha\beta}e^{i(\alpha-\beta-1)\theta} + b_{\alpha\beta}e^{-i(\alpha-\beta-1)\theta}\right]r^k.$$

For sufficiently small constant h, the solution of system (2.3) with initial condition $r|_{\theta=0} = h$ can be written as

$$r = \tilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} \nu_k(\theta)h^k,$$

and denote

$$\mathcal{T}(\vartheta, h) = \int_0^{\vartheta} \frac{d\theta}{\Theta(\tilde{r}(\theta,h),\theta)}.$$

In [24], the definitions of complex center and isochronous center of system (2.1) were given as follows.

Definition 2.1. For sufficiently small constant h, if

$$\tilde{r}(2\pi, h) \equiv h,$$

then the origin of system (2.1) is called a complex center.

If the origin of system (2.1) is a complex center and

$$\mathcal{T}(2\pi, h) \equiv 2\pi,$$

then the origin of system (2.1) is called a complex isochronous center.

Studying the center problem of system (2.1) is equivalent to considering the integrable problem in the neighborhood of the origin. Similarly, the isochronous center problem of system (2.1) is equivalent to its linearizable problem in the neighborhood of the origin. However, for a concrete system, it is difficult to find its first integral and the linearization transformation. An efficient method to prove the sufficiency of integrable and linearizable condition is to find invariant curves. For some special systems, the following theorems can be used to determine the complex isochronous center.

Theorem 2.1. *If there exists a regular integral in the neighborhood of the origin of system* (2.1) *and $Z(z,w) = z\varphi(z)$, where $\varphi(z)$ is analytic in the neighborhood of $z = 0$ and $\varphi(0) = 1$, then the origin of system* (2.1) *is a complex isochronous center.*

Proof. When the conditions hold, system (2.1) can be rewritten as

$$\frac{dz}{dT} = z\varphi(z), \quad \frac{dw}{dT} = -W(z, w). \tag{2.4}$$

Because $z = 0$ is a solution of (2.4), in the neighborhood of the origin, system (2.4) has a first integral

$$F(z, w) = zG(z, w),$$

where $G(z, w) = w + \text{h.o.t.}$ is analytic in the neighborhood of the origin. $\frac{dF}{dT} = 0$ shows that

$$\frac{dG}{dT} = -\varphi G. \tag{2.5}$$

Based on (2.5), it is easy to check that there exists a linearization transformation of system (2.4)

$$\xi = z \exp \int_0^z \frac{1 - \varphi(z)}{z\varphi(z)} dz, \quad \eta = G(z, w) \exp \int_0^z \frac{\varphi(z) - 1}{z\varphi(z)} dz$$

in the neighborhood of the origin, so the conclusion holds. □

Similarly, we have

Theorem 2.2. *If there exists a regular integral in the neighborhood of the origin of system (2.1) and $W(z, w) = w\psi(w)$, where $\psi(w)$ is analytic in the neighborhood of $w = 0$ and $\psi(0) = 1$, then the origin of system (2.1) is a complex isochronous center.*

Theorem 2.3. *Suppose that the origin of system (2.1) is a complex center and there exists an analytic function in the neighborhood of the origin, given by*

$$\eta = \psi(z, w) = w + \text{h.o.t.},$$

which satisfies

$$\frac{d\eta}{dT} = -\eta,$$

then the origin of system (2.1) is a complex isochronous center.

Similarly, we have the following theorem.

Theorem 2.4. *Suppose that the origin of system (2.1) is a complex center and there exists an analytic function in the neighborhood of the origin,*

$$\xi = \varphi(z, w) = z + \text{h.o.t.},$$

which satisfies

$$\frac{d\xi}{dT} = \xi,$$

then the origin of system (2.1) *is a complex isochronous center.*

Now, consider the following autonomous complex systems,

$$\frac{dz}{dT} = z + \text{h.o.t.}, \quad \frac{dw}{dT} = -wf(w), \tag{2.6}$$

and

$$\frac{dz}{dT} = zg(z), \quad \frac{dw}{dT} = -w + \text{h.o.t.}, \tag{2.7}$$

where $f(w)$ and $g(z)$ are power series with non-zero convergence radius and $f(0) = g(0) = 1$. The functions on the right-hand side of the above differential equations are assumed to be analytic in the neighborhood of the origin. Then the following results directly follow from Theorems 2.3 and 2.4.

Corollary 2.1. *If the origin of system* (2.6) *is a complex center, then it is a complex isochronous center.*

Corollary 2.2. *If the origin of system* (2.7) *is a complex center, then it is a complex isochronous center.*

3. Normal forms of Z_2-equivariant cubic systems

Consider the cubic Z_2-equivariant system,

$$\begin{aligned} \frac{dz}{dT} &= a_{10}z + a_{01}w + a_{30}z^3 + a_{21}z^2w + a_{12}zw^2 + a_{03}w^3 = Z(z,w), \\ \frac{dw}{dT} &= -b_{10}w - b_{01}z - b_{30}w^3 - b_{21}w^2z - b_{12}wz^2 - b_{03}z^3 = -W(z,w), \end{aligned} \tag{3.1}$$

where z, w, T are complex variables and a_{kj}, b_{kj} are complex coefficients.

Suppose that the functions on the right-hand side of system (3.1) have no common factors. Let (z_0, w_0), which is not the origin, be an isolated singular point of system (3.1), then $(-z_0, -w_0)$ is also an isolated singular point of system because system (3.1) is Z_2 equivariant.

Without loss of generality, let $z_0 \neq 0,\ w_0 \neq 0$ (otherwise, let $z = z' + w', w = z' - w'$). Further, let $z_0 = w_0 = 1$ (or let $z = z'z_0,\ w = w'w_0$). Then,

$$\begin{aligned} a_{01} &= -(a_{10} + a_{30} + a_{21} + a_{12} + a_{03}), \\ b_{01} &= -(b_{10} + b_{30} + b_{21} + b_{12} + b_{03}). \end{aligned} \tag{3.2}$$

The Jacobian of system (3.1) evaluated at $(\pm 1, \pm 1)$ is

$$
\begin{aligned}
J_0 &= \begin{pmatrix} \dfrac{\partial Z}{\partial z} & \dfrac{\partial Z}{\partial w} \\ -\dfrac{\partial W}{\partial z} & -\dfrac{\partial W}{\partial w} \end{pmatrix}_{(\pm 1,\ \pm 1)} \\
&= \begin{pmatrix} a_{10}+3a_{30}+2a_{21}+a_{12} & -a_{10}-a_{30}+a_{12}+2a_{03} \\ b_{10}+b_{30}-b_{12}-2b_{03} & -b_{10}-3b_{30}-2b_{21}-b_{12} \end{pmatrix}.
\end{aligned}
$$

In order to reduce the difficulties in the following analysis, we discuss how to transform system (3.1) to a simple new system by linear transformations. In the new system, the Jacobian evaluated at the two singular points become simple and the number of parameters is reduced.

Denote

$$
\begin{aligned}
A &= a_{30}+a_{21}+a_{12}+a_{03}, \\
B &= b_{30}+b_{21}+b_{12}+b_{03}.
\end{aligned}
$$

Suppose the two eigenvalues of the Jacobian J_0 are λ_1 and λ_2. The two singular points are elementary which yields that $\lambda_1\lambda_2 \neq 0$. Then, there are four cases for system (3.1):

$$
\begin{aligned}
&\text{Case } 1: \quad A+B \neq 0, \quad \lambda_1 \neq \lambda_2; \\
&\text{Case } 2: \quad A+B = 0, \quad \lambda_1 \neq \lambda_2; \\
&\text{Case } 3: \quad A+B \neq 0, \quad \lambda_1 = \lambda_2 = \lambda; \\
&\text{Case } 4: \quad A+B = 0, \quad \lambda_1 = \lambda_2 = \lambda.
\end{aligned}
$$

3.1. Case 1*:* $A+B \neq 0$*,* $\lambda_1 \neq \lambda_2$

When $A+B \neq 0$, by using

$$
\operatorname{trace}(J_0) = \lambda_1 + \lambda_2, \quad \det(J_0) = \lambda_1\lambda_2,
$$

we get

$$
\begin{aligned}
a_{10} &= -\frac{(2A-\lambda_2)(2A-\lambda_1)}{2(A+B)} + (2a_{03}+a_{12}-a_{30}), \\
b_{10} &= -\frac{(2B+\lambda_2)(2B+\lambda_1)}{2(A+B)} + (2b_{03}+b_{12}-b_{30}).
\end{aligned}
\tag{3.3}
$$

Then, introducing the linear transformation

$$
\begin{aligned}
u &= z - \frac{2A-\lambda_1}{2(A+B)}(z-w), \\
v &= w + \frac{2B+\lambda_2}{2(A+B)}(z-w),
\end{aligned}
\tag{3.4}
$$

into system (3.1) yields the following system,

$$
\begin{aligned}
\frac{dz}{dT} &= -\frac{1}{4}\lambda_1(z+w) + \frac{1}{2}(b_1+b_4+\lambda_1)(z-w) \\
&\quad + \frac{1}{16}\lambda_1(z+w)^3 - \frac{1}{8}(b_1+b_4)(z+w)^2(z-w) \\
&\quad + \frac{1}{8}(b_2-b_5)(z+w)(z-w)^2 + \frac{1}{8}(b_3+b_6)(z-w)^3, \\
\frac{dw}{dT} &= -\frac{1}{4}\lambda_2(z+w) + \frac{1}{2}(b_1-b_4-\lambda_2)(z-w) \\
&\quad + \frac{1}{16}\lambda_2(z+w)^3 - \frac{1}{8}(b_1-b_4)(z+w)^2(z-w) \\
&\quad + \frac{1}{8}(b_2+b_5)(z+w)(z-w)^2 + \frac{1}{8}(b_3-b_6)(z-w)^3.
\end{aligned} \tag{3.5}
$$

It is not difficult to verify that $(\pm 1, \pm 1)$ are two isolated singular points of system (3.5). Because the determinant of the Jacobian of transformation (3.4) is given by

$$
\det \begin{bmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial w} \end{bmatrix} = \frac{\lambda_1 - \lambda_2}{2(A+B)} \neq 0,
$$

indicating that the transformation (3.4) is non-degenerate, and so the Jacobian at $(\pm 1, \pm 1)$ becomes

$$
J_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.
$$

Remark 3.1. When $\lambda_1 = 1, \lambda_2 = -1$, the transformation (3.4) was used in [17] and [21] where cubic Z_2 equivariant systems with two real focus were studied.

Then we have the following result.

Lemma 3.1. *Suppose* $(\pm 1, \pm 1)$ *are isolated singular points of system* (3.1)*, and the two eigenvalues of the Jacobian evaluated at the two singular points are* λ_1 *and* λ_2*, and* $(A+B)(\lambda_1 - \lambda_2) \neq 0$*. Then, the Jacobian* J_0 *at the singular points* $(\pm 1, \pm 1)$ *is given by* J_1.

By (3.2) and $J_0 = J_1$, the number of parameters in (3.1) is reduced by half, from 12 to 6. The following result directly follows from Lemma 3.1.

Theorem 3.1. *Suppose* $(\pm 1, \pm 1)$ *are isolated singular points of system* (3.1)*, and the two eigenvalues of the Jacobian at the two singular points are* λ_1 *and* λ_2*, and* $(A+B)(\lambda_1 - \lambda_2) \neq 0$*, then system* (3.1) *can be transformed into system* (3.5) *which has only* 6 *independent parameters. (The variables* z *and* w *are still used for simplicity.)*

In fact, system (3.5) can be further simplified.

Theorem 3.2. *By*

$$z = x + iy, \quad w = x - iy, \quad T = it, \tag{3.6}$$

system (3.5) *can be changed into*

$$\begin{aligned} \frac{dx}{dt} &= -\frac{\delta}{2}x - (a_1 + \mu)y + \frac{\delta}{2}x^3 + a_1x^2y + a_2xy^2 + a_3y^3, \\ \frac{dy}{dt} &= -\frac{\mu}{2}x + (\delta - a_4)y + \frac{\mu}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3, \end{aligned} \tag{3.7}$$

where the coefficients of system (3.5) *and* (3.7) *have the following relations*

$$\begin{array}{lll} \delta = \frac{i}{2}(\lambda_1 + \lambda_2), & \mu = \frac{1}{2}(\lambda_1 - \lambda_2), & \\ a_1 = b_1, & a_3 = b_3, & a_5 = b_5, \\ a_2 = -ib_2, & a_4 = -ib_4, & a_6 = -ib_6. \end{array}$$

The Jacobian of system (3.7) evaluated at $(\pm 1, 0)$ is

$$J_1^* = \begin{pmatrix} \delta & -\mu \\ \mu & \delta \end{pmatrix}.$$

Remark 3.2. The $(\pm 1, 0)$ of system (3.7) are strong foci if $\delta \neq 0$, and weak foci when $\lambda_1 = -\lambda_2$. The integrability conditions of system (3.7) were completely obtained in [17].

3.2. Case 2: $A + B = 0$, $\lambda_1 \neq \lambda_2$

Suppose $(\pm 1, \pm 1)$ are isolated singular points of system (3.1), and $A + B = 0$, $\lambda_1 \neq \lambda_2$, then (3.2) holds. From $A + B = 0$ and (3.3), we have

$$\begin{aligned} b_{10} &= a_{10} + a_{12} + 2a_{21} + 3a_{30} - b_{12} - 2b_{21} - 3b_{30} - \lambda_1 - \lambda_2, \\ b_{03} &= \frac{1}{2}(-2b_{12} - 2b_{21} - 2b_{30} - \lambda_1), \\ a_{03} &= \frac{1}{2}(-2a_{12} - 2a_{21} - 2a_{30} + \lambda_1). \end{aligned}$$

Then, system (3.1) can be transformed into the following system,

$$\begin{aligned} \frac{dz}{dT} =& -\frac{1}{2}\lambda_1 z - a_1(z - w) + \frac{1}{2}\lambda_1 z^3 \\ &+ a_1z^2(z - w) + a_2z(z - w)^2 + a_3(z - w)^3, \\ \frac{dw}{dT} =& -\frac{1}{2}\lambda_1 z - (a_1 - a_4 + \lambda_2)(z - w) + \frac{1}{2}\lambda_1 z^3 \\ &+ (a_1 - a_4)z^2(z - w) + (a_2 - a_5)z(z - w)^2 \\ &+ (a_3 - a_6)(z - w)^3, \end{aligned} \tag{3.8}$$

by the following transformation,

$$
\begin{aligned}
u &= z + \frac{a_{10} + a_{12} + 2a_{21} + 3a_{30} - \lambda_1}{\lambda_1 - \lambda_2}(z - w), \\
v &= w + \frac{a_{10} + a_{12} + 2a_{21} + 3a_{30} - \lambda_1}{\lambda_1 - \lambda_2}(z - w).
\end{aligned} \tag{3.9}
$$

Because the determinant of the Jacobian of transformation (3.9) is

$$
\det \begin{bmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial w} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial w} \end{bmatrix} = 1,
$$

implying that the transformation (3.9) is non-degenerate. Obviously, $(\pm 1, \pm 1)$ are isolated elementary singular points of system (3.8), and the Jacobian of (3.8) evaluated at the singular points $(\pm 1, \pm 1)$ are

$$
J_2 = \begin{pmatrix} \lambda_1 & 0 \\ \lambda_1 - \lambda_2 & \lambda_2 \end{pmatrix}.
$$

So we have

Lemma 3.2. *Suppose* $(\pm 1, \pm 1)$ *are isolated singular points of system* (3.1)*, and the two eigenvalues of the Jacobian of the system evaluated at the two singular points are* λ_1 *and* λ_2*, with* $A + B = 0$, $\lambda_1 \neq \lambda_2$*. Then, the Jacobian* J_0 *at singular points* $(\pm 1, \pm 1)$ *is given by* J_2*.*

Based on Lemma 3.2, a similar proof to that for Theorem 3.1 leads to the following theorem.

Theorem 3.3. *Suppose* $(\pm 1, \pm 1)$ *are isolated singular points of system* (3.1)*, and the two eigenvalues of the Jacobian evaluated at the two singular points are* λ_1 *and* λ_2*, with* $A + B = 0$*,* $\lambda_1 \neq \lambda_2$*, then system* (3.1) *can be transformed into system* (3.8) *which has only* 6 *independent parameters.*

As a matter of fact, system (3.8) can be further simplified.

Theorem 3.4. *System* (3.8) *can be changed to*

$$
\begin{aligned}
\frac{dx}{dT} &= -\frac{1}{2}\lambda_1 x - a_1 y + \frac{1}{2}\lambda_1 x^3 + a_1 x^2 y + a_2 x y^2 + a_3 y^3, \\
\frac{dy}{dT} &= (\lambda_2 - a_4) y + a_4 x^2 y + a_5 x y^2 + a_6 y^3,
\end{aligned} \tag{3.10}
$$

by

$$
x = z, \quad y = z - w. \tag{3.11}
$$

The Jacobian of system (3.10) evaluated at $(\pm 1, 0)$ becomes

$$J_2^* = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

3.3. *Case* 3: $A + B \neq 0,\ \lambda_1 = \lambda_2$

Suppose $\lambda_1 = \lambda_2 = \lambda$, when $A + B \neq 0$, (3.3) becomes

$$a_{10} = -\frac{(2A-\lambda)^2}{2(A+B)} + (2a_{03} + a_{12} - a_{30}),$$
$$b_{10} = -\frac{(2B+\lambda)^2}{2(A+B)} + (2b_{03} + b_{12} - b_{30}).$$

Consider the following transformation

$$u = z - \frac{2A-\lambda}{2(A+B)}(z-w), \quad v = \frac{1}{2(A+B)}(z-w), \tag{3.12}$$

which is non-degenerate because the determinant of the Jacobian of transformation (3.12) is

$$\det \begin{bmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial w} \end{bmatrix} = \frac{-1}{2(A+B)} \neq 0.$$

It is easy to prove the following theorem.

Theorem 3.5. *Suppose* $(\pm 1, \pm 1)$ *are isolated singular points of system* (3.1)*, and the two eigenvalues of the Jacobian evaluated at the two singular points are* $\lambda_1 = \lambda_2 = \lambda$*, with* $A + B \neq 0$*, then system* (3.1) *can be transformed into the following form,*

$$\begin{aligned} \frac{du}{dT} &= -\frac{1}{2}\lambda u - a_1 v + \frac{1}{2}\lambda u^3 + a_1 u^2 v + a_2 u v^2 + a_3 v^3, \\ \frac{dv}{dT} &= -\frac{1}{2}u + (\lambda - a_4)v + \frac{1}{2}u^3 + a_4 u^2 v + a_5 u v^2 + a_6 v^3, \end{aligned} \tag{3.13}$$

by the transformation (3.12)*, which has only* 6 *independent parameters. Moreover,* $(\pm 1, 0)$ *are isolated elementary singular points of system* (3.13)*, and the Jacobian evaluated at the two singular points* $(\pm 1, 0)$ *is*

$$J_3 = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}.$$

Remark 3.3. When $\lambda \neq 0$, the two singular points $(\pm 1, 0)$ of system (3.13) are elementary node points, they are integrable and linearizable. When $\lambda = 0$, system (3.13) has been studied in [22], with six integrability conditions obtained, three of them are center conditions of system (3.13).

3.4. *Case* 4*:* $A+B=0,\ \lambda_1=\lambda_2$

Suppose $A+B=0,\ \lambda_1=\lambda_2=\lambda$. Denote

$$s=-2b_{03}+b_{10}-b_{12}+b_{30},$$

and

$$\operatorname{trace}(J_0)=2\lambda,\quad \det(J_0)=\lambda^2,$$

which yield that

$$\begin{aligned}
a_{10}&=-a_{12}-2a_{21}-3a_{30}+\lambda+s,\\
b_{10}&=-b_{12}-2b_{21}-3b_{30}-\lambda+s,\\
a_{03}&=-\frac{1}{2}(2a_{12}+2a_{21}+2a_{30}-\lambda),\\
b_{03}&=-\frac{1}{2}(2b_{12}+2b_{21}+2b_{30}+\lambda).
\end{aligned}$$

So the Jacobian of system (3.1) evaluated at $(\pm1,\pm1)$ can be written as

$$J_0=\begin{pmatrix}\lambda+s & -s\\ s & \lambda-s\end{pmatrix}.$$

Obviously, if $s=0$, $J_0=\lambda E$. System (3.1) is a special case of system (3.5) when $\lambda_1=\lambda_2=\lambda$. If $s\neq0$, we consider the following transformation

$$u=z,\quad v=s(z-w), \tag{3.14}$$

which yields the following result.

Theorem 3.6. *Suppose* $(\pm1,\pm1)$ *are isolated singular points of system* (3.1)*, and the two eigenvalues of the Jacobian evaluated at the two singular points are* $\lambda_1=\lambda_2=\lambda$*, with* $A+B=0$*. Then, when* $s\neq0$*, system* (3.1) *can be transformed into the following system,*

$$\begin{aligned}
\frac{du}{dT}&=-\frac{1}{2}\lambda u+(1-a_1)v+\frac{1}{2}\lambda u^3+a_1u^2v+a_2uv^2+a_3v^3,\\
\frac{dv}{dT}&=(\lambda-a_4)v+a_4u^2v+a_5uv^2+a_6v^3,
\end{aligned} \tag{3.15}$$

by the transformation (3.14)*, which has only* 6 *independent parameters.*

It is easy to verify that the transformation (3.14) is non-degenerate because the determinant of the Jacobian of transformation (3.14) is

$$\det\begin{bmatrix} \dfrac{\partial u}{\partial z} & \dfrac{\partial u}{\partial w} \\ \dfrac{\partial v}{\partial z} & \dfrac{\partial v}{\partial w} \end{bmatrix} = -s \neq 0.$$

Obviously, $(\pm 1, 0)$ are isolate elementary singular points of system (3.15), and the Jacobian evaluated at the singular points $(\pm 1, 0)$ is

$$J_4 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Remark 3.4. When $\lambda = 0$, $(\pm 1, 0)$ are not isolate singular points in system (3.15) which yields that $\lambda \neq 0$ in system (3.15), so systems (3.13) and (3.15) can not be transformed with each other.

4. The integrability and linearizability of cubic Z_2 systems with non-resonant singular points

Now, we consider the Z_2-equivariant cubic system (3.1) with two finite singular points and the two eigenvalues of Jacobian of the system evaluated at the two singular points $(\pm 1, 0)$ satisfy $\frac{\lambda_1}{\lambda_2} = -1$. Without loss of generality, let $\lambda_1 = 1$, $\lambda_2 = -1$. There are two types of systems: one is $(3.5)|_{\lambda_1=1,\lambda_2=-1}$ and the other $(3.10)|_{\lambda_1=1,\lambda_2=-1}$. The system $(3.5)|_{\lambda_1=1,\lambda_2=-1}$ can be changed into $(3.7)|_{\delta=0,\mu=1}$ by the transformation (3.6). The center problem of system $(3.7)|_{\delta=0,\mu=1}$ has been solved completely in [17], while the linearizability problem of this system in real domain was investigated in [20], and two isochronous center conditions were obtained. Recently, the linearizability problem of system $(3.7)|_{\delta=0,\mu=1}$ in complex domain was studied in [21], and 54 complex isochronous center conditions were obtained.

So, the integrability and linearizability problem for Cases 1 and 3 has been completely solved, while the problem for Cases 2 and 4 are still open. In the following, we first summarize the results for Cases 1 and 3 for convenience, and then study the two open cases.

Theorem 4.1. [21] *The origin of system* (3.5) *is a complex center if and only if one of the following* 11 *conditions holds:*

$$(C_1):\ b_1 = -b_5, \quad b_4 = 0, \quad b_6 = \frac{-1}{3} b_2;$$

$$(C_2):\ b_2 = 0, \quad b_4 = 0, \quad b_6 = 0;$$

$$\begin{aligned}(C_3):\ & 3(b_1 + b_5)(2 + 2b_1 - b_3 + 2b_5 + 2b_1 b_5) \\ & \quad + 2b_4(2b_2 + b_1 b_2 + 2b_4 + b_2 b_5 + 2b_4 b_5) = 0, \\ & b_6 - \frac{1}{3}(-b_2 - 2b_1 b_2 + 2b_4 - 2b_2 b_5 + 2b_4 b_5) = 0, \\ & 2(1 + b_1)(b_1 + b_5)^2 + b_4^2(1 + 2b_1 + 2b_5) = 0, \quad b_1 + b_5 \neq 0;\end{aligned}$$

$$\begin{aligned}(C_4):\ & -2b_4(1 + b_5) - b_2(2 + b_1 + b_5) = 0, \quad b_3 - 2(1 + b_1)(1 + b_5) = 0, \\ & b_6 - \frac{1}{3}(-b_2 - 2b_1 b_2 + 2b_4 - 2b_2 b_5 + 2b_4 b_5) = 0;\end{aligned}$$

$$
\begin{aligned}
&(C_5): \ b_1 = \frac{-1}{2}(2+3b_4^2), \ \ b_2 = b_4, \ \ b_3 = -b_4^2(1+b_4^2+b_5), \ \ b_6 = b_4(1+b_4^2);\\
&(C_6): \ b_1 = \frac{-1}{8}(8+5b_4^2), \quad b_2 = \frac{1}{2}b_4, \quad b_3 = \frac{-5}{32}b_4^4,\\
&\qquad b_5 = \frac{1}{8}(-8+b_4^2), \qquad b_6 = \frac{1}{4}b_4(2+b_4^2);\\
&(C_7): \ b_1 = \frac{1}{32}(-32+15b_4^2), \quad b_2 = \frac{1}{4}b_4, \quad b_3 = \frac{-1}{512}b_4^2(64-15b_4^2),\\
&\qquad b_5 = \frac{1}{32}(-96+17b_4^2), \quad b_6 = \frac{-3}{16}b_4(4-b_4^2);\\
&(C_8): \ b_1 = \frac{1}{50}(-50+21b_4^2), \quad b_2 = \frac{1}{5}b_4, \quad b_3 = \frac{-1}{1250}b_4^2(250-63b_4^2),\\
&\qquad b_5 = \frac{1}{50}(-200+39b_4^2), \quad b_6 = \frac{-1}{25}b_4(35-9b_4^2);\\
&(C_9): \ b_1 = \frac{-1}{2}(2+3b_4^2), \ \ b_2 = b_4, \ \ b_3 = -b_4^2(1+b_4^2+b_5), \ \ b_6 = b_4(1+b_4^2);\\
&(C_{10}): \ b_1 = \frac{1}{8}(-8+3b_4^2), \quad b_2 = \frac{-1}{2}b_4,\\
&\qquad b_3 = \frac{-3}{16}b_4^2(4-b_4^2+4b_5), \quad b_6 = \frac{1}{8}b_4(4+b_4^2+8b_5);\\
&(C_{11}): \ b_1 = \frac{-1}{32}(32-15b_4^2), \quad b_2 = \frac{-1}{4}b_4, \quad b_3 = \frac{-1}{512}b_4^2(832-495b_4^2),\\
&\qquad b_5 = \frac{1}{32}(160-111b_4^2), \quad b_6 = \frac{1}{16}b_4(76-45b_4^2).
\end{aligned} \tag{4.1}
$$

Theorem 4.2. *The origin of system* (3.5) *is a real isochronous center if and only if one of the following two conditions holds:*

$$
\begin{aligned}
&L_1: \quad b_1 = -3, \quad b_2 = 0, \quad b_3 = 0, \quad b_4 = 0, \quad b_5 = -9, \quad b_6 = 0;\\
&L_2: \quad b_1 = -\frac{3}{2}, \quad b_2 = 0, \quad b_3 = \frac{1}{2}, \quad b_4 = 0, \quad b_5 = -\frac{3}{2}, \quad b_6 = 0.
\end{aligned}
$$

Furthermore, complex linearizability conditions of system (3.5) were also obtained in [21].

For Case 3, it is easy to know that the two singular points $(\pm 1, 0)$ are degenerate nodes if $\lambda \neq 0$, and they are integrable and linearizable. When $\lambda = 0$, the integrability problem has been solved in [22], with the integrability conditions given below.

Theorem 4.3. [22] *The two singular points* $(\pm 1, 0)$ *of system* (3.13) *are integrable if and only if one of the following conditions holds:*

$$\mathrm{I}_1: \ A_{21} = -B_{12}, \ A_{30} = \frac{-2}{3}A_{12}B_{12}, \ A_{12}^2 = \frac{9}{4B_{12}(3+2B_{12}^2)};$$

$$\mathrm{I}_2: \ A_{30} = 0, \ A_{12} = 0;$$

$$\mathrm{I}_3: \ A_{21} = \frac{-9}{4A_{12}^2}, \ A_{30} = \frac{-3}{2A_{12}}, \ B_{12} = \frac{-4A_{12}^2}{9};$$

$$\mathrm{I}_4: \ B_{12} = \frac{-3A_{12}^2}{8}, \ A_{30} = -\frac{16+3A_{12}^4}{12A_{12}}, \ A_{21} = -\frac{32+3A_{12}^4}{24A_{12}^2};$$

$$\mathrm{I}_5: \ A_{21} = \frac{2-A_{12}^4}{2A_{12}^2}, \ A_{30} = -A_{12}^3, \ B_{12} = \frac{3}{2}A_{12}^2.$$

For case 2, when $\lambda_1\lambda_2 > 0$, the singular points $(\pm 1, 0)$ are node points, they are integrable and linearizable. When $\lambda_1\lambda_2 < 0$ and $\lambda_1 : \lambda_2 \neq 1 : -1$, the singular points $(\pm 1, 0)$ are strong saddle points, they are also integrable and linearizable. When $\lambda_1\lambda_2 = 0$, the singular points $(\pm 1, 0)$ are degenerate singular points, we do not consider them here. The difficult case about the integrability and linearizability of system (3.8) is $\lambda_1 = 1, \lambda_2 = -1$, namely, the singular points $(\pm 1, \pm 1)$ are weak saddle points. Now, we study the integrability and linearizability of system $(3.8)|_{\lambda_1=1,\lambda_2=-1}$.

4.1. Saddle quantities of a class of Z_2-equivariant cubic system with two weak saddle points

When $\lambda_1 = 1, \lambda_2 = -1$, system (3.8) becomes

$$\begin{aligned}
\frac{dz}{dT} =& -\frac{1}{2}z - a_1(z-w) + \frac{1}{2}z^3 + a_1 z^2(z-w) \\
&+ a_2 z(z-w)^2 + a_3(z-w)^3, \\
\frac{dw}{dT} =& -\frac{1}{2}z - (a_1 - a_4 - 1)(z-w) + \frac{1}{2}z^3 + (a_1 - a_4)z^2(z-w) \\
&+ (a_2 - a_5)z(z-w)^2 + (a_3 - a_6)(z-w)^3,
\end{aligned}$$

which can be further transformed into the form,

$$\begin{aligned}
\frac{dx}{dT} &= -\frac{1}{2}x - a_1 y + \frac{1}{2}x^3 + a_1 x^2 y + a_2 x y^2 + a_3 y^3, \\
\frac{dy}{dT} &= -(1+a_4)y + a_4 x^2 y + a_5 x y^2 + a_6 y^3,
\end{aligned} \tag{4.2}$$

by the transformation (3.11).

Next, in order to study its integrability conditions of system (4.2), we compute the saddle values at $(\pm 1, 0)$. Let

$$u = x - 1, \quad v = y,$$

under which system (4.2) becomes

$$
\begin{aligned}
\frac{du}{dT} &= u + \frac{3}{2}u^2 + 2a_1uv + a_2v^2 + \frac{1}{2}u^3 + a_1u^2v + a_2uv^2 + a_3v^3, \\
\frac{dv}{dT} &= -v + 2a_4uv + a_5v^2 + a_4u^2v + a_5uv^2 + a_6v^3.
\end{aligned}
\tag{4.3}
$$

Then the singular points $(\pm 1, 0)$ of system (4.2) have been shifted to the singular point $(0, 0)$ of system (4.1).

Theorem 4.4. *The first seven saddle values at the origin of system* (4.3) *are*

$$
\begin{aligned}
\mu_1 &= -2a_1 + a_5 + 2a_4a_5, \\
\mu_2 &= -\frac{4}{3}a_4(2a_2 + 2a_2a_4 - 3a_6)g_1, \\
\mu_3 &= \frac{1}{24}(3a_3 - 108a_3a_4^2 - 7a_2a_5 - 10a_2a_4a_5 - 36a_2a_4^2a_5 \\
&\quad + 72a_2a_4^3a_5 + 12a_5a_6)g_1g_2, \\
\mu_4 &= \frac{32}{81}a_4a_5(3 + 4a_4)g_1g_2g_3, \\
\mu_5 &= \frac{7}{419904}(1458a_2 + 3662a_2a_4 + 24048a_2a_4^2 + 17496a_5^2 + 2187a_6)g_1g_2g_3, \\
\mu_6 &= 0, \\
\mu_7 &= \frac{165}{4096}a_5(-a_3 + 5a_2a_5)g_1g_2g_3,
\end{aligned}
$$

where

$$
g_1 = 1 + 2a_4, \quad g_2 = 3 + 2a_4, \quad g_3 = -3a_3 - a_2a_5 + 2a_2a_4a_5.
$$

The following two theorems directly follow Theorem 4.4.

Theorem 4.5. *The origin of system* (4.3) *is a seventh order weak saddle if and only if*

$$
\begin{aligned}
&a_1 = \frac{1}{2}a_5, \quad a_2 = -\frac{3}{2}(8a_5^2 + a_6), \quad a_3 = -\frac{1}{2}a_5(56a_5^2 + 15a_6), \\
&a_4 = 0, \qquad a_5(4a_5^2 + a_6) \neq 0.
\end{aligned}
$$

Theorem 4.6. *The first seven saddle values at the origin of system* (4.3) *are all zero if and only if one of the following six conditions holds:*

$$
\begin{aligned}
&C_1: \ a_1 = 0, \ \ a_4 = -\frac{1}{2}; \\
&C_2: \ a_1 = -a_5, \ a_2 = -3a_6, \ a_4 = -\frac{3}{2}; \\
&C_3: \ a_1 = \frac{1}{2}(1 + 2a_4)a_5, \ a_3 = -\frac{1}{3}a_2(1 - 2a_4)a_5, \ a_6 = \frac{2}{3}a_2(1 + a_4);
\end{aligned}
$$

$$C_4:\ a_1=0,\ a_3=0,\ a_4=0,\ a_5=0;$$

$$C_5:\ a_1=0,\ a_2=0,\ a_4=\frac{1}{6},\ a_5=0,\ a_6=0;$$

$$C_6:\ a_1=0,\ a_2=0,\ a_4=-\frac{1}{6},\ a_5=0,\ a_6=0.$$

4.2. Complex center conditions of system (4.3)

Theorem 4.6 implies that the origin of system (4.3) is a complex center if one of the six necessary conditions in the theorem holds. Next, we prove that these conditions are also sufficient.

Theorem 4.7. *The origin of system (4.3) is a complex center if and only if one of the six conditions in Theorem 4.6 holds.*

Proof. The necessity of these conditions have been shown in Theorem 4.6. So we only need to prove sufficiency. First, consider the condition C_1, under which system (4.3) can be rewritten as

$$\begin{aligned}\frac{du}{dT}&=u+\frac{3}{2}u^2+a_2v^2+\frac{1}{2}u^3+a_2uv^2+a_3v^3,\\ \frac{dv}{dT}&=-v\Big(1+u-a_5v+\frac{1}{2}u^2-a_5uv-a_6v^2\Big),\end{aligned}\tag{4.4}$$

which can be further transformed into

$$\begin{aligned}\frac{d\xi}{dT}&=(1+2\xi)\Big[(a_2+a_3\eta)\eta^2+(1+2a_2\eta^2+2a_3\eta^3)\xi\Big],\\ \frac{d\eta}{dT}&=-(1+2\xi)\eta[1-a_5\eta+(a_2-a_6)\eta^2+a_3\eta^3],\end{aligned}\tag{4.5}$$

by

$$\xi=u\Big(1+\frac{1}{2}u\Big),\qquad \eta=\frac{v}{1+u}.$$

System (4.5) has an inverse integrating factor,

$$M_1(\xi,\eta)=(1+2\xi)\exp\int_0^{\eta}\frac{-2a_5+(a_2-3a_6)\eta+2a_3\eta^2}{1-a_5\eta+(a_2-a_6)\eta^2+a_3\eta^3}\,d\eta,$$

which implies that the conclusion is true for the condition C_1.

If the condition C_2 in Theorem 4.6 holds, system (4.3) can be simplified to

$$\begin{aligned}\frac{du}{dT}&=u+\frac{3}{2}u^2-2a_5uv-3a_6v^2+\frac{1}{2}u^3-a_5u^2v-3a_6uv^2+a_3v^3,\\ \frac{dv}{dT}&=-v\Big(1+3u-a_5v+\frac{3}{2}u^2-a_5uv-a_6v^2\Big),\end{aligned}\tag{4.6}$$

which is a Hamilton system, having a first integral,

$$F_1 = v\Big(u + \frac{3}{2}u^2 - a_5 uv - a_6 v^2 + \frac{1}{2}u^3 - \frac{1}{2}a_5 u^2 v - a_6 uv^2 + \frac{1}{4}a_3 v^3\Big).$$

When the condition C_3 holds, system (4.3) can be rewritten as

$$\begin{aligned} \frac{du}{dT} &= u + \frac{3}{2}u^2 + (1+2a_4)a_5 uv + a_2 v^2 \\ &\quad + \frac{1}{2}u^3 + \frac{1}{2}(1+2a_4)a_5 u^2 v + a_2 uv^2 + \frac{1}{3}(-1+2a_4)a_2 a_5 v^3, \\ \frac{dv}{dT} &= -v + 2a_4 uv + a_5 v^2 + a_4 u^2 v + a_5 uv^2 + \frac{2}{3}(1+2a_4)a_2 v^3, \end{aligned} \tag{4.7}$$

which has a first integral

$$F_2 = v\Big(u + \frac{1}{2}u^2 + \frac{1}{3}a_2 v^2\Big)(1 + u - a_5 v)^{-2(1+a_4)},$$

and so the conclusion holds under the condition C_3.

If the condition C_4 holds, system (4.3) is simplified to

$$\frac{du}{dT} = (1+u)\Big(u + \frac{1}{2}u^2 + a_2 v^2\Big), \quad \frac{dv}{dT} = -v(1 - a_6 v^2). \tag{4.8}$$

By finding invariant algebraic curves of system (4.8), we get an inverse integrating factor for system (4.8),

$$M_3 = (1+u)^3 g_1^{\frac{1}{2}(-2a_2+3a_6)},$$

where

$$g_1 = \begin{cases} (1 - a_6 v^2)^{\frac{1}{a_6}}, & if\ a_6 \neq 0, \\ \exp(-v^2), & if\ a_6 = 0. \end{cases}$$

This indicates that the condition C_4 is sufficient.

When the condition C_5 holds, system (4.3) can be rewritten as

$$\frac{du}{dT} = u + \frac{3}{2}u^2 + \frac{1}{2}u^3 + a_3 v^3, \quad \frac{dv}{dT} = -v\Big(1 - \frac{1}{3}u - \frac{1}{6}u^2\Big), \tag{4.9}$$

which admits an inverse integrating factor

$$M_4 = g_2 g_3^{-2},$$

where

$$g_2 = u + \frac{1}{2}u^2 + \frac{1}{4}a_3 v^3 + \frac{1}{4}a_3 u v^3 + \frac{1}{28}a_3^2 v^6,$$
$$g_3 = 1 + u + \frac{1}{3}a_3 v^3,$$

and so the conclusion is true for the condition C_5.

Finally, when the condition C_6 holds, system (4.3) becomes

$$\frac{du}{dT} = u + \frac{3}{2}u^2 + \frac{1}{2}u^3 + a_3 v^3, \quad \frac{dv}{dT} = -v\Big(1 + \frac{1}{3}u + \frac{1}{6}u^2\Big). \tag{4.10}$$

By using the transformation,

$$\xi = \frac{u(2+u)}{2(1+u)^2}, \quad \eta = (1+u)^{\frac{1}{3}} v,$$

system (4.10) can be brought into

$$\frac{d\xi}{dT} = \xi + a_3(1-2\xi)^2\eta^3, \quad \frac{d\eta}{dT} = -\eta + \frac{1}{3}a_3(1-2\xi)\eta^4. \tag{4.11}$$

According to Theorem 2.7 in [24], the origin of system (4.11) is a complex center, so the conclusion is true for system (4.3) under the condition C_6.

This finishes the proof of Theorem 4.7. □

4.3. Complex isochronous center conditions of system (4.3)

Having obtained the six conditions in Theorem 4.6 under which the origin of system (4.3) is a complex center, we now consider the complex isochronous center conditions under the six conditions. We have the following theorem.

Theorem 4.8. *The origin of system* (4.3) *is a complex isochronous center if and only if one of the following nine conditions holds:*

$$C_1,\ C_{2-1},\ C_{3-1},\ C_{3-2},\ C_{3-3},\ C_{3-4},\ C_4,\ C_5,\ C_6,$$

where the conditions C_1, C_4, C_5 and C_6 are given in Theorem 4.6, and

$$C_{2-1}: \quad a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = -\frac{3}{2},\ a_5 = 0,\ a_6 = 0;$$
$$C_{3-1}: \quad a_1 = a_5,\ a_3 = 0,\ a_4 = \frac{1}{2},\ a_6 = a_2;$$
$$C_{3-2}: \quad a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = 0,\ a_6 = 0.$$

Proof. First, consider the condition C_1. It is easy to verify that system (4.4) has two invariant algebraic curves:

$$f_1 = v,$$
$$f_2 = (1+u)^3 - a_5(1+u)^2 v + (a_2 - a_6)(1+u)v^2 + a_3 v^3,$$

which satisfy

$$\frac{df_1}{dT} = h_1 f_1, \quad \frac{df_2}{dT} = h_2 f_2,$$

where

$$h_1 = -1 - u + a_5 v - \frac{1}{2}u^2 + a_5 uv + a_6 v^2,$$
$$h_2 = 3u + a_5 v + \frac{3}{2}u^2 + a_5 uv + (a_2 + 2a_6)v^2.$$

Since the origin of system (4.4) is a complex center under the condition C_1, there exists an analytic inverse integrating factor $M_1(u, v)$ in the neighborhood of the origin, which satisfies that

$$M_1(0,0) = 1, \quad \frac{dM_1}{dT} = h_3 M_1,$$

where

$$h_3 = 2u + 2a_5 v + u^2 + 2a_5 uv + (a_2 + 3a_6)v^2.$$

Let

$$\zeta = f_1 f_2 M_2^{-1},$$

then, it follows from above discussion that

$$\frac{d\zeta}{dT} = (h_1 + h_2 - h_3)\zeta = -\zeta. \tag{4.12}$$

According to Theorems 2.3 and equation (4.12), we have shown that the conclusion is true for the condition C_1.

Next, consider the condition C_2. A direct computation gives the first three periodic constants of system (4.6) at the origin:

$$\tau_1 = 16a_5, \quad \tau_2|_{\tau_1=0} = 192a_6, \quad \tau_3|_{\tau_1=\tau_2=0} = 480a_3.$$

Obviously, when the condition C_2 holds, the first three periodic constants at the origin of system (4.3) are zero if the condition C_{2-1} hold. This proves the necessity.

Next, we show that the condition C_{2-1} is also sufficient. When the condition C_{2-1} holds, system (4.6) becomes

$$\frac{du}{dT} = \frac{1}{2}u(1+u)(2+u), \quad \frac{dv}{dT} = -\frac{1}{2}(2 + 6u + 3u^2)v.$$

Then the sufficiency directly follows Corollary 2.2.

When the condition C_3 in Theorem 4.6 holds, similarly we use system (4.7) to discuss the complex isochronous center at the origin. With the aid of Mathematica, it is not difficult to obtain the first two periodic constants of system (4.7), given by

$$\tau_1 = 2(-1+2a_4)(1+2a_4)a_5,$$
$$\tau_2|_{\tau_1=0} = \frac{16}{3}a_2a_4(-1+2a_4)(1+2a_4).$$

It is easy to see that when the condition C_3 holds, the first two periodic constants at the origin of system (4.7) becomes zero if and only if one of the four conditions

$$\begin{aligned}
&C_{3-1}: \quad a_1 = 0,\ a_3 = \frac{-2}{3}a_2a_5,\ a_4 = -\frac{1}{2},\ a_6 = \frac{1}{3}a_2;\\
&C_{3-2}: \quad a_1 = a_5,\ a_3 = 0,\ a_4 = \frac{1}{2},\ a_6 = a_2;\\
&C_{3-3}: \quad a_1 = 0,\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = \frac{2}{3}a_2;\\
&C_{3-4}: \quad a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_5 = 0,\ a_6 = 0,
\end{aligned}$$

holds.

Now, we prove these conditions are also sufficient. When the condition C_{3-1} in Theorem 4.8 holds, system (4.7) take the form of

$$\begin{aligned}
\frac{du}{dT} &= u + \frac{3}{2}u^2 + a_2v^2 + \frac{1}{2}u^3 + a_2uv^2 - \frac{2}{3}a_2a_5v^3,\\
\frac{dv}{dT} &= -v\Big(1 + u - a_5v + \frac{1}{2}u^2 - a_5uv - \frac{1}{3}a_2v^2\Big),
\end{aligned} \tag{4.13}$$

which has a linearization transformation in the neighborhood of the origin of system (4.13), given by

$$\xi = f_4 f_6^{-1}, \quad \eta = v f_5^{-1} f_6,$$

where

$$\begin{aligned}
f_4 &= u + \frac{1}{2}u^2 + \frac{1}{3}a_2v^2,\\
f_5 &= 1 + u - a_5v,\\
f_6 &= 1 + 2u + u^2 + \frac{2}{3}a_2v^2.
\end{aligned}$$

When the condition C_{3-2} in Theorem 4.8 holds, system (4.7) becomes

$$\begin{aligned}
\frac{du}{dT} &= u + \frac{3}{2}u^2 + 2a_5uv + a_2v^2 + \frac{1}{2}u^3 + a_5u^2v + a_2uv^2,\\
\frac{dv}{dT} &= -v\Big(1 - u - a_5v - \frac{1}{2}u^2 - a_5uv - a_2v^2\Big),
\end{aligned} \tag{4.14}$$

which has a linearization transformation,

$$\xi = f_4 f_5^{-2}, \quad \eta = v f_5^{-1},$$

in the neighborhood of the origin of system (4.14).

When the condition C_{3-3} in Theorem 4.8 holds, then system (4.7) can be changed into

$$\frac{du}{dT} = (1+u)\Big(u + \frac{1}{2}u^2 + a_2 v^2\Big), \quad \frac{dv}{dT} = -v\Big(1 - \frac{2}{3}a_2 v^2\Big).$$

Then the conclusion for this condition follows Corollary 2.1.

When the condition C_{3-4} in Theorem 4.8 holds, system (4.7) can be rewritten as

$$\frac{du}{dT} = u(1+u)\left(1 + \frac{1}{2}u\right), \quad \frac{dv}{dT} = -v,$$

and so the conclusion is true by Corollary 2.1.

For the condition C_4, we use Theorem 2.1 or Corollary 2.2 to show that the origin of system (4.8) is a complex isochronous center if and if it is a complex center.

Now, consider the condition C_5, under which system (4.9) can be rewritten as

$$\frac{du}{dT} = u + \frac{3}{2}u^2 + \frac{1}{2}u^3 + a_3 v^3, \quad \frac{dv}{dT} = -v\Big(1 - \frac{1}{3}u - \frac{1}{6}u^2\Big),$$

which can be further transformed into a linear system by the linear transformation,

$$\xi = f_7 f_8^{-2}, \quad \eta = v f_8^{\frac{-1}{3}},$$

where

$$f_7 = u + \frac{1}{2}u^2 + \frac{1}{4}a_3 v^3 + \frac{1}{4}a_3 u v^3 + \frac{1}{28}a_3^2 v^6,$$
$$f_8 = 1 + u + \frac{1}{3}a_3 v^3.$$

Finally, when the condition C_6 holds, system (4.3) becomes (4.10). By the transformation,

$$\xi = \frac{u(2+u)}{2(1+u)^2}, \quad \eta = (1+u)^{\frac{1}{3}} v,$$

system (4.10) can be brought into

$$\frac{d\xi}{dT} = \xi + a_3(1-2\xi)^2\eta^3, \quad \frac{d\eta}{dT} = -\eta + \frac{1}{3}a_3(1-2\xi)\eta^4. \tag{4.15}$$

According to Theorem 2.7 in [24], the origin of system (4.15) is a complex isochronous center, and so the conclusion is true for system (4.10) under the condition C_6.

However, the conditions C_{3-3} and C_{3-4} were contained in condition C_1 and C_4 respectively. The proof is complete. □

4.4. *Integrability and linearizability of system* (3.15)

To end this section, we consider integrability and linearizability of system (3.15), namely, the case 4. We point out $\lambda \neq 0$ in system (3.15). Otherwise, suppose $\lambda = 0$, then system (3.15) can be rewritten as

$$\frac{du}{dT} = (1-a_1)v + a_1u^2v + a_2uv^2 + a_3v^3,$$
$$\frac{dv}{dT} = -a_4v + a_4u^2v + a_5uv^2 + a_6v^3,$$

for which the two singular points $(\pm 1, 0)$ are not isolated singular points. So we always suppose $\lambda \neq 0$. That is, the $(\pm 1, 0)$ of system (3.15) are degenerate nodes, which are integrable and linearizable. Namely, for Case 4, if the two singular points $(\pm 1, 0)$ are isolated singular points, they are integrable and linearizable.

5. Conclusion

In this paper, integrability and linearizability of cubic Z_2 systems with non-resonant and elementary singular points are investigated thoroughly. Based on the coefficients and eigenvalues of cubic Z_2 systems, four simple norm forms are obtained. Then, for each of the cases, the integrable and linearizable conditions are classified. We briefly summarize the existing results for Cases 1 and 3, and then completely solved the integrable and linearizable problem for Cases 2 and 4. The integrability and linearizability of cubic Z_2 systems with resonant singular points are left for future study.

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