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Bi-center problem and bifurcation of limit cycles from nilpotent singular points in Z_2-equivariant cubic vector fields

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Abstract

In this paper, bi-center problem and bifurcation of limit cycles from nilpotent singular points in Z_2-equivariant cubic vector fields are studied. First, the system is simplified by using some proper transformations and the first five Lyapunov constants at a nilpotent singular point are calculated by applying the inverse integrating factor method. Then, sufficient and necessary conditions are obtained for two nilpotent singular points of the system being centers. A new perturbation scheme is present to prove the existence of 12 small-amplitude limit cycles in cubic Z_2-equivariant vector fields, which bifurcate from two nilpotent singular points. This is a new lower bound of the number of limit cycles bifurcating in such systems.

Keywords: Nilpotent singular point; Center–focus problem; Bi-center problem; Lyapunov constant; Limit cycle

1. Introduction

In qualitative theory of planar vector fields, the analysis on the existence, number and distribution of limit cycles for planar polynomial differential systems,

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$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{1.1}$$

is closely related to the second part of Hilbert's 16th problem.

Let $M(n)$ denote the maximal number of small-amplitude limit cycles bifurcating from either an elementary focus or a center. There have been many results in the literature. $M(2) = 3$ was obtained by Bautin in 1952 [6]. For $n = 3$, Yu and Tian [40] proved existence of 12 small-amplitude limit cycles around a singular point, which is the best result so far for cubic systems. The center problem which is closely related to Hopf bifurcation has also been intensively studied. The study started from the quadratic polynomial differential systems with linear type singular points, for example, Dulac [12], Bautin [7], and Żołądek [51]; see also Schlomiuk [36] for an update of these works. But the center–focus problem for polynomial differential systems with degree larger than two remains open.

It is well known that when the origin of a dynamical system is a degenerate critical point, the center problem becomes more difficult. However, for nilpotent critical point, some methods have been developed to investigate the center problem. The origin of a system is called a nilpotent critical point if it is an isolated critical point, and the linear part of the system has a double zero eigenvalue but the matrix of the linearized system at the origin is not identically null. There are many different kinds of topological phase constructions around a nilpotent critical point, for example, see [49]. Early results can be found in Sections 17–19 of [4]. Recently, more and more attentions have been paid to the center problem and bifurcation of limit cycles in systems with a nilpotent critical point, which is more challenging compared to the study for systems with an element critical point.

With a proper linear transformation, planar autonomous analytic systems with a nilpotent critical point can always be given in the form of

$$\begin{aligned} \frac{dx}{dt} &= \Phi(x, y) = y + \sum_{k+j=2}^{\infty} a_{kj} x^k y^j, \\ \frac{dy}{dt} &= \Psi(x, y) = \sum_{k+j=2}^{\infty} b_{kj} x^k y^j, \end{aligned} \tag{1.2}$$

where $\Phi(x, y)$, $\Psi(x, y)$ are analytic in the neighborhood of the origin.

The results given in [4] show that the origin of system (1.2) is a monodromic critical point if the following conditions hold:

$$\begin{aligned} &\Psi(x, f(x)) = \alpha x^{2n-1} + o(x^{2n-1}), \quad \alpha \neq 0, \\ &\left[\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\right]_{y=f(x)} = \beta x^{n-1} + o(x^{n-1}), \\ &\beta^2 + 4n\alpha < 0, \end{aligned} \tag{1.3}$$

where n is a positive integer.

Suppose that the conditions in (1.3) are satisfied. The authors of [4] introduced a transformation,

$$x = (-\alpha)^{\frac{-1}{2n-2}} x_1, \qquad y = (-\alpha)^{\frac{-1}{2n-2}} y_1 + f(x), \tag{1.4}$$

into system (1.2), which, with the Lyapunov polar coordinates

$$x_1 = rCs\ \vartheta, \qquad y_1 = -r^n Sn\ \vartheta, \tag{1.5}$$

leads to that the solution of the system, satisfying $r|_{\vartheta=0} = h$, can be written as

$$r = \tilde{r}(\vartheta, h) = \sum_{k=1}^{\infty} \nu_k(\vartheta) h^k. \tag{1.6}$$

Here,

$$\begin{aligned}
&Cs\ \vartheta = \cos\varphi, \\
&Sn\ \vartheta = \frac{\sin\varphi}{\sqrt{n}}\sqrt{1+\cos^2\varphi+\cdots+\cos^{2n-2}\varphi}, \\
&\nu_1(\vartheta) = \exp\int_0^{\vartheta} \frac{\beta Sn^2\ \vartheta Cs^{n-1}\ \vartheta}{1+\beta Sn\ \vartheta Cs^n\ \vartheta}\, d\vartheta.
\end{aligned} \tag{1.7}$$

Consequently, the successor function, focal values and kth-order focus value were defined by the author in [4] because every $\nu_k(\vartheta)$ could be solved, and a method was developed to compute the focal values in order to solve the center–focus problem. Therefore, the related theory of successor function may be considered as a basic theory to solve the center–focus problem associated with the nilpotent critical point and bifurcation of limit circle of (1.2). However, unfortunately, it is difficult to use this method to solve every $\nu_k(\vartheta)$ for a given polynomial system.

In fact, Theorem 19.10 in [4] shows that when the conditions in (1.3) hold, there exist the following formal transformations,

$$\begin{aligned}
u &= x + \sum_{k+j=2}^{\infty} a'_{kj} x^k y^j, \\
v &= y + \sum_{k+j=2}^{\infty} b'_{kj} x^k y^j, \\
\frac{dt}{d\tau} &= 1 + \sum_{k+j=1}^{\infty} c'_{kj} x^k y^j,
\end{aligned} \tag{1.8}$$

such that (1.2) becomes the Liénard equation,

$$\frac{du}{d\tau} = v + F(u), \qquad \frac{dv}{d\tau} = \alpha u^{2n-1}, \tag{1.9}$$

where

$$F(u) = \frac{1}{n}\beta u^n + o(u^n). \tag{1.10}$$

Furthermore, if conditions in (1.3) are satisfied, there exist analytic transformations in the neighborhood of the origin of (1.8), which change (1.2) into a Liénard equation,

$$\frac{du}{d\tau}=v,\quad \frac{dv}{d\tau}=\alpha u^{2n-1}+v\sum_{k=n-1}^{\infty}B_k u^k,\quad B_{n-1}=\beta. \tag{1.11}$$

Thus, B_{2k} can be considered as Lyapunov constants of (1.2).

For a special case, assume system (1.2) is symmetric with the origin. Then (1.2) can be written as

$$\frac{dx}{dt}=y+\sum_{k=1}^{\infty}X_{2k+1}(x,y),\quad \frac{dy}{dt}=\sum_{k=1}^{\infty}Y_{2k+1}(x,y), \tag{1.12}$$

where $X_{2k+1}(x,y)$ and $Y_{2k+1}(x,y)$ are homogeneous polynomials of degree $2k+1$ in x and y. Amelikin has claimed (see [4]) that if the conditions in (1.3) hold for system (1.12), one can construct successively a positive formal power series $F(x,y)$ in the neighborhood of the origin, satisfying

$$\left.\frac{dF}{dt}\right|_{(1.12)}=\sum_{k=[\frac{3n+1}{2}]}^{\infty}V_k x^{2k}. \tag{1.13}$$

From then, normal form theory was also applied to solve the center–focus problem for monodromic planar nilpotent singularities. In [4,39,38,34,2], by considering the normal forms of (1.2), the authors tried to study the computation problem of focal values. In [14], the authors obtained the expressions of the coefficients of normal forms and investigated the possibilities of simplifying classical normal forms by means of a recursive algorithm well suited to symbolic computation, leading to the simpler and higher order normal forms. Furthermore, it is clear as shown in [3] that the method of normal forms can also be applied to study bifurcation of limit cycles around nilpotent singularities. In [9], Hamiltonian linear type centers and nilpotent centers of the linear and cubic polynomial terms were considered. Twelve normal forms were obtained for all the Hamiltonian planar polynomial vector fields with linear and cubic homogeneous terms, which possess either a linear type center or a nilpotent center at the origin. Moreover, the global phase portraits were present on the Poincaré disk.

For cubic-order nilpotent critical points of planar dynamical systems, the center–focus problem was solved by using the integral factor method in [27,28] where the quasi-Lyapunov constants were defined and their computation method was developed. For a class of cubic systems, under small perturbations, existence of 8 small-amplitude limit cycles bifurcating from a nilpotent critical point was proved in [29]. Furthermore, a new kind of bifurcation phenomena was discussed in [26], showing that a cubic-order nilpotent focus of planar dynamical systems can be broken into two element foci and an element saddle, yielding limit cycles bifurcating from the two element foci. As an example, a class of cubic systems with 3-multiple nilpotent foci was investigated to show that nine limit cycles can bifurcate from the origin when the origin is a weak focus of order 8. For cubic-order nilpotent critical points of planar dynamical systems, the analytic center problem was completely solved by using the integrating factor method [33].

By using this method, some special systems were investigated in [19,21]. In [16], local behavior of an isolated nilpotent critical point for polynomial Hamiltonian systems was investigated, proving that there are exact three cases: a center, a cusp or a saddle. Then for quadratic and cubic Hamiltonian systems, necessary and sufficient conditions were obtained for classifying a nilpotent critical point as a center, a cusp or a saddle. Some special systems were studied in [18,43] by using this method. Furthermore, limit cycle bifurcation near a double homoclinic loop passing through a nilpotent saddle was studied in [17] by applying the analytical property of the first order Melnikov functions to general near-Hamiltonian systems and the conditions were obtained for the perturbed system to have 8, 10 or 12 limit cycles in a neighborhood of the loop with seven different distributions.

Recently, explicit expansion of the first Melnikov function was obtained [5] by perturbing an integrable and reversible system with a homoclinic loop passing through a nilpotent singular point, and the first three coefficients of the expansion were obtained. Zhao [45] studied the limit cycles of a class of cubic Hamiltonian systems under polynomial perturbations, with the assumption that the corresponding Hamiltonian system has finite singular points with at least one center and is symmetric with respect to both the x- and y-axes, and the origin is a nilpotent singular point. In [11], the authors provide normal forms and the global phase portraits in the Poincaré disk for all Hamiltonian planar polynomial vector fields of degree 3, symmetric with respect to the x-axis, having a nilpotent center at the origin. Llibre [10] studied bifurcation diagrams for Hamiltonian nilpotent centers of polynomial vector fields with linear and cubic terms. Normal form theory was also applied to compute the generalized Lyapunov constants and to prove the existence of at least 9 and 10 small-amplitude limit cycles in the neighborhood of a nilpotent critical point in [44] and [48], respectively. The research in this direction attracts more and more researchers.

Some other methods or systems were also considered recently. For example, it was proved in [15] that all the nilpotent centers are limit of linear type centers and consequently the Poincaré-Liapunov method to study linear type centers can be also used to consider nilpotent centers. A quasi-homogeneous vector field with a nilpotent and monodromic isolated singular point was investigated in [1] to prove the existence of a Lyapunov function and to theoretically solve the center problem.

As far as limit cycles are concerned, there have been many results obtained in the last decade. Let $H(n)$ be the maximal number of limit cycles of (1.1) when P and Q are polynomials of degree at most n. The best results published so far are as follows: In [37,8], it was shown that $H(2) \geqslant 4$. $H(3) \geqslant 13$ was proved in [22,23] and $H(4) \geqslant 16$ was obtained in [41,42]. In addition, a study was given in [47] on the limit cycle bifurcation of Z_q-equivariant polynomial vector fields with degrees 3 and 4. The Z_2-equivariant system with degree 3 and its bifurcation problem were studied in [46,25]. Furthermore, Liu and Li [27,28] obtained 13 limit cycles in Z_2-equivariant systems with degree 3. It has been noticed from the above results that better results were often obtained from Z_n-equivariant vector fields. In fact, as far as an isolated focus is concerned, it is difficult to compute higher-order focal values with simpler expression. Thus, it is difficult to obtain more limit cycles by calculating the focal values of a single focus and very few results have been achieved for higher-order polynomial differential systems. Recently, a complete study on bi-center problem for Z_2-equivariant cubic vector fields has been given in [32] and bi-center problem for some Z_2-equivariant quintic systems was studied in [35].

It is even more difficult to analyze non-analytic systems. As far as bifurcation of limit cycles and conditions of centers at the origin are concerned, the following Z_2-equivariant polynomial vector field,

$$\begin{aligned}\frac{dx}{dt} &= -y + (x^2+y^2)^{\frac{d-3}{2}}(3x^2y - 12B_{03}xy^2 - 4B_{12}xy^2 \\ &\quad - y^3 - 4A_{03}y^3 - 4A_{12}y^3), \\ \frac{dy}{dt} &= x + (x^2+y^2)^{\frac{d-3}{2}}(-x^3 + 3xy^2 + 12A_{03}xy^2 + 4A_{12}xy^2 \\ &\quad - 4B_{03}y^3 - 4B_{12}y^3),\end{aligned} \tag{1.14}$$

was investigated in [20]

Note that the origin of system (1.14) is either an elementary focus or a center. For degenerate singular points, because of the difficulty, there are very few results obtained for Z_2-equivalent system of degree 3 with two nilpotent singular points. Hence, in this paper, we study bifurcation of limit circles in a class of Z_2-equivalent cubic planar differential systems with two nilpotent singular points, described by

$$\begin{aligned}\frac{dx}{dt} &= A_{10}x + A_{01}y + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3 = X(x,y), \\ \frac{dy}{dt} &= B_{10}x + B_{01}y + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3 = Y(x,y).\end{aligned} \tag{1.15}$$

Sufficient and necessary conditions for the critical points of system (1.15) to be centers are derived. In addition, the existence of 12 small-amplitude limit cycles bifurcating from the critical points is proved.

The rest of the paper is organized as follows. In the next section, we simplify system (1.15) for the convenience of analysis. In Section 3, the first five Lyapunov constants at a nilpotent singular point are computed by using the inverse integrating factor method or the method of normal forms. Section 4 is devoted to discuss the integrability and center condition in Z_2-equivariant vector fields, with five possible integral conditions obtained, three of them are proved true center conditions. Furthermore, bifurcation of limit cycles will be discussed in Section 5, and a new perturbation scheme is present to obtain 12 limit cycles. Finally, conclusion is drawn in Section 6.

2. Simplification of system (1.15)

Suppose $(0, \pm 1)$ are singular points of system (1.15). Then,

$$A_{01} = -A_{03}, \qquad B_{01} = -B_{03}, \tag{2.1}$$

and the Jacobin matrix of system (1.15) at $(0, \pm 1)$ is given by

$$J_0 = \begin{bmatrix} \frac{\partial X}{\partial x}, & \frac{\partial X}{\partial y} \\ \frac{\partial Y}{\partial x}, & \frac{\partial Y}{\partial y} \end{bmatrix}_{(0,\ \pm 1)} = \begin{bmatrix} A_{10}+A_{12}, & 2A_{03} \\ B_{10}+B_{12}, & 2B_{03} \end{bmatrix}. \tag{2.2}$$

We have the following result.

Lemma 2.1. *The necessary condition for* $(0, \pm 1)$ *being isolated nilpotent singular points of* (1.15) *is*

$$A_{03} \neq 0.$$

Proof. Suppose $A_{03} = 0$, then J_0 is a triangular matrix, having two characteristic roots, given by

$$\lambda_1 = A_{10} + A_{12}, \quad \lambda_2 = 2B_{03}.$$

Since $(0, \pm 1)$ are nilpotent singular points, we have

$$\lambda_1 = \lambda_2 = 0, \tag{2.3}$$

under which together with (2.1), system (1.15) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= x(-A_{12} + A_{30}x^2 + A_{21}xy + A_{12}y^2), \\ \frac{dy}{dt} &= x(B_{10} + B_{30}x^2 + B_{21}xy + B_{12}y^2), \end{aligned} \tag{2.4}$$

which has a common factor x in the two equations, implying that $(0, \pm 1)$ are not isolated singular points, and so Lemma 2.1 is proved. □

Now suppose $(0, \pm 1)$ are isolated nilpotent singular points of system (1.15). Consider the following transformation,

$$x = 2A_{03}\xi, \quad y = 2B_{03}\xi + \eta. \tag{2.5}$$

It is obvious that transformation (2.5) is not degenerate since $A_{03} \neq 0$, and $(0, 0)$ and $(0, \pm 1)$ are fixed points of (2.5). By applying transformation (2.5), it is easy to find that the Jacobin matrix of system (1.15) evaluated at $(0, \pm 1)$ is given by

$$J_1 = \begin{bmatrix} \mathrm{Tr}(J_0) & 1 \\ -\mathrm{Det}(J_0) & 0 \end{bmatrix}. \tag{2.6}$$

Because $(0, \pm 1)$ are nilpotent singular points of system (1.15), we have

$$\mathrm{Tr}(J_0) = \mathrm{Det}(J_0) = 0,$$

under which (2.6) becomes

$$J_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

One immediate consequence is given below.

Lemma 2.2. *Suppose* $(0, \pm 1)$ *are nilpotent singular points of system* (1.15). *Without loss of generality, let*

$$J_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which yields

$$\begin{aligned} &A_{10} = -A_{12}, \quad A_{03} = \frac{1}{2}, \\ &B_{10} = -B_{12}, \quad B_{03} = 0. \end{aligned} \tag{2.7}$$

The following result directly follows from Lemma 2.2.

Lemma 2.3. *If* $(0, \pm 1)$ *are nilpotent singular points of system* (1.15)*, for simplicity, system* (1.15) *can be written as*

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}y + \frac{1}{2}y^3 - A_{12}x + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2, \\ \frac{dy}{dt} &= -B_{12}x + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2. \end{aligned} \tag{2.8}$$

Remark 2.1. In fact, for any nonzero constant r, by the transformation,

$$x = rx', \quad y = y', \quad t = rt',$$

system (2.8) can be changed to

$$\begin{aligned} \frac{dx'}{dt'} &= -\frac{1}{2}y' + \frac{1}{2}y'^3 - A'_{12}x' + A'_{30}x'^3 + A'_{21}x'^2y' + A'_{12}x'y'^2, \\ \frac{dy'}{dt'} &= -B'_{12}x' + B'_{30}x'^3 + B'_{21}x'^2y' + B'_{12}x'y'^2, \end{aligned}$$

where

$$\begin{aligned} &A'_{12} = rA_{12}, \quad A'_{21} = r^2A_{21}, \quad A'_{30} = r^3A_{30}, \\ &B'_{12} = r^2B_{12}, \quad B'_{21} = r^3B_{21}, \quad B'_{30} = r^4B_{30}. \end{aligned}$$

In other words, system (2.8) is invariant under the following transformation:

$$(x, y, t, A_{12}, A_{21}, A_{30}, B_{12}, B_{21}, B_{30}) \to (x', y', t', A'_{12}, A'_{21}, A'_{30}, B'_{12}, B'_{21}, B'_{30}). \tag{2.9}$$

Now we discuss the multiplicity of nilpotent singular points $(0, \pm 1)$ of system (2.8). System (2.8) can be transformed into

$$\begin{aligned}\frac{d\xi}{dt} &= \frac{1}{2}\eta(1+\eta)(2+\eta) + A_{30}\xi^3 + A_{21}\xi^2(1+\eta) + A_{12}\xi\eta(2+\eta) = \Phi(\xi,\eta),\\ \frac{d\eta}{dt} &= B_{30}\xi^3 + B_{21}\xi^2(1+\eta) + B_{12}\xi\eta(2+\eta) = \Psi(\xi,\eta),\end{aligned} \tag{2.10}$$

via

$$\xi = \pm x, \quad \eta = \pm y - 1.$$

Suppose

$$\eta = f(\xi) = \sum_{k=2}^{\infty} c_k \xi^k$$

is the only solution of the implicit function equation,

$$\Phi(\xi,\eta) = 0, \quad \eta|_{\xi=0} = 0.$$

Denote

$$\begin{aligned}&\Psi(\xi, f(\xi)) = \sum_{k=2}^{\infty} \alpha_k \xi^k,\\ &\left[\frac{\partial\Phi}{\partial\xi} + \frac{\partial\Psi}{\partial\eta}\right]_{(\xi, f(\xi))} = \sum_{k=1}^{\infty} \beta_k \xi^k.\end{aligned} \tag{2.11}$$

It is easy to get

$$\begin{aligned}&\beta_1 = 2(A_{21} + B_{12}),\\ &\alpha_2 = B_{21}, \ \ \alpha_3 = B_{30} - 2A_{21}B_{12},\\ &\alpha_4|_{\alpha_2=\alpha_3=0} = 2(2A_{12}A_{21} - A_{30})B_{12}.\end{aligned} \tag{2.12}$$

Proposition 2.1. *The nilpotent singular points* $(0, \pm 1)$ *of* (2.8) *are degenerate singular points when* $\alpha_2 = B_{21} \neq 0$.

Proof. Theorems 7.2 and 7.3 in [50] show that the types of the origin of system (2.10) can be determined as follows.

When $k = 2m, \ \alpha_k \neq 0$,

$$\begin{cases} \beta_n = 0, & \text{degenerate point,} \\ \beta_n \neq 0 & \begin{cases} n \geq m, & \text{degenerate point,} \\ n < m, & \text{saddle-node point.} \end{cases} \end{cases}$$

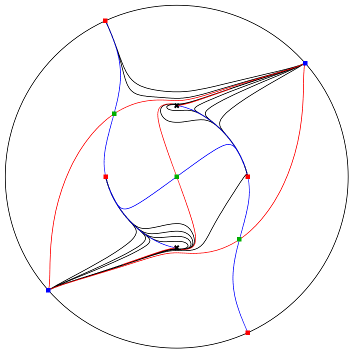


Fig. 1. Phase portrait of system (2.8) for $A_{30} = A_{21} = A_{12} = B_{30} = B_{21} = B_{12} = 1$, showing that $(0, \pm 1)$ are degenerate nilpotent singular points.

When $k = 2m + 1,\ \lambda = \beta_n^2 + 4(m + 1)\alpha_{2m+1}$,

$$\begin{cases} \alpha_{2m+1} > 0, & \text{saddle,} \\ \alpha_{2m+1} < 0, & \begin{cases} \beta_n = 0, & \text{center or focus,} \\ \beta_n \neq 0 & \begin{cases} n > m, \\ \text{or } n = m,\ \lambda < 0, \end{cases} & \text{center or focus,} \\ & \begin{cases} n < m, \\ \text{or } n = m,\ \lambda \geq 0, \end{cases} & \begin{cases} n & \text{even,} & \text{node,} \\ n & \text{odd,} & \text{degenerate point.} \end{cases} \end{cases} \end{cases}$$

So if $\alpha_2 = B_{21} \neq 0$, $m = 1$ and $n \geq 1$, the origin of system (2.9) is degenerate, namely, the nilpotent singular points $(0, \pm 1)$ of (2.8) are degenerate singular points. $\square$

With the help of Maple, phase portraits of some cases were given in [13]. In the following, we give more examples to illustrate different situations.

Example 2.1. When $A_{30} = A_{21} = A_{12} = B_{30} = B_{21} = B_{12} = 1$, obviously $\alpha_2 = B_{21} = 1 \neq 0$, so $(0, \pm 1)$ are degenerate singular points, see Fig. 1.

The multiple number of a critical point of system (1.2) has been defined by using the intersection number of algebraic curves (see [24]). According to the definition of the multiple number given in [24], the multiple number of element critical points is 1, and a non-element critical point can be broken into several complex critical points with lower multiple number. For nilpotent singular point, Definition 2.1 in [30] indicates that for any positive integer k, if $\alpha_2 = \alpha_3 = \cdots = \alpha_{k-1} = 0,\ \alpha_k \neq 0$, the multiplicity of the nilpotent singular point is exactly k.

Proposition 2.2. *The multiplicity of nilpotent singular points* $(0, \pm 1)$ *of* (2.8) *is* 4 *at most.*

Proof. In (2.12), if $\alpha_2 = \alpha_3 = 0$, then

$$B_{21} = 0, \quad B_{30} = 2A_{21}B_{12},$$
$$\Psi(\xi, \eta) = B_{12}\xi(2A_{21}\xi^2 + 2\eta + \eta^2).$$

When $\alpha_4 = 0$, there are two cases.

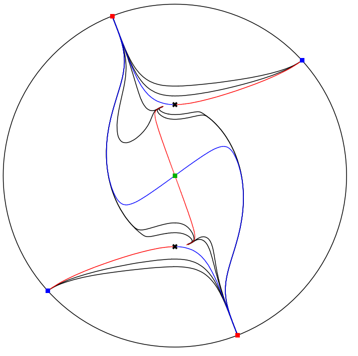


Fig. 2. Phase portrait of system (2.8) for $B_{21} = 0$, $A_{21} = B_{12} = A_{12} = A_{30} = 1$, $B_{30} = 2$, showing that $(0, \pm)$ are saddle-node nilpotent singular points.

If $B_{12} = 0$, $\Psi(\xi, \eta) \equiv 0$, $(0, \pm 1)$ are not isolated singular points.

If $2A_{12}A_{21} - A_{30} = 0$, $\Phi(\xi, \eta) = \frac{1}{2}(1 + 2A_{12}\xi + \eta)(2A_{21}\xi^2 + 2\eta + \eta^2)$, so there exists a common factor $2A_{21}\xi^2 + 2\eta + \eta^2$ in $\Phi(\xi, \eta)$ and $\Psi(\xi, \eta)$, implying that $(0, \pm 1)$ are not isolated singular points. $\square$

Remark 2.2. It follows from the Bezout theorem in Algebraic curve theory that the sum of numbers of all intersections of two cubic-degree polynomials in complex projective plane is exactly $3^2 = 9$. So the sum of multiplicity of all finite singular points of system (2.8) (real or complex) is not more than 9, and the multiplicity of nilpotent singular points $(0, \pm 1)$ of (2.8) is 4 at most.

Proposition 2.3. *The nilpotent singular points* $(0, \pm 1)$ *of* (2.8) *are saddle-node points if* $\alpha_2 = \alpha_3 = 0$, $\alpha_4\beta_1 \neq 0$, *degenerate singular points if* $\beta_1 = \alpha_2 = \alpha_3 = 0$, $\alpha_4 \neq 0$, *and saddle points if* $\alpha_2 = 0, \alpha_3 > 0$.

Example 2.2. When $B_{21} = 0, A_{21} = B_{12} = A_{12} = A_{30} = 1,\ B_{30} = 2$, obviously $\alpha_2 = \alpha_3 = 0$, $\alpha_4\beta_1 \neq 0$, so $(0, \pm 1)$ are saddle-node points, see Fig. 2. When $B_{21} = 0, A_{21} = A_{12} = A_{30} = 1$, $B_{12} = -1$, $B_{30} = -2$, obviously $\alpha_2 = \alpha_3 = \beta_1 = 0$, $\alpha_4 \neq 0$, so $(0, \pm 1)$ are degenerate singular points, see Fig. 3.

Because the multiplicity of a nilpotent focus or center is an odd positive integer which is greater than 1, Proposition 2.2 indicates that the multiplicity of $(0, \pm 1)$ is 3 if $(0, \pm 1)$ are nilpotent foci or centers of system (2.8). More precisely, we have the following result: $(0, \pm 1)$ are nilpotent foci or centers with multiplicity 3 of system (2.8) if and only if

$$\alpha_2 = 0, \quad \alpha_3 < 0, \quad \Delta = \beta_1^2 + 8\alpha_3 < 0,$$

namely

$$B_{21} = 0, \quad B_{30} - 2A_{21}B_{12} < 0, \quad 4(A_{21} - B_{12})^2 + 8B_{30} < 0.$$

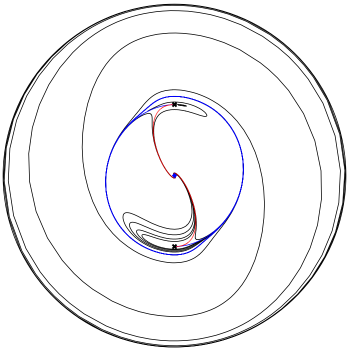


Fig. 3. Phase portrait of system (2.8) for $B_{21} = 0$, $A_{21} = A_{12} = A_{30} = 1$, $B_{12} = -1$, $B_{30} = -2$, showing that $(0, \pm)$ are degenerate nilpotent singular points.

Otherwise, $(0, \pm 1)$ are degenerate points with multiplicity 3 of system (2.8) if

$$B_{21} = 0, \quad B_{30} - 2A_{21}B_{12} < 0, \quad 4(A_{21} - B_{12})^2 + 8B_{30} \geq 0.$$

Remark 2.1 tells us that we can always choose proper r to satisfy that $\alpha_3 = -2$, yielding $B_{30} = -2 + 2A_{21}B_{12}$. Now, summarizing the above results, we have our first main theorem.

Theorem 2.1. *Suppose* $(0, \pm 1)$ *are nilpotent foci or centers of system* (1.15) *with multiplicity 3. By proper linear state variable transformation and time rescaling, system* (1.15) *can be transformed to*

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}y + \frac{1}{2}y^3 - A_{12}x + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2, \\ \frac{dy}{dt} &= -B_{12}x + (-2 + 2A_{21}B_{12})x^3 + B_{12}xy^2, \end{aligned} \tag{2.13}$$

and $(0, \pm 1)$ *are nilpotent foci or centers of system* (2.13) *with multiplicity 3 if and only if*

$$\Delta = 4(A_{21} + B_{12})^2 - 16 < 0. \tag{2.14}$$

3. Lyapunov constants

Consider the following system,

$$\begin{aligned} \frac{dx}{dt} &= y + a_{20}x^2 + \sum_{k+2j=3}^{\infty} a_{kj}x^k y^j = X(x, y), \\ \frac{dy}{dt} &= b_{11}xy + b_{30}x^3 + \sum_{k+2j=4}^{\infty} b_{kj}x^k y^j = Y(x, y), \end{aligned} \tag{3.1}$$

where $X(x, y)$ and $Y(x, y)$ are power series in x, y with nonzero radius. The origin of system (3.1) is a nilpotent singular point with multiplicity 3 if and only if $b_{30} - a_{20}b_{11} \neq 0$. When $b_{30} -$

$a_{20}b_{11} < 0$, the origin of system (3.1) is a nilpotent focus or center if $(2a_{20} - b_{11})^2 + 8b_{30} < 0$, and a degenerate singular point if $(2a_{20} - b_{11})^2 + 8b_{30} > 0$.

For system (3.1) with a nilpotent focus or center, Liu and Li [31] developed an inverse integrating factor method for computing the Lyapunov constants of the system, as stated in the following theorem.

Theorem 3.1. *For system* (3.1)*, a power series can be obtained as*

$$M(x, y) = \left[y^2 + \frac{1}{2}(2a_{20} - b_{11})x^2 y - \frac{1}{2}b_{30}x^4\right] + \sum_{k+2j=5}^{\infty} c_{kj}x^k y^j,$$

which satisfies

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}}\sum_{m=1}^{\infty} \nu_m(2m - 4s - 3)x^{2m+4},$$

where s is an integer.

The recursive formulas for computing c_{kj} and ν_m can be found in Theorem 4.5 in [31], ν_m is the mth Lyapunov constant of system (3.1) at the origin.

Now we compute the first five Lyapunov constants at $(0, \pm 1)$ of system (2.13). Denote

$$A_{21} = 2\mu - B_{12} \tag{3.2}$$

Theorem 2.1 shows that the singular points $(0, \pm 1)$ of system (2.13) are foci or centers if $\mu^2 < 1$ and degenerate singular points if $\mu^2 > 1$.

There are two cases in the calculations of the Lyapunov constants: $\mu = 0$ and $\mu \neq 0$.

3.1. $\mu = 0$

When $\mu = 0$, system (2.13) becomes

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}y + \frac{1}{2}y^3 - A_{12}x + A_{30}x^3 - B_{12}x^2 y + A_{12}xy^2, \\ \frac{dy}{dt} &= -B_{12}x - (2 + 2B_{12}^2)x^3 + B_{12}xy^2. \end{aligned} \tag{3.3}$$

Theorem 3.2. *The first two Lyapunov constants at* $(0, \pm 1)$ *of system* (3.3) *are*

$$\begin{aligned} \nu_1 &= \frac{1}{3}(3A_{30} + 2A_{12}B_{12}), \\ \nu_2 &= \frac{4}{45}A_{12}(-9 + 12A_{12}^2 B_{12} + 8A_{12}^2 B_{12}^3). \end{aligned}$$

Then, we have the following result.

Theorem 3.3. *The first two Lyapunov constants at* $(0, \pm 1)$ *of system* (3.3) *are zero if and only if one of the following conditions holds:*

$$A_{30} = A_{12} = 0; \tag{3.4}$$

$$A_{30} = \frac{-2}{3} A_{12} B_{12},\ A_{12}^2 = \frac{9}{4B_{12}(3 + 2B_{12}^2)}. \tag{3.5}$$

In Section 4, we will prove that the two conditions (3.4) and (3.5) are actually the conditions for $(0, \pm 1)$ of system (3.3) to be centers.

3.2. $\mu \neq 0$

In this case, the computation and simplification of Lyapunov constants at $(0, \pm 1)$ of system (2.13) can be done in several steps.

Step 1. By applying Theorem 3.1, we can get the Lyapunov constants as given below.

Proposition 3.1. *When* $\mu \neq 0$*, the first five Lyapunov constants at* $(0, \pm 1)$ *of system* (2.13) *are*

$$\begin{aligned}
\nu_1 &= \frac{H_1}{15}, \\
\nu_2 &= \frac{2H_2}{1575(9 - 8\mu^2)}, \\
\nu_3 &= \frac{H_3}{330750\mu(4 - 3\mu^2)(9 - 8\mu^2)}, \\
\nu_4 &= \frac{2H_4}{1031443875\mu(25 - 16\mu^2)(9 - 8\mu^2)(4 - 3\mu^2)(1 + 8\mu^2)}, \\
\nu_5 &= \frac{H_5}{424789845480000\mu^2(25 - 16\mu^2)(4 - 3\mu^2)(1 + 3\mu^2)(9 - 5\mu^2)(9 - 8\mu^2)(1 + 8\mu^2)},
\end{aligned}$$

where

$$H_1 = 3A_{30}(5 - 4B_{12}\mu) + 2A_{12}(5B_{12} - 18\mu - 12B_{12}^2\mu + 24B_{12}\mu^2), \tag{3.6}$$

and $H_2,\ H_3,\ H_4$ *and* H_5 *are polynomials in* $A_{30},\ A_{12},\ B_{12}$ *and* μ*, and contains respectively* 54*,* 259*,* 908*, and* 2445 *terms.*

Step 2. Computing the resultants of H_1 respectively with $H_2,\ H_3$ and H_4 about A_{12} yields

$$\begin{aligned}
\mathrm{Res}(H_1, H_2, A_{12}) &= 360A_{30}(9 - 8\mu^2)R_1, \\
\mathrm{Res}(H_1, H_3, A_{12}) &= 80640A_{30}\mu(4 - 3\mu^2)R_2, \\
\mathrm{Res}(H_1, H_4, A_{12}) &= 816480A_{30}(25 - 16\mu^2)(1 + 8\mu^2)R_3,
\end{aligned}$$

where $R_1,\ R_2$ and R_3 are polynomials in B_{12}, A_{30}^2 and μ, and have 30, 149 and 428 terms, respectively.

Step 3. Computing the resultants of R_1 respectively with R_2 and R_3 about A_{30}, we obtain

$$\begin{aligned}\mathrm{Res}(R_1, R_2, A_{30}) &= 4200h_0^4h_1h_2h_3\mu(9-8\mu^2)G_1,\\ \mathrm{Res}(R_1, R_3, A_{30}) &= 201600h_0^6h_1h_2h_3\mu^2(4-3\mu^2)G_2,\end{aligned}$$

where G_1 and G_2 are polynomials in B_{12} and μ, and have 22 and 82 terms, respectively, and

$$\begin{aligned}h_0 &= 5B_{12} - 18\mu - 12B_{12}^2\mu + 24B_{12}\mu^2,\\ h_1 &= 1 + B_{12}^2 - 2B_{12}\mu,\\ h_2 &= 3 + 8B_{12}^2 - 12B_{12}\mu,\\ h_3 &= 9 + 4B_{12}^2 - 12B_{12}\mu.\end{aligned} \tag{3.7}$$

Remark 3.1. Since

$$\mathrm{Res}(5-4B_{12}\mu, h_0, B_{12}) = \mu(25-24\mu^2),$$

$5-4B_{12}\mu$ and h_0 can not equal zero simultaneously when $0 \neq \mu^2 < 1$. Therefore, (3.6) yields $A_{30} = 0$ when $\nu_1 = h_0 = 0$.

Finally, we have

$$\begin{aligned}\mathrm{Res}(G_1, G_2, B_{12}) = {}& \mu^{18}(16+15\mu^2)(6-5\mu^2)^2(9-8\mu^2)^{14}(16-15\mu^2)^2\\ & \times (25-21\mu^2)(25-24\mu^2)^{28}(49-40\mu^2)^4 g(\mu),\end{aligned}$$

where

$$g(\mu) = -104976 + 288265\mu^2 - 253200\mu^4 + 72000\mu^6.$$

Above discussions lead to the following results.

Proposition 3.2. *If $A_{30}h_1h_2h_3 \neq 0$ and $0 \neq \mu^2 < 1$, then $g(\mu) = 0$ when $\nu_1 = \nu_2 = \nu_3 = \nu_4 = 0$.*

Proposition 3.3. $g(\mu) = \nu_1 = \nu_2 = \nu_3 = \nu_4 = 0$ *if and only if*

$$\begin{aligned}& g(\mu) = 0,\\ & B_{12} = \frac{\mu}{6480}(161791 - 323760\mu^2 + 158400\mu^4),\\ & A_{12} = \frac{-A_{30}\mu}{1456110}(41268311 - 86087760\mu^2 + 40190400\mu^4),\\ & A_{30}^2 = \frac{\mu}{105336450}(-27228293 - 878342520\mu^2 + 660556800\mu^4).\end{aligned} \tag{3.8}$$

$g(\mu) = 0$ has six real solutions $\mu_1^\pm,\ \mu_2^\pm,\ \mu_3^\pm$:

$$\begin{aligned}
\mu_1^\pm &= \pm\sqrt{\tfrac{211}{180} + \tfrac{\sqrt{205}}{36}\cos\left(\tfrac{1}{3}\arctan\tfrac{6\sqrt{2319}}{511}\right)} = \pm 0.88337\cdots, \\
\mu_2^\pm &= \pm\sqrt{\tfrac{211}{180} + \tfrac{\sqrt{205}}{36}\cos\left(\tfrac{1}{3}\arctan\tfrac{6\sqrt{2319}}{511} + \tfrac{\pi}{3}\right)} = \pm 1.14428\cdots, \\
\mu_3^\pm &= \pm\sqrt{\tfrac{211}{180} + \tfrac{\sqrt{205}}{36}\cos\left(\tfrac{1}{3}\arctan\tfrac{6\sqrt{2319}}{511} - \tfrac{\pi}{3}\right)} = \pm 1.19455\cdots.
\end{aligned} \tag{3.9}$$

So only $\mu_1^\pm$ satisfy $|\mu_1^\pm| < 1$. When $\mu = 0.88337\cdots$, $A_{30}^2 < 0$, and so it is not a solution. For the unique solution $\mu = -0.88337\cdots$, (3.8) yields that

$$\begin{aligned}
&\mu = -0.88337\cdots, \qquad B_{12} = -0.7638\cdots, \\
&A_{30} = \pm 1.6134\cdots, \qquad A_{12} = \mp 1.4058\cdots,
\end{aligned} \tag{3.10}$$

under which we obtain

$$v_5 = \mp 0.015\cdots \neq 0.$$

This clearly shows that

Theorem 3.4. *The two singular points* $(0, \pm 1)$ *of system* (2.13) *are* 5*th weak foci for the solutions given in* (3.10).

Remark 3.2. Normal form theory and the method of computation developed in [48] can be also applied to compute the Lyapunov constants at $(0, \pm 1)$ of system (2.13). A cross check using other computation method has verified the above obtained first five Lyapunov constants.

4. Integrability and center conditions

In this section, we are devoted to study the integrability of system (2.13) by using the inverse integrating factor method. First of all, regarding the integrability of system (2.13), we apply Theorem 3.3 and Proposition 3.2 to obtain the following result.

Theorem 4.1. *The first five Lyapunov constants at the two singular points* $(0, \pm 1)$ *of system* (2.13) *are all zero if and only if one of the following conditions holds:*

$$
\begin{aligned}
&\mathrm{I}_1: \ A_{21}=-B_{12},\ A_{30}=\frac{-2}{3}A_{12}B_{12},\ A_{12}^2=\frac{9}{4B_{12}(3+2B_{12}^2)};\\
&\mathrm{I}_2: \ A_{30}=0,\ A_{12}=0;\\
&\mathrm{I}_3: \ A_{21}=\frac{-9}{4A_{12}^2},\ A_{30}=\frac{-3}{2A_{12}},\ B_{12}=\frac{-4A_{12}^2}{9};\\
&\mathrm{I}_4: \ B_{12}=\frac{-3A_{12}^2}{8},\ A_{30}=-\frac{16+3A_{12}^4}{12A_{12}},\ A_{21}=-\frac{32+3A_{12}^4}{24A_{12}^2};\\
&\mathrm{I}_5: \ A_{21}=\frac{2-A_{12}^4}{2A_{12}^2},\ A_{30}=-A_{12}^3,\ B_{12}=\frac{3}{2}A_{12}^2.
\end{aligned}
\tag{4.1}
$$

If the condition I_1 in Theorem 4.1 holds, system (2.13) can be rewritten as

$$
\begin{aligned}
\frac{dx}{dt}&=\frac{1}{6}(-6A_{12}x-4A_{12}B_{12}x^3-3y-6B_{12}x^2y+6A_{12}xy^2+3y^3),\\
\frac{dy}{dt}&=-B_{12}x-2(1+B_{12}^2)x^3+B_{12}xy^2,
\end{aligned}
\tag{4.2}
$$

where

$$
A_{12}^2=\frac{9}{4B_{12}(3+2B_{12}^2)}. \tag{4.3}
$$

Proposition 4.1. *System* (4.2) *has the algebraic integral curve,*

$$
\begin{aligned}
f_1=&\,3(9+8B_{12}^2)(2B_{12}x^2-2A_{12}xy-y^2)\\
&+(3+4B_{12}^2)(12x^4+12B_{12}^2x^4-4A_{12}B_{12}x^3y-12B_{12}x^2y^2+6A_{12}xy^3+3y^4),
\end{aligned}
$$

and an inverse integrating factor

$$
M_1=f_1.
$$

It is easy to obtain the following result from Proposition 4.1.

Lemma 4.1. $(0, \pm 1)$ *are centers of system* (2.13) *when the condition* I_1 *holds.*

Proof. When the condition I_1 is satisfied, $A_{21}+B_{12}=0$, which implies that (2.14) holds, and so $(0, \pm 1)$ are monodromic critical points. Then it follows from Proposition 4.1 that system (2.13) has an inverse integrating factor. Thus, $(0, \pm 1)$ are centers of system (2.13). □

Example 4.1. An example, as depicted in Fig. 4, shows that $(0, \pm 1)$ are centers of (2.13) when the condition I_1 holds.

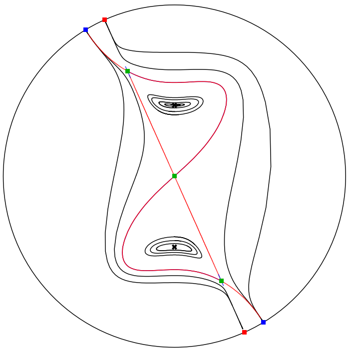


Fig. 4. Phase portrait of system (2.13), showing that $(0, \pm 1)$ are centers when the condition I_1 holds.

If the condition I_2 in Theorem 4.1 is satisfied, system (2.13) is reduced to

$$\begin{aligned} \frac{dx}{dt} &= \frac{-y}{2} + A_{21}x^2y + \frac{y^3}{2}, \\ \frac{dy}{dt} &= -B_{12}x^2 - 2(1 + A_{21}B_{12})x^3 + B_{12}xy^2. \end{aligned} \tag{4.4}$$

For system (4.4), we have the following result.

Proposition 4.2. *System* (4.2) *has two algebraic integral curves:*

$$f_2 = 1 - 2(A_{21} - B_{12})x^2 + 4(1 - A_{21}B_{12})x^4 - 2y^2 + 2(A_{21} - B_{12})x^2y^2 + y^4,$$
$$f_3 = \exp \arctan \tfrac{\sqrt{4-(A_{21}+B_{12})^2}x^2}{1-(A_{21}-B_{12})x^2-y^2}.$$

Moreover, system (4.4) *has an inverse integrating factor*

$$M_2 = f_2,$$

and a first integral

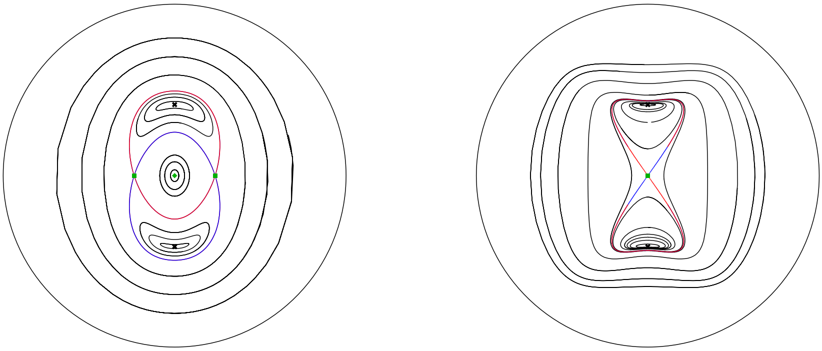
$$F_1 = f_2^{\sqrt{4-(A_{21}+B_{12})^2}} f_3^{2(A_{21}+B_{12})}. \tag{4.5}$$

One immediate consequence is obtained below.

Lemma 4.2. $(0, \pm 1)$ *are centers of* (2.14) *if both the conditions* I_2 *and* $|A_{21} + B_{12}| < 2$ *hold.*

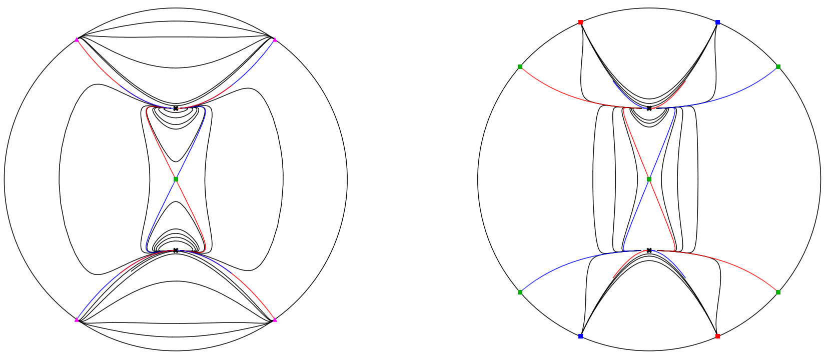
Proof. When the condition I_2 holds, system (4.4) has the first integral F_1 given in (4.5), which exists for $|A_{21} + B_{12}| < 2$. Hence, $(0, \pm 1)$ are centers of (2.14) if and only if both the conditions I_2 and $|A_{21} + B_{12}| < 2$ hold. $\square$

Example 4.2. Two cases are illustrated in Fig. 5, showing that $(0, \pm 1)$ are centers of (2.13) when both conditions I_2 and $|A_{21} + B_{12}| < 2$ are satisfied.



(a) (b)

Fig. 5. Phase portraits of system (2.13), showing that $(0, \pm 1)$ are centers when both the conditions I_2 and $|A_{21} + B_{12}| < 2$ hold: (a) $A_{21} = 1,\ B_{12} = -1$; and (b) $A_{21} = 0,\ B_{12} = 1$.



(a) (b)

Fig. 6. Phase portraits of system (2.13), showing that $(0, \pm 1)$ are not centers when the condition I_2 holds but the condition $|A_{21} + B_{12}| < 2$ is not satisfied: (a) $A_{21} = 0,\ B_{12} = 2$; and (b) $A_{21} = 0,\ B_{12} = 3$.

Example 4.3. Two cases are depicted in Fig. 6, indicating that $(0, \pm 1)$ are not centers of (2.13) if the condition I_2 is satisfied, but the condition $|A_{21} + B_{12}| < 2$ does not hold.

If the condition I_3 in Theorem 4.1 holds, system (2.13) becomes

$$\begin{aligned} \frac{dx}{dt} &= -A_{12}x - \frac{3x^3}{2A_{12}} - \frac{y}{2} - \frac{9x^2y}{4A_{12}^2} + A_{12}xy^2 + \frac{y^3}{2}, \\ \frac{dy}{dt} &= \frac{4}{9}A_{12}^2 x(1-y)(1+y). \end{aligned} \tag{4.6}$$

Proposition 4.3. *System (4.6) has algebraic integral curves:*

$$f_4 = 2A_{12}x + 3y, \qquad f_5 = 1 - y, \qquad f_6 = 1 + y, \tag{4.7}$$

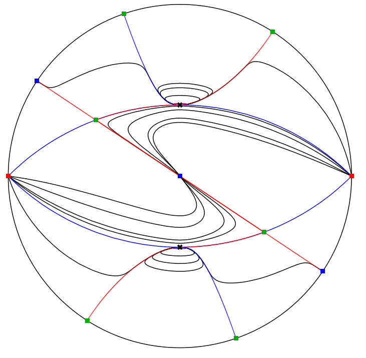


Fig. 7. Phase portrait of system (2.13), showing that $(0, \pm 1)$ are not centers when the condition I_3 holds.

which allow to construct an inverse integrating factor,

$$M_3 = f_4^3 (f_5 f_6)^{\frac{81+16A_{12}^4}{16A_{12}^4}},$$

and a first integral,

$$F_2 = \frac{(4A_{12}x + 3y)}{9(2A_{12}x + 3y)^2}(1 - y^2)^{\frac{-81}{16A_{12}^4}} - \frac{3}{8A_{12}^4} \int (1 - y^2)^{\frac{-(81+16A_{12}^4)}{16A_{12}^4}} dy.$$

Remark 4.1. It follows from (4.7) that there are two straight line solutions $y = \pm 1$ which pass through $(0, \pm 1)$, so $(0, \pm 1)$ are not centers of system (4.6).

Example 4.4. An example, shown in Fig. 7, indicates that $(0, \pm 1)$ are not centers of (2.13) when the condition I_3 holds.

In fact, if the condition I_3 in Theorem 4.1 holds, it is easy to get that $A_{21} + B_{12} = -(\frac{9}{4A_{12}^2} + \frac{4A_{12}^2}{9}) \le -2$ and $\Delta = 4(A_{21} + B_{12})^2 - 16 > 0$. So $(0, \pm 1)$ are not centers or foci, but degenerate singular points.

If the condition I_4 in Theorem 4.1 holds, system (2.13) takes the form:

$$\begin{aligned} \frac{dx}{dt} &= -A_{12}x - \frac{(16 + 3A_{12}^4)x^3}{12A_{12}} - \frac{y}{2} - \frac{(32 + 3A_{12}^4)x^2 y}{24A_{12}^2} + A_{12}xy^2 + \frac{y^3}{2}, \\ \frac{dy}{dt} &= \frac{3A_{12}^2 x}{8} + \frac{(-32 + 3A_{12}^4)x^3}{32} - \frac{3}{8}A_{12}^2 xy^2. \end{aligned} \tag{4.8}$$

Proposition 4.4. *The two singular points* $(0, \pm 1)$ *are centers of* (4.8) *if* $\Delta = 4(A_{21} + B_{12})^2 - 16 < 0$ *with*

$$A_{12} \in (-1.77615\cdots, -0.919402\cdots) \bigcup (0.919402\cdots, 1.77615\cdots).$$

Proof. Consider the following transformations,

$$\begin{aligned} u &= \frac{2A_{12}x}{A_{12}x+2y}, \\ v &= \left[3A_{12}^4 - 3A_{12}^4 u - \frac{3}{4}A_{12}^4(A_{12}x-2y)^2 + 2u^2(A_{12}x-2y)^2\right]h(u), \\ \frac{dt}{d\tau} &= \frac{6A_{12}^3(1-u)}{1+u}\,h(u), \end{aligned} \tag{4.9}$$

where

$$h(u) = \begin{cases} (1+u)\left(1-\dfrac{8u^2}{3A_{12}^4}\right)^{\frac{-(16+3A_{12}^4)}{2(8-3A_{12}^4)}}(1-u^2)^{\frac{-2(2-3A_{12}^4)}{8-3A_{12}^4}}, & \text{if } 8-3A_{12}^4 \neq 0, \\ \dfrac{1+u}{(1-u^2)^{\frac{3}{2}}}\exp\dfrac{3u^2}{2(1-u^2)}, & \text{if } 8-3A_{12}^4 = 0, \end{cases}$$

satisfying

$$h'(u) = \frac{3A_{12}^4 + (16+9A_{12}^4)u - 8u^2 - 16u^3}{(1-u)(1+u)(3A_{12}^4-8u^2)}h(u), \quad h(0)=1.$$

Under the transformation (4.9), the nilpotent singular points $(0, \pm 1)$ of system (4.8) in the x–y plane are transformed to the origin in the u–v plane, and system (4.8) becomes a Liénard system,

$$\begin{aligned} \frac{du}{d\tau} &= -v, \\ \frac{dv}{d\tau} &= f(u)v + g(u), \end{aligned} \tag{4.10}$$

where

$$f(u) = \begin{cases} \dfrac{-2(8+3A_{12}^4)u\left(1+\frac{8u^2}{8+3A_{12}^4}\right)}{\left(1-\frac{8u^2}{3A_{12}^4}\right)^{\frac{32-3A_{12}^4}{2(8-3A_{12}^4)}}(1-u^2)^{\frac{2(2-3A_{12}^4)}{8-3A_{12}^4}}}, & \text{if } 8-3A_{12}^4 \neq 0, \\ \dfrac{-16u(2+u^2)}{(1-u^2)^{\frac{5}{2}}}\exp\dfrac{3u^2}{2(1-u^2)}, & \text{if } 8-3A_{12}^4 = 0, \end{cases}$$

$$g(u) = \begin{cases} 72A_{12}^4u^3\left(1-\dfrac{8u^2}{3A_{12}^4}\right)^{\frac{-24}{8-3A_{12}^4}}(1-u^2)^{\frac{9A_{12}^4}{8-3A_{12}^4}}, & \text{if } 8-3A_{12}^4 \neq 0, \\ \dfrac{192u^3}{(1-u^2)^3}\exp\dfrac{3u^2}{1-u^2}, & \text{if } 8-3A_{12}^4 = 0. \end{cases}$$

Since $f(u)$ and $g(u)$ are odd functions, system (4.10) is symmetric with the v-axis, implying that the origin of system (4.10) is a nilpotent center [30]. That is, the singular points $(0, \pm 1)$ are nilpotent centers of (4.8). □

If the condition I_5 in Theorem 4.1 holds, system (2.13) is reduced to

$$\begin{aligned} \frac{dx}{dt} &= \frac{x^2 y}{A_{12}^2} - \frac{1}{2}(2A_{12}x + y)(1 + A_{12}^2 x^2 - y^2), \\ \frac{dy}{dt} &= x^3 - \frac{3}{2}A_{12}^2 x(1 + A_{12}^2 x^2 - y^2). \end{aligned} \tag{4.11}$$

Proposition 4.5. *System* (4.11) *has three algebraic integral curves,*

$$\begin{aligned} f_7 &= A_{12}x - y, \\ f_8 &= 1 + A_{12}^2 x^2 - y^2, \\ f_9 &= A_{12}^6 + A_{12}^2(A_{12}^2 x^2 - y^2) \\ &\quad + (1 - A_{12}^4)(A_{12}x - y)^2(2x^2 - 3A_{12}^4 x^2 + 2A_{12}^3 xy + A_{12}^2 y^2), \end{aligned}$$

yielding an inverse integrating factor,

$$M_4 = f_7^{-2} f_8 f_9, \tag{4.12}$$

and a first integral,

$$F_3 = f_8^{-1} f_9^{A_{12}^4}.$$

In fact, if the condition I_5 in Theorem 4.1 holds, it is easy to obtain that

$$A_{21} + B_{12} = -(\frac{9}{4A_{12}^2} + \frac{4A_{12}^2}{9}) \le -2, \quad \Delta = 4(A_{21} + B_{12})^2 - 16 > 0,$$

which clearly indicates that $(0, \pm 1)$ are not centers or foci, but degenerate singular points.

Remark 4.2. It follows from (4.12) that there are one straight line solution $y = A_{12}x$ and one hyperbola solution $1 + A_{12}^2 x^2 - y^2 = 0$ which pass through $(0, \pm 1)$, so $(0, \pm 1)$ are not centers of system (4.11).

Example 4.5. An example, as depicted in Fig. 8, shows that $(0, \pm 1)$ are not centers of (2.13) when the condition I_5 holds.

Summarizing the above discussions and results, we obtain the following theorem.

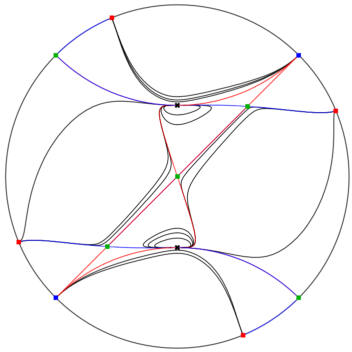


Fig. 8. Phase portrait of system (2.13), showing that $(0, \pm 1)$ are not centers when the condition I_5 holds.

Theorem 4.2. *The nilpotent singular points* $(0, \pm 1)$ *of system* (2.13) *are centers if one of the following conditions holds:*

$$\begin{aligned} \mathrm{C}_1: &\quad A_{12} = A_{30} = 0, \quad |A_{21} + B_{12}| < 2; \\ \mathrm{C}_2: &\quad A_{21} + B_{12} = 3A_{30} + 2A_{12}B_{12} = 8A_{12}^2 B_{12}^3 + 12A_{12}^2 B_{12} - 9 = 0; \\ \mathrm{C}_3: &\quad B_{12} = \frac{-3A_{12}^2}{8}, \ A_{30} = -\frac{16 + 3A_{12}^4}{12A_{12}}, \ A_{21} = -\frac{32 + 3A_{12}^4}{24A_{12}^2}, \\ &\quad A_{12} \in (-1.77615\cdots, \ -0.919402\cdots) \bigcup (0.919402\cdots, \ 1.77615\cdots). \end{aligned}$$

5. A new perturbation scheme for bifurcation of limit cycles from multiple nilpotent critical points

In [31], bifurcation of limit cycles from multiple nilpotent critical points of planar dynamical systems was discussed, and limit cycles were obtained by changing the stability of the multiple nilpotent critical points. In [26], an interesting bifurcation phenomenon was found, showing that a nilpotent focus of planar dynamical systems can be broken into two elementary weak foci and a saddle, and limit cycles can then bifurcate from the two weak foci. This bifurcation method was called double bifurcation. In this section, we generalize the double bifurcation method and introduce a new bifurcation scheme to obtain more limit cycles from multiple nilpotent critical points of system (2.13).

Under the condition (3.2), H_1 given in (3.6) is reduced to

$$\begin{aligned} H_1 = {} & 3A_{30}(5 - 2A_{21}B_{12} - 2B_{12}^2) \\ & - 2A_{12}(9A_{21} + 4B_{12} - 6A_{21}^2 B_{12} - 6A_{21}B_{12}^2). \end{aligned} \tag{5.1}$$

Suppose the singular points $(0, \pm 1)$ of system (2.13) are first order nilpotent foci. Then

$$(A_{21} + B_{12})^2 < 4 \quad \text{and} \quad H_1 \neq 0. \tag{5.2}$$

Further, assume that

$$A_{21} + B_{12} \neq 0, \tag{5.3}$$

and consider a perturbed system of (2.13),

$$\begin{aligned}\frac{dx}{dt} &= -\frac{1}{2}y + \frac{1}{2}y^3 - A_{12}x + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + \sigma^2\varphi(x, y),\\ \frac{dy}{dt} &= -B_{12}x + (-2 + 2A_{21}B_{12})x^3 + B_{12}xy^2 + \sigma^2\psi(x, y),\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}\varphi(x, y) &= 2\varepsilon_1 x,\\ \psi(x, y) &= -(1 + \sigma^2\varepsilon_1^2)x + \frac{(2A_{12}A_{21} - 3A_{30} - \varepsilon_2)}{2(A_{21} + B_{12})}x^2y,\end{aligned}$$

in which σ, ε_1 and ε_2 are small perturbation parameters.

The new bifurcation scheme can be divided into two steps. In the first step, let $\sigma = 0$, and suppose that the conditions given in (3.10) hold. Then, the two singular points $(0, \pm 1)$ of system (2.13) are 5th-order weak foci, and so four limit cycles can be obtained by changing the stability of the multiple nilpotent critical points with appropriate perturbations on the system coefficients.

In the second step, when $0 < |\sigma| \ll 1$, $(0, \pm 1)$ are element weak foci of system (5.4). There are four complex singular points $(\pm x_0, \pm y_0)$ of system (5.4) in the neighborhood of $(0, \pm 1)$ with

$$x_0 = \frac{i\sigma}{\sqrt{2}} + o(\sigma), \quad y_0 = 1 - \frac{A_{21}}{2}\sigma^2 + o(\sigma^2).$$

We introduce a change of state variables and a time rescaling,

$$x = \sigma\xi, \quad y = 1 - \sigma^3\varepsilon_1\xi - \sigma^2\eta, \quad t = \frac{\tau}{\sigma}, \tag{5.5}$$

into system (5.4) to obtain

$$\begin{aligned}\frac{d\xi}{d\tau} &= \sum_{k=0}^{7}\Phi_k(\xi, \eta)\sigma^k = \Phi(\xi, \eta),\\ \frac{d\eta}{d\tau} &= \sum_{k=0}^{8}\Psi_k(\xi, \eta)\sigma^k = \Psi(\xi, \eta),\end{aligned} \tag{5.6}$$

where $\Phi_k(\xi, \eta)$, $\Psi_k(\xi, \eta)$ are polynomials in ξ, η, ε_1 and ε_2. In particular,

$$\begin{aligned}\Phi_0(\xi, \eta) &= A_{21}\xi^2 - \eta,\\ \Psi_0(\xi, \eta) &= \xi + 2(1 - A_{21}B_{12})\xi^3 + 2B_{12}\xi\eta,\\ \Phi_1(\xi, \eta) &= \xi(\varepsilon_1 - 2A_{12}\eta + A_{30}\xi^2),\\ \Psi_1(\xi, \eta) &= \frac{(3A_{30} - 2A_{12}A_{21} + \varepsilon_2)\xi^2}{2(A_{21} + B_{12})} - \varepsilon_1(A_{21} - 2B_{12})\xi^2 + \varepsilon_1\eta.\end{aligned}$$

The linearized system of (5.6) in the neighborhood of the origin is given by

$$\frac{d\xi}{d\tau} = -\eta + \sigma\varepsilon_1\xi, \qquad \frac{d\eta}{d\tau} = \xi + \sigma\varepsilon_1\eta. \tag{5.7}$$

Consequently, when $0 < |\sigma| \ll 1$, the nilpotent singular point (0, 1) of system (5.4) is changed to the origin of system (5.6) by the transformation (5.5), and it is a strong focus when $\varepsilon_1 \neq 0$. The divergence of the origin of system (5.6) is

$$V_0 = 2\sigma\varepsilon_1, \tag{5.8}$$

showing that the origin of (5.6) is an elementary weak focus when $\varepsilon_1 = 0$.

Proposition 5.1. *The origin of system* (5.6) *is a center when* $\sigma = 0$.

Proof. As a matter of fact, when $\sigma = 0$, system (5.6) can be written as

$$\frac{d\xi}{d\tau} = \Phi_0(\xi, \eta), \qquad \frac{d\eta}{d\tau} = \Psi_0(\xi, \eta),$$

which is symmetric with the η-axis, and so the origin of system (5.6) is a symmetric center when $\sigma = 0$. □

Now we consider the first order approximation system of (5.6), given by

$$\frac{d\xi}{d\tau} = \Phi_0(\xi, \eta) + \Phi_1(\xi, \eta)\sigma, \qquad \frac{d\eta}{d\tau} = \Psi_0(\xi, \eta) + \Psi_1(\xi, \eta)\sigma. \tag{5.9}$$

It is easy to obtain the following result.

Proposition 5.2. *When* $\varepsilon_1 = 0$, *the first two focus values at the origin of system* (5.9) *are*

$$\begin{aligned} V_1 &= \frac{-1}{4}\varepsilon_2\sigma + o(\sigma), \\ V_2|_{V_1=0} &= \frac{1}{12}H_1\sigma + o(\sigma). \end{aligned} \tag{5.10}$$

The following theorem follows from (5.8) and (5.10).

Theorem 5.1. *Suppose* $o < |\sigma| \ll 1$. *The origin of system* (5.6) *is a second-order weak focus when* $\varepsilon_1 = \varepsilon_2 = 0$, *and there can have two limit cycles bifurcating in the neighborhood of the origin of system* (5.6) *for*

$$0 < \frac{\varepsilon_1}{H_1} \ll \frac{\varepsilon_2}{H_1} \ll 1.$$

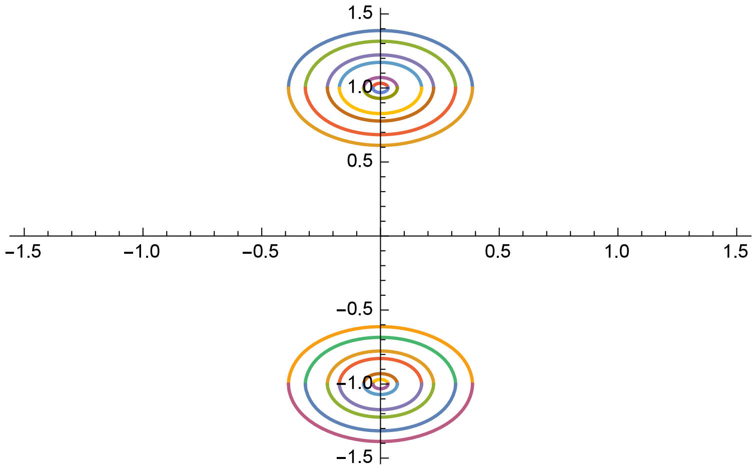


Fig. 9. Distribution of 12 limit cycles of system (5.4) when $A_{21} + B_{12} \neq 0$.

Summarizing the above results we conclude that for the 5th-order nilpotent foci $(0, \pm 1)$, four limit cycles can bifurcate around each of $(0, \pm 1)$ by changing their stability. Then, the first order nilpotent focus can be broken into an elementary second-order focus and two complex singular points. Two limit cycles can bifurcate from the elementary second order focus. Therefore, a total of six limit cycles can be obtained in each neighborhood of $(0, \pm 1)$, yielding a total of 12 small-amplitude limit cycles in such a Z_2-equivariant vector field by applying our new perturbation scheme which is different from that used for the case of elementary focus, see Fig. 9.

6. Conclusion

In this paper, we have studied a cubic Z_2-equivariant vector field with two isolated nilpotent singular points. We first introduce some transformations to simplify this system, and get a general form of the system which has two isolated nilpotent foci or centers at $(0, \pm 1)$. Then, we compute the first five Lyapunov constants of the system by using the inverse integrating factor method. Furthermore, the integrability of the system is discussed, and five conditions of integrability are obtained by using different approaches. Especially for the condition I_4, a technical transformation is developed to transform the system to a symmetric Liénard system. Among the five conditions, three of them are proved to be true center conditions for system (2.13). We present several examples for the other two conditions to illustrate that those two conditions are not sufficient for the two singular points to be centers. Finally, a new perturbation scheme is present to get two more limit cycles bifurcating from the elementary second-order focus. With the new method, 12 small-amplitude limit cycles are obtained for a cubic Z_2-equivariant vector field with isolated nilpotent singular points. This is a new lower bound obtained for such cubic systems.

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References

[1] A. Algaba, C. Garca, M. Reyes, The center problem for a family of systems of differential equations having a nilpotent singular point, J. Math. Anal. Appl. 340 (1) (2008) 32–43.

[2] M. Alvarez, A. Gasull, Monodromy and stability for nilpotent critical points, Internat. J. Bifur. Chaos 15 (4) (2005) 1253–1265.

[3] M. Alvarez, A. Gasull, Generating limits cycles from a nilpotent critical point via normal forms, J. Math. Anal. Appl. 318 (2006) 271–287.

[4] V.V. Amelbkin, N.A. Lukasevnky, A.P. Sadovski, Nonlinear Oscillations in Second Order Systems, BGY Lenin. B. I. Press, Minsk, 1992 (in Russian).

[5] Y. An, M. Han, On the number of limit cycles near a homoclinic loop with a nilpotent singular point, J. Differential Equations 258 (9) (2015) 3194–3247.

[6] N. Bautin, On the number of limit cycles appearing from an equilibrium point of the focus or center type under varying coefficients, Mat. Sb. 30 (1952) 181–196.

[7] N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Amer. Math. Soc. Transl. 100 (1954) 1–19.

[8] L. Chen, M. Wang, The relative position and number of limit cycles of a quadratic differential system, Acta Math. Sinica (Chin. Ser.) 22 (6) (1979) 751–758.

[9] I. Colak, Hamiltonian Linear Type Centers and Nilpotent Centers of Linear Plus Cubic Polynomial Vector Fields, PhD thesis, Universitat Autnoma de Barcelona, 2014.

[10] I. Colak, J. Llibre, C. Valls, Bifurcation diagrams for Hamiltonian nilpotent centers of linear plus cubic homogeneous polynomial vector fields, J. Differential Equations 262 (11) (2017) 5518–5533.

[11] F. Dias, J. Llibre, C. Valls, Polynomial Hamiltonian systems of degree 3 with symmetric nilpotent centers, Math. Comput. Simulation 144 (2018) 60–77.

[12] H. Dulac, Determination et integration dune certaine classe d'équations diffrentielles ayant par point sigulier un centre, Bull. Sci. Math. 32 (1908) 230–252.

[13] F. Dumortier, J. Llibre, C. Arts, Qualitative Theory of Planar Differential Systems, Universitext, Springer-Verlag, Berlin, 2006.

[14] E. Gamero, E. Freire, E. Ponce, Normal forms for planar systems with nilpotent linear part, in: Bifurcation and Chaos: Analysis, Algorithms, Applications, 1991, pp. 123–127.

[15] H. Giacomini, J. Giné, J. Llibre, The problem of distinguishing between a center and a focus for nilpotent and degenerate analytic systems, J. Differential Equations 227 (2) (2006) 406–426.

[16] M. Han, C. Shu, J. Yang, Polynomial Hamiltonian systems with a nilpotent critical point, Adv. Space Res. 46 (4) (2010) 521–525.

[17] M. Han, J. Yang, D. Xiao, Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle, Internat. J. Bifur. Chaos 22 (8) (2012), 10 pages.

[18] J. Jiang, J. Zhang, M. Han, Limit cycles for a class of quintic near-Hamiltonian systems near a nilpotent center, Internat. J. Bifur. Chaos 19 (6) (2009) 2107–2113.

[19] F. Li, Bifurcations of limit cycles in a quintic Lyapunov system with eleven parameters, Chaos Solitons Fractals 45 (11) (2012) 1417–1422.

[20] F. Li, Y. Liu, Y. Jin, Bifurcations of limit circles and center conditions for a class of non-analytic cubic Z_2 polynomial differential systems, Acta Math. Sinica 4 (2012) 1–14.

[21] F. Li, Y. Liu, H. Li, Center conditions and bifurcation of limit cycles at three-order nilpotent critical point in a septic Lyapunov system, Math. Comput. Simulation 81 (12) (2011) 2595–2607.

[22] J. Li, Y. Liu, New results on the study of Z_q-equivariant planar polynomial vector fields, Qual. Theory Dyn. Syst. 9 (1) (2010) 167–219.

[23] C. Li, C. Liu, J. Yang, A cubic system with thirteen limit cycles, J. Differential Equations 246 (9) (2009) 3609–3619.

[24] Y. Liu, Multiplicity of higher order singular point of differential autonomous system, J. Cent. South Univ. Technol. 30 (3) (1999) 325–326.

[25] Y. Liu, W. Huang, A cubic system with twelve small amplitude limit cycles, Bull. Sci. Math. 129 (2005) 83–98.

[26] Y. Liu, F. Li, Double bifurcation of nilpotent focus, Internat. J. Bifur. Chaos 25 (3) (2015) 1550036 (10 pages).

[27] Y. Liu, J. Li, New study on the center problem and bifurcations of limit cycles for the Lyapunov system (I), Internat. J. Bifur. Chaos 19 (9) (2009) 3087–3099.

[28] Y. Liu, J. Li, New study on the center problem and bifurcations of limit cycles for the Lyapunov system (II), Internat. J. Bifur. Chaos 19 (11) (2009) 3791–3801.
[29] Y. Liu, J. Li, Bifurcations of limit cycles and center problem for a class of cubic nilpotent system, Internat. J. Bifur. Chaos 20 (8) (2010) 2579–2584.
[30] Y. Liu, J. Li, Bifurcations of limit cycles created by a multiple nilpotent critical point of planar dynamical systems, Internat. J. Bifur. Chaos 21 (2) (2011) 497–504.
[31] Y. Liu, J. Li, On three-order nilpotent critical points: integral factor method, Internat. J. Bifur. Chaos 21 (5) (2011) 1293–1309.
[32] Y. Liu, J. Li, Complete study on a bi-center problem for the Z_2-equivariant cubic vector fields, Acta Math. Sinica 27 (7) (2011) 1379–1394.
[33] T. Liu, L. Wu, F. Li, Analytic center of nilpotent critical points, Internat. J. Bifur. Chaos 22 (08) (2012), 9 pages.
[34] R. Moussu, Symétrie et forme normale des centres et foyers dégénérés, Ergodic Theory Dynam. Systems 2 (1982) 241–251.
[35] V. Romanovski, W. Fernandes, R. Oliveira, Bi-center problem for some classes of Z_2-equivariant systems, J. Comput. Appl. Math. 320 (2017) 61–75.
[36] D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, Trans. Amer. Math. Soc. 338 (1993) 799–841.
[37] S. Shi, A concrete example of the existence of four limit cycles for quadratic systems, Sci. Sinica 23 (1980) 16–21.
[38] E. Strozyna, H. Żołądek, The analytic normal for the nilpotent singularity, J. Differential Equations 179 (2012) 479–537.
[39] F. Takens, Singularities of vector fields, Publ. Math. l'IHÉS 43 (1974) 47–100.
[40] Y. Tian, P. Yu, Bifurcation of ten small-amplitude limit cycles by perturbing a quadratic Hamiltonian system with cubic polynomials, J. Differential Equations 260 (2) (2016) 971–990.
[41] Q. Wang, Y. Liu, C. Du, Small limit cycles bifurcating from fine focus points in quartic order Z_3-equivariant vector fields, J. Math. Anal. Appl. 337 (2008) 524–536.
[42] Y. Wu, Y. Gao, M. Han, Bifurcations of the limit cycles in a Z_3-equivariant quartic planar vector field, Chaos Solitons Fractals 38 (2008) 1177–1186.
[43] J. Yang, M. Han, Limit cycle bifurcations of some Línard systems with a nilpotent cusp, Internat. J. Bifur. Chaos 20 (11) (2010) 3829–3839.
[44] J. Yang, P. Yu, Nine limit cycles around a singular point by perturbing a cubic Hamiltonian system with a nilpotent center, Appl. Math. Comput. 298 (2017) 141–152.
[45] J. Yang, L. Zhao, The cyclicity of period annuli for a class of cubic Hamiltonian systems with nilpotent singular points, J. Differential Equations 263 (9) (2017) 5554–5581.
[46] P. Yu, M. Han, Small limit cycles bifurcation from fine focus points in cubic order Z_2-equivariant vector fields, Chaos Solitons Fractals 24 (2005) 329–348.
[47] P. Yu, M. Han, Y. Yuan, Analysis on limit cycles of Z_q-equivariant polynomial vector fields with degree 3 or 4, J. Math. Anal. Appl. 322 (2006) 51–65.
[48] P. Yu, F. Li, Bifurcation of limit cycles in a cubic-order planar system around a nilpotent critical point, J. Math. Anal. Appl. 453 (2) (2017) 645–667.
[49] Z. Zhang, C. Li, On the number of limit cycles of a class of quadratic Hamiltonian systems under quadratic perturbations, Adv. Math. 26 (5) (1992) 445–460.
[50] Z. Zhang, T. Ding, W. Huang, Z. Dong, The Qualitative Theory of Differential Equations, Science Press, Beijing, 1997.
[51] H. Żołądek, Quadratic systems with center and their perturbations, J. Differential Equations 109 (2) (1994) 223–273.