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Complex isochronous centers and linearization transformations for cubic Z_2-equivariant planar systems

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Abstract

In this paper, we study complex isochronous center problem for cubic complex planar vector fields, which are assumed to be Z_2-equivariant with two symmetric centers. Such integrable systems can be classified as 11 cases. A complete classification is given on the complex isochronous centers and proven to have a total of 54 cases. All the algebraic conditions for the 54 cases are derived and, moreover, all the corresponding linearization transformations are obtained. This problem for the Z_2-equivariant with two symmetric centers has been completely solved.

Keywords: Center–focus problem; Isochronous center; Periodic constant; Linearization

1. Introduction

It is well known that the Hilbert's 16th problem is far from being solved, which is considered as perhaps the most difficult problem among the 23 mathematical problems proposed by D. Hilbert in 1900 [18]. Many good results on this problem have been obtained in the past half century. As far as the maximal number of small-amplitude limit cycles is concerned, bifurcating from an elementary center or focus, the best known result obtained by Bautin in 1952 [2] for

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quadratic polynomial systems is $M(2) = 3$. Here, $M(n)$ denotes the maximal number of small-amplitude limit cycles around a singular point, and n is the degree of the polynomials in the vector field. For $n = 3$, many results have also been obtained. Around an isolated focus, in 2009 Yu and Corless [52] constructed a cubic system to prove the existence of 9 limit cycles using both symbolic and numerical computations. Later, Chen et al. [8] reconsidered this system and used purely symbolic computation (based on the regular chain technique) to find all real solutions. In 2012, Lloyd and Pearson [39] constructed another cubic system to show 9 limit cycles with purely symbolic computation. Around an isolated center, in 1995, Żołądek [56] first proposed a rational Darboux integral, and claimed the existence of 11 small-amplitude limit cycles around a center, which was reinvestigated recently and proved that this system can actually have only 9 limit cycles using up to second-order Melnikov functions [49,53]. After more than ten years, another two cubic-order systems were constructed to show 11 limit cycles [3,11]. Recently, the system considered in [11] was used by Yu and Tian [54] to show the existence of 12 small-amplitude limit cycles around a singular point. To the best of our knowledge, this is the best result obtained so far for cubic polynomial systems with all limit cycles around a single singular point. For $n \geq 4$, there are very few results because of the difficulty in computing the focal values. An example was constructed by Huang et al. [19] to show 8 limit cycles bifurcating from a fine focus of a quartic system. Using the inverse integrating method, the results for quartic and quintic systems have been improved: 11 limit cycles around a nilpotent focus of quartic system was given in [44] and 14 limit cycles around a nilpotent focus of quintic system was obtained in [24].

On the other hand, studying global bifurcation of limit cycles in planar differential systems is more difficult. So far the best results obtained are: $H(2) \geq 4$ [10,48], $H(3) \geq 13$ [22,31,51], and $H(4) \geqslant 22$ [11]. For quintic system, $H(5) \geq 28$ was proved in [50]. Here, $H(n)$ denotes the maximal number of limit cycles bifurcating in planar polynomial differential systems.

Noticing from the above mentioned results, better results were often obtained from Z_q-equivariant vector fields. In fact, when an isolated focus is concerned, it is difficult to compute higher-order focal values with simpler expression. Thus, it is hard to obtain more limit cycles based on the calculation of focal values, and very few results have been achieved for higher-order polynomial differential systems. In 2012, some new results have been obtained for Z_q-equivariant planar polynomial vector fields [21]. Recently, a complete classification on bi-center problem for Z_2-equivariant cubic vector fields has been given in [30], and bi-center problem for some Z_2-equivariant quintic systems is studied in [46].

The linearization problem, closely related to the Hilbert's 16th problem, is to decide whether a given differential system can be transformed to a linear one by means of the formal change of the phase variables. This problem plays an important role in the study of dynamical systems, and has also been intensively investigated over the past three decades. The study on the isochronous center problem is also interesting, and in fact many results have been rediscovered several times, see for instance [15,16,55] and references therein. Several methods have been developed for deriving the necessary conditions under which a center becomes an isochronous center, see [5,7,25,27,28] and references therein. An efficient method for computing the period constants of a planar vector field was proposed in [14]. There are only a few families of polynomial differential systems for which a complete classification on the isochronous centers is known, see for instance [5,7,25,40,41,43,45,47]. In particular, in 1964 Loud [40] classified isochronous centers of systems with homogeneous polynomials of degree two; and in 1969, Pleshkan [43] found all isochronous centers for cubic systems with only linear and cubic degree polynomials. However, the classification on isochronous centers in the form of a linear center perturbed by homogeneous polynomials of degree four or five turned out to be much more difficult.

A time-reversible system is invariant with respect to a line passing through the origin of the system. Time-reversible cubic vector fields were studied in [4,6]. In [5,7], Giné found the isochronous centers for time-reversible systems with a linear center and 4th- or 5th-degee polynomials. More recently, all linearizable centers have been classified in [9] for time-reversible systems with a linear center and 4th-degee polynomials.

A complete classification on the isochronous centers for a linear center perturbed by 5th-degree homogeneous polynomials has been obtained in [45]. However, complete classification on the isochronous and linearizable centers for a linear center perturbed by 4th-degree homogeneous polynomials is still open. The linearizability problem for the complex Lotka-Volterra system was solved in [13]. In 2003, a new method was developed by Liu and Huang [27] for determining isochronous centers of polynomial differential systems, and later period constants and time-angle of isochronous centers for complex analytic systems were investigated in [29]. The methodology and results can also be found in the book [32]. Isochronicity and linearizability of planar polynomial Hamiltonian systems were studied in [34], and isochronicity for trivial quintic and septic planar polynomial Hamiltonian systems was discussed in [42].

If a system is not analytic, then the center and isochronous center problems become very difficult because the classical theorem is no longer applicable. Recently, the following systems,

$$
\begin{aligned}
&\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-5}{2}}(Az^{4+j}\bar{z}^{1-j} + Bz^3\bar{z}^2 + Cz^{2-j}\bar{z}^{3+j} + D\bar{z}^5),\\
&\qquad (d = 2m+1 \geq 5),\\
&\dot{z} = iz + (z\bar{z})^{\frac{d-4}{2}}(Az^3\bar{z} + Bz^2\bar{z}^2 + C\bar{z}^4),\\
&\qquad (d = 2m \geq 4),\\
&\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-3}{2}}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3),\\
&\qquad (d = 2m+1 \geq 3),\\
&\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-2}{2}}(Az^2 + Bz\bar{z} + C\bar{z}^2),\\
&\qquad (d = 2m \geq 2),\\
&\dot{z} = (\lambda + i)z + (z\bar{z})^{\frac{d-5}{2}}(Az^5 + Bz^4\bar{z} + Cz^3\bar{z}^2 + Dz^2\bar{z}^3 + Ez^3\bar{z}^4 + F\bar{z}^5),\\
&\qquad (d = 2m+1 \geq 5),
\end{aligned}
\tag{1.1}
$$

have been investigated, see [17,35–38]. The conditions of centers and isochronous centers for the above systems were obtained, with a restriction on d such that the system becomes a polynomial system. For the general case when the system is non-analytic, the analysis becomes much more difficult, and very few results have been obtained. As far as center and isochronous center conditions at the origin of the system are concerned, several special systems have been studied, see [26,32]. For non-analytic quartic and quintic systems, some results have been obtained in [20,23] on the conditions of center and isochronous center at the origin of the system.

In [30], a class of Z_2 cubic-degree systems,

$$\begin{aligned}\frac{dx}{dt} &= -(a_1+1)y + a_1x^2y + a_2xy^2 + a_3y^3,\\ \frac{dy}{dt} &= -\frac{1}{2}x - a_4y + \frac{1}{2}x^3 + a_4x^2y + a_5xy^2 + a_6y^3,\end{aligned} \tag{1.2}$$

was studied, and the first six focus values at $(\pm 1, 0)$ of this system were obtained. Further, 11 center conditions were derived, and a complete study on bi-center problem has been carried out. In 2017, the bi-isochronous center problem for a cubic Z_2-equivariant vector field with real coefficients was considered in [12], and two real isochronous center conditions are obtained. However, the complex isochronous center problem at $(\pm 1, 0)$ of system (1.2) is still open. There are two main difficulties in solving this problem. The first one is the computation of period constants, and the second one is to find all linearizability transformations. So in this paper we will study the linearizability problem at $(\pm 1, 0)$ of system (1.2) in complex domain. Necessary and sufficient conditions for $(\pm 1, 0)$ of system (1.2) to be isochronous centers are derived. A complete classification on the bi-isochronous center problem in complex domain for Z_2-equivariant cubic vector fields is achieved, and so this open problem is completely solved.

The rest of the paper is organized as follows. In the next section, some preliminary results are presented, which will be used in the following sections. In Section 3, some new methods used to determine complex isochronous center are described, and an example is given to illustrate the efficiency of our method. In Section 4, complex center conditions of (1.2) are obtained, which are classified into 11 cases. Then, in Section 5, the period constants at $(\pm 1, 0)$ of system (1.2) are computed, and complex isochronous center conditions are obtained for all the 11 cases. Further, the linearizability transformation for each of the 11 cases is obtained. Finally, conclusion is drawn in Section 6.

2. Preliminary results

In order to study the linearizability problem in complex domain, we present some preliminary results in this section, which will be used in the following sections. Consider the following complex system,

$$\begin{aligned}\frac{dz}{dT} &= z + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} z^{\alpha} w^{\beta} = Z(z,w),\\ \frac{dw}{dT} &= -w - \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} w^{\alpha} z^{\beta} = -W(z,w),\end{aligned} \tag{2.1}$$

where z, w, T are complex variables and $a_{\alpha\beta}, b_{\alpha\beta}$ are complex coefficients, $Z(z,w)$ and $W(z,w)$ are analytic functions in the neighborhood of the origin. The origin of system (2.1) is called a weak saddle. By transformation

$$z = x + iy, \quad w = x - iy, \quad T = it, \quad i = \sqrt{-1}, \tag{2.2}$$

system (2.1) can be brought to

$$\frac{dx}{dt} = -y + \sum_{\alpha+\beta=2}^{\infty} A_{\alpha\beta} x^{\alpha} y^{\beta} = X(x, y),$$
$$\frac{dy}{dt} = x + \sum_{\alpha+\beta=2}^{\infty} B_{\alpha\beta} x^{\alpha} y^{\beta} = Y(x, y). \tag{2.3}$$

In the following, we give two theorems about the normal form of system (2.1), which have been known for a long time but explicitly formatted in [1] and [27].

Theorem 2.1. [1] *For given $\{c_{k+1,k}\}$ and $\{d_{k+1,k}\}$, by using a formal change of variables,*

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \quad \eta = w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j, \tag{2.4}$$

system (2.1) can be reduced to the formal form of

$$\frac{d\xi}{dT} = \xi + \xi \sum_{k=1}^{\infty} p_k \xi^k \eta^k, \quad \frac{d\eta}{dT} = -\eta - \eta \sum_{k=1}^{\infty} q_k \xi^k \eta^k. \tag{2.5}$$

Definition 2.1. Denote

$$\mu_k = p_k - q_k, \quad \tau_k = p_k + q_k, \quad k = 1,\ 2,\ \cdots. \tag{2.6}$$

Then μ_k is called the kth complex singular point value of the origin of system (2.1), and τ_k is called the kth complex periodic constant of the origin of system (2.1).

For system (2.1), we consider the two parameter transformation,

$$z = re^{i\theta}, \quad w = re^{-i\theta}, \quad T = it, \tag{2.7}$$

which, with (2.2), is equivalent to

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{2.8}$$

Then, by transformation (2.7) (or (2.8)), system (2.1) (or (2.3)) is reduced to

$$\frac{dr}{dt} = i\,\frac{wZ - zW}{2r} = R(r, \theta), \quad \frac{d\theta}{dt} = \frac{wZ + zW}{2zw} = \Theta(r, \theta), \tag{2.9}$$

where

$$R(r, \theta) = \frac{ir}{2} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=k+1}^{\infty} \left[a_{\alpha\beta} e^{i(\alpha-\beta-1)\theta} - b_{\alpha\beta} e^{-i(\alpha-\beta-1)\theta} \right] r^k,$$
$$\Theta(r, \theta) = 1 + \frac{1}{2} \sum_{k=1}^{\infty} \sum_{\alpha+\beta=k+1}^{\infty} \left[a_{\alpha\beta} e^{i(\alpha-\beta-1)\theta} + b_{\alpha\beta} e^{-i(\alpha-\beta-1)\theta} \right] r^k. \tag{2.10}$$

For sufficiently small h, denote $\tilde{r}(\theta, h)$ as the solution of system (2.9), satisfying the initial condition $r|_{\theta=0} = h$,

$$r = \tilde{r}(\theta, h) = h + \sum_{k=2}^{\infty} \nu_k(\theta) h^k, \tag{2.11}$$

and

$$\mathcal{T}(\vartheta, h) = \int_0^{\vartheta} \frac{d\theta}{\Theta(\tilde{r}(\theta, h), \theta)}. \tag{2.12}$$

The definition of complex center and isochronous center were given in [27].

Definition 2.2. [27] For sufficiently small complex constant h, if

$$\tilde{r}(2\pi, h) \equiv h,$$

then the origin of system (2.1) is called a complex center. If, in addition, the following

$$\mathcal{T}(2\pi, h) \equiv 2\pi$$

holds, then the origin of system (2.1) is called a complex isochronous center.

It is well known that the origin of system (2.1) is a complex isochronous center if and only if all complex singular point values μ_k and period constants τ_k are zero, namely the normal form (2.5) of system (2.1) is a linear system. Moreover, it is known that

Theorem 2.2. [27] *The origin of system* (2.1) *is a complex isochronous center if and only if it is linearizable in the neighborhood of the origin. That is, there exist power series with non-zero convergence radius, given by*

$$\varphi(z, w) = z + h.o.t., \quad \psi(z, w) = w + h.o.t.,$$

where "h.o.t." denotes higher-order terms, together with the transformation,

$$\xi = \varphi(z, w), \quad \eta = \psi(z, w), \tag{2.13}$$

such that system (2.1) *can be changed into a linear one,*

$$\frac{d\xi}{dT} = \xi, \quad \frac{d\eta}{dT} = -\eta.$$

However, it is very difficult to find the transformation (2.13) under which the origin of the system becomes an isochronous center.

3. Some new results about complex isochronous center problem

As far as we know, there are many methods for computing periodic constants. In particular, the so called determination methods were developed in the past three decades. However, none of them can be used to solve all isochronous center problems. In this section, we present a new determination method for determining isochronous centers.

3.1. A new determination method for determining isochronous centers

We assume that the origin of system (2.1) is a complex center. We have the following theorem.

Theorem 3.1. *Suppose that the origin of system* (2.1) *is a complex center and there exists an analytic function in the neighborhood of the origin, given by*

$$\eta = \psi(z, w) = w + h.o.t., \tag{3.1}$$

which satisfies

$$\frac{d\eta}{dT} = -\eta, \tag{3.2}$$

then the origin of system (2.1) *is a complex isochronous center.*

Proof. With implicit function theorem, it follows from (3.1) that $w = f(z, \eta) = \eta + h.o.t.$, which is a power series of η and z with non-zero convergence radius. Thus, system (2.1) can be transformed into

$$\frac{dz}{dT} = Z(z, f(z, \eta)) = z + h.o.t., \quad \frac{d\eta}{dT} = -\eta. \tag{3.3}$$

By the transformation,

$$z = z, \quad \eta = \psi(z, w).$$

Because the origin of system (2.1) is a complex center, and so is the origin of system (3.3). Therefore, there exists a first integral $F(z, \eta) = z\eta + h.o.t.$ in the neighborhood of the origin of system (3.3). Moreover, since $\eta = 0$ is a solution of system (3.3), $F(z, \eta)$ can be rewritten as

$$F(z, \eta) = \eta\varphi(z, \eta),$$

where $\varphi(z, \eta) = z + h.o.t.$ is a power series of η and z with non-zero convergence radius. Let $\xi = \varphi(z, \eta)$. Then, $\frac{dF}{dT} = 0$ and (3.2) yield

$$\frac{d\xi}{dT} = \xi. \tag{3.4}$$

Obviously, (3.2) and (3.4) imply that system (2.1) is linearizable. □

Similarly, we have the following theorem.

Theorem 3.2. *Suppose that the origin of system* (2.1) *is a complex center and there exists an analytic function in the neighborhood of the origin,*

$$\xi = \varphi(z, w) = z + h.o.t.,$$

which satisfies

$$\frac{d\xi}{dT} = \xi,$$

then the origin of system (2.1) *is a complex isochronous center.*

Now, consider the following autonomous complex systems,

$$\frac{dz}{dT} = z + h.o.t., \quad \frac{dw}{dT} = -wf(w), \tag{3.5}$$

and

$$\frac{dz}{dT} = zg(z), \quad \frac{dw}{dT} = -w + h.o.t., \tag{3.6}$$

where $f(w)$ and $g(z)$ are power series with non-zero convergence radius and $f(0) = g(0) = 1$. The functions on the right-hand side of the above differential equations are assumed to be analytic in the neighborhood of the origin. Then the following corollaries directly follow from Theorems 3.1 and 3.2.

Corollary 3.1. *If the origin of system* (3.5) *is a complex center, then it is a complex isochronous center.*

Corollary 3.2. *If the origin of system* (3.6) *is a complex center, then it is a complex isochronous center.*

3.2. Simple integral curve and linearization transformation

For the following system,

$$\frac{dz}{dT} = \sum_{\alpha+\beta=0}^{n} a_{\alpha\beta} z^{\alpha} w^{\beta}, \quad \frac{dw}{dT} = \sum_{\alpha+\beta=0}^{n} b_{\alpha\beta} z^{\alpha} w^{\beta}, \tag{3.7}$$

whose right-hand sides are polynomials, its linearizability transformation is often found from invariant algebraic curves. Simple integral curve is defined in [33].

Definition 3.1. [33] Suppose $f(z, w)$ is differentiable in $\mathcal{D}$. If there exists a polynomial $h(z, w)$ whose degree is no more than $n - 1$, and it satisfies

$$\left.\frac{df}{dT}\right|_{(3.7)} = h(z,w)f(z,w),$$

in $\mathcal{D}$, then $f(z,w)$ is called a simple integral curve.

Obviously, an invariant algebraic curve is a special case of the simple integral curve. For example, an integrating factor of system (3.7) is a simple integral curve but not invariant algebraic curve. So simple integral curves can be used to construct a Darboux integral.

In the rest of this section, we give an example to illustrate the above definition. The example is described by the following differential equations,

$$\begin{aligned}\frac{dx}{dT} &= \frac{1}{2}(-x + x^3 + 2a_2xy^2 + 2a_3y^3),\\ \frac{dy}{dT} &= \frac{1}{2}y(-1 - x^2 + 2a_5xy + 2a_6y^2),\end{aligned} \tag{3.8}$$

which has weak saddles at $(\pm 1, 0)$. Under the transformation,

$$\xi = x^2 - 1, \quad \eta = \frac{y}{x}, \quad \frac{dt}{d\tau} = \frac{1}{1+\xi},$$

system (3.8) can be changed into

$$\begin{aligned}\frac{d\xi}{d\tau} &= \xi + 2a_2\eta^2 + 2a_2\xi\eta^2 + 2a_3\eta^3 + 2a_3\xi\eta^3,\\ \frac{d\eta}{d\tau} &= -\eta[1 - a_5\eta + (a_2 - a_6)\eta^2 + a_3\eta^3],\end{aligned} \tag{3.9}$$

which has an inverse integrating factor,

$$M_1 = e^{\int_0^\eta \frac{-2a_5 + (a_2 - 3a_6)\eta + 2a_3\eta^2}{1 - a_5\eta + (a_2 - a_6)\eta^2 + a_3\eta^3}\, d\eta},$$

so system (3.8) has complex centers at $(\pm 1, 0)$.

Now, introducing $u = x - 1$, $v = y$ into (3.8) yields

$$\begin{aligned}\frac{du}{dT} &= u + \frac{3}{2}u^2 + A_2v^2 + \frac{1}{2}u^3 + A_2uv^2 + A_3v^3 \equiv U(u,v),\\ \frac{dv}{dT} &= -v\left(1 + u - A_5v + \frac{1}{2}u^2 - A_5uv - A_6v^2\right) \equiv V(u,v).\end{aligned} \tag{3.10}$$

Then, the origin of system (3.10) is a complex center. So there exists an analytic inverse integrating factor $g_1(u,v)$, satisfying the following equation,

$$\frac{dg_1}{dT} = \left(\frac{\partial U}{\partial u} + \frac{\partial V}{\partial v}\right) g_1, \tag{3.11}$$

with $g_1(0,0) = 1$ in the neighborhood of the origin of (3.10).

Further, it can be shown that system (3.10) has two simple integral curves,

$$\begin{aligned} g_2 &= v, \\ g_3 &= 1 + 3u - A_5 v + 3u^2 - 2A_5 uv + (A_2 - A_6)v^2 \\ &\quad + u^3 - A_5 u^2 v + (A_2 - A_6)uv^2 + A_3 v^3, \end{aligned}$$

which satisfy the equations:

$$\begin{aligned} \frac{dg_2}{dT} &= -\Big(1 + u - A_5 v + \frac{1}{2}u^2 - A_5 uv - A_6 v^2\Big) g_2, \\ \frac{dg_3}{dT} &= \Big(3u + A_5 v + \frac{3}{2}u^2 + A_5 uv + (A_2 + 2A_6)v^2\Big) g_3. \end{aligned} \tag{3.12}$$

Finally, let

$$\eta = v g_3 g_1^{-1}. \tag{3.13}$$

Then, equation (3.2) can be obtained from (3.11), (3.12) and (3.13). Thus, it follows from Theorem 3.1 that the origin of system (3.10) is a complex isochronous center.

4. Complex center conditions for system (1.2)

By (2.2), system (1.2) can be transformed to

$$\begin{aligned} \frac{dz}{dT} = &\ \frac{-1}{4}(z + w) + \frac{1}{2}(1 + b_1 + b_4)(z - w) \\ &+ \frac{1}{16}(z + w)^2\big[(z + w) - 2(b_1 + b_4)(z - w)\big] \\ &+ \frac{1}{8}(z - w)^2\big[(b_2 - b_5)(z + w) + (b_3 + b_6)(z - w)\big], \\ \frac{dw}{dT} = &\ \frac{1}{4}(z + w) + \frac{1}{2}(1 + b_1 - b_4)(z - w) \\ &- \frac{1}{16}(z + w)^2\big[(z + w) + 2(b_1 - b_4)(z - w)\big] \\ &+ \frac{1}{8}(z - w)^2\big[(b_2 + b_5)(z + w) + (b_3 - b_6)(z - w)\big], \end{aligned} \tag{4.1}$$

where

$$b_1 = a_1, \quad b_2 = ia_2, \quad b_3 = a_3, \quad b_4 = ia_4, \quad b_5 = a_5, \quad b_6 = ia_6. \tag{4.2}$$

Remark 4.1. If z, w, T are real variables and b_i's are real coefficients, then system (4.1) is a planar autonomous differential system with a weak saddle at the origin.

For system (4.1), replacing i by $-i$ in (4.2) is equivalent to the coefficient transformation,

$$\{b_1, b_2, b_3, b_4, b_5, b_6\} \longrightarrow \{b_1, -b_2, b_3, -b_4, b_5, -b_6\}, \tag{4.3}$$

or the following transformation,

$$z \to w, \quad w \to z, \quad T \to -T. \tag{4.4}$$

Definition 4.1. The transformation (4.4) is called conjugated symmetric transformation, and the systems (4.1) and (4.4) are said to be conjugated symmetric systems for each other.

It is clear that the conjugated symmetric transformation does not change the isochronous center. In order to study the complex isochronous centers at $(\pm 1, \pm 1)$ of system (4.1), let

$$u = z - 1, \;\; v = w - 1, \tag{4.5}$$

under which system (4.1) becomes

$$\begin{aligned}
\frac{du}{dT} = {} & \frac{-1}{4}(u+v+2) + \frac{1}{2}(1+b_1+b_4)(u-v) \\
& + \frac{1}{16}(u+v+2)^2\big[(u+v+2) - 2(b_1+b_4)(u-v)\big] \\
& + \frac{1}{8}(u-v)^2\big[(b_2-b_5)(u+v+2) + (b_3+b_6)(u-v)\big], \\
\frac{dv}{dT} = {} & \frac{1}{4}(u+v+2) + \frac{1}{2}(1+b_1-b_4)(u-v) \\
& - \frac{1}{16}(u+v+2)^2\big[(u+v+2) + 2(b_1-b_4)(u-v)\big] \\
& + \frac{1}{8}(u-v)^2\big[(b_2+b_5)(u+v+2) + (b_3-b_6)(u-v)\big].
\end{aligned} \tag{4.6}$$

The following theorem directly follows from Theorem 11 in [30].

Theorem 4.1. *The origin of system* (4.6) *is a complex center if and only if one of the following* 11 *conditions holds:*

$$(C_1):\; b_1 = -b_5, \quad b_4 = 0, \quad b_6 = \frac{-1}{3}b_2;$$

$$(C_2):\; b_2 = 0, \qquad b_4 = 0, \quad b_6 = 0;$$

$$\begin{aligned}
(C_3):\; & 3(b_1+b_5)(2+2b_1-b_3+2b_5+2b_1b_5) \\
& \quad + 2b_4(2b_2+b_1b_2+2b_4+b_2b_5+2b_4b_5) = 0, \\
& b_6 - \frac{1}{3}(-b_2-2b_1b_2+2b_4-2b_2b_5+2b_4b_5) = 0, \\
& 2(1+b_1)(b_1+b_5)^2 + b_4^2(1+2b_1+2b_5) = 0, \quad b_1+b_5 \neq 0;
\end{aligned}$$

$$\begin{aligned}
(C_4):\; & -2b_4(1+b_5) - b_2(2+b_1+b_5) = 0, \quad b_3 - 2(1+b_1)(1+b_5) = 0, \\
& b_6 - \frac{1}{3}(-b_2-2b_1b_2+2b_4-2b_2b_5+2b_4b_5) = 0;
\end{aligned}$$

$$(C_5):\ b_1=\frac{-1}{2}(2+3b_4^2),\ \ b_2=b_4,\ \ b_3=-b_4^2(1+b_4^2+b_5),\ \ b_6=b_4(1+b_4^2);$$

$$(C_6):\ b_1=\frac{-1}{8}(8+5b_4^2),\quad b_2=\frac{1}{2}b_4,\quad b_3=\frac{-5}{32}b_4^4,$$
$$b_5=\frac{1}{8}(-8+b_4^2),\qquad b_6=\frac{1}{4}b_4(2+b_4^2);$$

$$(C_7):\ b_1=\frac{1}{32}(-32+15b_4^2),\quad b_2=\frac{1}{4}b_4,\quad b_3=\frac{-1}{512}b_4^2(64-15b_4^2),$$
$$b_5=\frac{1}{32}(-96+17b_4^2),\quad b_6=\frac{-3}{16}b_4(4-b_4^2);$$

$$(C_8):\ b_1=\frac{1}{50}(-50+21b_4^2),\quad b_2=\frac{1}{5}b_4,\quad b_3=\frac{-1}{1250}b_4^2(250-63b_4^2),$$
$$b_5=\frac{1}{50}(-200+39b_4^2),\quad b_6=\frac{-1}{25}b_4(35-9b_4^2);$$

$$(C_9):\ b_1=\frac{-1}{2}(2+3b_4^2),\ \ b_2=b_4,\ \ b_3=-b_4^2(1+b_4^2+b_5),\ \ b_6=b_4(1+b_4^2);$$

$$(C_{10}):\ b_1=\frac{1}{8}(-8+3b_4^2),\quad b_2=\frac{-1}{2}b_4,$$
$$b_3=\frac{-3}{16}b_4^2(4-b_4^2+4b_5),\quad b_6=\frac{1}{8}b_4(4+b_4^2+8b_5);$$

$$(C_{11}):\ b_1=\frac{-1}{32}(32-15b_4^2),\quad b_2=\frac{-1}{4}b_4,\quad b_3=\frac{-1}{512}b_4^2(832-495b_4^2),$$
$$b_5=\frac{1}{32}(160-111b_4^2),\quad b_6=\frac{1}{16}b_4(76-45b_4^2).$$

5. Conditions for complex isochronous center

In this section, we discuss the conditions on complex isochronous center for the 11 different cases listed in Theorem (4.1), and obtain a total of 54 complex isochronous center conditions. Moreover, for each of the isochronous center conditions, its corresponding linearization transformation is given.

5.1. *On complex center condition C_1*

Computing and analysing period constants at the origin of system (4.6), the following complex isochronous conditions are obtained.

Lemma 5.1. *If the condition C_1 in* Theorem 4.1 *holds, then all the first four period constants at the origin of system* (4.6) *vanish if and only if one of following two conditions holds:*

$$\begin{aligned}&L_1:\ b_1=-\frac{3}{2},\ b_2=-3,\ b_3=\frac{3}{2},\ b_4=0,\ b_5=\frac{3}{2},\ b_6=1;\\&L_1^*:\ b_1=-\frac{3}{2},\ b_2=3,\quad b_3=\frac{3}{2},\ b_4=0,\ b_5=\frac{3}{2},\ b_6=-1.\end{aligned}\tag{5.1}$$

Theorem 5.1. *If the condition C_1 in* Theorem 4.1 *holds, then the origin of system* (4.6) *is a complex isochronous center if and only if one of the conditions in* Lemma 5.1 *is satisfied.*

Proof. When the condition L_1 in Lemma 5.1 holds, system (4.6) becomes

$$\begin{aligned} \frac{du}{dT} &= u + 3uv - \frac{3}{2}v^2 + \frac{3}{2}uv^2 - v^3, \\ \frac{dv}{dT} &= -v(1+v)\left(1 + \frac{1}{2}v\right). \end{aligned} \tag{5.2}$$

There exists a linearizability transformation,

$$\xi = f_1^2 f_5, \quad \eta = v f_1^{-2} f_3,$$

where

$$f_1 = 1 + v, \quad f_3 = 1 + \frac{1}{2}v, \quad f_5 = u + uv - \frac{1}{2}v^2,$$

in the neighborhood of the origin of system (5.2), yielding

$$\frac{d\xi}{dT} = \eta, \quad \frac{\eta}{dT} = -\xi. \tag{5.3}$$

Similarly when the condition L_1^* in Lemma 5.1 holds, system (4.6) can be reduced to

$$\begin{aligned} \frac{du}{dT} &= u(1+u)\left(1 + \frac{1}{2}u\right), \\ \frac{dv}{dT} &= -v + \frac{3}{2}u^2 - 3uv + u^3 - \frac{3}{2}u^2 v. \end{aligned} \tag{5.4}$$

There exists a linearizability transformation,

$$\xi = u f_2^{-2} f_4, \quad \eta = f_2^2 f_6, \tag{5.5}$$

where

$$f_2 = 1 + u, \quad f_4 = 1 + \frac{1}{2}u, \quad f_6 = v - \frac{1}{2}u^2 + uv,$$

such that system (5.4) becomes (5.3). □

Remark 5.1. (i) By using the transformation (4.3), the condition L_1 can be transformed into L_1^*. So the two systems under the conditions L_1 and L_1^* are conjugated symmetric systems, and we only need to discuss one of them. Thus, in the remaining of the paper, we only discuss the complex isochronous center under the condition L_k, without showing the conditions L_k^*.

(ii) In fact, by Corollaries *3.1* and *3.2*, it is easy to show that the origin of the system (5.2) or (5.4) is a complex isochronous center. However, it is more intuitive to prove it by finding a linearizability transformation.

Table 1
Linearizability transformations of system (5.7) and (5.8).

	Linearizability transformation	Integral curves
L_2:	$\xi = f_7 f_9^{\frac{-1}{2}}, \quad \eta = f_8 f_9^{\frac{-1}{2}}$	$f_7 = u + \frac{3}{8}(u+v)^2 + \frac{1}{16}(u+v)^3$ $f_8 = v + \frac{3}{8}(u+v)^2 + \frac{1}{16}(u+v)^3$ $f_9 = 1 + 9(u+v) + \frac{9}{4}(13u^2 - 22uv + 13v^2)$
L_3:	$\xi = u f_1^{-2} f_3, \quad \eta = v f_2^{-2} f_4$	

5.2. On complex center condition C_2

For this case, we have the following result.

Lemma 5.2. *If the condition C_2 in* Theorem 4.1 *holds, then all the first four period constants at the origin of system* (4.6) *become zero if and only if one of following two conditions holds:*

$$
\begin{aligned}
&L_2: \ b_1 = -3, \ \ b_2 = 0, \ b_3 = 0, \ \ b_4 = 0, \ b_5 = -9, \ \ b_6 = 0; \\
&L_3: \ b_1 = -\frac{3}{2}, \ b_2 = 0, \ b_3 = \frac{1}{2}, \ b_4 = 0, \ b_5 = -\frac{3}{2}, \ b_6 = 0.
\end{aligned}
\tag{5.6}
$$

When the condition L_2 is satisfied, system (4.6) can be rewritten as

$$
\begin{aligned}
\frac{du}{dT} &= u + \frac{3}{8}(11u^2 - 10uv + 3v^2) + \frac{1}{16}(u+v)(25u^2 - 34uv + 13v^2), \\
\frac{dv}{dT} &= -v - \frac{3}{8}(3u^2 - 10uv + 11v^2) - \frac{1}{16}(u+v)(13u^2 - 34uv + 25v^2).
\end{aligned}
\tag{5.7}
$$

If the condition L_3 holds, system (4.6) can be brought to

$$
\frac{du}{dT} = u(1+u)\Big(1 + \frac{1}{2}u\Big), \qquad \frac{dv}{dT} = v(1+v)\Big(1 + \frac{1}{2}v\Big).
\tag{5.8}
$$

Linearizability transformations of system (5.7) and (5.8) can be obtained by finding simple integral curves, as listed in Table 1.

Remark 5.2. In the conditions L_2 and L_3, $b_2 = b_4 = b_6 = 0$. So the two conditions are symmetric, namely, the L_2^* and L_3^* (which are not listed) are identical to L_2 and L_3, respectively. So the conditions L_2 and L_3 yield real isochronous center of system (4.6), which is the same as that obtained in [12].

Theorem 5.2. *When the condition C_2 in Theorem* 4.1 *holds, the origin of system* (4.6) *is a complex isochronous center if and only if either the condition L_2 or L_3 in* Lemma 5.2 *is true.*

5.3. On complex center condition C_3

Lemma 5.3. *If the condition C_2 in* Theorem 4.1 *holds, then all the first four period constants at the origin of system* (4.6) *vanish if and only if one of following sixteen* (*eight for corresponding L_k^**) *conditions holds:*

$$
\begin{array}{llllll}
L_4 : b_2 = -2b_1, & b_3 = -b_1, & b_4 = \frac{-(3+2b_1)}{2}, & b_5 = \frac{-(3+4b_1)}{2}, & b_6 = \frac{1+2b_1}{2}; & \\
L_5 : b_1 = -\frac{5}{2}, & b_2 = -3, & b_3 = -\frac{27}{2}, & b_4 = 1, & b_5 = \frac{7}{2}, & b_6 = 6; \\
L_6 : b_1 = \frac{-1}{8}, & b_2 = 1, & b_3 = \frac{-17}{32}, & b_4 = 2, & b_5 = \frac{-31}{8}, & b_6 = \frac{-3}{2}; \\
L_7 : b_1 = -\frac{1}{2}, & b_2 = \frac{-1}{3}, & b_3 = -\frac{61}{54}, & b_4 = \frac{5}{3}, & b_5 = \frac{-9}{2}, & b_6 = \frac{-44}{9}; \\
L_8 : b_1 = -1, & b_2 = -1, & b_3 = -2, & b_4 = 1, & b_5 = \frac{1}{2}, & b_6 = 1; \\
L_9 : b_1 = 0, & b_2 = 0, & b_3 = 0, & b_4 = \frac{3}{2}, & b_5 = \frac{-3}{2}, & b_6 = \frac{-1}{2}; \\
L_{10} : b_1 = \frac{-3}{4}, & b_2 = \frac{-3}{4}, & b_3 = \frac{1}{8}, & b_4 = \frac{3}{4}, & b_5 = \frac{-3}{4}, & b_6 = \frac{-3}{8}; \\
L_{11} : b_1 = \frac{1}{2}, & b_2 = \frac{1}{3}, & b_3 = \frac{-1}{18}, & b_4 = 2, & b_5 = \frac{-7}{6}, & b_6 = \frac{-5}{27}.
\end{array}
$$

Proposition 5.1. *When the condition L_4 is satisfied, system (4.6) becomes*

$$
\begin{aligned}
\frac{du}{dT} &= \frac{1}{2}u(1+u)(2+u), \\
\frac{dv}{dT} &= -v - \frac{1}{2}(3+4b_1)u^2 + 2b_1uv - \frac{1}{2}(1+2b_1)u^3 + b_1u^2v,
\end{aligned}
\tag{5.9}
$$

which has a linearizability transformation

$$
\xi = uf_1^{-2}f_3, \quad \eta = f_1^{-2b_1}f_3^{-1}f_{10},
$$

where

$$
f_{10} = c_2\,(1-e^{T+2c_1})^{-\frac{1}{2}-b_1}\,e^{-\frac{T}{2}\pm\tanh^{-1}\sqrt{1-e^{T+2c_1}}},
$$

in which c_1 and c_2 are arbitrary constants. f_{10} satisfies the differential equation,

$$
\frac{df_{10}}{dT} = \frac{1}{2}(2+u)(-1+u+2b_1u)f_{10}. \tag{5.10}
$$

Remark 5.3. The solution f_{10} obtained from equation (5.10) is a simple integral curve of system (5.9), but it is not an algebraic integral curve unless $(1+b_1)$ is a positive number. When $(1+b_1)$ is a positive number, the degree of f_{10} will increase with respect to b_1, which shows that there exist arbitrary degree algebraic polynomial solutions for the cubic-degree planar autonomous system (5.9).

We summarize the results for the cases L_5 to L_{11} in the following proposition.

Proposition 5.2. *Under each of the conditions L_5-L_{11}, system (4.6) respectively has a linearizability transformation* $\mathrm{T_k}$, $k = 5, 6, \ldots, 11$ *as follows:*

$$L_5:\ \begin{cases} \dfrac{du}{dT}=(1+u)\Big(u-\dfrac{3}{2}u^2+4uv-2v^2\Big),\\ \dfrac{dv}{dT}=-v+\dfrac{1}{2}(3u^2-2uv-4v^2)-\dfrac{1}{2}(u-2v)(4u^2-7uv+2v^2),\end{cases}$$

$$\mathrm{T}_5:\ \xi=f_1f_{11}f_{12},\quad \eta=f_1^{-1}f_6f_{11}^{-1};$$

$$L_6:\ \begin{cases} \dfrac{du}{dT}=u+\dfrac{3}{32}(7u^2-18uv+27v^2)+\dfrac{1}{256}(u+3v)(47u^2-114uv+99v^2),\\ \dfrac{dv}{dT}=-v-\dfrac{1}{32}(u^2-22uv+69v^2)-\dfrac{1}{256}(u+3v)(9u^2-46uv+69v^2),\end{cases}$$

$$\mathrm{T}_6:\ \xi=f_{13}f_{15}^{-3},\quad \eta=f_{14}f_{15}^{-4};$$

$$L_7:\ \begin{cases} \dfrac{du}{dT}=u+\dfrac{1}{6}(5u^2-8uv+12v^2)-\dfrac{1}{54}(u-4v)(17u^2-28uv+20v^2),\\ \dfrac{dv}{dT}=-v-\dfrac{1}{6}(12u^2-8uv+5v^2)-\dfrac{1}{54}(4u-v)(20u^2-28uv+17v^2),\end{cases}$$

$$\mathrm{T}_7:\ \xi=f_{16}f_{18}^{-3},\quad \eta=f_{17}f_{18}^{-6};$$

$$L_8:\ \begin{cases} \dfrac{du}{dT}=u\Big(1+\dfrac{1}{2}u\Big)\Big(1-\dfrac{1}{2}u+\dfrac{3}{2}v\Big),\\ \dfrac{dv}{dT}=-\Big(v-\dfrac{1}{2}u^2+uv\Big)\Big(1-\dfrac{1}{2}u+\dfrac{3}{2}v\Big),\end{cases}$$

$$\mathrm{T}_8:\ \xi=uf_3f_{19},\quad \eta=f_3^{-1}f_{19}^{-1}f_{20};$$

$$L_9:\ \begin{cases} \dfrac{du}{dT}=u+\dfrac{3}{2}v^2+\dfrac{1}{2}v^3,\\ \dfrac{dv}{dT}=-v(1+v)\Big(1+\dfrac{1}{2}v\Big),\end{cases}$$

$$\mathrm{T}_9:\ \xi=f_4^{-1}f_{21},\quad \eta=vf_2^{-2}f_4;$$

$$L_{10}:\begin{cases} \dfrac{du}{dT}=u+\dfrac{3}{8}(u+v)^2+\dfrac{1}{32}(u^3+9u^2v+3uv^2+3v^3),\\ \dfrac{dv}{dT}=-v(1+v)\Big(1+\dfrac{1}{2}v\Big),\end{cases}$$

$$\mathrm{T}_{10}:\xi=f_2^2f_4^{-1}f_{15}^{-2}f_{22},\quad \eta=vf_2^{-2}f_4;$$

$$L_{11}:\begin{cases} \dfrac{du}{dT}=u-\dfrac{1}{2}(u-2v)(u+2v)-\dfrac{1}{54}(5u-8v)(u+2v)^2,\\ \dfrac{dv}{dT}=-v+\dfrac{1}{6}(u-4v)(u+2v)-\dfrac{1}{54}(2u-5v)(u+2v)^2,\end{cases}$$

$$\mathrm{T}_{11}:\xi=f_{16}f_{24}^{-1},\quad \eta=f_{23}f_{24}^{-2}.$$

Here,

$$f_{11}=1-u+2v,$$

$$f_{12}=u+\frac{1}{6}(3u^2+12uv-4v^2)-\frac{1}{3}u(3u^2-11uv+4v^2)-\frac{1}{3}u^2(u-2v)(2u-v),$$

$$
\begin{aligned}
f_{13} &= u + \frac{1}{128}(u+3v)^2(12+u+3v),\\
f_{14} &= v + \frac{1}{1536}(u+3v)\big[16(u+27v)+(u+3v)^2(16+u+3v)\big],\\
f_{15} &= 1 + \frac{1}{4}(u+3v),\\
f_{16} &= u + \frac{1}{54}(u+2v)^2(9+u+2v),\\
f_{17} &= v + \frac{1}{6}(u^2+2uv+8v^2) + \frac{1}{27}(u+2v)^2(u+5v)\\
&\quad + \frac{1}{8748}(u+2v)^3\big[108(u+3v)+(u+2v)^2(18+u+2v)\big],\\
f_{18} &= 1 + \frac{1}{3}(u+2v),\\
f_{19} &= 1 - \frac{1}{2}(u-3v),\\
f_{20} &= v - \frac{1}{6}u(u-3v),\\
f_{21} &= u + \frac{1}{2}v(u+v),\\
f_{22} &= u + \frac{1}{8}(u^2+6uv+v^2),\\
f_{23} &= v - \frac{1}{18}u(u+6v) - \frac{1}{243}(u+2v)^2\big[18v-u(u+2v)\big]\\
&\quad + \frac{1}{26244}(u+2v)^5(18+u+2v),\\
f_{24} &= 1 - \frac{1}{3}(u-2v) - \frac{1}{81}(u+2v)^2(9+u+2v).
\end{aligned}
$$

We have the following theorem.

Theorem 5.3. *When the condition C_3 in* Theorem 4.1 *holds, the origin of system* (4.6) *is a complex isochronous center if and only if one of the 16 conditions in* Lemma 5.3 *is satisfied.*

5.4. On complex center condition C_4

Lemma 5.4. *If the condition C_4 in* Theorem 4.1 *holds, then all the first four period constants at the origin of system* (4.6) *vanish if and only if one of following ten (five for corresponding L_k^*) conditions holds:*

$$
\begin{aligned}
&L_{12}:\ b_1=\frac{1}{2},\quad b_2=0,\quad b_3=0,\quad b_4=2,\quad b_5=-1,\quad b_6=0;\\
&L_{13}:\ b_1=\frac{-1}{3},\ b_2=0,\quad b_3=0,\quad b_4=\frac{4}{3},\quad b_5=-1,\quad b_6=0;
\end{aligned}
$$

$$L_{14}:\ b_1=b_5,\quad b_2=\frac{-1}{2}(3+2b_5),\quad b_3=2(1+b_5)^2,\quad b_4=\frac{1}{2}(3+2b_5),$$
$$b_6=\frac{1}{2}(1+2b_5)(3+2b_5);$$
$$L_{15}:\ b_2=1,\ b_3=1+b_1,\ b_4=\frac{-1}{2}(3+2b_1),\ b_5=\frac{-1}{2},\ b_6=\frac{-1}{2}(1+2b_1);$$
$$L_{16}:\ b_1=\frac{-1}{2}(2+2b_2+b_2^2),\quad b_3=\frac{-1}{2}b_2^3(2+b_2),\quad b_4=1,$$
$$b_5=\frac{-1}{2}(2-b_2^2),\quad b_6=b_2(1+b_2).$$

For the above conditions, we have the following result.

Proposition 5.3. *Under each of the conditions L_{12}-L_{16}, system* (4.6) *respectively has a linearizability transformation* $\mathrm{T_k}$, $k=12,13,\ldots,16$ *as follows:*

$$L_{12}:\begin{cases}\dfrac{du}{dT}=u-\dfrac{1}{8}(5u^2-2uv-15v^2)-\dfrac{1}{8}(u+v)(u^2+uv-4v^2),\\ \dfrac{dv}{dT}=-(1+v)\Big(v-\dfrac{1}{8}u^2+\dfrac{1}{4}uv+\dfrac{3}{8}v^2\Big),\end{cases}$$
$$\mathrm{T}_{12}:\xi=f_2^{-1}f_{25}^{-1}f_{26},\quad \eta=f_2^{-2}f_{25}^{-1}f_{27};$$

$$L_{13}:\begin{cases}\dfrac{du}{dT}=u+\dfrac{1}{16}\big[2(u^2+2uv+9v^2)+(u+v)(u^2-2uv+5v^2)\big],\\ \dfrac{dv}{dT}=-v+\dfrac{1}{48}\big[2(5u^2-6uv-35v^2)+(u+v)(u^2+6uv-19v^2)\big],\end{cases}$$
$$\mathrm{T}_{13}:\xi=f_{19}^{\frac{1}{2}}f_{28}^{\frac{-3}{2}}f_7,\quad \eta=f_{19}^{\frac{-1}{2}}f_{28}^{\frac{-3}{2}}f_{29};$$

$$L_{14}:\begin{cases}\dfrac{du}{dT}=u-\dfrac{1}{4}\big[3(1+2b_5)u^2-2(3+2b_5)uv-(3+2b_5)v^2\big]\\ \qquad+\dfrac{1}{8}u^2\big[(1+2b_5)^2u-3(1+2b_5)(3+2b_5)v\big]\\ \qquad+\dfrac{1}{8}v^2\big[(3+2b_5)(5+6b_5)u-(1+2b_5)(3+2b_5)v\big],\\ \dfrac{dv}{dT}=-v(1+v)\Big(1+\dfrac{1}{2}v\Big),\end{cases}$$
$$\mathrm{T}_{14}:\xi=f_2^{-2(1+2b_5)}f_4^{-1}f_{30}^{-1}f_{31},\quad \eta=vf_2^{-2}f_4;$$

$$L_{15}:\begin{cases}\dfrac{du}{dT}=u(1+u)\Big(1+\dfrac{1}{2}u\Big),\\ \dfrac{dv}{dT}=-v-uv-(1+b_1)u^2(1+v)+\dfrac{1}{2}(1+2b_1)v^2(1+u),\end{cases}$$
$$\mathrm{T}_{15}:\xi=uf_1^{-2}f_3,\quad \eta=f_1^2f_3^{-1}f_{32};$$

$$L_{16}:\begin{cases}\dfrac{du}{dT}= u+\dfrac{1}{8}\big[(1+b_2)(5+b_2)u^2+2(1-b_2)^2uv+(1-b_2)(5+3b_2)v^2\big]\\ \qquad +\dfrac{1}{16}u^2\big[(1+b_2)^2(3-b_2^2)u+(1-b_2^2)(1-6b_2-3b_2^2)v\big]\\ \qquad +\dfrac{1}{16}v^2(1-b_2)\big[(1+3b_2+9b_2^2+3b_2^3)u+(1-b_2^2)(3+b_2)v\big],\\ \dfrac{dv}{dT}= -v+\dfrac{1}{8}\big[3(1+b_2)^2u^2-2(1+b_2)^2uv-(13+2b_2+b_2^2)v^2\big]\\ \qquad +\dfrac{1}{16}u^2(1+b_2)^2\big[(1-b_2^2)u+3(1+b_2^2)v\big]\\ \qquad -\dfrac{1}{16}v^2\big[(1+b_2)^2(5+3b_2^2)u-(1-b_2)(7+5b_2+3b_2^2+b_2^3)v\big],\end{cases}$$

$$\mathrm{T}_{16}:\xi=f_{33}f_{34}f_{36},\quad \eta=f_{33}^{-1}f_{34}^{-1}f_{35}.$$

Here,

$$\begin{aligned}
f_{25}&=1-\frac{1}{8}(u+v)(4+u+v),\\
f_{26}&=u+\frac{1}{8}(u+v)\big[(u+5v)+v(u+v)\big],\\
f_{27}&=v-\frac{1}{24}(u^2+6uv-3v^2)-\frac{1}{24}v(u+v)\big[2(u+3v)+v(u+v)\big],\\
f_{28}&=1+\frac{1}{6}(u+5v),\\
f_{29}&=v-\frac{1}{72}(u^2-18uv-39v^2)-\frac{1}{288}(u+v)^2\big[4(u-5v)+(u-3v)(u+v)\big],\\
f_{30}&=1-(1+2b_5)(u+v)+\frac{1}{4}(1+2b_5)\big[(1+2b_5)(u^2+v^2)-2(3+2b_5)uv\big],\\
f_{31}&=\frac{-(1+2b_5)e^T+(1+2b_5)e^{2c_1}-(3+2b_5)\sqrt{e^T(e^T-e^{2c_1})}}{(1+2b_5)(e^T-e^{2c_1})}\\
&\quad\pm\frac{e^{(1+b_5)T}(e^T-e^{2c_1})^{-\frac{3}{2}-b_5}}{\sqrt{c_2+\frac{(1+2b_5)^2e^{2(T+b_5T-c_1)}(e^T-e^{2c_1})^{-2(1+b_5)}}{8(1+b_5)}}},\\
f_{32}&=c_2\sqrt{1-e^{T+2c_1}}\,e^{-\frac{T}{2}\pm\tanh\sqrt{1-e^{T+2c_1}}},\\
f_{33}&=1+\frac{1}{2}\big[(1+b_2)u+(1-b_2)v\big],\\
f_{34}&=1-\frac{1}{2}\big[(1+b_2)u-(3+b_2)v\big],\\
f_{35}&=v-\frac{1}{8}\big[(1+b_2)^2u^2-2(1+b_2)^2uv-(1-b_2)(3+b_2)v^2\big],\\
f_{36}&=\frac{\ln(1+2f_{35})}{2(1+b_2)f_{35}}-\frac{\ln(f_{33}^2+2b_2f_{35})-2\ln f_{33}}{2b_2(1+b_2)f_{35}}+\cdots,
\end{aligned}$$

where c_1, c_2 are arbitrary constants.

Remark 5.4. (i) For system L_{14}, f_{31} satisfies

$$\frac{df_{31}}{dT} = h_{31} f_{31},$$

where

$$\begin{aligned} h_{31} =& 1 - (1+2b_5)u - \frac{1}{2}(3+4b_5)v + \frac{1}{2}(1+2b_5)^2 u^2 \\ & - \frac{1}{2}(1+2b_5)(3+2b_5)uv - \frac{1}{4}(1-4b_5-4b_5^2)v^2. \end{aligned}$$

(ii) For system L_{15}, f_{32} satisfies

$$\frac{df_{32}}{dT} = \frac{-1}{2}(1+u)(2+u) f_{32}.$$

(iii) $f_{36} = u + h.o.t.$ is a power series of u, v with non-zero convergence radius, with coefficients being polynomials in b_2. f_{36} is a simple integral curve of the system with L_{16} and satisfy

$$\frac{df_{36}}{dT} = f_{33} f_{34} f_{36}.$$

To sum up, we have

Theorem 5.4. *When the condition C_4 in Theorem 4.1 holds, the origin of system (4.6) is a complex isochronous center if and only if one of the ten conditions in* Lemma 5.4 *is satisfied.*

5.5. On complex center condition C_5

Lemma 5.5. *If the condition C_5 in* Theorem 4.1 *is satisfied, then all the first four period constants at the origin of system (4.6) are zero if and only if one of following four (two for corresponding L_k^*) conditions holds:*

$$\begin{aligned} L_{17}: \ & b_1 = \frac{-5}{2},\ b_2 = 1,\ b_3 = \frac{-3}{2},\ b_4 = 1,\ b_5 = \frac{-1}{2}, \quad b_6 = 2; \\ L_{18}: \ & b_1 = \frac{-5}{2},\ b_2 = 1,\ b_3 = \frac{9}{2}, \quad b_4 = 1,\ b_5 = \frac{-13}{2},\ b_6 = 2. \end{aligned}$$

Similarly we have the following result.

When the condition L_{17} is satisfied, system (4.6) can be rewritten as

$$\begin{aligned} \frac{du}{dT} &= u(1+u)\left(1 + \frac{1}{2}u\right), \\ \frac{dv}{dT} &= -v + \frac{1}{2}(3u^2 - 2uv - 4v^2) + \frac{1}{2}uv(3u - 4v). \end{aligned} \tag{5.11}$$

Table 2
Linearizability transformations of system (5.11) and (5.12).

	Linearizability Transformation	Integral curves
L_{17}:	$\xi = uf_1^{-2}f_3, \quad \eta = f_1^2 f_6 f_{37}^{-1}$	$f_{37} = 1+2u+2v+2uv$
L_{18}:	$\xi = f_1 f_{37}^{-2} f_{38}, \quad \eta = vf_1^2 f_4 f_{37}^{-2}$	$f_{38} = u+uv+\frac{1}{2}v^2+\frac{1}{2}uv^2$

When the condition L_{18} holds, system (4.6) is reduced to

$$\begin{aligned} \frac{du}{dT} &= (1+u)\Big(u+2u^2-3uv+\frac{3}{2}v^2\Big), \\ \frac{dv}{dT} &= -v\Big(1+\frac{1}{2}v\Big)(1-2u+3v), \end{aligned} \tag{5.12}$$

Linearization transformations of systems (5.11) and (5.12) are given in Table 2.
Then, we have the following theorem.

Theorem 5.5. *When the condition C_5 in Theorem 4.1 holds, the origin of system (4.6) is a complex isochronous center if and only if one of the four conditions in Lemma 5.5 is satisfied.*

5.6. On complex center condition C_6

Lemma 5.6. *If the condition C_6 in Theorem 4.1 is satisfied, then all the first four period constants at the origin of system (4.6) vanish if and only if one of following four (two for corresponding L_k^*) conditions holds:*

$$\begin{aligned} &L_{19}: b_1 = \frac{-13}{8},\ b_2 = \frac{1}{2},\ b_3 = \frac{-5}{32},\ b_4 = 1,\ b_5 = \frac{-7}{8},\ b_6 = \frac{3}{4}; \\ &L_{20}: b_1 = \frac{-7}{2},\quad b_2 = 1,\quad b_3 = \frac{-5}{2},\ b_4 = 2,\ b_5 = \frac{-1}{2},\ b_6 = 3. \end{aligned}$$

Under each of the conditions L_{19}-L_{20}, system (4.6) becomes

$$L_{19}: \begin{cases} \dfrac{du}{dT} = u + \frac{1}{32}(33u^2+2uv+13v^2) + \dfrac{1}{256}(3u+v)(33u^2-22uv+21v^2), \\ \dfrac{dv}{dT} = -v + \dfrac{3}{32}(9u^2-6uv-19v^2) + \dfrac{1}{256}(3u+v)(9u^2+42uv-83v^2), \end{cases}$$

$$L_{20}: \begin{cases} \dfrac{du}{dT} = u(1+u)\Big(1+\dfrac{1}{2}u\Big), \\ \dfrac{dv}{dT} = -v + \dfrac{1}{2}(5u^2-2uv-6v^2) + \dfrac{1}{2}uv(5u-6v). \end{cases}$$

When one of the conditions L_{19} and L_{20} holds, the linearization transformations of system (4.6) are obtained, as listed in Table 3.

Then, we have the following result.

Theorem 5.6. *When the condition C_6 in Theorem 4.1 holds, the origin of system (4.6) is a complex isochronous center if and only if one of the four conditions in* Lemma *5.6 is satisfied.*

Table 3
Linearizability transformations of system (4.6).

	Linearizability Transformation	Integral curves
L_{19}	$\xi = f_{15}^{-4} f_{40}, \quad \eta = f_{15}^{-1} f_{39}^{-1} f_{41}$	$f_{39} = 1 - \frac{3}{4}u + \frac{7}{4}v$
		$f_{40} = u + \frac{1}{96}(3u+v)(63u+13v) + \frac{1}{24}(3u+v)^3 + \frac{1}{384}(3u+v)^4$
		$f_{41} = v - \frac{1}{32}(3u-7v)(3u+v)$
L_{20}	$\xi = u f_1^{-2} f_3, \quad \eta = f_1^{-3} f_{42}^{-1} f_{43}$	$f_{42} = 1 - 2u + 3v$
		$f_{43} = v - \frac{1}{6}u(5u-12v) - \frac{1}{6}u^2(5u-9v) - \frac{1}{6}u^3(2u-3v)$

Table 4
Linearization transformation of system (5.13).

	Linearization Transformation	Integral curves
L_{21}	$\xi = f_{44}^{-1} f_{45}, \quad \eta = v f_4 f_{44}^{\frac{-2}{3}}$	$f_{44} = 1 - 6u - 3v - 15v^2 - 5v^3$
		$f_{45} = u - \frac{3}{2}v^2 + \frac{1}{2}v^3$

5.7. On complex center condition C_7

Lemma 5.7. *If the condition C_7 in Theorem 4.1 is satisfied, then the first two period constants at the origin of system (4.6) are zero if and only if one of following two (one for L_{21}^*) conditions holds:*

$$L_{21}: \, b_1 = \frac{13}{2}, \; b_2 = 1, \; b_3 = \frac{11}{2}, \; b_4 = 4, \; b_5 = \frac{11}{2}, \; b_6 = 9.$$

When the condition L_{21} holds, system (4.6) becomes

$$\begin{aligned} \frac{du}{dT} &= u - \frac{3}{2}(4u^2 - 2uv - 3v^2) - \frac{1}{2}v(12u^2 - 15uv + 2v^2), \\ \frac{dv}{dT} &= -v\Big(1 + \frac{1}{2}v\Big)(1 + 4u - 3v). \end{aligned} \tag{5.13}$$

A linearization transformation of system (5.13) is given in Table 4.

Then, the following result is obtained.

Theorem 5.7. *When the condition C_7 in Theorem 4.1 holds, the origin of system (4.6) is a complex isochronous center if and only if one of the two conditions in* Lemma 5.7 *is satisfied.*

5.8. On complex center condition C_8

Lemma 5.8. *If the condition C_8 in* Theorem 4.1 *is satisfied, then the first two period constants at the origin of system (4.6) equal zero if and only if one of following two (one for L_{22}^*) conditions holds:*

$$L_{22}: \, b_1 = \frac{1}{6}, \; b_2 = \frac{1}{3}, \; b_3 = \frac{-1}{6}, \; b_4 = \frac{5}{3}, \; b_5 = \frac{-11}{6}, \; b_6 = \frac{-2}{3}.$$

Table 5
Linearization transformation of system (5.14).

	Linearization transformation	Integral curves
L_{22}	$\xi = f_2^{-5} f_{46}, \quad \eta = v f_4 f_2^2$	$f_{46} = u + \frac{1}{18}(84u + 11v)v + \frac{1}{18}(8u^2 + 154uv + 47v^2)v + \frac{1}{54}(68u^2 + 440uv + 227v^2)v^2 + \frac{1}{486}(32u^3 + 588u^2v + 2184uv^2 + 1543v^3)v^2 + \frac{4}{243}(u + 2v)^2 \times (4u + 17v)v^3 + \frac{4}{243}(u + 2v)^3 v^4$

When the condition L_{22} holds, system (4.6) can be rewritten as

$$\begin{aligned}\frac{du}{dT} &= u - \frac{1}{6}(2u - 11v)v - \frac{1}{6}(u - 4v)v^2,\\ \frac{dv}{dT} &= -v\left(1 + \frac{1}{2}v\right)(1 + v).\end{aligned} \tag{5.14}$$

When the condition L_{22} holds, the linearization transformations of system (5.14) is obtained, as listed in Table 5.

Summarizing the above results gives the following theorem.

Theorem 5.8. *When the condition C_8 in Theorem 4.1 holds, the origin of system (4.6) is a complex isochronous center if and only if one of the two conditions in* Lemma 5.8 *is satisfied.*

5.9. On complex center condition C_9

Similarly, we have the following lemma.

Lemma 5.9. *If the condition C_9 in* Theorem 4.1 *holds, then all the first four period constants at the origin of system (4.6) are zero if and only if one of following two (one for L_{23}^*) conditions holds:*

$$L_{23}: \; b_1 = 3, \; b_2 = b_3 = 0, \; b_4 = 3, \; b_5 = 3, \; b_6 = 8.$$

Proposition 5.4. *When the condition L_{23} holds, system (4.6) can be rewritten as*

$$\begin{aligned}\frac{du}{dT} &= u - \frac{3}{8}(9u^2 - 6uv - 7v^2) - \frac{1}{16}(u^3 + 51u^2v - 69uv^2 + 9v^3),\\ \frac{dv}{dT} &= -v + \frac{3}{8}(u^2 - 6uv + v^2) - \frac{1}{16}(11u^3 - 39u^2v + 57uv^2 - 21v^3),\end{aligned} \tag{5.15}$$

which becomes (5.3) under the transformation,

$$\xi = f_{20}^{-1} f_{47}^{-1} f_{48}, \quad \eta = f_{20}^{-1} f_{47}^{\frac{-1}{2}} f_{49},$$

where

$$f_{47} = 1 - \frac{3}{4}(u+v)(4+u+v),$$
$$f_{48} = u - \frac{1}{16}(u+v)\big[2(u-7v)+(u-3v)(u+v)\big],$$
$$f_{49} = v - \frac{1}{8}(u-3v)(u+v).$$

Then, we have the following theorem.

Theorem 5.9. *When the condition C_9 in Theorem 4.1 is satisfied, the origin of system (4.6) is a complex isochronous center if and only if one of the two conditions in* Lemma 5.9 *holds.*

5.10. On complex center condition C_{10}

Lemma 5.10. *If the condition C_{10} in* Theorem 4.1 *holds, then all the first four period constants at the origin of system (4.6) vanish if and only if one of following eight (four for corresponding L_k^*) conditions is satisfied:*

$$\begin{aligned}
&L_{24}: b_1 = \tfrac{-5}{8},\ b_2 = \tfrac{-1}{2},\ b_3 = \tfrac{3}{32},\ \ b_4 = 1,\ b_5 = \tfrac{-7}{8},\ b_6 = \tfrac{-1}{4};\\
&L_{25}: b_1 = \tfrac{1}{2},\ \ b_2 = -1,\ b_3 = \tfrac{3}{2},\ \ b_4 = 2,\ b_5 = \tfrac{-1}{2},\ b_6 = 1;\\
&L_{26}: b_1 = \tfrac{1}{2},\ \ b_2 = -1,\ b_3 = \tfrac{-33}{2},\ b_4 = 2,\ b_5 = \tfrac{11}{2},\ b_6 = 13;\\
&L_{27}: b_1 = \tfrac{-5}{6},\ b_2 = \tfrac{-1}{3},\ b_3 = \tfrac{5}{54},\ \ b_4 = \tfrac{2}{3},\ b_5 = \tfrac{-7}{6},\ b_6 = \tfrac{-11}{27}.
\end{aligned}$$

Under each of the conditions L_{24}-L_{27}, system (4.6) can be reduced into one of the following systems:

$$L_{24}: \begin{cases} \dfrac{du}{dT} = u + \dfrac{3}{32}(3u^2+6uv+7v^2) + \dfrac{1}{256}(u+3v)(11u^2+6uv+15v^2),\\ \dfrac{dv}{dT} = -v + \dfrac{1}{32}(3u^2-2uv-49v^2) + \dfrac{1}{256}(u+3v)(3u^2+6uv-41v^2), \end{cases}$$

$$L_{25}: \begin{cases} \dfrac{du}{dT} = u - \dfrac{1}{2}(2u^2-2uv-3v^2) - \dfrac{1}{2}uv(2u-3v),\\ \dfrac{dv}{dT} = -v(1+v)\Big(1+\dfrac{1}{2}v\Big), \end{cases}$$

$$L_{26}: \begin{cases} \dfrac{du}{dT} = u(1-3u+4v)\Big(1+\dfrac{1}{2}u\Big),\\ \dfrac{dv}{dT} = -v + \dfrac{3}{2}u(u-2v) - \dfrac{1}{2}(6u^3-21u^2v+24uv^2-8v^3), \end{cases}$$

$$L_{27}: \begin{cases} \dfrac{du}{dT} = u + \dfrac{1}{6}(4u^2+2uv+3v^2) + \dfrac{1}{54}(u+2v)(8u^2-4uv+5v^2),\\ \dfrac{dv}{dT} = -v(1+v)\Big(1+\dfrac{1}{2}v\Big). \end{cases}$$

Their linearization transformations can be summarized in Table 6.

Table 6
Linearization transformations of system (4.6).

	Linearization Transformation	Integral curves
L_{24}	$\xi = f_{15}^{-2} f_{50} f_{51}, \quad \eta = f_{15}^{-1} f_{50}^{-1} f_{52}$	$f_{50} = 1 - \frac{1}{4}u + \frac{5}{4}v$ $f_{51} = u + \frac{1}{32}(3u+v)(5u+7v) + \frac{1}{128}(3u+v)^2(u+3v)$ $f_{52} = v - \frac{1}{32}(u-5v)(u+3v)$
L_{25}	$\xi = f_2^2 f_{11}^{-1} f_{38}, \quad \eta = v f_2^{-2} f_4$	
L_{26}	$\xi = u f_3 f_{11} f_{53}^{-1}, \quad \eta = f_6 f_{11}^{\frac{-1}{2}} f_{53}^{\frac{-1}{2}}$	$f_{53} = 1 - (3u+2v) + 3u(u-4v) + 3u^2(u-2v)$
L_{27}	$\xi = f_2^2 f_{54} f_{55}^{\frac{-3}{2}} f_{56}^{\frac{-3}{2}} f_{57}^{-\sqrt{3}} f_{58}^{-\sqrt{3}},$ $\eta = v f_2^{-2} f_4$	$f_{54} = u + \frac{1}{6}(2u+v)^2 + \frac{1}{54}(2u+v)^3$ $f_{55} = \frac{1}{27}(3+2u+v)^2 + \frac{1}{9}(3+2u+v)(1+v)^{\frac{1}{3}} + \frac{1}{3}(1+v)^{\frac{2}{3}}$ $f_{56} = \frac{1}{3}\left[1 + (1+v)^{\frac{1}{3}} + v(1+v)^{\frac{1}{3}} + (1+v)^{\frac{2}{3}}\right]$ $f_{57} = e^{\tan^{-1}\left[\frac{3+2u+v-3(1+v)^{1/3}}{\sqrt{3}\left(3+2u+v+3(1+v)^{1/3}\right)}\right]}$ $f_{58} = e^{\tan^{-1}\left[\frac{(1+v)^{2/3}-1}{\sqrt{3}\left((1+v)^{2/3}+1\right)}\right]}$

Remark 5.5. Let

$$\begin{aligned} x_1 &= \frac{1}{6}\left[3 + 2u + v + 3(1+v)^{\frac{1}{3}}\right], \\ y_1 &= \frac{1}{6\sqrt{3}}\left[3 + 2u + v - 3(1+v)^{\frac{1}{3}}\right], \\ x_2 &= \frac{1}{2}\left[1 + (1+v)^{\frac{2}{3}}\right], \\ y_2 &= -\frac{1}{2\sqrt{3}}\left[1 - (1+v)^{\frac{2}{3}}\right], \end{aligned} \tag{5.16}$$

and

$$G_1 = (x_1 + iy_1)(x_1 - iy_1)(x_2 + iy_2)(x_2 - iy_2) = (x_1^2 + y_1^2)(x_2^2 + y_2^2),$$

$$G_2 = \left(\frac{(x_1 + iy_1)(x_2 + iy_2)}{(x_1 - iy_1)(x_2 - iy_2)}\right)^{\frac{-i}{2}} = e^{\tan^{-1}\left(\frac{x_2 y_1 + x_1 y_2}{x_1 x_2 + y_1 y_2}\right)}.$$

Then,

$$f_{55} f_{56} = G_1, \quad f_{57} f_{58} = G_2,$$

and (5.16) becomes

$$\xi = f_2^2 f_{54} G_1^{\frac{-3}{2}} G_2^{-\sqrt{3}}, \quad \eta = v f_2^{-2} f_4,$$

where

$$G = G_1^{\frac{-3}{2}} G_2^{-\sqrt{3}}$$

is a simple integral curve of system (5.16) and satisfies

$$\frac{dG}{dT} = \frac{1}{18}(-24u + 42v - 8u^2 - 8uv + 25v^2)G.$$

The above results yield the following theorem.

Theorem 5.10. *When the condition C_{10} in Theorem 4.1 holds, then the origin of system (4.6) is a complex isochronous center if and only if one of the eight conditions in* Lemma 5.10 *is satisfied.*

5.11. On complex center condition C_{11}

Lemma 5.11. *If the condition C_{11} in* Theorem 4.1 *is satisfied, then both the first two period constants at the origin of system (4.6) equal zero if and only if one of following two (one for L_{28}^*) conditions holds:*

$$L_{28}: b_1 = \frac{-1}{6},\ b_2 = \frac{-1}{3},\ b_3 = \frac{1}{6},\ b_4 = \frac{4}{3},\ b_5 = \frac{-7}{6},\ b_6 = \frac{-1}{3}.$$

Proposition 5.5. *When the condition L_{28} holds, system (4.6) can be rewritten as*

$$\begin{aligned}\frac{du}{dT} &= u + \frac{1}{6}(2u + 7v)v + \frac{1}{6}(u + 2v)v^2,\\ \frac{dv}{dT} &= -v(1 + v)\Big(1 + \frac{1}{2}v\Big),\end{aligned}$$

which has a linearization transformation,

$$\xi = f_2^{-3} f_{59},\quad \eta = v f_2^{-2} f_4,$$

where

$$\begin{aligned}f_{59} = u &+ \frac{1}{18}(60u + 7v)v + \frac{1}{18}(8u^2 + 70uv + 19v^2)v\\ &+ \frac{1}{27}(26u^2 + 56uv + 23v^2)v^2 + \frac{1}{243}(2u + v)(8u^2 + 71uv + 47v^2)v^2\\ &+ \frac{1}{486}(2u + v)^2 v^3\big[(8u + 13v) + (2u + v)v\big].\end{aligned}$$

The following theorem directly follows from Proposition 5.5.

Theorem 5.11. *When the condition C_{11} in Theorem 4.1 is satisfied, the origin of system (4.6) is a complex isochronous center if and only if one of the two conditions in* Lemma 5.11 *holds.*

6. Conclusion

In this paper, we have studied complex isochronous center problem for cubic complex planar vector fields, which has Z_2-equivariant property with two symmetric centers. Such integrable systems have been classified into 11 cases in previous work. We have obtained a total of 54 complex isochronous center conditions in two categories:

$$L_1 \sim L_{28}, \qquad L_1^*,\ L_4^* \sim L_{28}^*,$$

as well as all corresponding linearization transformations given explicitly in terms of system parameters. When the system coefficients take real values, there are only two real isochronous center conditions L_2 and L_3, which are the same as that given in [12].

It has been noted that commutating the periodic constant at the origin of system (4.6) is a very tedious work, and deriving the linearization transformation for each case is not easy, some of the cases are actually difficult. The methodology developed in this paper can be extended to consider other dynamical systems.

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