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Complex integrability and linearizability of cubic Z_2-equivariant systems with two 1:q resonant singular points

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Abstract

In this paper, complex integrability and linearizability of cubic Z_2-equivariant systems with two 1:q resonant singular points are investigated, and the necessary and sufficient conditions on complex integrability and linearizability of the systems with two 1:$(-q)$ resonant saddles are obtained for $q = 1, 2, 3, 4$. Moreover, for general positive integer q, the complex integrability and linearizability conditions are classified, and the sufficiency of the conditions is proved. Further, the linearizability conditions of cubic Z_2-equivariant systems with two 1:q resonant node points are also classified.

Keywords: Integrability; Linearizability; Saddle value; Periodic constant; Resonant node

1. Introduction

The Hilbert's 16th problem is far from being solved after one hundred years since it was proposed by Hilbert in the Second World Congress of Mathematicians, which has attracted many mathematicians and promoted great progress of mathematics in the 20th century. As far as the

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lower bound of limit cycles is concerned, the best results obtained so far are $H(2) \geq 4$ and $H(3) \geq 13$, where $H(n)$ denotes the number of limit cycles bifurcating in planar systems with degree n, see [3,13,14,27,30,33]. The symmetry property of vector fields of planar systems plays an important role in studying the Hilbert's 16th problem. Generally speaking, an efficient approach to obtain more limit cycles is to perturb symmetric systems which have centers as many as possible.

If a system

$$\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y), \tag{1.1}$$

is invariant under a real planar counter-clockwise rotation with angle $\frac{2\pi}{q}$, it is called Z_q-equivariant. The Z_q-equivariant system has been investigated intensively, and many significant results on the number of limit cycles of polynomial differential systems were obtained, see for instance [8,9,18–23]. In particular, the system (1.1) is Z_2-equivariant if the following conditions on the vector field hold:

$$X(-x, -y) = -X(x, y), \quad Y(-x, -y) = -Y(x, y),$$

under which (1.1) can be rewritten as (if it is of C^∞)

$$\frac{dx}{dt} = \sum_{k=0}^{\infty} X_{2k+1}(x, y), \quad \frac{dy}{dt} = \sum_{k=0}^{\infty} Y_{2k+1}(x, y). \tag{1.2}$$

For cubic Z_2-equivariant systems with two non-resonant singular points, there are four classes of normal forms, see [17]. The first case of the systems has two elementary foci, and its bi-center and bi-isochronous center problems have been considered in [6,26]. When the systems have two isolated elementary foci at $(1, 0)$ and $(-1, 0)$, an example was first constructed by Yu and Han [34–36] to obtain at least 12 small-amplitude limit cycles. Then, Liu and Huang [25] confirmed the result with simpler expressions of the Lyapunov constants. Furthermore, in [26], a class of Z_2-equivariant cubic systems given in the form of

$$\begin{aligned}
\frac{dx}{dt} &= -(a_{21}+1)y + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\frac{dy}{dt} &= -\frac{1}{2}x - b_{21}y + \frac{1}{2}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3,
\end{aligned} \tag{1.3}$$

was studied, and the first six focus values at $(\pm 1, 0)$ of this system were obtained. Then, 11 center conditions were derived, and a complete study on bi-center problem was carried out to obtain the necessary and sufficient conditions for the existence of the two centers. The bi-center problem for some Z_2-equivariant quintic systems was studied in [29], and the simultaneous existence of centers for two families of planar Z_q-equivariant systems was investigated in [10]. The isochronous bi-center problem for a cubic Z_2-equivariant vector field with real coefficients was considered in [6], and two real isochronous center conditions were obtained. For the complex isochronous center problem of system (1.3) with two centers at $(\pm 1, 0)$, there exist two difficulties: the first one is related to the computation of periodic constants, and the second one is in finding all linearizability transformations. In [16], these two difficulties were overcome and the problem was completely solved, with 54 complex linearization centers being identified.

For the second and third cases, system (1.1) can be simplified to

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{2}\lambda u - a_{21}v + \frac{1}{2}\lambda u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3, \\ \frac{dv}{dt} &= -\frac{1}{2}u + (\lambda - b_{21})v + \frac{1}{2}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3, \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{2}\lambda u + (1 - a_{21})v + \frac{1}{2}\lambda u^3 + a_{21}u^2v + a_{12}uv^2 + a_{03}v^3, \\ \frac{dv}{dt} &= (\lambda - b_{21})v + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3, \end{aligned} \tag{1.5}$$

respectively, which have two nilpotent singular points when $\lambda = 0$. For the above two systems, the integrability problem has been completely solved in [15] in which the bi-center problem and bifurcation of limit cycles from the two nilpotent singular points were also studied. Moreover, a new perturbation scheme was presented in [1] to prove the existence of 12 small-amplitude limit cycles, which bifurcate from the two nilpotent singular points, and the center problem was also studied.

For the last case, the cubic system can be written as

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}\lambda_1 x - a_{21}y + \frac{1}{2}\lambda_1 x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \frac{dy}{dt} &= (\lambda_2 - b_{21})y + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{aligned} \tag{1.6}$$

System (1.6) with $\lambda_1 = 1$ and $\lambda_2 = -1$ has been investigated in [17], where sufficient and necessary conditions on complex center and complex isochronous center were obtained. However, some open problems still exist, for example, when $(\pm 1, 0)$ of system (1.6) are $1{:}q$ resonant singular points, namely, $\lambda_1 = 1$, $\lambda_2 = q$, the conditions on integrability and linearizability have not been obtained due to the difficulty of computing the saddle values and periodic constants associated with $1{:}q$ resonant saddle points.

In this paper, we will focus on the integrability and linearizability problems of the cubic Z_2-equivariant systems with $1{:}q$ resonant singular points at $(\pm 1, 0)$, namely, $\lambda_1 = 1$, $\lambda_2 = q$, where q is an integer. Both cases for positive q, yielding two nodes at $(\pm 1, 0)$, and for negative q, giving two saddle points at $(\pm 1, 0)$, will be discussed.

The rest of the paper is organized as follows. In the next section, we present some definitions and lemmas which are needed to prove our main results in following sections. In section 3, the cubic Z_2-equivariant systems with $1{:}(-q)$ $(q = 2, 3, 4)$ resonant singular points are studied, and the integrability and linearization conditions are derived. Section 4 is devoted to studying the integrability and linearization conditions for general integer $q \geqslant 5$, and a fairly general set of sufficient conditions for the cubic Z_2-equivariant systems with $1{:}q$ resonant singular points are obtained. Finally, in Section 5, the linearization conditions for the cubic Z_2-equivariant systems with $1{:}q$ resonant node are completely solved.

2. Preliminary

A polynomial differential system with its linear part in the form of the p:$(-q)$ resonant saddle point can be expressed in the form of

$$\frac{dz}{dt} = pz + P(z, w), \quad \frac{dw}{dt} = -qw + Q(z, w), \tag{2.1}$$

where $p, q \in \mathbb{Z}^+$, $z, w, t \in \mathbb{R}$, $P(z, w)$ and $Q(z, w)$ are polynomials. By a time scaling $t \to p^{-1}t$, system (2.1) can be rewritten as

$$\frac{dz}{dt} = z + P(z, w), \quad \frac{dw}{dt} = -\lambda w + Q(z, w), \tag{2.2}$$

where $\lambda = \frac{q}{p} \in \mathbb{Q}^+$. Related to the integrability problem, the only approach of finding the necessary conditions of the integrability for system (2.1) or (2.2) is to compute the p:$(-q)$ saddle values, which is a natural generalization of the focal value computation [37].

Several classes of the system (2.1) or (2.2) have been studied. For the 1:(−2) quadratic polynomial systems, the integrability problem was completely solved in [5,7,37]. For the Lotka-Volterra systems, necessary and sufficient conditions for the integrability of the case $\lambda \in \mathbb{N}$, that is, the 1:$(-n)$ resonant cases, were obtained in [5,37]. Some sufficient conditions were given in [11] for general λ, where the necessary and sufficient conditions for integrable systems were derived for $\lambda = \frac{p}{2}$ and $\frac{2}{p}$ with $p \in \mathbb{Z}^+$. In [24], some sufficient conditions for the integrable Lotka-Volterra systems with 3:$(-q)$ resonance were given, and the integrability problem was investigated for the two particular cases, 3:(−4) and 3:(−5). The 1:$(-q)$ resonant center problem for certain cubic Lotka-Volterra systems was considered in [4].

In [2,11], for the complex polynomial differential system given in the form of

$$\frac{dz}{dT} = pz + \sum_{\alpha+\beta=2}^{\infty} a_{\alpha\beta} z^{\alpha} w^{\beta}, \quad \frac{dw}{dT} = -qw - \sum_{\alpha+\beta=2}^{\infty} b_{\alpha\beta} w^{\alpha} z^{\beta}, \tag{2.3}$$

the saddle quantity and generalized period constant were defined and a computation method was also provided.

Lemma 2.1. [2,28] *System* (2.3) *can be transformed into the normal form,*

$$\frac{d\xi}{dT} = p\xi \sum_{i=0}^{\infty} p_i (\xi^q \eta^p)^i, \quad \frac{d\eta}{dT} = -q\eta \sum_{i=0}^{\infty} q_i (\xi^q \eta^p)^i, \tag{2.4}$$

by the unique formal series,

$$\xi = z + \sum_{k+j=2}^{\infty} c_{kj} z^k w^j, \quad \eta = w + \sum_{k+j=2}^{\infty} d_{kj} w^k z^j, \tag{2.5}$$

where $p_0 = q_0 = 1$, $c_{k+1,k} = d_{k+1,k} = 0$, $k = 1, 2, \cdots$.

Let $\mu_0 = \tau_0 = 0$, $\mu_k = p_k - q_k$, $\tau_k = p_k + q_k$, $k = 1, 2, \cdots$. Then, the saddle quantity and generalized period constant are defined below [31,32].

Definition 2.1. For any positive integer k, μ_k is called the kth singular point quantity of the origin of system (2.3). If system (2.3) is a real planar differential system, then μ_k is the kth saddle quantity. Moreover, the origin of system (2.3) is called a generalized complex center if $\mu_k = 0$, $k = 1, 2, \cdots$.

Definition 2.2. For any positive integer k, τ_k is called the kth generalized period constant of the origin of system (2.3), and the origin of system (2.3) is called a generalized complex isochronous center if $\mu_k = \tau_k = 0$, $k = 1, 2, \cdots$.

Remark 2.1. If system (2.3) is a real system, then the μ_k defined in Definition 2.1 is the "saddle quantity of order k", as defined in [37].

Integrability and linearizability were also discussed in [32]. Using the results given in [32] we have the following lemma.

Lemma 2.2. *System (2.3) is integrable at the origin if and only if the origin is a generalized complex center, namely $\mu_k = 0$, $k = 1, 2, \cdots$. System (2.3) is linearizable at the origin if and only if the origin is a generalized complex isochronous center, namely $\mu_k = \tau_k = 0$, $k = 1, 2, \cdots$.*

The normal form (2.4) can be simplified further when the origin of system (2.3) is a generalized complex center or complex isochronous center. In particular, we have the following two lemmas.

Lemma 2.3. *The origin of system (2.3) is a generalized complex center if and only if there exists a unique formal series (2.5) such that $p_i = q_i$ $(i = 1, 2, \cdots)$ in (2.4).*

Lemma 2.4. *System (2.3) is linearizable at the origin if and only if there exists the unique formal series (2.5) such that*

$$\frac{d\xi}{dT} = p\xi, \quad \frac{d\eta}{dT} = -q\eta,$$

namely, $p_i = q_i = 0$ $(i = 1, 2, \cdots)$ in (2.4).

A new method for computing the singular point quantity was also presented in [32], which is given below for convenience.

Lemma 2.5. *For system (2.3), the following successive formal series holds:*

$$F(z, w) = \sum_{\alpha+\beta=p+q}^{\infty} c_{\alpha\beta} z^{\alpha} w^{\beta} = z^q w^p + \text{h.o.t.},$$

where $c_{qp} = 1$, $c_{kq,kp} = 0$, $k = 2, 3, \cdots$, h.o.t. *stands for high order terms satisfying*

$$\left.\frac{dF}{dT}\right|_{(2.3)} = \sum_{m=1}^{\infty} \lambda_m (z^q w^p)^{m+1}.$$

If $\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = 0,\ \lambda_m \neq 0$, *then* $\mu_1 = \mu_2 = \cdots = \mu_{m-1} = 0,\ \mu_m \neq 0$, *and* $\lambda_m \sim pq\mu_m,\ m = 1, 2, \cdots$, *where* $\sim$ *represents algebraic equivalence.*

In order to study linearizability, many methods were developed, for example, see [12]. Especially, the following result has been obtained for $p = q = 1$ [28].

Theorem 2.1. [28] *The origin of following complex analytic system,*

$$\frac{dz}{dT} = z + a_{11}zw + \sum_{k=2}^{\infty} f_k(z)w^k, \quad \frac{dw}{dT} = -w - \sum_{k=2}^{\infty} g_k(z)w^k,$$

or its symmetric system,

$$\frac{dz}{dT} = z + \sum_{k=2}^{\infty} g_k(w)z^k, \quad \frac{dw}{dT} = -w - b_{11}zw + \sum_{k=2}^{\infty} f_k(w)z^k,$$

is a complex isochronous center if

$$\deg(f_k) \leq k \quad \text{and} \quad \deg(g_k) \leq k - 2. \tag{2.6}$$

3. Integrability and linearizability conditions for cubic Z_2-equivariant systems with 1:($-q$) ($q = 2, 3, 4$) resonant saddles

In this section, we consider the integrability and linearizability conditions for cubic Z_2-equivariant systems with 1:($-q$) resonant saddles when $q = 2, 3, 4$. The case $q = 1$ (i.e., $\lambda_1 = 1$, $\lambda_2 = -1$), which is called 1:(-1) weak saddle, has been solved [17], and the results are summarized in the following two theorems.

Theorem 3.1. [17] *When* $\lambda_1 = 1,\ \lambda_2 = -1$, *system* (1.6) *is integrable at* $(\pm 1, 0)$ *if and only if one of the following five conditions holds:*

$$\begin{aligned}
&\widetilde{C}_1:\ a_{21} = 0,\ \ b_{21} = -\frac{1}{2};\\
&\widetilde{C}_2:\ a_{21} = -b_{12},\ a_{12} = -3b_{03},\ b_{21} = -\frac{3}{2};\\
&\widetilde{C}_3:\ a_{21} = \frac{1}{2}(1 + 2b_{21})b_{12},\ a_{03} = -\frac{1}{3}a_{12}(1 - 2b_{21})b_{12},\ b_{03} = \frac{2}{3}a_{12}(1 + b_{21});\\
&\widetilde{C}_4:\ a_{21} = 0,\ a_{03} = 0,\ b_{21} = 0,\ b_{12} = 0;\\
&\widetilde{C}_5:\ a_{21} = 0,\ a_{12} = 0,\ b_{12} = 0,\ b_{03} = 0,\ b_{21}^2 = \frac{1}{36}.
\end{aligned}$$

Theorem 3.2. [17] *When $\lambda_1 = 1,\ \lambda_2 = -1$, system (1.6) is linearizable at $(\pm 1, 0)$ if and only if one of the following six conditions is satisfied:*

$$
\begin{aligned}
&\widetilde{C}_1, \quad \widetilde{C}_4, \quad \widetilde{C}_5, \ \textit{(given in Theorem 3.1)}; \quad \textit{and} \\
&\widetilde{C}_2^*: \ a_{21} = 0,\ a_{12} = 0,\ a_{03} = 0,\ b_{21} = -\frac{3}{2},\ b_{12} = 0,\ b_{03} = 0; \\
&\widetilde{C}_3^*: \ a_{21} = b_{12},\ a_{03} = 0,\ b_{21} = \frac{1}{2},\ b_{03} = a_{12}; \\
&\widetilde{C}_3^{**}: \ a_{21} = 0,\ a_{12} = 0,\ a_{03} = 0,\ b_{12} = 0,\ b_{03} = 0.
\end{aligned}
$$

Now, we consider the case $q = 2$. We first derive the necessary integrability conditions based on the computation of saddle quantities, and then prove that they are also sufficient. Further, the necessary linearization conditions are derived by computing the periodic constants, and then their sufficiency is also proved by using various different approaches.

3.1. Saddle quantities for a class of Z_2-equivariant cubic systems with 1:(−2) resonant saddles

When $\lambda_1 = 1,\ \lambda_2 = -2$, system (1.6) becomes

$$
\begin{aligned}
\frac{dx}{dt} &= -\frac{1}{2}x - a_{21}y + \frac{1}{2}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\frac{dy}{dt} &= (-2 - b_{21})y + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.
\end{aligned}
\tag{3.1}
$$

To compute the saddle quantities of system (3.1) at $(\pm 1, 0)$, introducing the transformation, $z = \pm x - 1$, into system (3.1) yields

$$
\begin{aligned}
\frac{dz}{dt} &= z + \frac{3}{2}z^2 + 2a_{21}zy + a_{12}y^2 + \frac{1}{2}z^3 + a_{21}z^2y + a_{12}zy^2 + a_{03}y^3, \\
\frac{dy}{dt} &= -2y + 2b_{21}zy + b_{12}y^2 + b_{21}z^2y + b_{12}zy^2 + b_{03}y^3,
\end{aligned}
\tag{3.2}
$$

with the singular point $(\pm 1, 0)$ of (3.1) shifted to the origin $(0, 0)$ of (3.2).

Theorem 3.3. *The first seven saddle quantities at the origin of system (3.2) are given as follows:*

$$
\begin{aligned}
\mu_1 &= -\frac{1}{2}(4a_{21} - 3b_{12} - 2b_{12}b_{21})g_1, \\
\mu_2 &= -\frac{8}{15}b_{21}(1 + b_{21})(4a_{12} - 5b_{03} + 2a_{12}b_{21})g_1g_2,
\end{aligned}
\tag{3.3}
$$

and if $b_{21}(1 + b_{21}) \neq 0$,

$$
\begin{aligned}
\mu_3 &= -\frac{1}{2800}(-1 + 6b_{21})(1 + 6b_{21})(5 + 6b_{21})(7 + 6b_{21})g_1g_2g_3, \\
\mu_4 &= -\frac{128}{295245}b_{12}(7 + 4b_{21})(10 + 189b_{21} + 189b_{21}^2)g_1g_2g_3,
\end{aligned}
$$

$$\mu_5 = \frac{32}{4428675} a_{12}(17+10b_{21})(367+1500b_{21}+1500b_{21}^2) g_1 g_2 g_3,$$

or if $b_{21}=0$,

$$\begin{aligned}
\mu_3 &= -\frac{1}{32}(12a_{03}-38a_{12}b_{12}+49b_{03}b_{12}) g_1 g_2,\\
\mu_4 &= 0,\\
\mu_5 &= -\frac{221}{3072}(4a_{12}-5b_{03})b_{12}(4a_{12}+7b_{03}+63b_{12}^2) g_1 g_2,\\
\mu_6 &= 0,\\
\mu_7 &= \frac{2860165}{49152}(4a_{12}-5b_{03})b_{12}^5 g_1 g_2,
\end{aligned}$$

or if $b_{21}=-1$,

$$\begin{aligned}
\mu_3 &= -\frac{1}{32}(8a_{03}-6a_{12}b_{12}+21b_{03}b_{12}) g_1 g_2,\\
\mu_4 &= 0,\\
\mu_5 &= -\frac{91}{2048}(2a_{12}-5b_{03})b_{12}(2a_{12}+3b_{03}+8b_{12}^2) g_1 g_2,\\
\mu_6 &= 0,\\
\mu_7 &= \frac{10659}{16384}(2a_{12}-5b_{03})b_{12}^5 g_1 g_2.
\end{aligned}$$

Here, the polynomials g_1, g_2 *and* g_3 *are given by*

$$g_1 = 1+2b_{21}, \quad g_2 = 3+2b_{21}, \quad g_3 = (10a_{03}+a_{12}b_{12}-2a_{12}b_{12}b_{21})(3+b_{21}).$$

The following result directly follows Theorem 3.1.

Theorem 3.4. *All the first seven saddle quantities at the origin of system* (3.2) *vanish if and only if one of the following six conditions holds:*

$$\begin{aligned}
&C_1: \ b_{21} = -\frac{1}{2};\\
&C_2: \ a_{21}=0, \ b_{21} = -\frac{3}{2};\\
&C_3: \ a_{21} = \frac{1}{4}b_{12}(3+2b_{21}), \ b_{03} = \frac{2}{5}a_{12}(2+b_{21}), b_{21}=-3;\\
&C_4: \ a_{21} = \frac{1}{4}b_{12}(3+2b_{21}), \ b_{03} = \frac{2}{5}a_{12}(2+b_{21}), \ a_{03} = \frac{1}{10}a_{12}b_{12}(2b_{21}-1);\\
&C_5: \ a_{21}=b_{03}=a_{12}=b_{12}=0, \ (6b_{21}-1)(6b_{21}+1)(5+6b_{21})(7+6b_{21})=0;\\
&C_6: \ a_{21}=0, \ a_{03}=0, \ b_{12}=0, \ b_{21}(b_{21}+1)=0.
\end{aligned}$$

3.2. Generalized complex center of system (3.2)

Theorem 3.3 implies that any of the six conditions in this theorem is necessary for the origin of system (3.2) to be a generalized complex center. Next, we will show that these six conditions are also sufficient.

3.2.1. Sufficiency of C_1, C_2, C_3 and C_4

Proposition 3.1. *If one of the conditions C_1, C_2, C_3 and C_4 in Theorem 3.4 holds, then $(\pm 1, 0)$ of system (3.1) are generalized complex centers.*

Proof. When the condition C_1 is satisfied, system (3.1) can be rewritten as

$$\begin{aligned}\frac{dx}{dt}&=\frac{1}{2}\left(-x+x^3-2a_{21}y+2a_{21}x^2y+2a_{12}xy^2+2a_{03}y^3\right),\\ \frac{dy}{dt}&=\frac{1}{2}y\left(-3-x^2+2b_{12}xy+2b_{03}y^2\right),\end{aligned}\tag{3.4}$$

which can be transformed to

$$\begin{aligned}\frac{du}{dt}&=u+2a_{21}uv-2a_{21}u^2v+2a_{12}v^2-4a_{12}uv^2+2a_{12}u^2v^2\\ &\quad+2a_{03}v^3-6a_{03}uv^3+6a_{03}u^2v^3-2a_{03}u^3v^3,\\ \frac{dv}{dt}&=v(-2+b_{12}v+a_{21}uv+a_{12}v^2+b_{03}v^2-a_{12}uv^2-b_{03}uv^2\\ &\quad+a_{03}v^3-2a_{03}uv^3+a_{03}u^2v^3),\end{aligned}\tag{3.5}$$

by

$$u=\frac{x^2-1}{x^2},\qquad v=xy.$$

Further, introducing $v=z^2$ into system (3.5) we obtain

$$\begin{aligned}\frac{du}{dt}&=u-2a_{21}(u^2-u)z^2+2a_{12}(u-1)^2z^4-2a_{03}(u-1)^3z^6\\ &\triangleq u+\sum_{k=2}f_k(u)z^k,\\ \frac{dz}{dt}&=-z+\frac{1}{2}(b_{12}+A_{21}u)z^3-\frac{1}{2}(b_{12}+b_{03})(u-1)z^5+\frac{1}{2}a_{03}(u-1)^2z^7\\ &\triangleq -z+\sum_{k=2}g_k(u)z^k.\end{aligned}\tag{3.6}$$

It is easy to verify that $\deg(f_k)\leqslant k$, $\deg(g_k)\leqslant k-2$. So according to Theorem 2.1, the origin of system (3.6) is a complex isochronous center, which implies that the singular points $(\pm 1, 0)$ of system (3.1) are generalized complex isochronous centers when the condition C_1 holds.

If the condition C_2 in Theorem 3.4 holds, system (3.1) can be simplified as

$$\begin{aligned}\frac{dx}{dt}&=\frac{1}{2}\left(-x+x^3+2a_{12}xy^2+2a_{03}y^3\right),\\ \frac{dy}{dt}&=\frac{1}{2}y\left(-1-3x^2+2b_{12}xy+2b_{03}y^2\right).\end{aligned}\tag{3.7}$$

System (3.7) can be transformed into

$$\begin{aligned}\frac{du}{dT}&=u+2a_{12}v^2+2a_{12}uv^2+2a_{03}v^3+2a_{03}uv^3,\\ \frac{dv}{dT}&=-v(2-b_{12}v+a_{12}v^2-b_{03}v^2+a_{03}v^3),\end{aligned}\tag{3.8}$$

by

$$u=x^2-1,\quad v=\frac{y}{x},\quad t=(1+u)T.$$

Further, under the transformation $v=z^2$, system (3.8) is changed to

$$\begin{aligned}\frac{du}{dT}&=u+2a_{12}(1+u)z^4+2a_{03}(1+u)z^6,\\ \frac{dz}{dT}&=-z+\frac{1}{2}b_{12}z^3-\frac{1}{2}(a_{12}-b_{03})z^5-\frac{1}{2}a_{03}z^7.\end{aligned}\tag{3.9}$$

According to Theorem 2.1, the origin of system (3.9) is a complex isochronous center, and so are the $(\pm 1,0)$ of system (3.1).

If the condition C_3 in Theorem 3.4 holds, system (3.1) can be reduced to

$$\begin{aligned}\frac{dx}{dt}&=\frac{1}{4}\left(-2x+2x^3+3b_{12}y-3b_{12}x^2y+4a_{12}xy^2+4a_{03}y^3\right),\\ \frac{dy}{dt}&=-\frac{1}{5}y\left(-5+15x^2-5b_{12}xy+2a_{12}y^2\right),\end{aligned}\tag{3.10}$$

which has an invariant curve, $f_1=y$, admitting an integrating factor, $F=y^{-\frac{1}{2}}$, and hence the origin of system (3.10) is a complex center, implying that $(\pm 1,0)$ of system (3.1) are generalized complex centers.

If the condition C_4 in Theorem 3.4 holds, system (3.1) can be rewritten as

$$\begin{aligned}\frac{dx}{dt}=&\frac{1}{20}(-10x+10x^3-15b_{12}y-10b_{12}b_{21}y+15b_{12}x^2y\\ &+10b_{12}b_{21}x^2y+20a_{12}xy^2-2a_{12}b_{12}y^3+4a_{12}b_{12}b_{21}y^3),\\ \frac{dy}{dt}=&-\frac{1}{5}y(-10-5b_{21}+5b_{21}x^2+5b_{12}xy+4a_{12}y^2+2a_{12}b_{21}y^2).\end{aligned}\tag{3.11}$$

By computing the invariant algebraic curves of system (3.11), we get a first integral of the system, given by

$$M_3 = f_1 f_2^{-2(2+b_{21})} f_3^2,$$

where

$$f_1 = y, \quad f_2 = 2x - b_{12}y, \quad f_3 = -2 + 2x^2 + \frac{4}{5}a_{12}y^2.$$

Hence, the singular points $(\pm 1, 0)$ of system (3.11), i.e., the system (3.1), are generalized complex centers. $\square$

3.2.2. Sufficiency of C_5

Proposition 3.2. *If the condition C_5 holds, then the singular points $(\pm 1, 0)$ of system (3.1) are generalized complex centers.*

Proof. When the condition C_5 in Theorem 3.4 holds, there exist four sub-cases, as listed below.

$$\begin{aligned} &C_5^1: \ a_{21} = b_{03} = a_{12} = b_{12} = 0, \ b_{21} = \frac{1}{6}; \\ &C_5^2: \ a_{21} = b_{03} = a_{12} = b_{12} = 0, \ b_{21} = -\frac{1}{6}; \\ &C_5^3: \ a_{21} = b_{03} = a_{12} = b_{12} = 0, \ b_{21} = -\frac{5}{6}; \\ &C_5^4: \ a_{21} = b_{03} = a_{12} = b_{12} = 0, \ b_{21} = -\frac{7}{6}. \end{aligned} \tag{3.12}$$

When the condition C_5^1 holds, system (3.1) can be rewritten as

$$\frac{dx}{dt} = \frac{1}{2}(-x + x^3 + 2a_{03}y^3), \quad \frac{dy}{dt} = \frac{1}{6}y(-13 + x^2), \tag{3.13}$$

which admits a first integral $M_4 = f_1^3 f_4^{-13} f_5^6$, where

$$f_4 = 6x + a_{03}y^3, \quad f_5 = -91 + 91x^2 + 26a_{03}xy^3 + 2a_{03}^2y^6.$$

Thus, according to Theorem 2.1, $(\pm 1, 0)$ of the system (3.13) are generalized complex centers, and so $(\pm 1, 0)$ of system (3.1).

When the condition C_5^2 holds, system (3.2) becomes

$$\frac{dz}{dt} = \frac{1}{2}(2z + 3z^2 + z^3 + 2a_{03}y^3), \quad \frac{dy}{dt} = -\frac{1}{6}y(12 + 2z + z^2),$$

which can be transformed to

$$\frac{dz}{dt} = \frac{1}{2}(2z + 3z^2 + z^3 + 2a_{03}w^6), \quad \frac{dw}{dt} = -\frac{1}{12}(12 + 2z + z^2)w, \tag{3.14}$$

by $y = w^2$. Further, with another transformation,

$$\widetilde{x}=\frac{z(2+z)}{2(1+z)^2},\quad \widetilde{y}=(1+z)^{\frac{1}{6}}w,$$

system (3.14) can be brought into

$$\frac{d\widetilde{x}}{dt}=\widetilde{x}+a_{03}(1-4\widetilde{x}+4\widetilde{x}^2)\widetilde{y}^6,\quad \frac{d\widetilde{y}}{dt}=-\widetilde{y}+\frac{1}{6}a_{03}(2\widetilde{x}-1)\widetilde{y}^7, \tag{3.15}$$

which has a complex isochronous center at the origin by Theorem 2.1, and so the origin of system (3.2) is a generalized complex isochronous center, namely, the two singular points $(\pm 1,0)$ of the system (3.1) are generalized complex isochronous centers.

When the condition C_5^3 holds, system (3.2) can be rewritten as

$$\frac{dz}{dt}=\frac{1}{2}(2z+3z^2+z^3+2a_{03}y^3),\quad \frac{dy}{dt}=-\frac{1}{6}y(12+14z+7z^2),$$

which can be transformed to

$$\frac{dz}{dt}=\frac{1}{2}(2z+3z^2+z^3+2a_{03}w^6),\quad \frac{dw}{dt}=-\frac{1}{12}(12+14z+7z^2)w,$$

by $y=w^2$. Further, the above system is changed to

$$\frac{d\widetilde{x}}{dt}=\widetilde{x}-a_{03}(2\widetilde{x}-1)^5\widetilde{y}^6,\quad \frac{d\widetilde{y}}{dt}=-\widetilde{y}+\frac{7}{6}a_{03}(2\widetilde{x}-1)^4\widetilde{y}^7,$$

under the transformation,

$$\widetilde{x}=\frac{z(z+2)}{2(1+z)^2},\quad \widetilde{y}=w(1+z)^{\frac{7}{6}}.$$

Thus, according to Theorem 2.1, the origin of above system is a complex isochronous center, implying that $(\pm 1,0)$ of system (3.1) are generalized complex isochronous centers.

When the condition C_5^4 holds, system (3.2) can be rewritten as

$$\frac{dz}{dt}=\frac{1}{2}(2z+3z^2+z^3+2a_{03}y^3),\quad \frac{dy}{dt}=-\frac{1}{6}y(12+10z+5z^2), \tag{3.16}$$

which can be transformed to

$$\frac{dz}{dt}=\frac{1}{2}(2z+3z^2+z^3+2a_{03}w^6),\quad \frac{dw}{dt}=-\frac{1}{12}(12+10z+5z^2)w,$$

by $y=w^2$, and can be further changed to

$$\frac{d\widetilde{x}}{dt}=\widetilde{x}-a_{03}(2\widetilde{x}-1)^4\widetilde{y}^6,\quad \frac{d\widetilde{y}}{dt}=-\widetilde{y}-\frac{5}{6}a_{03}(2\widetilde{x}-1)^3\widetilde{y}^7,$$

by the transformation,

$$\widetilde{x} = \frac{z(z+2)}{2(1+z)^2}, \quad \widetilde{y} = w(1+z)^{\frac{5}{6}}.$$

Therefore, the origin of above system is a complex isochronous center by Theorem 2.1, that is, $(\pm 1, 0)$ of system (3.1) are generalized complex isochronous centers. □

3.2.3. *Sufficiency of C_6*

Proposition 3.3. *If the condition C_6 is satisfied, then $(\pm 1, 0)$ of system (3.1) are generalized complex centers.*

Proof. When the condition C_6 in Theorem 3.4 holds, there are two sub-cases, namely,

$$\begin{aligned} &C_6^1: \ a_{21} = 0, \ a_{03} = 0, \ b_{12} = 0, \ b_{21} = 0; \\ &C_6^2: \ a_{21} = 0, \ a_{03} = 0, \ b_{12} = 0, \ b_{21} = -1. \end{aligned} \tag{3.17}$$

When the condition C_6^1 holds, system (3.2) is simplified to

$$\frac{dz}{dt} = (1+z)\Big(z + \frac{1}{2}z^2 + a_{12}y^2\Big), \quad \frac{dy}{dt} = -y(2 - b_{03}y^2). \tag{3.18}$$

It can be shown by finding the invariant algebraic curves of system (3.18) that there exists an inverse integrating factor for the system,

$$M_3 = y^{\frac{1}{2}}(1+z)^3 g_1^{\frac{1}{4}(-4a_{12}+5b_{03})},$$

where

$$g_1 = \begin{cases} (1 - b_{03}y^2)^{\frac{1}{b_{03}}}, & \text{if } b_{03} \neq 0, \\ \exp(-y^2), & \text{if } b_{03} = 0. \end{cases}$$

This indicates that the origin of system (3.18) is a complex center under the condition C_6^1, and so $(\pm 1, 0)$ of system (3.1) are generalized complex centers.

When the condition C_6^2 holds, system (3.2) can be rewritten as

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{2}(1+z)(2z + z^2 + 2a_{12}y^2), \\ \frac{dy}{dt} &= y(-2 + b_{03}y^2 - 2z - z^2), \end{aligned}$$

which can be transformed to

$$\begin{aligned} \frac{dz}{dt} &= z + \frac{1}{2}(3z + z^2)z + a_{12}(1+z)w^4, \\ \frac{dw}{dt} &= -w - \frac{1}{2}w(2z + z^2) + \frac{b_{03}}{2}w^5, \end{aligned}$$

by $y = w^2$. Further, the above system can be changed to

$$\begin{aligned}\frac{d\widetilde{x}}{dt} &= \widetilde{x} - a_{12}(2\widetilde{x}-1)^3\widetilde{y}^4,\\ \frac{d\widetilde{y}}{dt} &= -\widetilde{y} + \frac{1}{2}(2a_{12}+b_{03})(-1+2\widetilde{x})^2\widetilde{y}^5,\end{aligned} \tag{3.19}$$

by the transformation,

$$\widetilde{x} = \frac{z(z+2)}{2(1+z)^2}, \quad \widetilde{y} = w(1+z).$$

Therefore, according to Theorem 2.1, the origin of system (3.19) is a complex isochronous center, and so is the origin of system (3.2), which means that $(\pm 1, 0)$ of system (3.1) are generalized complex isochronous centers. □

Summarizing the above results we have the following theorem.

Theorem 3.5. *The origin of system* (3.2) *is a generalized complex center if and only if one of the six conditions in Theorem* 3.4 *holds. Namely,* $(\pm 1, 0)$ *of system* (3.1) *are generalized complex centers if and only if one of the six conditions in Theorem* 3.4 *holds.*

3.3. Generalized complex isochronous center conditions for system (3.2)

We have shown that the origin of system (3.2) is a generalized complex center if and only if one of the six conditions in Theorem 3.4 holds. In the following, we study the generalized complex isochronous center problem of system (3.2) by considering the six complex center conditions one by one. Namely, the linearizable problem of system (3.2) will be considered.

3.3.1. Conditions C_1 and C_2

When the conditions C_1 and C_2 in Theorem 3.4 hold, we have shown in the above proof for Proposition 3.1 that the origin of system (3.2) is a generalized complex isochronous center, that is,

Proposition 3.4. *If the conditions C_1 and C_2 in Theorem* 3.4 *are satisfied, then the origin of system* (3.2) *is a generalized complex isochronous center.*

Now, we consider other conditions in Theorem 3.4.

3.3.2. Condition C_3

When the condition C_3 in Theorem 3.4 holds, a direct computation shows that the first three periodic constants at the origin of system (3.2) are given by

$$\tau_1 = -\frac{105}{2}b_{12}, \quad \tau_2|_{\tau_1=0} = -1344a_{12}, \quad \tau_3|_{\tau_1=\tau_2=0} = \frac{415701}{8}a_{03}. \tag{3.20}$$

It is easy to see that when the condition C_3 holds, the first three periodic constants at the origin of system (3.2) are zero if and only if the following conditions hold:

$$C_{3-1}: \quad a_{21}=a_{12}=a_{03}=b_{12}=b_{03}=0, \quad b_{21}=-3. \tag{3.21}$$

This leads to the following proposition.

Proposition 3.5. *The origin of system (3.2) is a generalized complex isochronous center if and only if the condition C_{3-1} in (3.21) holds.*

Proof. The necessity has been proved. To prove the sufficiency, we note that when the condition C_{3-1} holds, the system (3.2) is reduced to

$$\frac{dz}{dt}=\frac{1}{2}z(1+z)(2+z), \quad \frac{dy}{dt}=-y(2+6z+3z^2),$$

which has a linearization transformation,

$$\xi=yz^3(2+z)^3, \quad \eta=yz(1+z)^4(2+z),$$

and hence the conclusion is true. □

3.3.3. Condition C_4

When the condition C_4 in Theorem 3.4 holds, we can similarly discuss the generalized complex isochronous center at the origin of system (3.2). With the aid of computer algebraic system Mathematica, it is not difficult to get the first two periodic constants at the origin of system (3.2), given by

$$\begin{aligned}
\tau_1 &= \frac{1}{2}b_{12}(-1+2b_{21})(1+2b_{21})(3+2b_{21}),\\
\tau_2|_{\tau_1=0} &= \frac{32}{15}a_{12}b_{21}(1+b_{21})(-1+2b_{21})(1+2b_{21})(3+2b_{21}).
\end{aligned}$$

Thus, under the condition C_4, the first two periodic constants given above become zero if and only if one of the following six conditions is satisfied:

$$\begin{aligned}
&C_{4-1}: \ a_{21}=b_{12}, \ a_{03}=0, \ b_{03}=a_{12}, \ b_{21}=\frac{1}{2};\\
&C_{4-2}: \ a_{21}=0, \ a_{03}=0, \ b_{03}=0, \ a_{12}=0, \ b_{12}=0;\\
&C_{4-3}: \ a_{21}=\frac{1}{2}b_{12}, \ a_{03}=\frac{3}{5}a_{12}, \ b_{03}=-\frac{1}{5}a_{12}b_{12}, \ b_{21}=-\frac{1}{2};\\
&C_{4-4}: \ a_{21}=0, \ a_{03}=-\frac{2}{5}a_{12}b_{12}, \ b_{03}=-\frac{1}{5}a_{12}, \ b_{21}=-\frac{3}{2};\\
&C_{4-5}: \ a_{21}=0, \ a_{03}=0, \ b_{03}=\frac{4}{5}a_{12}, \ b_{12}=0, \ b_{21}=0;\\
&C_{4-6}: \ a_{21}=0, \ a_{03}=0, \ b_{03}=\frac{2}{5}a_{12}, \ b_{12}=0, \ b_{21}=-1.
\end{aligned} \tag{3.22}$$

Obviously, the conditions C_{4-3}, C_{4-4}, C_{4-5} and C_{4-6} are contained in the conditions C_1, C_2, C_6^1 and C_6^2, respectively. When one of the conditions C_1, C_2 and C_6^2 holds, we have proved that

$(\pm 1, 0)$ of system (3.1) are generalized complex isochronous centers. So we only need to prove the sufficiency for the conditions C_{4-1}, C_{4-2} and C_{4-5}.

Proposition 3.6. *The origin of system (3.2) is a generalized complex isochronous center if one of the conditions C_{4-1}, C_{4-2} and C_{4-5} holds.*

Proof. When the condition C_{4-1} holds, system (3.2) becomes

$$\begin{aligned}\frac{dz}{dt} &= z + \frac{1}{2}(2a_{12}y^2 + 4b_{12}yz + 2a_{12}y^2z + 3z^2 + 2b_{12}yz^2 + z^3),\\ \frac{dy}{dt} &= -2y + \frac{1}{2}y(2b_{12}y + 2a_{12}y^2 + 2z + 2b_{12}yz + z^2),\end{aligned} \tag{3.23}$$

which has a linearization transformation,

$$\xi = f_7 f_6^{-2}, \quad \eta = w^{\frac{2}{5}} f_7^{-\frac{1}{5}}$$

in the neighborhood of the origin of system (3.23), where

$$f_6 = 2 - b_{12}y + 2z, \quad \text{and} \quad f_7 = 2a_{12}y^2 + 10z + 5z^2.$$

So the origin of system (3.2) is a generalized complex isochronous center under the condition C_{4-1}.

When the condition C_{4-2} holds, system (3.2) can be rewritten as

$$\frac{dz}{dt} = \frac{1}{2}z(1+z)(2+z), \quad \frac{dy}{dt} = y(-2 + 2b_{21}z + b_{21}z^2),$$

which can be transformed to

$$\frac{dz}{dt} = z + \frac{3}{2}z^2 + \frac{1}{2}z^3, \quad \frac{dw}{dt} = -w + \frac{1}{2}(2b_{21}z + b_{21}z^2)w,$$

by $y = w^2$. Thus, by Theorem 2.1, the origin of above system is a complex isochronous center, and so the origin of system (3.2) is a generalized complex isochronous center if the condition C_{4-2} holds.

When the condition C_{4-5} is satisfied, system (3.2) can be rewritten as

$$\begin{aligned}\frac{dz}{dt} &= z + \frac{1}{2}(3z^2 + z^3 + 2a_{12}y^2 + 2a_{12}zy^2),\\ \frac{dy}{dt} &= -2y + \frac{4}{5}a_{12}y^3,\end{aligned} \tag{3.24}$$

which can be transformed to

$$\begin{aligned}\frac{dz}{dt} &= z + \frac{1}{2}(3z^2 + z^3 + 2a_{12}w^4 + 2a_{12}zw^4),\\ \frac{dw}{dt} &= -w + \frac{2}{5}a_{12}w^5,\end{aligned}$$

by $y = w^2$, and can be further changed to

$$\frac{d\widetilde{x}}{dt} = \widetilde{x} - a_{12}(2x-1)w^4, \quad \frac{dw}{dt} = -w + \frac{2}{5}a_{12}w^5,$$

under the transformation,

$$\widetilde{x} = \frac{z(z+2)}{2(1+z)^2}.$$

Thus, according to the Theorem 2.1, the origin of above system is a complex isochronous center, implying that the origin of system (3.2) is a generalized complex isochronous center under the condition C_{4-5}. $\square$

For the condition C_{4-4}, we have an alternative to prove the sufficiency. When the condition C_{4-4} holds, system (3.2) can be written as

$$\begin{aligned} \frac{dz}{dt} &= z + \frac{1}{2}(2a_{12}y^2 - \frac{4}{5}a_{12}b_{12}y^3 + 2a_{12}y^2 z + 3z^2 + z^3), \\ \frac{dy}{dt} &= -y + y(b_{12}y + \frac{a_{12}}{5}y^2 - 3z + b_{12}yz - \frac{3}{2}z^2), \end{aligned} \tag{3.25}$$

which has a linearization transformation,

$$\xi = f_7 f_8^{-1}, \quad \eta = -w f_7 f_8 f_9^{-1}$$

in the neighborhood of the origin, where

$$\begin{aligned} f_7 &= 2a_{12}y^2 + 10z + 5z^2, \\ f_8 &= 5 + 2a_{12}y^2 + 10z + 5z^2, \\ f_9 &= 2 - b_{12}y + 2z, \end{aligned} \tag{3.26}$$

and so the origin of above system is a generalized complex isochronous center.

3.3.4. Condition C_5

Proposition 3.7. *The origin of system* (3.2) *is a generalized complex isochronous center if the condition C_5 in* (3.12) *holds.*

Proof. There are four subcases: C_5^k, $k = 1, 2, 3, 4$. The proofs for the three subcases C_5^k, $k = 2, 3, 4$ have been given in the proof for Proposition 3.2. So we only need to prove the case C_5^1.

When the condition C_5^1 holds, system (3.2) can be rewritten as

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{2}(2z + 3z^2 + z^3 + 2a_{03}y^3), \\ \frac{dy}{dt} &= \frac{1}{6}y(-12 + 2z + z^2), \end{aligned}$$

which can be transformed to

$$\frac{dz}{dt} = \frac{1}{2}(2z + 3z^2 + z^3 + 2a_{03}w^6),$$
$$\frac{dw}{dt} = \frac{1}{12}(-12 + 2z + z^2)w,$$

by $y = w^2$. Further, the above system can be changed to

$$\frac{d\widetilde{x}}{dt} = \widetilde{x} - a_{03}\widetilde{y}^6 - 2a_{03}\widetilde{x}\widetilde{y}^6, \quad \frac{d\widetilde{y}}{dt} = -\widetilde{y} + \frac{1}{6}a_{03}\widetilde{y}^7,$$

by the transformation,

$$\widetilde{x} = \frac{z(z+2)}{2(1+z)^2}, \quad \widetilde{y} = w(1+z)^{-\frac{1}{6}}.$$

Hence, according to Theorem 2.1, the origin of the above system is a complex isochronous center, and so the origin of system (3.2) is a generalized complex isochronous center. □

3.3.5. Condition C_6

Proposition 3.8. *The origin of system (3.2) is a generalized complex isochronous center if the condition C_6 in (3.12) holds.*

Proof. There are two subcases C_6^1 and C_6^2. The proof for the case C_6^2 has been given in the proof for Proposition 3.3.

When the condition C_6^1 holds, system (3.2) can be rewritten as

$$\frac{dz}{dt} = (1+z)\Big(z + \frac{1}{2}z^2 + a_{12}y^2\Big),$$
$$\frac{dy}{dt} = -y(2 - b_{03}y^2),$$

which can be transformed to

$$\frac{dz}{dt} = \frac{1}{2}(2z + 3z^2 + z^3) + 2a_{12}(1+z)w^4,$$
$$\frac{dw}{dt} = -w - \frac{b_{03}}{2}w^5,$$

by $y = w^2$, and can be further changed to

$$\frac{d\widetilde{x}}{dt} = \widetilde{x} - a_{12}(2\widetilde{x} - 1)w^4, \quad \frac{dw}{dt} = -w - \frac{b_{03}}{2}w^5,$$

by

$$\widetilde{x} = \frac{z(z+2)}{2(1+z)^2}.$$

So according to Theorem 2.1, the conclusion is true. □

Since the conditions C_{4-3}, C_{4-4} and C_{4-5} and C_{4-6} are contained in C_1, C_5 and C_6, respectively, we directly have the following theorem.

Theorem 3.6. *The origin of system (3.2) is a generalized complex isochronous center if and only if one of the following conditions holds:*

$$C_1,\ \ C_{2-1},\ \ C_{3-1},\ \ C_{4-1},\ \ C_{4-2},\ \ C_5,\ \ C_6.$$

3.4. Integrability and linearization conditions for cubic Z_2-equivariant systems with 1:(−3) and 1:(−4) resonant saddles

In this subsection, we present the results for the 1:(−3) and 1:(−4) cases without proofs for brevity, since the proofs are similar to that for the 1:2 case.

When $\lambda_1 = 1$, $\lambda_2 = -3$, system (1.6) becomes

$$\begin{aligned} \frac{dx}{dT} &= -\frac{1}{2}x - a_{21}y + \frac{1}{2}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \frac{dy}{dT} &= (-3 - b_{21})y + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{aligned} \tag{3.27}$$

Theorem 3.7. *The two singular points $(\pm 1, 0)$ of system (3.27) are generalized complex centers if and only if one of the following six conditions holds:*

$$\begin{aligned}
&D_1 : \left(b_{21} + \frac{1}{2}\right)\left(b_{21} + \frac{3}{2}\right) = 0;\\
&D_2 : a_{21} = 0,\ b_{21} = -\frac{5}{2};\\
&D_3 : a_{21} = -\frac{1}{6}b_{12}(5 + 2b_{21}),\ b_{03} = \frac{2}{7}a_{12}(3 + b_{21}),\ b_{21} = -\frac{9}{2};\\
&D_4 : a_{21} = -\frac{1}{6}b_{12}(5 + 2b_{21}),\ b_{03} = \frac{2}{7}a_{12}(3 + b_{21}),\ a_{03} = \frac{1}{21}a_{12}b_{12}(2b_{21} - 1);\\
&D_5 : a_{21} = b_{03} = a_{12} = b_{12} = 0,\\
&\qquad (-1 + 6b_{21})(1 + 6b_{21})(5 + 6b_{21})(7 + 6b_{21})(11 + 6b_{21})(13 + 6b_{21}) = 0,\\
&D_6 : a_{21} = 0,\ a_{03} = 0,\ b_{12} = 0,\ b_{21}(b_{21} + 1)(b_{21} + 2) = 0.
\end{aligned}$$

Theorem 3.8. *The two singular points $(\pm 1, 0)$ of system (3.27) are generalized complex isochronous centers if and only if one of the following seven conditions holds:*

$$\begin{aligned}
&D_1, \quad D_2, \quad D_5, \quad D_6 \ \ (\textit{in Theorem } 3.7), \ \textit{and} \\
&D_{3-1}: \ \ a_{21} = b_{12} = b_{03} = a_{12} = a_{03} = 0, b_{21} = -\frac{9}{2}; \\
&D_{4-1}: \ \ a_{21} = b_{12}, a_{03} = 0, b_{03} = a_{12}, b_{21} = \frac{1}{2}; \\
&D_{4-2}: \ \ a_{21} = 0, \ a_{03} = 0, \ b_{03} = 0, \ a_{12} = 0, \ b_{12} = 0.
\end{aligned} \tag{3.28}$$

When $\lambda_1 = 1$, $\lambda_2 = -4$, system (1.6) becomes

$$\begin{aligned}
\frac{dx}{dT} &= -\frac{1}{2}x - a_{21}y + \frac{1}{2}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\frac{dy}{dT} &= (-4 - b_{21})y + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.
\end{aligned} \tag{3.29}$$

For system (3.29), we have the following two theorems.

Theorem 3.9. *The two singular points* $(\pm 1, 0)$ *of system* (3.29) *are generalized complex centers if and only if one of the following six conditions holds:*

$$\begin{aligned}
&E_1 : \Big(b_{21} + \frac{1}{2}\Big)\Big(b_{21} + \frac{3}{2}\Big)\Big(b_{21} + \frac{5}{2}\Big) = 0; \\
&E_2 : a_{21} = 0, \ b_{21} = -\frac{7}{2}; \\
&E_3 : a_{21} = \frac{1}{8}b_{12}(7 + 2b_{21}), \ b_{03} = \frac{2}{9}a_{12}(4 + b_{21}), \ b_{21} = -6; \\
&E_4 : a_{21} = \frac{1}{8}b_{12}(7 + 2b_{21}), \ b_{03} = \frac{2}{9}a_{12}(4 + b_{21}), \ a_{03} = \frac{1}{36}a_{12}b_{12}(2b_{21} - 1); \\
&E_5 : a_{21} = b_{03} = a_{12} = b_{12} = 0, \\
&\qquad (-1 + 6b_{21})(1 + 6b_{21})(5 + 6b_{21})(7 + 6b_{21})(11 + 6b_{21}) \\
&\qquad \times (13 + 6b_{21})(17 + 6b_{21})(19 + 6b_{21}) = 0; \\
&E_6 : a_{21} = 0, \ a_{03} = 0, \ b_{12} = 0, \ b_{21}(b_{21} + 1)(b_{21} + 2)(b_{21} + 3) = 0.
\end{aligned}$$

Theorem 3.10. *The two singular points* $(\pm 1, 0)$ *of system* (3.29) *are generalized complex isochronous centers if and only if one of the following six conditions holds:*

$$\begin{aligned}
&E_1, \quad E_2, \quad E_5, \quad E_6 \ \ (\textit{in Theorem } 3.9), \ \ \textit{and} \\
&E_{3-1}: \ \ a_{21} = b_{12} = b_{03} = a_{12} = a_{03} = 0, \ b_{21} = -6; \\
&E_{4-1}: \ \ a_{21} = b_{12}, \ a_{03} = 0, \ b_{03} = a_{12}, \ b_{21} = \frac{1}{2}; \\
&E_{4-2}: \ \ a_{21} = 0, \ a_{03} = 0, \ b_{03} = 0, \ a_{12} = 0, \ b_{12} = 0.
\end{aligned} \tag{3.30}$$

4. Integrability and linearizability conditions for cubic Z_2-equivariant systems with 1:($-q$) ($q \geqslant 5$) resonant saddles

Finally, for general positive integer $q \geqslant 5$, we consider the integrability and linearizability conditions for cubic Z_2-equivariant systems with 1:($-q$) resonant saddles. When $\lambda_1 = 1,\ \lambda_2 = -q\ (q \geqslant 5)$, system (1.6) becomes

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}x - a_{21}y + \frac{1}{2}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \frac{dy}{dt} &= y(-q + b_{12}y + b_{03}y^2 + 2b_{21}z + b_{12}yz + b_{21}z^2), \end{aligned} \tag{4.1}$$

which can be transformed into

$$\begin{aligned} \frac{dz}{dt} &= z + \frac{1}{2}(2a_{12}y^2 + 2a_{03}y^3 + 4a_{21}yz + 2a_{12}y^2z + 3z^2 + 2a_{21}yz^2 + z^3), \\ \frac{dy}{dt} &= (-q - b_{21})y + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{4.2}$$

by $x = \pm z + 1$.

For a concrete q, we can compute the saddle values and periodic constants, from which the necessary conditions for integrability and linearizability may be obtained. However, for general positive integer q, it is difficult to obtain a consistent formula for the saddle values and periodic constants. Therefore, necessary conditions for general integer q are difficult to derive. The following slightly general sufficient conditions for arbitrary integer $q \geqslant 5$ are obtained following the pattern of the results given in Section 3.

Theorem 4.1. *The two singular points* $(\pm 1, 0)$ *of system* (4.1) *are generalized complex centers if one of the following six conditions holds:*

$$G_1:\ \Big(b_{21} + \frac{1}{2}\Big)\Big(b_{21} + \frac{3}{2}\Big)\Big(b_{21} + \frac{5}{2}\Big)\cdots\Big(b_{21} + \frac{2q-3}{2}\Big) = 0;$$

$$G_2:\ a_{21} = 0,\ b_{21} = -\frac{2q-1}{2};$$

$$G_3:\ a_{21} = \frac{1}{2q}b_{12}((2q-1) + 2b_{21}),\ b_{03} = \frac{2}{2q+1}a_{12}(q + b_{21}),\ b_{21} = -\frac{3q}{2};$$

$$\begin{aligned} G_4:\ & a_{21} = \frac{1}{2q}b_{12}((2q-1) + 2b_{21}),\ b_{03} = \frac{2}{2q+1}a_{12}(q + b_{21}), \\ & a_{03} = \frac{1}{q(2q+1)}a_{12}b_{12}(2b_{21} - 1); \end{aligned}$$

$$\begin{aligned} G_5:\ & a_{21} = b_{03} = a_{12} = b_{12} = 0, \quad (-1 + 6b_{21})(1 + 6b_{21}) \\ & \times (5 + 6b_{21})\cdots(11 + 6b_{21})(13 + 6b_{21})(17 + 6b_{21})[(4q+3) + 6b_{21}] = 0; \end{aligned}$$

$$\begin{aligned} G_6:\ & a_{21} = 0,\ a_{03} = 0,\ b_{12} = 0, \\ & b_{21}(b_{21} + 1)(b_{21} + 2)(b_{21} + 3)\cdots[b_{21} + (q-1)] = 0. \end{aligned}$$

Proof. To prove this theorem, we will apply either Theorem 2.1 for some conditions or other approaches such as finding invariant algebraic curves for other conditions. Although Theorem 2.1 can be used to prove complex isochronous centers, we in general can only claim complex center since some of the conditions are proved not using Theorem 2.1.

When the condition G_1 in Theorem 4.1 holds, let $b_{21} = -\frac{2k-1}{2},\ k = 1 \cdots q-1$. Then, system (4.1) can be rewritten as

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\left(-x + x^3 - 2a_{21}y + 2a_{21}x^2y + 2a_{12}xy^2 + 2a_{03}y^3\right),\\
\frac{dy}{dt} &= \frac{1}{2}y\left(1 - 2k + 2q - x^2 + kx^2 + 2b_{12}xy + 2b_{03}y^2\right),
\end{aligned}$$

which can be transformed into

$$\begin{aligned}
\frac{du}{dt} &= u + 2a_{21}(1-u)^k uv + 2(a_{12} + a_{03}(1-u)^k v)(1-u)^{2k}v^2,\\
\frac{dv}{dt} &= -qv + v\big\{(1-u)^{k-1}v\big[b_{12} + a_{21}(-1+2k)u\\
&\quad + (1-u)^k v(b_{03} + a_{12}(-1+2k) + a_{03}(-1+2k)(1-u)^k v)\big]\big\},
\end{aligned}$$

by

$$u = \frac{x^2-1}{x^2}, \quad v = x^{2k-1}y.$$

Further, introducing $v = z^q$ into the above system we obtain

$$\begin{aligned}
\frac{du}{dt} &= u + 2a_{21}(1-u)^k uz^q + 2(1-u)^{2k}z^{2q}(a_{12} + a_{03}(1-u)^k z^q),\\
\frac{dz}{dt} &= -z + \frac{1}{q}z\big\{(1-u)^{k-1}z^q\big[b_{12} + a_{21}(-1+2k)u + (1-u)^k z^q(b_{03}\\
&\quad + a_{12}(-1+2k) + a_{03}(-1+2k)(1-u)^k z^q)\big]\big\}.
\end{aligned}$$

Since for a fixed k ($1 \le k \le q-1$) it is easy to verify that the condition (2.6) in Theorem 2.1 is true under the condition G_1, the origin of above system is a complex isochronous center, implying that $(\pm 1, 0)$ of system (4.1) are generalized complex isochronous centers under the condition G_1.

When the condition G_2 in Theorem 4.1 holds, system (4.1) can be rewritten as

$$\begin{aligned}
\frac{dx}{dt} &= \frac{1}{2}\left(-x + x^3 + 2a_{12}xy^2 + 2a_{03}y^3\right),\\
\frac{dy}{dt} &= -\frac{1}{2}y\left(1 - x^2 + 2qx^2 - 2b_{12}xy - 2b_{03}y^2\right),
\end{aligned}$$

which can be transformed to

$$\frac{du}{dT} = u + 2(a_{12} + a_{03}v)(1+u)v^2,$$
$$\frac{dv}{dT} = -qv - v(-b_{12}v + a_{12}v^2 - b_{03}v^2 + a_{03}v^3), \tag{4.3}$$

by

$$u = x^2 - 1, \quad v = \frac{y}{x}, \quad t = (u+1)T.$$

Further, introducing $v = z^q$ into system (4.3) yields

$$\frac{du}{dT} = u + 2a_{12}(1+u)z^{2q} + 2a_{03}(1+u)z^{3q},$$
$$\frac{dz}{dT} = -z + \frac{1}{q}z(b_{12}z^q - (a_{12} - b_{03})z^{2q} - a_{03}z^{3q}). \tag{4.4}$$

Thus, according to Theorem 2.1, the origin of system (4.4) is complex isochronous center when the condition G_2 is satisfied, implying that the conclusion holds.

If the condition G_3 in Theorem 4.1 holds, system (4.1) can be simplified to

$$\frac{dx}{dt} = \frac{1}{4}\left(-2x + 2x^3 + 3b_{12}y - 3b_{12}x^2y + 4a_{12}xy^2 + 4a_{03}y^3\right),$$
$$\frac{dy}{dt} = -\frac{1}{5}y\left(-5 + 15x^2 - 5b_{12}xy + 2a_{12}y^2\right). \tag{4.5}$$

It can be shown that system (4.5) has an invariant curve $f_1 = y$, which admits an integrating factor, $F_1 = y^{-\frac{1}{q}}$. So the origin of system (4.5) is a generalized complex center.

When the condition G_4 in Theorem 4.1 holds, system (4.1) becomes

$$\begin{aligned}\frac{dx}{dt} =& \frac{1}{2q(2q+1)}\big(-qx - 2q^2x + qx^3 + 2q^2x^3 + b_{12}y + 3b_{12}qy \\ &+ 2b_{12}q^2y - b_{12}x^2y - 3b_{12}qx^2y - 2b_{12}q^2x^2y \\ &+ 2a_{12}qxy^2 + 4a_{12}q^2xy^2 - 2a_{12}b_{12}y^3 - 6a_{12}b_{12}qy^3\big), \\ \frac{dy}{dt} =& -\frac{1}{2(2q+1)}\,y\big(-q - 2q^2 + 3qx^2 + 6q^2x^2 - 2b_{12}xy \\ &- 4b_{12}qxy + 2a_{12}qy^2\big).\end{aligned} \tag{4.6}$$

By computing the invariant algebraic curves of system (4.6), we obtain a first integral for system (4.6), given by

$$M_3 = f_1 f_{10}^{-2(q+b_{21})} f_{11}^{q},$$

where

$$f_1 = y, \quad f_{10} = qx - b_{12}y, \quad f_{11} = -1 - 2q + x^2 + 2qx^2 + 2a_{12}y^2,$$

which indicates that the origin of system (4.6) is a generalized complex center under the condition G_4.

When the condition G_5 in Theorem 4.1 holds, we let $b_{21} = -\frac{2k-1}{6},\ k = 1 \cdots q-1$. Then, system (4.2) can be rewritten as

$$\frac{dz}{dt} = \frac{1}{2}\left(2z + 3z^2 + z^3 + 2a_{03}y^2\right),$$
$$\frac{dy}{dt} = -\frac{1}{6}y\left(6q - 2z + 4kz - z^2 + 2kz^2\right),$$

which can be transformed to

$$\begin{aligned}\frac{dz}{dt} &= \frac{1}{2}\left(2z + 3z^2 + z^3 + 2a_{03}w^{3q}\right),\\ \frac{dw}{dt} &= -\frac{1}{6q}w\left(6q - 2z + 4kz - z^2 + 2kz^2\right),\end{aligned} \tag{4.7}$$

by $y = w^q$. Further, with the transformation,

$$u = \frac{z(2+z)}{2(1+z)^2}, \quad v = w(1+z)^{\frac{2k-1}{3q}},$$

system (4.7) can be changed to

$$\begin{aligned}\frac{du}{dt} &= u + a_{03}(1-2u)^{k+1}v^{3q},\\ \frac{dv}{dt} &= -v - \frac{a_{03}(1-2k)}{3q}(1-2u)^k v^{3q+1},\end{aligned} \tag{4.8}$$

which shows that the origin of the above system is integrable according to Theorem 2.1, implying that the origin of system (4.8) is a complex isochronous center and so the conclusion is true.

When the condition G_6 in Theorem 4.1 holds, let $b_{21} = -(m-1),\ m = 1 \cdots q$. Then, system (4.2) can be rewritten as

$$\frac{dz}{dt} = \frac{1}{2}(z+1)\left(2z + z^2 + 2a_{12}y^2\right),$$
$$\frac{dy}{dt} = -y\left(q - 2z + 2mz - z^2 + mz^2 - b_{03}y^2\right),$$

which can be transformed to

$$\frac{dz}{dt} = \frac{1}{2}(1+z)(2z + z^2 + 2a_{12}w^{2q}),$$
$$\frac{dw}{dt} = -w + \frac{1}{q}w(2z - 2mz + z^2 - mz^2 + b_{03}w^{2q}),$$

by $y = w^q$, and can be further changed to

$$\begin{aligned}\frac{du}{dt} &= u + a_{12}(1-2u)^{2m-1}v^{2q},\\ \frac{dv}{dt} &= -v - \frac{2a_{12} - b_{03} - 2a_{12}m}{q}(1-2u)^{2m-2}v^{2q+1},\end{aligned} \tag{4.9}$$

under the transformation,

$$u = \frac{z(2+z)}{2(1+z)^2}, \quad v = w(1+z)^{\frac{2(m-1)}{q}}.$$

This shows that the origin of system (4.9) is a complex isochronous center according to Theorem 2.1, and so the conclusion is true. □

Furthermore, we also have

Theorem 4.2. *The two singular points* $(\pm 1, 0)$ *of system* (4.1) *are generalized complex isochronous centers if one of the following seven conditions holds:*

$$\begin{aligned}&G_1, \quad G_2, \quad G_5, \quad G_6 \ \ (\textit{in Theorem } 4.1) \ \ \textit{and}\\ &G_{3-1}: \ a_{21} = b_{12} = b_{03} = a_{12} = a_{03} = 0, \ b_{21} = -6, b_{21} = -\frac{3q}{2};\\ &G_{4-1}: \ a_{21} = b_{12}, \ a_{03} = 0, \ b_{03} = a_{12}, \ b_{21} = \frac{1}{2};\\ &G_{4-2}: \ a_{21} = 0, \ a_{03} = 0, \ b_{03} = 0, \ a_{12} = 0, \ b_{12} = 0.\end{aligned} \tag{4.10}$$

Proof. According to Theorem 2.1, as one of the conditions G_1, G_2, G_5 and G_6 holds, the origin of system (4.2) is a generalized complex isochronous center. So any of the conditions G_1, G_2, G_5 and G_6 is sufficient such that $(\pm 1, 0)$ of system (4.1) are generalized complex isochronous centers.

When the condition G_{3-1} in Theorem 4.2 holds, system (4.2) can be rewritten as

$$\frac{dz}{dt} = \frac{1}{2}z(z+1)(z+2), \quad \frac{dy}{dt} = -\frac{q}{2}y\left(2 + 6z + 3z^2\right),$$

which can be transformed to

$$\frac{dz}{dt} = \frac{1}{2}z(1+z)(2+z), \quad \frac{dw}{dt} = -w - \frac{1}{2}w(6z + 3z^2),$$

by $y = w^q$. Further, introducing the transformation,

$$u = \frac{z(2+z)}{2(1+z)^2}, \quad v = w(1+z)^3,$$

into the above system yields

$$\frac{du}{dt} = u, \quad \frac{dv}{dt} = -v, \tag{4.11}$$

which implies that the origin of the above system is linearizable.

When the condition G_{4-1} in Theorem 4.2 holds, system (4.2) becomes

$$\begin{aligned}\frac{dz}{dt}&=\frac{1}{2}(2a_{12}w^2+2z+4b_{12}wz+2a_{12}w^2z+3z^2+2b_{12}wz^2+z^3),\\ \frac{dy}{dt}&=\frac{1}{2}y\left(-2q+2b_{12}y+2a_{12}y^2+2z+2b_{12}yz+z^2\right).\end{aligned}\tag{4.12}$$

It can be shown that system (4.12) has three invariant algebraic curves:

$$f_1=y,\quad f_{12}=q-b_{12}w+qz,\quad f_{13}=2a_{12}y^2+2(1+2q)z+(1+2q)z^2,$$

which admit a first integral $F=f_1^2f_{12}^{-2-4q}f_{13}^{2q}$, and a linearization transformation,

$$\xi=f_{12}f_{13}^{-2},\quad \eta=w^{-\frac{1}{q}}f_{12}^{-\frac{1}{q}}.$$

When the condition G_{4-2} in Theorem 4.2 holds, system (4.2) can be rewritten as

$$\frac{dz}{dt}=\frac{1}{2}z(z+1)(z+2),\quad \frac{dy}{dt}=y\left(-q+2b_{21}z+b_{21}z^2\right),$$

which can be transformed to

$$\frac{dz}{dt}=\frac{1}{2}z(1+z)(2+z),\quad \frac{dw}{dt}=-w+\frac{1}{q}w(2b_{21}z+b_{21}z^2),\tag{4.13}$$

by $y=w^q$. Further, with the transformation

$$u=\frac{z(2+z)}{2(1+z)^2},\quad v=w(1+z)^{-\frac{2b_{21}}{q}},$$

system (4.13) can be changed to the linear system (4.11).

The proof is complete. □

Although we cannot prove that the conditions given in Theorems 4.1 and 4.2 are necessary, we have the following Conjecture, based on the results for the cases $q=1,2,3,4$.

Conjecture 4.1. *The two singular points* $(\pm1,0)$ *of system* (4.1) *are integrable if and only if one of the six conditions in Theorem* 4.1 *holds, and moreover, they are linearizable if and only if one of the seven conditions in Theorem* 4.2 *holds.*

5. Integrability and linearizability conditions for cubic Z_2-equivariant systems with 1:q resonant nodes

Cubic Z_2-equivariant systems with two λ_1:$(n\lambda_1)$ resonant nodes can always be written as

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{2}\lambda_1 x - a_{21}y + \frac{1}{2}\lambda_1 x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \frac{dy}{dt} &= (n\lambda_1 - b_{21})y + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{5.1}$$

which can be further changed to

$$\begin{aligned} \frac{dz}{dt} &= \frac{1}{2}(2\lambda_1 z + 4a_{21}zy + 2a_{21}z^2y + 2a_{12}y^2 \\ &\quad + 2a_{12}zy^2 + 2a_{03}y^3 + 3\lambda_1 z^2 + \lambda_1 z^3), \\ \frac{dy}{dt} &= y(n\lambda_1 + 2b_{21}z + b_{21}z^2 + b_{12}y + b_{12}zy + b_{03}y^2 \end{aligned} \tag{5.2}$$

by $z = \pm x - 1$.

The following theorem directly follows Corollary 1.6.1 in [28].

Theorem 5.1. *The origin of system (5.2) is linearizable, and therefore if a cubic Z_2-equivariant system has two resonant nodes, it is linearizable.*

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