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Bifurcation of limit cycles at infinity in a class of switching systems

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Abstract In this paper, we present a method to compute focal values and periodic constants at infinity of a class of switching systems and apply it to study a cubic system. We prove that such a cubic system can have 7 limit cycles in the sufficiently small neighborhood of infinity. Moreover, we consider a quintic switching system to obtain 14 limit cycles at infinity, while continuous quintic systems can have only 11 limit cycles in the sufficiently small neighborhood of infinity. This indicates that switching systems or discontinuous systems can exhibit more complex dynamics compared to smooth systems.

Keywords Switching system · Infinity · Lyapunov constant · Limit cycle · Center · Quasi-isochronous center

1 Introduction

In recent years, bifurcation of limit cycles in planar differential systems with small perturbations has received a great deal of attention. This is closely related to the so-called weakened Hilbert's 16th problem [1], which asks for the number and distribution of limit cycles in dynamical systems around a singular point or near a closed orbit. Limit cycles theory has been widely applied to consider real practical problems, and in recent years, researchers have paid attention to an important and interesting problem related to multiple stability (e.g., see [2–5]). However, it has been noticed that most of the known results are concerned with the bifurcation of limit cycles in the finite region of the plane. However, the study of limit cycles bifurcation from infinity is important, particularly for the weakened Hilbert's 16th problem. There have been some results [6–10] on limit cycles bifurcation at infinity for certain special continuous systems. Bifurcation of periodic orbits from infinity has also been studied for polynomial planar vector fields, see for instance the work of Sotomayor and Paterlini [11], Blows and Rousseau [7] and Gunez et al. [12]; as well as for special systems such as the Rayleigh equation studied by Keith and Rand [13], the Van der Pol system by Malaguti [14] and the Liénard system by Sabatini [15]. In general, it is hard to solve the center problem for a system with a singular point at infinity and to determine the number of limit cycles at infinity. A special system with a singular point at infinity, described by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (\delta x - y)\left(x^2 + y^2\right)^n + \sum_{k=0}^{2n} F_k(x, y),$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = (x + \delta y)\left(x^2 + y^2\right)^n + \sum_{k=0}^{2n} G_k(x, y), \qquad (1.1)$$

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has been considered by Liu [16]. Here, $F_k(x, y)$ and $G_k(x, y)$ are homogeneous polynomials of order k. The importance of study of the limit cycles bifurcating from infinity of nonlinear systems is not only because it is related to the well-known Hilbert's 16th problem which considers possible bifurcation of limit cycles in the whole plane, but also because it helps people to understand global structure of nonlinear systems. To solve the problem, one may use a proper transformation to change the singular point of system (1.1) at infinity into the origin which is a weak focus, so that classical methods can be applied to determine the number of limit cycles.

Recently, many practical problems in science and engineering are modeled by using ordinary differential equations with discontinuity, see for example [17,18]. These modeling and studies are of great importance in direct control theory [19,20], in switching circuits of power electronics [21] and in impact and dry frictions of mechanical engineering [22,23] and so on. Study of switching systems associated with Hopf bifurcation has also attracted many researchers. Leine and Nijmeijer [24] and Zou et al. [25] considered nonsmooth Hopf bifurcation. Freire et al. [26] discussed the focus-center limit cycle bifurcation in a symmetric 3-dimensional, piecewise linear system. Furthermore, Chen and Du presented a quadratic switching system with nine limit cycles in [27]. Recently, Tian and Yu constructed a Bautin switching system to show the existence of ten limit cycles in [28]. Llibre et al. [29] studied the maximum number of limit cycles that bifurcate from the family of the periodic solutions arising from isochronous centers in a cubic polynomial system, and they also studied the maximum number of limit cycles which bifurcate from the periodic orbits associated with isochronous centers in discontinuous quadratic polynomial differential systems [30]. The number of limit cycles which can bifurcate from the periodic orbits of a linear center perturbed by nonlinear functions in the form of all classical polynomial Liénard differential equations with discontinuities was discussed in [31].

For piecewise continuous systems, Llibre et al. [32] established sufficient and necessary conditions for bifurcation of limit cycles from the periodic orbit at infinity in symmetric piecewise linear bidimensional systems. Later, the same authors studied the bifurcation of limit cycles from infinity for a nonsmooth but continuous piecewise differential system [33]. But for nonlinear systems, to date, none of the methods developed for studying the center problem at infinity is perfect to be used directly and thus needs further improving. In this paper, we consider the bifurcation of limit cycles from infinity for switching bidimensional systems. This problem can be treated as a kind of generalized Hopf bifurcation at infinity. In particular, we study a class of discontinuous planar systems of ordinary differential equations, with the upper system given by

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= (\delta x - y)\left(x^2+y^2\right)^n + \sum_{k=0}^{2n} F_k^+(x, y),\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= (x + \delta y)\left(x^2+y^2\right)^n + \sum_{k=0}^{2n} G_k^+(x, y),
\end{aligned}
\tag{1.2}
$$

and the lower system described by

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= (\delta x - y)\left(x^2+y^2\right)^n + \sum_{k=0}^{2n} F_k^-(x, y),\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= (x + \delta y)\left(x^2+y^2\right)^n + \sum_{k=0}^{2n} G_k^-(x, y),
\end{aligned}
\tag{1.3}
$$

where $F_k^{\pm}(x, y)$ and $G_k^{\pm}(x, y)$ are kth-order homogeneous polynomials. Note that the x-axis ($y = 0$) is the discontinuity boundary. As a class of discontinuous vector fields, the classical Hopf bifurcation theorem is not applicable so that specific techniques are needed. The aim of this paper is to modify an existing method on the study of the Poincaré Map in a neighborhood of infinity for switching systems with a singular point at infinity. In general, for such systems, it is hard to solve the center problem and to determine the number of limit cycles. However, for the system described by (1.2) and (1.3), classical Bendixson transformation could be used to transform the singular point at infinity into the origin, which is a weak focus and thus existing methods may be applied or modified to overcome the difficulty.

The rest of the paper is organized as follows. In the next section, a method to compute Lyapunov constants and periodic constants at infinity of the above switching system is presented. As examples, two cubic switching systems with a singular point at infinity are investigated in Sects. 3 and 4. Section 5 summarizes the results of this paper.

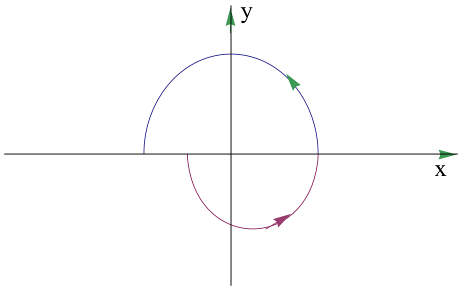

Fig. 1 Positive half-return map of system (1.2) and negative half-return map of system (1.2). (Color figure online)

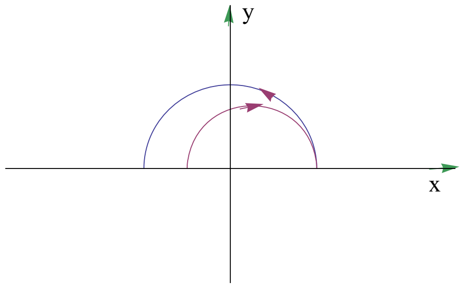

Fig. 2 The lower half-plane changed to the upper half-plane. (Color figure online)

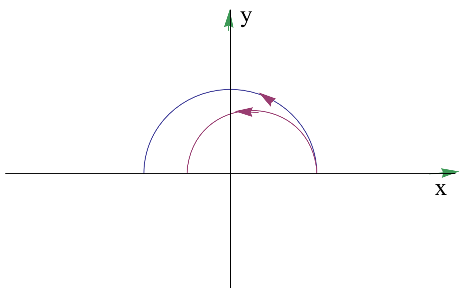

Fig. 3 The vector fields of systems (1.2) and (2.1) with positive half-return maps. (Color figure online)

2 Lyapunov constants of singular point at infinity for a class of switching systems

The study of the Poincaré map in a neighborhood of infinity for planar vector fields, when it is well defined, can be conveniently carried out by using the Bendixson transformation. This reduces the problem to a similar study in a neighborhood of the origin for the transformed system, see for instance Andronov and others [34]. But for switching systems, classical methods cannot be used, and we thus must develop some new techniques. Similarly, we may define half-return maps as that used in [35] and may generalize Lemma 2.1 in [35] to compute the positive half-return maps of (1.2) and (1.3), respectively.

Firstly, we define the positive half-return map of system (1.2), and then with a time changing, we can define the positive half-return map for the lower half-plane. This is illustrated in Figs. 1, 2 and 3. Figure 1 shows the positive half-return map of system (1.2) (in green color) and the negative half-return map of system (1.3) (in red color). Then, applying the transformation $y \to -y$ to the lower half-plane moves the negative half-return map of system (1.3) into the upper half-plane, as shown in Fig. 2 (see the red curve with the arrow in the counter clockwise direction). Finally, with the reversing time $t \to -t$, the negative half-return map of system (1.3) becomes a positive half-return map, see the red curve in Fig. 3 (i.e. the small semicircle), and the resulting system of (1.3) is now given by

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= (\delta x - y)\left(x^2+y^2\right)^n + \sum_{k=0}^{2n} F_k^-(x,-y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= (x+\delta y)\left(x^2+y^2\right)^n + \sum_{k=0}^{2n} G_k^-(x,-y). \end{aligned} \tag{2.1}$$

Therefore, we only need to consider the positive half-return maps of systems (1.2) and (2.1). More details can be found in [36]. To find the positive half-return maps, we may apply the Bendixson transformation in polar coordinates,

$$x = \frac{\cos\theta}{r}, \quad y = \frac{\sin\theta}{r}, \tag{2.2}$$

and the timescale $t = r^{2n}\tau$, to systems (1.2) and (2.1) to obtain

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\tau} &= -r\left[\pm\delta + \sum_{k=1}^{2n+1} \varphi_{2n+2-k}(\theta) r^k\right], \\ \frac{\mathrm{d}\theta}{\mathrm{d}\tau} &= 1 + \sum_{k=1}^{2n+1} \psi_{2n+2-k}(\theta) r^k, \end{aligned} \tag{2.3}$$

where $\varphi_k(\theta)$, $\psi_k(\theta)$ are polynomials of $\cos\theta$ and $\sin\theta$, given by

$$\begin{aligned} \varphi_{2n+2}(\theta) &= \cos\theta X_{2n+1}(\cos\theta, \sin\theta) \\ &\quad + \sin\theta Y_{2n+1}(\cos\theta, \sin\theta), \\ \psi_{2n+2}(\theta) &= \cos\theta Y_{2n+1}(\cos\theta, \sin\theta) \\ &\quad - \sin\theta X_{2n+1}(\cos\theta, \sin\theta). \end{aligned}$$

Combining the two equations in (2.3) yields that

$$\frac{dr}{d\theta} = -r\,\frac{\pm\delta + \Sigma_{k=1}^{\infty}\varphi_{2n+2-k}(\theta)\,r^k}{1+\Sigma_{k=1}^{\infty}\psi_{2n+2-k}(\theta)\,r^k}, \tag{2.4}$$

which is a special case of the equation,

$$\frac{dr}{d\theta} = r\sum_{k=1}^{\infty} R_k(\theta)\,r^k. \tag{2.5}$$

By the method of small parameter of Poincaré, the solutions of (2.5) can be written as

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k,$$

where $v_1(0) = 1$, $v_k(0) = 0$, $\forall k \geq 2$. So the successive function for the switching system can be written as

$$\Delta(h) = \Delta_1(h) - \Delta_2(h) = \tilde{r}_1(\pi, h) - \tilde{r}_2(\pi, h),$$

where

$$\Delta_1(h) = \tilde{r}_1(\pi, h) - h, \quad \Delta_2(h) = \tilde{r}_2(\pi, h) - h.$$

The period function for (1.2) and (2.1) can be defined as

$$T = T_1(\theta, h) + T_2(\theta, h) = 2\pi + \sum_{k=1}^{n} T_k h^k,$$

where

$$T_1(\theta, h) = \int_0^{\pi} \frac{d\vartheta}{1+\Sigma_{k=1}^{\infty}\psi_{2n+2-k}(\theta)\tilde{r}_1^k(\vartheta, h)},$$
$$T_2(\theta, h) = \int_0^{\pi} \frac{d\vartheta}{1+\Sigma_{k=1}^{\infty}\psi_{2n+2-k}(\theta)\tilde{r}_2^k(\vartheta, h)}. \tag{2.6}$$

In the following, we present some definitions which will be used in the next section.

Definition 2.1 $\Delta(h) = \sum_1^n (u_1(\theta) - v_1(\theta))h^k = \sum_{k=1}^n V_k h^k$, where V_k is called the kth-order focal values of the switching system at infinity.

Definition 2.2 $T(h) = T_1(\theta, h) + T_2(\theta, h) = \sum_{k=1}^n T_k h^k$, where T_k is called the kth-order periodic constant of the switching system at infinity.

Definition 2.3 The infinity is said to be a center if every solution curve of the system in a neighborhood of infinity (the equator) is a closed orbit. The infinity is called a quasi-isochronous center if all periodic constants at the origin of system (2.3) are zero.

The following steps describe how to compute the focal values and the periodic constants.

1. Introduce the transformations: $y \to -y$ and $t \to -t$ to the negative half-plane.
2. Apply the Bendixson transformation in polar coordinates $x = \frac{\cos\theta}{r}$, $y = \frac{\sin\theta}{r}$, and the timescale $t = r^{2n}\tau$ to systems (1.2) and (2.1), and write the solutions of the systems as
$$r_1 = \tilde{r}_1(\theta, h) = \sum_{k=1}^{\infty} u_k(\theta)h^k \quad \text{and}$$
$$r_2 = \tilde{r}_2(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta)h^k,$$
respectively, satisfying $u_1(0) = v_1(0) = 1$, $u_k(0) = v_k(0) = 0$, $\forall k \geq 2$.
3. Solve $u_k(\theta)$ and $v_k(\theta)$.
4. Compute the successive function for the switching system using the formula,
$$\Delta(h) = \Delta_1(h) - \Delta_2(h) = \tilde{r}_1(\pi, h) - \tilde{r}_2(\pi, h).$$
5. Compute the periodic constants for the switching system using the formula,
$$T = T_1(\pi, h) + T_2(\pi, h).$$

To end this section, in order to show the fundamental difference in the center conditions between smooth and switching systems, we give an example to demonstrate that even if both the upper and lower half-planes have analytic first integrals at infinity, the infinity of the switching system may not be a center.

Example 2.1

$$\begin{cases} \frac{dx}{dt} = -y\left(x^2+y^2\right), \\ \frac{dy}{dt} = x\left(x^2+y^2\right) + 3x^2, \end{cases} (y > 0), \quad \begin{cases} \frac{dx}{dt} = -y\left(x^2+y^2\right), \\ \frac{dy}{dt} = x\left(x^2+y^2\right), \end{cases} (y < 0). \tag{2.7}$$

Obviously, the upper half-plane has a first integral,

$$H(x, y) = \frac{1}{4}x^4 + \frac{1}{2}x^2y^2 + \frac{1}{4}y^4 + x^3,$$

and the lower half-plane also has a first integral,

$$H(x, y) = x^2 + y^2.$$

However, the singular point at infinity is not a center, as shown in Fig. 4.

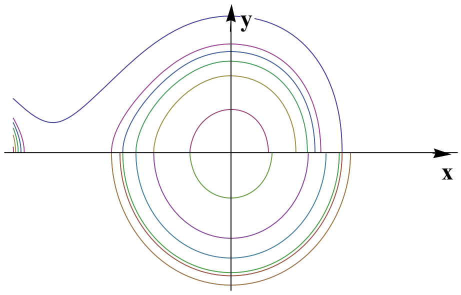

Fig. 4 The phase portrait of system (2.7)

3 A switching cubic system having seven limit cycles at infinity

In this section, we consider the following switching cubic system,

$$\begin{aligned} \frac{dx}{dt} &= (\delta x - y)\left(x^2+y^2\right) + a_{10}x + a_{01}y, \\ \frac{dy}{dt} &= (x + \delta y)\left(x^2+y^2\right) + b_{10}x + b_{01}y, \end{aligned} \quad (y > 0),$$
$$\begin{aligned} \frac{dx}{dt} &= (\delta x - y)\left(x^2+y^2\right) + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \frac{dy}{dt} &= (x + \delta y)\left(x^2+y^2\right) + b_{11}xy + b_{02}y^2, \end{aligned} \quad (y < 0). \tag{3.1}$$

The aim of this section is to solve the center and the pseudo-isochronous center problem of system (3.1). Moreover, we shall prove that seven limit cycles can bifurcate from infinity.

3.1 Center conditions and limit cycles for system (3.1)

A direct computation yields the following theorem.

Theorem 3.1 *For system (3.1), the first eight Lyapunov constants at infinity are given by*

$$\begin{aligned}
\lambda_0 &= 2\pi\delta, \\
\lambda_1 &= \tfrac{2}{3}\left(a_{11}+2b_{02}\right), \\
\lambda_2 &= -\tfrac{\pi}{8}\left[4\left(a_{10}+b_{01}\right)+b_{02}\left(b_{11}+2a_{20}\right)\right], \\
\lambda_3 &= -\tfrac{4}{315}\,b_{02}\left(2a_{20}+b_{11}\right)\left(a_{20}+20a_{02}-22b_{11}\right), \\
\lambda_4 &= \tfrac{\pi}{6400}\,b_{02}\left(2a_{20}+b_{11}\right)\left[400\left(b_{10}-a_{01}+b_{02}^2\right)\right. \\
&\quad \left.-7a_{20}^2-247a_{20}b_{11}+122b_{11}^2\right], \\
\lambda_5 &= -\tfrac{2}{716625}\,b_{02}\left(2a_{20}+b_{11}\right)\left[238a_{20}^3+26509a_{20}^2b_{11}\right. \\
&\quad -8a_{20}\left(5415b_{02}^2+2009b_{11}^2\right) \\
&\quad \left.-4b_{11}\left(1740b_{02}^2+749b_{11}^2\right)\right], \\
\lambda_6 &= \tfrac{\pi}{23592960000(361a_{20}+58b_{11})^2}\,b_{02}\left(2a_{20}+b_{11}\right)f_1\left(a_{20},b_{11}\right), \\
\lambda_7 &= \tfrac{2}{938765953125(361a_{20}+58b_{11})^2}\,b_{02}\left(2a_{20}+b_{11}\right)f_2\left(a_{20},b_{11}\right),
\end{aligned}$$

where f_1 and f_2 are lengthy polynomials in a_{20} and b_{11}, which are omitted here for brevity. In the above expressions of λ_k, we have set $\lambda_0 = \lambda_1 = \cdots = \lambda_{k-1} = 0$ for $k = 1, 2, \ldots, 7$.

The following assertion directly follows Theorem 3.1.

Proposition 3.1 *The first eight Lyapunov constants at infinity of system (3.1) are zero if and only if one of the following conditions is satisfied:*

$$\delta = a_{10} + b_{01} = a_{11} = b_{02} = 0, \tag{3.2}$$
$$\delta = a_{10} + b_{01} = a_{11} + 2b_{02} = b_{11} + 2a_{20} = 0. \tag{3.3}$$

They are also the center conditions of system (3.1) at infinity.

Proof The *necessity* can be easily obtained from setting the Lyapunov constants $\lambda_i = 0$, $i = 0, 1, 2, \ldots, 7$. To prove the *sufficiency* of these conditions, first we consider (3.2) under which system (3.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= -y\left(x^2+y^2\right) - b_{01}x + a_{01}y, \\ \frac{dy}{dt} &= x\left(x^2+y^2\right) + b_{10}x + b_{01}y, \end{aligned} \quad (y > 0),$$
$$\begin{aligned} \frac{dx}{dt} &= -y\left(x^2+y^2\right) + a_{20}x^2 + a_{02}y^2, \\ \frac{dy}{dt} &= x\left(x^2+y^2\right) + b_{11}xy, \end{aligned} \quad (y < 0). \tag{3.4}$$

Obviously, the upper half-plane has a first integral,

$$\begin{aligned} H_1(x,y) &= \frac{1}{4}\left(x^4+2x^2y^2+y^4\right) + b_{01}xy \\ &\quad + \frac{1}{2}\left(b_{10}x^2 - a_{01}y^2\right), \end{aligned}$$

which is an even function of x when $y = 0$; the lower half-plane is symmetric with the y-axis. So the singular point at infinity is a center [36].

Next, when condition (3.3) is satisfied, system (3.1) can be rewritten as

$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = -y\left(x^2+y^2\right) - b_{01}x + a_{01}y, \\
&\frac{\mathrm{d}y}{\mathrm{d}t} = x\left(x^2+y^2\right) + b_{10}x + b_{01}y,
\end{aligned} \quad (y>0),$$
$$\begin{aligned}
&\frac{\mathrm{d}x}{\mathrm{d}t} = -y\left(x^2+y^2\right) + a_{20}x^2 - 2b_{02}xy + a_{02}y^2, \\
&\frac{\mathrm{d}y}{\mathrm{d}t} = x\left(x^2+y^2\right) - 2a_{20}xy + b_{02}y^2,
\end{aligned} \quad (y<0). \tag{3.5}$$

It is easy to find that the upper half-plane has a first integral,

$$H_2(x,y) = \frac{1}{4}\left(x^4+2x^2y^2+y^4\right) + b_{01}xy + \frac{1}{2}\left(b_{10}x^2 - a_{01}y^2\right),$$

and the lower half-plane has a first integral,

$$H_3(x,y) = \frac{1}{4}\left(x^4+2x^2y^2+y^4\right) - a_{20}x^2y + b_{02}xy^2 + \frac{1}{3}a_{02}y^3,$$

and both H_2 and H_3 are even functions of x when $y = 0$. So the singular point at infinity is a center [36]. □

As far as bifurcation of limit cycles is concerned, it follows from Theorem 3.1 that seven limit cycles can bifurcate from the sufficiently small neighborhood of infinity of system (3.1). We have the following result.

Theorem 3.2 *When the singular point at infinity is a 7th-order weak focus of system (3.1), for $0 < \delta \ll 1$, seven limit cycles can bifurcate from the sufficiently small neighborhood of infinity of system (3.1).*

Proof When the singular point at infinity is a 7th-order weak focus of system (3.1), the following conditions should be satisfied (obtained by setting $\lambda_i = 0$, $i = 0, 1, \ldots, 5$):

$$\begin{aligned}
&\delta = 0, \\
&a_{11} = -2b_{02}, \\
&a_{10} = -\tfrac{1}{4}\left[4b_{01} + b_{02}(b_{11}+2a_{20})\right], \\
&a_{01} = -\tfrac{1}{400}\left[7a_{20}^2 - 400\left(b_{10}+b_{02}^2\right) + b_{11}\left(247a_{20} - 122b_{11}\right)\right], \\
&a_{02} = -\tfrac{1}{20}\left(a_{20} - 22\,b_{11}\right), \\
&b_{02}^2 = \frac{7\left(34a_{20}^3 + 3787a_{20}^2b_{11} - 2296a_{20}b_{11}^2 - 428b_{11}^3\right)}{120(361a_{20}+58b_{11})}.
\end{aligned} \tag{3.6}$$

Furthermore, let $a_{20} = kb_{11}$. Then, the function $f_2(a_{20}, b_{11})$

$$\begin{aligned}
f_2 = b_{11}^7\,(&116937939968 + 4287138780192k \\
&+ 37797150073792k^2 \\
&+ 103765059007440k^3 + 127218646021480k^4 \\
&+ 121994388459846k^5 \\
&+ 105515070638794k^6 + 35510995814089k^7).
\end{aligned} \tag{3.7}$$

We could choose other parameters like b_{10} such that $f_1 = 0$ and $f_2 \neq 0$, implying that the singular point at infinity is a 7th-order critical point. In fact, if $f_2 = 0$, it is easy to verify that $b_{02}^2 < 0$ for any values of k which satisfies $f_2 = 0$, so 7 is the highest order of the critical point at the infinity singular point.

When the singular point at infinity is a 7th-order weak focus, a direct computation shows that $J = \frac{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)}{\partial(a_{11},a_{10},a_{02},a_{01},b_{02},b_{10})} \neq 0$. In fact, it is easy to see that perturbing $\lambda_6, \lambda_5, \ldots, \lambda_0$ by using $b_{10}, b_{02}, a_{01}, a_{02}, a_{10}, a_{11}$ and δ one by one at each step guarantees the existence of seven limit cycles bifurcating from the sufficiently small neighborhood of infinity.

The proof is complete. □

3.2 Quasi-isochronous centers of system (3.1)

Now, we discuss the quasi-isochronous center conditions of system (3.1) under the center conditions (3.2) and (3.3). First, if condition (3.2) holds, the periodic constants are obtained as follows:

$$\begin{aligned}
\tau_1 &= \tfrac{\pi}{24}\left[12\left(a_{01}-b_{10}\right) + 5a_{02}\left(a_{02}-b_{11}\right) - a_{20}\left(a_{20}+b_{11}+4a_{02}\right) + 2b_{11}^2\right], \\
\tau_2 &= \tfrac{2}{2835}\left[320a_{02}^3 - 12a_{02}^2\left(104a_{20}+67b_{11}\right) + 6a_{02}\left(4a_{20}^2 + 151a_{20}b_{11} + 97b_{11}^2\right) - 41a_{20}^3 + 276a_{20}^2b_{11} - 51a_{20}b_{11}^2 - 184b_{11}^3\right], \\
\tau_3 &= -\tfrac{\pi}{768}\,f_3\left(a_{20},a_{02},b_{11}\right), \\
\tau_4 &= -\tfrac{2}{11609325}\,f_4\left(a_{20},a_{02},b_{11}\right),
\end{aligned} \tag{3.8}$$

where f_3 and f_4 are polynomials in a_{20}, a_{02} and b_{11} (which are omitted here for brevity).

Theorem 3.3 *The singular point at infinity of system* (3.1) *is a quasi-isochronous center if and only if one of the following conditions holds:*

$$\delta = a_{11} = b_{02} = a_{10} + b_{01} = a_{01} - b_{10} = a_{02} = b_{11} - a_{20} = 0; \tag{3.9}$$

$$\delta = a_{11} = b_{02} = a_{10} + b_{01} = a_{01} - b_{10} = a_{02} - a_{20} = b_{11} - 3a_{20} = 0. \tag{3.10}$$

Proof The *necessity* can be shown by setting $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$, which yields either $a_{02} = 0,\ b_{11} = a_{20}$ or $a_{02} = a_{20},\ b_{11} = 3a_{20}$. These two sets of conditions are then combined with (3.2) to yield (3.9) and (3.10). To prove the *sufficiency*, we first consider condition (3.9) and rewrite system (3.1) under (3.9) as

$$\begin{aligned} &\frac{dx}{dt} = -y\left(x^2+y^2\right) - b_{01}x + b_{10}y, \\ &\frac{dy}{dt} = x\left(x^2+y^2\right) + b_{10}x + b_{01}y, \end{aligned} \quad (y>0),$$
$$\begin{aligned} &\frac{dx}{dt} = -y\left(x^2+y^2\right) + b_{11}x^2, \\ &\frac{dy}{dt} = x\left(x^2+y^2\right) + b_{11}xy, \end{aligned} \quad (y<0). \tag{3.11}$$

By using the transformations,

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}, \quad \tau = \left(x^2+y^2\right)t,$$

the upper half-plane of system (3.11) becomes

$$\begin{aligned} \frac{du}{d\tau} &= b_{01}u^3 - v - 3b_{10}u^2v - 3b_{01}uv^2 + b_{10}v^3, \\ \frac{dy}{d\tau} &= u + b_{10}u^3 + 3b_{01}u^2v - 3b_{10}uv^2 - b_{01}v^3, \end{aligned} \tag{3.12}$$

which can be further transformed to

$$\begin{aligned} \frac{dz}{d\tau} &= i\,z\left[1 + (b_{10} - ib_{01})z^2\right], \\ \frac{d\bar{z}}{d\tau} &= -i\,\bar{z}\left[1 + (b_{10} + ib_{01})\bar{z}^2\right], \end{aligned} \tag{3.13}$$

under the transformations: $z = u + iv,\ \bar{z} = u - iv$, where $\bar{z}$ is the complex conjugate of z. Now, introducing another transformation,

$$F = \frac{z}{\sqrt{1 + (b_{10} - ib_{01})\,z^2}}, \qquad \overline{F} = \frac{\bar{z}}{\sqrt{1 + (b_{10} + ib_{01})\,\bar{z}^2}},$$

into (3.13) we obtain

$$\frac{dF}{d\tau} = i\,F, \qquad \frac{d\overline{F}}{d\tau} = -i\,\overline{F},$$

which clearly shows that infinity of system (3.1) is a quasi-isochronous center (e.g., see Sect. 3 in [37]).

For the lower half-plane of system (3.11), we apply a simple time rescaling $\tau = \left(x^2+y^2\right)t$ into the system to obtain

$$\begin{aligned} \frac{dx}{d\tau} &= -y + \frac{b_{11}x^2}{\left(x^2+y^2\right)}, \\ \frac{dy}{d\tau} &= x + \frac{b_{11}xy}{\left(x^2+y^2\right)}, \end{aligned} \tag{3.14}$$

which, by using the Bendixson transformation (2.2), yields

$$\frac{d\theta}{d\tau} = 1.$$

So the singular point at infinity of system (3.1) is a quasi-isochronous center.

Next, consider condition (3.10) in Theorem 3.3. If this condition holds, system (3.1) can be rewritten as

$$\begin{aligned} &\frac{dx}{dt} = -y\,(x^2+y^2) - b_{01}x + b_{10}y, \\ &\frac{dy}{dt} = x\,(x^2+y^2) + b_{10}x + b_{01}y, \end{aligned} \quad (y>0),$$
$$\begin{aligned} &\frac{dx}{dt} = -y\,(x^2+y^2) + \frac{1}{3}b_{11}x^2 + \frac{1}{3}b_{11}y^2, \\ &\frac{dy}{dt} = x\,(x^2+y^2) + b_{11}xy, \end{aligned} \quad (y<0). \tag{3.15}$$

We only need to consider the liberalization problem of the lower system, since the upper system is the same as that discussed above in (3.11). By using the following transformations,

$$u = x\left(x^2+y^2\right)^2, \quad v = y\left(x^2+y^2\right)^2, \quad \tau = \left(x^2+y^2\right)^3 t,$$

the lower system of (3.15) becomes

$$\begin{aligned} \frac{du}{d\tau} &= -\frac{1}{9}\left[9v + b_{11}\left(u^4 + 10u^2v^2 - 3v^4\right)\right], \\ \frac{dv}{d\tau} &= \frac{1}{9}u\left(9 + 5b_{11}u^2v - 7b_{11}v^3\right), \end{aligned} \tag{3.16}$$

which has an inverse integral factor

$$\mu(u,v) = \left(u^2+v^2\right)^2 f_5^{\frac{5}{6}},$$

where

$$f_5 = 9\left[1 + 2b_{11}\left(u^2 - v^2\right)v + b_{11}^2\left(u^2 + v^2\right)^2 v^2\right].$$

Thus, we obtain a first integral,

$$H(u,v) = \frac{u^2+v^2}{f_5^{\frac{1}{6}} + 16b_{11}\left(u^2+v^2\right)\int u f_5^{-\frac{5}{6}}\mathrm{d}u},$$

Then, the system has a transversal system,

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}t} &= u\left(3 - 9b_{11}^2 v^3\right)\frac{u^2+v^2}{f_5^{\frac{1}{6}}H(u,v)},\\ \frac{\mathrm{d}v}{\mathrm{d}t} &= v\left(3 + 9b_{11}u^2v - 3b_{11}v^3\right)\frac{u^2+v^2}{f_5^{\frac{1}{6}}H(u,v)},\end{aligned} \tag{3.17}$$

which indicates that the singular point at infinity of system (2.2) is a quasi-isochronous center. □

Finally, we consider the conditions in (3.3), under which the periodic constants of system (3.1) are given as follows:

$$\begin{aligned}\tau_1 &= \tfrac{\pi}{24}\left[5a_{02}^2 + 12(a_{01} - b_{10}) + 6a_{02}a_{20} + 9\left(a_{20}^2 + b_{02}^2\right)\right],\\ \tau_2 &= \tfrac{2}{567}\left(64a_{02}^3 + 72a_{02}^2a_{20} + 108a_{02}a_{20}^2\right.\\ &\quad\left. + 135a_{20}^3 + 216a_{02}b_{02}^2 + 324a_{20}b_{02}^2\right),\\ \tau_3 &= -\tfrac{\pi}{24}b_{10}f_6, \qquad \tau_4 = -\tfrac{2}{47385}f_7.\end{aligned} \tag{3.18}$$

where

$$\begin{aligned}f_6 &= 5a_{02}^2 + 6a_{02}a_{20} + 9a_{20}^2 + 9b_{02}^2,\\ f_7 &= 1792a_{02}^5 + 1920a_{02}^4a_{20} + 2880a_{02}^3\left(a_{20}^2 + 4b_{02}^2\right)\\ &\quad + 4320a_{02}^2a_{20}\left(a_{20}^2 + 6b_{02}^2\right)\\ &\quad + 270a_{02}\left(21a_{20}^4 + 144a_{20}^2b_{02}^2 + 48b_{02}^4\right)\\ &\quad + 243a_{20}\left(21a_{20}^4 + 140a_{20}^2b_{02}^2 + 80b_{02}^4\right).\end{aligned}$$

We have the following result.

Theorem 3.4 *Under condition (3.3), the infinity of system (3.1) is a quasi-isochronous center if and only if* $\delta = a_{20} = a_{02} = a_{11} = b_{11} = b_{02} = a_{10} + b_{01} = a_{01} - b_{10} = 0$.

Proof To prove the *necessity*, we need to find the conditions such that $\tau_1 = \tau_2 = \cdots = 0$. We start from τ_2.

Case 1. Assume $2a_{02} + 3a_{20} \neq 0$. Then $\tau_2 = 0$ yields $b_{02}^2 = -\frac{64a_{02}^3+72a_{02}^2a_{20}+108a_{02}a_{20}^2+135a_{20}^3}{108(2a_{02}+3a_{20})}$.

(1a) If $b_{10} \neq 0$, then Resultant$[f_6, f_7] = -4504037556848431104\,a_{20}^{12}$. Thus, when $a_{20} = 0$, we have $f_6 = f_7 = 0$, yielding $a_{02} = 0$, a contradiction with the assumption.

(1b) If $b_{10} = 0$, then $f_7 = 0$ becomes

$$\begin{aligned}&2176a_{02}^4 + 13056a_{02}^3a_{20} + 22896a_{02}^2a_{20}^2\\ &\quad + 16416a_{02}a_{20}^3 + 17091a_{20}^4 = 0.\end{aligned}$$

Hence, if $a_{20} = 0$, we would have $a_{02} = 0$ which again contradicts the assumption. So $a_{20} \neq 0$. Let $a_{02} = ka_{20}$ $(k \neq 0)$. Then, $f_7 = 0$ is reduced to

$$17091 + 16416k + 22896k^2 + 13056k^3 + 2176k^4 = 0,$$

which has two real solutions: $k = -3.24077\ldots$, $-2.51419\ldots$, for which $b_{02}^2 < 0$.

Therefore, there does not exist isochronous center condition when $2a_{02} + 3a_{20} \neq 0$.

Case 2. Assume $a_{02} = a_{20} = 0$, leading to $\tau_2 = 0$, and higher-order periodic constants become

$$\begin{aligned}\tau_3 &= -\tfrac{3\pi}{8}a_{01}b_{02}^2, \qquad \tau_4 = 0,\\ \tau_5 &= \tfrac{3\pi}{16384}b_{02}^2\left(1536b_{01}^2 - 25b_{02}^4\right), \; \tau_6 = \tau_7 = \tau_8 = 0,\\ \tau_9 &= \tfrac{9\pi}{67108864}b_{02}^2\\ &\quad\times\left(1966080b_{01}^4 + 1290240b_{01}^2b_{02}^4 + 6336149b_{02}^8\right).\end{aligned} \tag{3.19}$$

It is clear that $\tau_3 = \tau_5 = \tau_9 = 0$ yields $b_{02} = 0$, and so $a_{20} = a_{02} = b_{02} = a_{11} = b_{22} = 0$. Then, in addition, $\tau_1 = 0$ gives $a_{01} = b_{10}$. Combining these conditions with (3.3) yields $\delta = a_{20} = a_{02} = a_{11} = b_{11} = b_{02} = a_{10} + b_{01} = a_{01} - b_{10} = 0$. Next, to prove *sufficiency*, substituting these solutions into (3.1) we obtain

$$\begin{aligned}&\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -y\left(x^2+y^2\right) - b_{01}x + b_{10}y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x\left(x^2+y^2\right) + b_{10}x + b_{01}y,\end{aligned}\right\} \quad (y > 0),\\ &\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -y\left(x^2+y^2\right),\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= x\left(x^2+y^2\right),\end{aligned}\right\} \quad (y < 0).\end{aligned} \tag{3.20}$$

It is easy to see that system (3.20) is a special case of system (3.15). Therefore, under condition (3.3), the infinity of system (3.1) is not a quasi-isochronous center.

4 A quintic switching system to yield 14 limit cycles at infinity

In this section, we consider the following switching quintic system,

$$\begin{aligned}
\frac{dx}{dt} &= (\delta x - y)\left(x^2+y^2\right)^2 + a_{10}x + a_{01}y \\
&\quad + x^3 + a_{12}xy^2 + a_{03}y^3, \\
\frac{dy}{dt} &= (x+\delta y)\left(x^2+y^2\right)^2 + b_{10}x \\
&\quad + \left(2048 - \tfrac{237a_{03}-211b_{12}}{256}\right)y + b_{12}xy^2,
\end{aligned} \qquad (y>0),$$

$$\begin{aligned}
\frac{dx}{dt} &= (\delta x - y)\left(x^2+y^2\right)^2 - a_{20}x^2 + a_{11}xy - y^2 \\
&\quad + a_{13}xy^3, \\
\frac{dy}{dt} &= (x+\delta y)\left(x^2+y^2\right)^2 + b_{20}x^2 - b_{11}xy \\
&\quad - b_{31}x^3y + b_{02}y^2,
\end{aligned} \qquad (y<0). \tag{4.1}$$

and show that 14 limit cycles can bifurcate from the sufficiently small neighborhood of infinity of this system. Compared to continuous systems which can have only 11 limit cycles bifurcating at infinity, this example shows that switching systems exhibit more complex dynamical behavior.

A direct computation of the Lyapunov constants of system (4.1) yields the following theorem.

Theorem 4.1 *For system* (4.1)*, the first fifteen Lyapunov constants at infinity are given by*

$$\lambda_0 = 2\pi\delta, \quad \lambda_1 = \tfrac{4}{15}\,a_{13}, \quad \lambda_2 = -\tfrac{\pi}{8}\,(3+2a_{12}),$$

$$\lambda_3 = -\frac{2}{3}\,(a_{11}+2b_{02}+b_{20}),$$

$$\lambda_4 = \tfrac{\pi}{128}\,(21a_{03} - 64a_{10} + 5b_{12} - 8(b_{02}+b_{20})b_{31}),$$

$$\lambda_5 = \tfrac{664}{5775}\,(b_{02}+b_{20})\,b_{31}^2,$$

$$\begin{aligned}
\lambda_6 = \tfrac{\pi}{1024}\big[&36 + 288a_{01} + 128b_{10} + 35a_{03}^2 \\
&+32a_{03}(4b_{01} - 3b_{12}) \\
&+128b_{01}b_{12} - 67b_{12}^2 - 64(5+2a_{20})b_{20}\big],
\end{aligned}$$

$$\lambda_7 = \tfrac{8}{3465}\,(305 + 11a_{20})\,b_{20}b_{31},$$

$$\begin{aligned}
\lambda_8 = \tfrac{\pi}{1441792}\big[&253755392 + 369098752a_{01} \\
&-129261a_{03} - 17301504a_{03}^2 + 5511a_{03}^3 \\
&+11b_{12}(3801 - 36700160a_{03} + 9187a_{03}^2) \\
&-11b_{12}^2(35127296 - 7847a_{03}) \\
&-65549b_{12}^3 - 16b_{20}\,(13320a_{03} \\
&-57720b_{12} + 43223b_{31}^2\big)\big],
\end{aligned}$$

$$\begin{aligned}
\lambda_9 = \tfrac{2b_{20}}{1029888834975}\big(&-153919703280915 \\
&+1597888131840b_{11} \\
&+156935441520b_{20}^2 + 622726450432b_{31}^3\big),
\end{aligned}$$

$$\lambda_{10} = \tfrac{\pi}{5936904925985848637603708928 0}\,g_1,$$

$$\begin{aligned}
\lambda_{11} = \tfrac{2b_{20}b_{31}^2}{41153503709639611736048728125}&\\
(95271775612080363611807654126 25&\\
-116395583632124464652032620 00b_{20}^2&\\
-539690261041636616620283338 24b_{31}^3),&
\end{aligned}$$

$$\lambda_{12} = -\tfrac{\pi}{387520830110098903178823514428801024 00}\,g_2,$$

$$\begin{aligned}
\lambda_{13} = \tfrac{268578950576625\,b_{20}b_{31}}{19167980314145962807822724045203687119 0000000}&\\
\times\big[118138908984525&\\
\times(28264368438698196019 9&\\
-362868449933687 36b_{20}^2&\\
+562572489343744b_{20}^4)&\\
+512(863223938878025421260532496b_{20}^2&\\
-544024870948329059898852403421)b_{31}^3&\\
+157380377633100171016499201792086 4&\\
22228238336\,b_{31}^6\big],&
\end{aligned}$$

$$\lambda_{14} = \tfrac{\pi}{5222346479186729871505348160325105232587522048 0000}\,g_3,$$

where $\lambda_0 = \lambda_1 = \cdots = \lambda_{k-1} = 0$ *have been set in* λ_k *for* $k = 1, 2, \ldots, 14$ *and* g_1, g_2 *and* g_3 *are lengthy polynomial functions in* a_{03}, b_{12}, b_{20} *and* b_{31} *(which are omitted for brevity).*

As far as bifurcation of limit cycles is concerned, it follows from Theorem 4.2 that 14 limit cycles can bifurcate from the sufficiently small neighborhood of infinity of system (4.1). We have the following theorem.

Theorem 4.2 *When the singular point at infinity is a* 14*th-order weak focus, when* $0 < \delta \ll 1$*, fourteen limit cycles can bifurcate in the sufficiently small neighborhood of infinity of system* (4.1).

Proof When the singular point at infinity of system (4.1) is a 14th-order weak focus, the following conditions should be satisfied (solved from $\lambda_i = 0$, $i = 0, 1, 2, \ldots, 9$):

$$\delta = 0,$$
$$a_{13} = 0,$$
$$a_{12} = -\tfrac{3}{2},$$
$$a_{11} = -2b_{02} - b_{20},$$
$$a_{10} = \tfrac{1}{64}\big[21a_{02} + 5b_{12} - 5b_{12} - 8b_{31}(b_{02} + b_{20})\big],$$
$$b_{02} = -b_{20},$$
$$a_{20} = -\tfrac{305}{11},$$
$$b_{10} = \tfrac{1}{128}(-36 - 288a_{01} - 35a_{03}^2 - 128a_{03}b_{01} + 96a_{03}b_{12} - 128b_{01}b_{12} + 67b_{12}^2 + 320b_{20} + 128a_{20}b_{20}),$$
$$a_{01} = \tfrac{1}{369098752}\big[-5511a_{03}^3 + 11(1572864 - 9187b_{12})a_{03}^2 + (129261 + 213120b_{20} + 403701760b_{12} - 86317b_{12}^2)\,a_{03} - 253755392 + 65549b_{12}^3 + 386400256b_{12}^2 - (41811 + 923520b_{20})\,b_{12} + 691568b_{20}b_{31}^2\big],$$
$$b_{11} = \tfrac{153919703280915 - 156935441520b_{20}^2 - 622726450432b_{31}^3}{1597888131840},$$

Furthermore, letting $\lambda_{11} = 0$ yields

$$b_{20}^2 = \tfrac{128509892013953977}{157003307510192} - \tfrac{337306413151022885387677086 4}{72747239770077790407520387 5}\,b_{31}^3,$$

and then setting $\lambda_{13} = 0$ results in two solutions $b_{31}^{\pm}$. Since b_{31}^{+} yields $b_{20}^2 < 0$, we choose $b_{31} = b_{31}^{-}$ and use the above formula to obtain two solutions $b_{20}^{\pm}$. which gives two sets of solutions $S^{\pm} = (b_{31}, b_{20}^{\pm})$. Then substituting $S^{\pm}$ into g_1 and g_2 yields two polynomial functions $g_1^{\pm}$ and $g_2^{\pm}$ in a_{03} and b_{12}. Next, eliminating a_{03} from the equations $g_1^{\pm} = g_2^{\pm} = 0$ gives two solutions $a_{03}^{\pm} = a_{30}^{\pm}(b_{12})$ and two corresponding resultants $\mathrm{R}^{\pm}(b_{12})$, which are polynomial functions in b_{12} with irrational coefficients. Solving each of the two polynomial equations $\mathrm{R}^{\pm}(b_{12}) = 0$ we obtain 12 solutions for each one, given as follows:

$$\begin{aligned}(a_{03}, b_{12})^{+} = {} & (-2278.60904\cdots,\ -10568.8937\cdots),\\ & (2903.39028\ldots,\ -1947.39814\ldots),\\ & (1477.26646\ldots,\ -1776.61419\ldots),\\ & (1466.15641\ldots,\ -1282.04274\ldots),\\ & (1699.70968\ldots,\ -1054.45506\ldots),\\ & (53.5803033\ldots,\ -54.5457756\ldots),\\ & (34.2903757\ldots,\ -35.5482793\ldots),\\ & (6.30984655\ldots,\ -7.49406686\ldots),\\ & (-1.17343427\ldots,\ 1.46604612\ldots),\\ & (-50.1357624\ldots,\ 49.0129578\ldots),\\ & (-53.7987065\ldots,\ 52.7599527\ldots),\\ & (4116.44805\ldots,\ 507.713023\ldots),\end{aligned}$$

solved from $\mathrm{R}^{+}(b_{12}) = 0$ and a_{03} is given by $a_{03}^{+}(b_{12})$, for which $b_{20} = b_{20}^{+}$;

$$\begin{aligned}(a_{03}, b_{12})^{-} = {} & (-2278.32617\cdots,\ -10568.2354\cdots),\\ & (2798.40543\ldots,\ -1944.84558\ldots),\\ & (1863.32971\ldots,\ -1769.11783\ldots),\\ & (1451.63272\ldots,\ -1281.60011\ldots),\\ & (1721.35086\ldots,\ -1053.84434\ldots),\\ & (66.5963110\ldots,\ -66.6264036\ldots),\\ & (58.3903841\ldots,\ -58.8504509\ldots),\\ & (3.55510841\ldots,\ -2.38488604\ldots),\\ & (-1.07623960\ldots,\ 0.78491381\ldots),\\ & (-60.8126771\ldots,\ 60.2313023\ldots),\\ & (-70.2233376\ldots,\ 70.1263827\ldots),\\ & (4116.41945\ldots,\ 508.190205\ldots),\end{aligned}$$

solved from $\mathrm{R}^{-}(b_{12}) = 0$ and a_{03} is given by $a_{03}^{-}(b_{12})$, for which $b_{20} = b_{20}^{-}$. Moreover, it has been verified that $g_3 \neq 0$ for all of these 24 solutions. So there exist in a total 24 solutions such that the infinity singular point is a critical point with the highest order 14.

Further, when the singular point at infinity is a 14th-order weak focus, a direct computation shows that for these 24 solutions, the following condition holds:

$$J = \frac{\partial(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8, \lambda_9, \lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13})}{\partial(\delta, a_{13}, a_{12}, a_{11}, a_{10}, b_{02}, b_{10}, a_{20}, a_{01}, b_{11}, a_{03}, b_{20}, b_{12}, b_{31})} \neq 0.$$

In fact, we only need to verify $\tilde{J} = \frac{\partial(\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13})}{\partial(a_{03},b_{20},b_{12},b_{31})} \neq 0$, since all the equations $\lambda_i = 0,\ i = 0, 1, \ldots, 9$ have been solved one by one with one coefficient at each step. For example, choosing the solution

$$\begin{aligned}&(a_{03}, b_{12}, b_{31}, b_{20})\\ &= (-1.173434278\ldots,\ 1.466046122\ldots,\\ &\quad 4.433629948\ldots,\ 20.357292941\ldots)\end{aligned}$$

yields $\tilde{J} = \frac{\partial(\lambda_{10},\lambda_{11},\lambda_{12},\lambda_{13})}{\partial(a_{03},b_{20},b_{12},b_{31})} = -0.1548824843\cdots 10^{21}\pi^2$. This clearly shows that there exist 14 limit cycles which can bifurcate in the sufficiently small neighborhood of infinity of system (4.1).

The proof is complete.

5 Conclusion

In this paper, we present a method to compute the Poincaré map in a neighborhood of infinity for switching systems. When the map is well defined, one can use the Bendixson transformation to obtain a new system and then consider the limit cycles around the origin of the new system. As applications, two classes of cubic and quintic switching systems are investigated by using our method. These examples have shown that there exist more limit cycles at infinity of switching systems than that in continuous systems, and the dynamical behavior of these systems is more complex.

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