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A new formal series method and its application to the center problem in Z_2-equivariant nilpotent vector fields

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In this paper, center problems and bifurcation of limit cycles are considered for Z_2-equivariant nilpotent vector fields. A new formal series method is developed for computing the focal values of the vector fields, which can be conveniently implemented using a computer algebraic system. As an application, the new method is applied to classify the centers for a class of quintic-order systems, which contains four conditions associated with a nilpotent singular point at the origin and two center conditions associated with an elementary center at infinity. Moreover, eight small-amplitude limit cycles in the neighborhood of the origin and nine large-amplitude limit cycles at infinity are obtained. This is the first time to investigate the synchronous bifurcation problem associated with a nilpotent singular point at the origin and a Hopf singular point at infinity.

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1. Introduction

Limit cycles, describing self-sustained oscillations, may appear in almost all fields in nonlinear science and engineering. The study of limit cycles is of great significance theoretically and practically. Theoretically, the study is closely related to the well-known Hilbert's 16th problem, one of the 23 mathematical problems proposed by D. Hilbert in 1900 [16]. It is also proposed by S. Smale as one of the 18 most challenging mathematical problems for the 21st century [31]. In practical applications, many complex dynamical behaviors are triggered by the bifurcation of limit cycles. In the study of dynamical systems, center problems, which are closely related to the Hilbert's 16th problem, are far from being completely solved. It is very difficult to obtain all center conditions for a given system because it involves a lot of computations. Even for a simple system, its focal values may have tens of thousands of terms, which are usually difficult dealt with by the existing approaches. Therefore, previous researches on the center problem of singular points mainly focus on two aspects: one is to develop methodologies for calculating the focal values of general systems; and the other is to apply existing approaches to investigate some specific systems to obtain all possible center conditions.

When a singular point is elementary, there have been many methods which can be used to study the center problem, such as formal series, successive function, normal form and so on. In the past several decades, some specific systems have been investigated, for example, the one given in the form of

$$\begin{aligned}\frac{dx}{dt} &= y + P_m(x,y),\\ \frac{dy}{dt} &= -x + Q_m(x,y),\end{aligned}\tag{1.1}$$

where $P_m(x,y)$ and $Q_m(x,y)$ are homogeneous polynomials of degree m, as studied for $m = 3,4$ and 5 in [4,6] and [7,15], respectively. Nevertheless, the classification for the centers of cubic systems, described by

$$\begin{aligned}\frac{dx}{dt} &= y + P_2(x,y) + P_3(x,y),\\ \frac{dy}{dt} &= -x + Q_2(x,y) + Q_3(x,y),\end{aligned}\tag{1.2}$$

has not been completely solved. The big obstacle lies in the computation and simplification of the focus quantity.

For degenerate singular points, the center problem is more complicated because the classical methods for studying elementary singular points are no longer applicable. Moreover, a degenerate center may not be an analytical center, and so do not have analytical first integral, yielding much more difficulties in studying the center problem of degenerate singular points.

As a special degenerate singular point, nilpotent singular point is the most interesting case and receives increasing attention from mathematicians. Consider the following system,

$$\begin{aligned}\frac{dx}{dt} &= X(x,y) = y + \sum_{k+j=2}^{\infty} a_{kj}x^k y^j,\\ \frac{dy}{dt} &= Y(x,y) = \sum_{k+j=2}^{\infty} b_{kj}x^k y^j.\end{aligned} \tag{1.3}$$

The origin of system (1.3) is said to be a nilpotent singular point if the linear part of the system has a double-zero eigenvalue but the matrix of the linearized system at the origin is not identically null. Suppose that $y = f(x) = -a_{20}x^2 + o(x^2)$ is the unique solution of the implicit equation $X(x, f(x)) = 0$ with $f(0) = 0$ in the neighborhood of the origin. Further, assume that there exists a positive number m and a non-zero constant α satisfying

$$Y(x, f(x)) = \alpha x^m + o(x^m), \tag{1.4}$$

then the origin of system (1.3) is defined as a singular point with multiplicity m.

It is easy to prove that if the origin of system (1.3) is a nilpotent singular point with multiplicity m, it can be broken into m complex elementary singular points in the neighborhood of the origin of the system by small parametric perturbation. Therefore, there may appear many new bifurcation phenomena in the neighborhood of the nilpotent singular point, which has promoted many new developments in recent few years, see some results for the system (1.3) in [2].

At present, research is mainly conducted on nilpotent singular points along two lines. One is to develop general methods for calculating focal points, and the other is to find efficient algorithms suitable for certain special systems. Along the first main line, numerous scholars have studied the algorithm of focus quantity. The basic idea is to study the displacement map on a small line segment with its endpoint at the nilpotent singular point. A direct method is to generalize trigonometric functions, defined for the first time by Lyapunov in [19]. Some classical specific systems have been investigated by this method such as

$$\begin{aligned}\frac{dx}{dt} &= y + \sum X_{2n+1}(x,y),\\ \frac{dy}{dt} &= Y_{2n+1}(x,y),\end{aligned}$$

where $b_{2n+1} < 0$ and $n = 1, 2, 3$.

However, the computation of the generalized trigonometric functions is tedious and difficult. Afterwards, many researchers developed alternatives for computing the focal values of nilpotent singular points. For the case of the origin being a three-multiple monodromic critical point, some results were obtained in [2]. According to the results in [25,26], the origin of system (1.3) is a three-multiple monodromic critical point if and only if the system can be written in the form of

$$\begin{aligned}\frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij}x^i y^j \triangleq X(x,y),\\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij}x^i y^j \triangleq Y(x,y).\end{aligned} \tag{1.5}$$

Moreover, an inverse integrating method was proposed in [25,26], where quasi-Lyapunov constants were defined, with a computation method given. However, the inverse integrating factor method can only be used for the three-multiple nilpotent singular point, not applicable for higher-multiple nilpotent singular points because one cannot prove the existence of the inverse integrating factor. As a result, many researchers endeavored to consider the computation of focal values for higher-multiple nilpotent singular points. Via investigating the Taylor expansion of the return map near the origin of this system, focal values can be computed to determine small-amplitude limit cycles bifurcating from the critical point [1]. As an application, they characterized centers, and proved the existence of small-amplitude limit cycles bifurcating from the origin of several families. Based on computer algebra techniques, a new method was developed [18] for determining a minimal basis of the associated polynomial Bautin ideal in the parameter space of the family, and applied to solve the center problem of the following two systems,

$$\begin{aligned}\frac{dx}{dt} &= y + a_1x^5 + a_2x^2y,\\ \frac{dy}{dt} &= -x^7 + b_1x^4y + b_2xy^2;\end{aligned}$$

and

$$\begin{aligned}\frac{dx}{dt} &= y + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 + a_6xy^4 + a_7y^5,\\ \frac{dy}{dt} &= -x^3 + b_1x^4y + b_2xy^2 + b_6x^9 + b_3x^6y + b_4x^3y^2 + b_5y^3.\end{aligned}$$

Center and analytical center conditions were obtained for these two systems.

Following the second main line, scholars have studied the center conditions of many specific systems. Symmetric systems are one of the most interesting systems, since breaking of symmetry destroys the underlying order of nature. As a class of special symmetric systems, the time-reversible systems play an important role in qualitative analysis, and were studied by many researchers, see for example [10,11].

Another important kind of symmetric systems is the Z_2-equivariant system, with many fruitful results obtained on the center problems. Recently, a complete study on the bi-center problem for Z_2-equivariant cubic vector fields was given in [32,33,28], and the bi-center problem for some Z_2-equivariant quintic systems was studied in [30]. In 2017, the bi-isochronous center problem for cubic systems in Z_2-equivariant vector fields with real coefficients was considered in [13]. In 2020, the isochronous center problem for the Z_2-equivariant cubic vector fields with complex coefficients is completely solved in [20]. The Z_2-equivariant cubic vector fields with weak saddles or resonant saddles were studied in [21–23], and the Z_2-equivariant cubic polynomial Hamiltonian systems with bi-center were investigated in [12].

Some higher-order Z_2-equivariant systems were also explored including the following systems:

$$
\begin{aligned}
\frac{dx}{dt} &= y + a_1x^5 + a_2x^2y + a_3x^7 + a_4x^4y + a_5xy^2,\\
\frac{dy}{dt} &= -x^7 + b_1x^4y + b_2xy^2 + b_3y^3 + b_5xy^4 + b_6y^5;
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx}{dt} &= y + a_1x^9 + a_2x^6y + a_3x^3y^2 + a_4y^3,\\
\frac{dy}{dt} &= -x^{11} + b_1x^8y + b_2x^5y^2 + b_3x^2y^3;
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{dx}{dt} &= y + Ax^5 + Bx^4y + Cx^3y^2 + Dx^2y^3 + Exy^4 + Fy^5,\\
\frac{dy}{dt} &= -x^3 + Gx^2y + Hxy^2 + Iy^3.
\end{aligned}
$$

The classification of the center and analytical center for these systems was completed, see [5,8,9].

Although different methods have been proposed to study the center problem of higher-multiple nilpotent singular points, the computations are still tedious and challenging even for the symmetric systems. Therefore, it is necessary to develop new methods for computing the focal values of the Z_2-equivariant systems.

In this paper, the computation method for computing the focal values of the Z_2-equivariant systems with a nilpotent singular point will be improved, leading to a new formal series method which can be used to compute the focal values of the Z_2-equivariant systems. Compared with the existing algorithms, this method only involves the algebraic operation of the coefficients of the system, which is easier to be realized by the computer algebraic system and requires less computation.

The rest of the paper is organized as follows. Section 2 is devoted to describe a new efficient method for computing the focal values of Z_2-equivariant systems with a nilpotent singular point. As an application, a class of Z_2-equivariant quintic-order systems is studied to illustrate the efficiency of the new method in Section 3, with eight small-amplitude limit cycles obtained in the neighborhood of the origin. In the last section, simultaneous bifurcation of limit cycles from the origin and the infinity is discussed, and nine large-amplitude limit cycles at infinity are obtained.

2. A new method for computing the focal values of Z_2-equivariant systems with a nilpotent singular point

In this section, we consider a class of Z_2-equivariant systems described by (1.3), namely, the following conditions are satisfied:

$$X(-x,-y) = -X(x,y), \quad Y(-x,-y) = -Y(x,y).$$

We first consider a three-multiple nilpotent singular point for which $b_{30} \neq 0$. Then, system (1.3) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= X(x,y) = y + \sum_{m=1}^{\infty} X_{2m+1}(x,y), \\ \frac{dy}{dt} &= Y(x,y) = \sum_{l=1}^{\infty} Y_{2l+1}(x,y), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} X_{2m+1}(x,y) &= \sum_{k+j=2m+1} a_{kj} x^k y^j, \\ Y_{2l+1}(x,y) &= \sum_{k+j=2l+1} b_{kj} x^k y^j. \end{aligned} \tag{2.2}$$

Without loss of generality, for convenience we can always choose $b_{30} = -2$ by a simple rescaling. The following results are given in [2].

Lemma 2.1. *For the system (2.1), a formal power series can be constructed successively as follows:*

$$F(x,y) = y^2 + \sum_{m=2}^{\infty} \sum_{\alpha+\beta=2m} c_{\alpha\beta} x^{\alpha} y^{\beta}, \tag{2.3}$$

where $c_{40} = 1$, and F satisfies

$$\frac{dF}{dt} = x^4 \sum_{k=1}^{\infty} V_{2k} x^{2k}. \tag{2.4}$$

Definition 2.1. For the system (2.1), V_{2k} is called the k-th Liapunov constant of system (2.1).

Now, we give the linear recursive formulas for computing of the k-th Liapunov constant of the system (2.1) in the following theorem.

Theorem 2.1. *In (2.3), let*

$$c_{0,2m} = 0, \ m = 2, 3, \cdots . \tag{2.5}$$

Then, for any positive number m, when $\alpha + \beta = 2m$, $\alpha \neq 0$, $k \geq 0, j \geq 0$, $c_{\alpha\beta}$ have the following recursive formulas:

$$c_{\alpha\beta} = \frac{-1}{\alpha} \sum_{k+j=4}^{\alpha+\beta} \left[(\alpha - k)a_{k,j-1} + (\beta - j + 2)b_{k-1,j}\ \right] c_{\alpha-k,\ \beta-j+2}. \tag{2.6}$$

And for any positive number m, V_{2m} have the following recursive formulas:

$$V_{2m} = \sum_{k=3}^{2m+4} \left[(2m - k + 5)a_{k,0}\ c_{2m+5-k,0} + b_{k,0}\ c_{2m+4-k,1}\right], \tag{2.7}$$

where

$$c_{20} = c_{11} = 0, \ c_{02} = c_{40} = 1. \tag{2.8}$$

In particular, $a_{\alpha\beta} = b_{\alpha\beta} = c_{\alpha\beta} = 0$ for $\alpha < 0$ or $\beta < 0$ and $c_{\alpha\beta} = 0$ for $\alpha + \beta = 2m + 1$.

Proof. Denote

$$\begin{aligned} X_{2n+1}(x, y) &= \sum_{k+j=2n+2} a_{k,j-1} x^k y^{j-1}, \\ Y_{2n+1}(x, y) &= \sum_{k+j=2n+2} b_{k-1,j} x^{k-1} y^j. \end{aligned} \tag{2.9}$$

A direct calculation gives

$$\frac{dF}{dt} = y\frac{\partial F}{\partial x} + H(x, y), \tag{2.10}$$

where

$$y\frac{\partial F}{\partial x} = \sum_{m=2}^{\infty} \sum_{\alpha+\beta=2m} \alpha c_{\alpha\beta} x^{\alpha-1} y^{\beta+1}, \tag{2.11}$$

and

$$\begin{aligned}
H(x,y) &= \frac{\partial F}{\partial x}\sum_{m=1}^{\infty} X_{2m+1}(x,y) + \frac{\partial F}{\partial y}\sum_{m=1}^{\infty} Y_{2m+1}(x,y) \\
&= \sum_{\substack{m=1\\n=1}}^{\infty}\ \sum_{\substack{\alpha+\beta=2m\\k+j=2n+2}}^{\infty} \left(\alpha a_{k,j-1}+\beta b_{k-1,j}\right) c_{\alpha\beta}x^{\alpha+k-1}y^{\beta+j-1} \\
&= \sum_{m=2}^{\infty}\ \sum_{\alpha+\beta=2m} \Delta_{\alpha\beta}x^{\alpha}y^{\beta},
\end{aligned} \tag{2.12}$$

in which

$$\Delta_{\alpha\beta} = \sum_{k+j=4}^{\alpha+\beta}\left[(\alpha-k+1)a_{k,j-1}+(\beta-j+1)b_{k-1,j}\ \right] c_{\alpha-k+1,\ \beta-j+1}. \tag{2.13}$$

It follows from (2.10), (2.11) and (2.12) that

$$\frac{dF}{dt} = \sum_{m=2}^{\infty} G_{2m}(x,y), \tag{2.14}$$

where $G_{2m}(x,y)$ is a homogeneous polynomial in x and y with degree $2m$,

$$G_{2m}(x,y) = \sum_{\alpha+\beta=2m} \alpha c_{\alpha\beta}x^{\alpha-1}y^{\beta+1} + \Delta_{\alpha\beta}x^{\alpha}y^{\beta}. \tag{2.15}$$

In particular,

$$\begin{aligned}
G_4 &= \frac{\partial F_4}{\partial x}\ y + \frac{\partial F_2}{\partial y}\ Y_3 \\
&= \Delta_{40}x^4+(4c_{40}+\Delta_{31})x^3y+(3c_{31}+\Delta_{22})x^2y^2+(2c_{22}+\Delta_{13})xy^3+(c_{13}+\Delta_{04})y^4.
\end{aligned}$$

According to (2.13), we have

$$\begin{aligned}
\Delta_{40} &= \sum_{k+j=4}^{4}\left[(5-k)a_{k,j-1}+(1-j)b_{k-1,j}\right] c_{5-k,1-j} = 0, \\
\Delta_{31} &= \sum_{k+j=4}^{4}\left[(4-k)a_{k,j-1}+(2-j)b_{k-1,j}\right] c_{4-k,2-j} = 2b_{3,0},
\end{aligned}$$

$$\Delta_{22} = \sum_{k+j=4}^{4} \left[(3-k)a_{k,j-1} + (3-j)b_{k-1,j}\right] c_{3-k,3-j} = 2b_{2,1},$$

$$\Delta_{13} = \sum_{k+j=4}^{4} \left[(2-k)a_{k,j-1} + (4-j)b_{k-1,j}\right] c_{2-k,4-j} = 2b_{1,2},$$

$$\Delta_{04} = \sum_{k+j=4}^{4} \left[(1-k)a_{k,j-1} + (5-j)b_{k-1,j}\right] c_{1-k,5-j} = 2b_{0,3}.$$

Thus, we obtain

$$G_4 = (4c_{40} + 2b_{30})x^3y + (3c_{31} + 2b_{21})x^2y^2 + 2(c_{22} + b_{12})xy^3 + (c_{13} + 2b_{03})y^4, \tag{2.16}$$

showing that G_4 is independent of c_{04}, so one can set $c_{04} = 0$. Moreover, because $b_{30} = -2$, we can let

$$c_{40} = 1, \quad c_{31} = -\frac{2}{3}b_{21}, \quad c_{22} = -b_{12}, \quad c_{13} = -2b_{03}, \tag{2.17}$$

which satisfies $G_4(x, y) = 0$.

According to (2.14) and (2.15), we obtain that

$$\begin{aligned} &c_{0,2m} = 0, \ \ m = 2, 3, \cdots, \\ &c_{\alpha\beta} = \frac{-1}{\alpha}\Delta_{\alpha-1,\beta+1}, \quad \alpha \neq 0, \ \ \alpha + \beta \geq 4, \\ &V_{2m} = \Delta_{2m+4,0}, \quad m = 1, 2, \cdots . \end{aligned} \tag{2.18}$$

Furthermore,

$$\begin{aligned} V_{2m} &= \Delta_{2m+4,0} \\ &= \sum_{k+j=4}^{2m+4} \left[(2m+5-k)\ a_{k,j-1} + (1-j)\ b_{k-1,j}\ \right] c_{2m+5-k,\ 1-j} \\ &= \sum_{k=4}^{2m+4} b_{k-1,0}\ c_{2m+5-k,\ 1} + \sum_{k=3}^{2m+4} (2m+5-k)\ a_{k,0}c_{2m+5-k,\ 0} \\ &= \sum_{k=3}^{2m+4} \left[(2m-k+5)\ a_{k,0}\ c_{2m+5-k,0} + b_{k,0}\ c_{2m+4-k,1}\right] \end{aligned}$$

So (2.14) leads to (2.22), and the proof is complete. □

Remark 2.1. As a matter of fact, by using (2.6), we can show that $c_{40} = 1$:

$$\begin{aligned}
c_{40} &= -\frac{1}{4}\sum_{k+j=4}^{4}\left[(4-k)a_{k,j-1} + (2-j)b_{k-1,j}\right]c_{4-k,2-j} \\
&= -\frac{1}{4}\left[2b_{3,0}c_{0,2} + (a_{3,0}+b_{2,1})c_{1,1} + 2a_{2,1}c_{2,0} + (3a_{1,2}-b_{0,3})c_{1,-1}\right. \\
&\quad \left. + (4a_{0,3}-2b_{-1,4})c_{0,-2}\right] \\
&= -\frac{1}{4}\times 2b_{3,0}c_{0,2} \\
&= 1.
\end{aligned}$$

When the origin is a nilpotent singular point with multiplicity $2l+1$, similarly, we can always suppose $b_{2l+1,0} = -l-1$, the system can be written as

$$\begin{aligned}
\frac{dx}{dt} &= X(x,y) = y + \sum_{m=k}^{\infty} X_{2m+1}(x,y), \\
\frac{dy}{dt} &= Y(x,y) = \sum_{n=l}^{\infty} Y_{2n+1}(x,y),
\end{aligned} \tag{2.19}$$

where

$$\begin{aligned}
X_{2m+1}(x,y) &= \sum_{k+j=2m+1} a_{kj}x^k y^j, \\
Y_{2m+1}(x,y) &= \sum_{k+j=2l+1} b_{kj}x^k y^j,
\end{aligned} \tag{2.20}$$

we have a similar result as given below.

Lemma 2.2. *For the system (2.19), a formal power series can be constructed successively as*

$$F(x,y) = y^2 + \sum_{m=2}^{\infty}\sum_{\alpha+\beta=2m} c_{\alpha\beta}x^{\alpha}y^{\beta}, \tag{2.21}$$

where $c_{40} = 1$, and F satisfies

$$\frac{dF}{dt} = x^{\lambda}\sum_{k=1}^{\infty} V_{2k}x^{2k}, \tag{2.22}$$

where $\lambda = \min\{2(k+l), 4l\}$.

The relation between V_{2k} and $v_{2k}(-2\pi)$ has been given in the following definition [25].

Definition 2.2. [25] Let f_k and g_k be polynomials with respect to μ and a_{kj}, b_{kj}, $k = 1, 2, \cdots$. If for a positive integer m, there exist polynomials $\xi_1^{(m)}$, $\xi_2^{(m)}$, $\cdots$, $\xi_{m-1}^{(m)}$, with respect to μ and a_{kj}, b_{kj}, such that

$$f_m = g_m + \left(\xi_1^{(m)} f_1 + \xi_2^{(m)} f_2 + \cdots + \xi_{m-1}^{(m)} f_{m-1}\right), \tag{2.23}$$

then, f_m is said to be algebraic equivalent to g_m, denoted by $f_m \sim g_m$. Furthermore, if for any positive integer m, $f_m \sim g_m$ holds, then the sequences of the function f_m are said to be algebraic equivalent to g_m, written as $f_m \sim g_m$.

Obviously, the algebraic equivalent relationship of the sequences of the functions is self-reciprocal, symmetric and transmissible. In the sense of algebraic equivalence, the two quantities V_{2k} and $v_{2k}(-2\pi)$ differ only by a very constant.

Based on Theorem 3.1 in [25], we have the following result.

Theorem 2.2. *For a positive integer k, the k-th Liapunov constant of the system* (2.19) *and the k-th focal value of system* (2.19) *are algebraic equivalent:*

$$\left\{v_{2k}(-2\pi)\right\} \sim \left\{\frac{1}{8}\sigma_k V_{2k}\right\}, \tag{2.24}$$

where

$$\sigma_k = \int_0^{2\pi} \frac{(1+\sin^2\theta)\cos^{2k+4}\theta}{(\cos^4\theta + \sin^2\theta)^{\frac{2k+7}{4}}} d\theta > 0, \quad k = 1, 2, \cdots. \tag{2.25}$$

3. Center problem for a class of Z_2-equivariant quintic systems with a nilpotent singular point

The synchronized bifurcation problem for a class of Z_2-equivariant quintic systems with elementary singular points at the origin and infinity, described by

$$\begin{aligned}
\frac{dx}{dt} &= -y + \delta_1 x + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \lambda(\delta_2 x - y)(x^2+y^2)^2, \\
\frac{dy}{dt} &= 2\delta_1 y + x + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + \lambda(x + \delta_2 y)(x^2+y^2)^2,
\end{aligned}$$

has been discussed in [3], and some interesting results have been obtained. However, no relevant results have been published for the case when the origin is a nilpotent singular point. In the following, we consider a class of Z_2-equivariant quintic systems with a nilpotent singular point at the origin, given by

$$
\begin{aligned}
\frac{dx}{dt} &= -y + \delta_1 x + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \lambda(\delta_2 x - y)(x^2+y^2)^2, \\
\frac{dy}{dt} &= 2\delta_1 y - 2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + \lambda(x + \delta_2 y)(x^2+y^2)^2.
\end{aligned} \tag{3.1}
$$

When $\delta_1 = \delta_2 = 0$, the system (3.1) is reduced to

$$
\begin{aligned}
\frac{dx}{dt} &= -y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 - \lambda y(x^2+y^2)^2, \\
\frac{dy}{dt} &= -2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + \lambda x(x^2+y^2)^2.
\end{aligned} \tag{3.2}
$$

For the special case of system (3.2) when $\lambda = 0$, without loss of generality, suppose $a_{30} = 0$. Otherwise, a_{30} can be removed by the transformation: $\xi = x + \frac{1}{2}a_{30}y$, $\eta = y$. This system was studied in [2], where the first three Liapunov constants were computed, and three center conditions were obtained. In this paper, λ is supposed to be real, non zero.

3.1. The Liapunov constants and bifurcation of limit cycles of system (3.2)

By using the method given in Theorem 2.1, we obtain the following result.

Theorem 3.1. *The first 8 Liapunov constants of system (3.2) are given by*

$$
\begin{aligned}
V_2 &= \frac{4}{3}(3a_{30} + b_{21}), \\
V_4 &= \frac{8}{5}(a_{12} + 3b_{03} - a_{30}a_{21} - a_{30}b_{12}), \\
V_6 &= \frac{4}{7}(a_{21} + b_{12})(4b_{03} - 2a_{30}^3 - 2a_{30}b_{12} - a_{30}\lambda), \\
V_8 &= -\frac{2}{45}a_{30}\lambda(a_{21} + b_{12})(24 + 2a_{21} + 2b_{12} + 10a_{30}^2 + 5\lambda), \\
V_{10} &= \frac{4}{1925}a_{30}\lambda(a_{21} + b_{12})f_5, \\
V_{12} &= \frac{2}{2925}a_{30}\lambda^2(a_{21} + b_{12})f_6, \\
V_{14} &= \frac{1}{721875}a_{30}\lambda^2(a_{21} + b_{12})(4a_{21} - b_{12} - 2)f_7, \\
V_{16} &= \frac{2}{248625}a_{30}\lambda^3(a_{21} + b_{12})(4a_{21} - b_{12} - 2)f_8,
\end{aligned} \tag{3.3}
$$

where

$$
\begin{aligned}
f_5 = &- 1148 - 250a_{03} + 42a_{21} - 78b_{12} - 70a_{30}^2 + 8a_{21}^2 + 6a_{21}b_{12} \\
&- 2b_{12}^2 + 140a_{30}^2a_{21} - 35a_{30}^2b_{12},
\end{aligned}
$$

$$
\begin{aligned}
f_6 =& -376 + 148a_{21} + 16b_{12} - 18a_{30}^2 + 8a_{21}^2 - 10a_{21}b_{12} + 2b_{12}^2 \\
&+ 36a_{30}^2a_{21} - 9a_{30}^2b_{12}, \\
f_7 =&\ 7472 + 4680a_{21} - 264b_{12} + 8464a_{30}^2 + 352a_{21}^2 + 112a_{21}b_{12} \\
&+ 1922a30^2a_{21} - 18a_{30}^2b_{12} + 1925a_{30}^4, \\
f_8 =&\ 2208 + 880a_{21} - 56b_{12} + 1644a_{30}^2 + 72a21^2 + 338a_{30}^2a_{21} \\
&- 16a_{30}^2b_{12} + 327a_{30}^4.
\end{aligned}
\tag{3.4}
$$

Lemma 3.1. *A necessary condition for the origin of the system (3.2) to be a center is*

$$
a_{30}\lambda(a_{21} + b_{12})(4a_{21} - b_{12} - 2) = 0. \tag{3.5}
$$

Proof. Let

$$
\begin{aligned}
m_1 =& (132 - 56a_{21} + 9a_{30}^2)^2, \\
m_2 =& (4792 + 1232a_{21} + 132b_{12} + 3710a_{30}^2 + 456a_{21}^2 - 56a_{21}b_{12} + 1168a_{30}^2a_{21} \\
&+ 9a_{30}^2b_{12} + 922a_{30}^4), \\
m_3 =& -784(19284 + 1232a_{21} + 392b_{12} + 6925a_{30}^2), \\
m_4 =&\ 4191644224 + 852340608a_{21} + 4827640368a_{30}^2 + 183907584a_{21}^2 \\
&+ 887506368a_{30}^2a_{21} + 10185728a_{21}^3 + 1920703404a_{30}^4 + 78051904a_{30}^2a_{21}^2 \\
&+ 199772664a_{30}^4a_{21} + 252910513a_{30}^6, \\
m_5 =& -4(7 + 2a_{30}^2), \\
m_6 =&\ 160400 + 109864a_{30}^2 + 17585a_{30}^4.
\end{aligned}
\tag{3.6}
$$

Then, we have that

$$
\begin{aligned}
&m_1f_6 + m_2f_7 = 50R_1, \\
&m_3R_1 + m_4f_7 = 25m_6R_2, \\
&m_5R_2 + m_6f_8 = R_3,
\end{aligned}
\tag{3.7}
$$

where

$$
\begin{aligned}
R_1 =&\ 585088 + 795392a_{21} + 152640a_{21}^2 + 58272a_{21}^3 + 3712a_{21}^4 + 1341472a_{30}^2 \\
&+ 947040a_{30}^2a_{21} + 245616a_{30}^2a_{21}^2 + 27848a_{30}^2a_{21}^3 + 948840a_{30}^4 \\
&+ 476376a_{30}^4a_{21} + 68232a_{30}^4a_{21}^2 + 298882a_{30}^6 + 80468a_{30}^6a_{21} + 35497a_{30}^8, \\
R_2 =& -2(160400 + 109864a_{30}^2 + 17585a_{30}^4)b_{12} + 5604544 + 3340448a_{21} \\
&+ 8154352a_{30}^2 + 938304a_{21}^2 + 3837520a_{30}^2a_{21} + 51968a_{21}^3 + 4187684a_{30}^4 \\
&+ 398224a_{30}^2a_{21}^2 + 908714a_{30}^4a_{21} + 669053a_{30}^6,
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
R_3 =& 197235968 + 47619456a_{21} + 233119104a_{30}^2 - 14723712a_{21}^2 \\
&+ 16721376a_{30}^2a_{21} - 1455104a_{21}^3 + 89404928a_{30}^4 - 10746496a_{30}^2a_{21}^2 \\
&- 3535320a_{30}^4a_{21} - 1919672a_{30}^4a_{21}^2 - 415744a_{30}^2a_{21}^3 + 12600312a_{30}^6 \\
&- 1325982a_{30}^6a_{21} + 397871a_{30}^8,
\end{aligned}
$$

and

$$
\mathrm{Res}(R_1, R_3, a_{21}) = m_6^4 R_4, \tag{3.9}
$$

in which

$$
\begin{aligned}
R_4 =& \ 13551838143692800 + 19500026467946496a_{30}^2 \\
&+ 12444744099234816a_{30}^4 + 4605635683715840a_{30}^6 \\
&+ 1082430568495680a_{30}^8 + 165639977586800a_{30}^{10} \\
&+ 16129850492700a_{30}^{12} + 913409335125a_{30}^{14} + 22966006000a_{30}^{16}.
\end{aligned} \tag{3.10}
$$

It is clear that R_4 has no real solutions for a_{30}^2, implying that no real solutions exist for the equations: $f_6 = f_7 = f_8 = 0$.

The proof is complete. □

The following theorem directly follows from Theorem 3.1 and Lemma 3.1.

Theorem 3.2. *The first 8 Liapunov constants of system (3.2) are all zero if and only if one of the following four conditions holds:*

$$
a_{30} = 0, \quad b_{21} = 0, \quad a_{12} = 0, \quad b_{03} = 0; \tag{3.11}
$$

$$
\begin{cases}
\lambda = 0, \quad b_{21} = -3a_{30}, \quad a_{12} = \dfrac{1}{2}(2a_{21} - b_{12} - 3a_{30}^2)a_{30}, \\
b_{03} = \dfrac{1}{2}(b_{12} + a_{30}^2)a_{30};
\end{cases} \tag{3.12}
$$

$$
b_{12} = -a_{21}, \quad b_{21} = -3a_{30}, \quad a_{12} = -3b_{03}; \tag{3.13}
$$

$$
\begin{cases}
b_{12} = 6, \quad a_{21} = 2, b_{21} = -3a_{30}, \quad \lambda = -2(4 + a_{30}^2), \quad a_{12} = 5a_{30}, \\
b_{03} = a_{30}, \quad a_{03} = -6.
\end{cases} \tag{3.14}
$$

Further, we have the following result.

Theorem 3.3. *The origin of the system (3.2) is a center if and only if one of the four conditions in Theorem 3.2 holds.*

Proof. The necessity has been proved in Theorem 3.2. We only need to prove the sufficiency. If the condition (3.11) holds, system (3.2) can be simplified to

$$\begin{aligned}\frac{dx}{dt} &= y\big[1 + a_{21}x^2 + a_{03}y^2 - \lambda(x^2+y^2)^2\big],\\ \frac{dy}{dt} &= x\big[-2x^2 + b_{12}y^2 + \lambda(x^2+y^2)^2\big].\end{aligned} \tag{3.15}$$

Obviously, system (3.15) is symmetric with respect to both axes, which means that the origin is a center.

When the condition (3.12) holds, the system (3.2) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y + a_{30}x^3 + a_{21}x^2y + \frac{1}{2}(2a_{21} - b_{12} - 3a_{30}^2)a_{30}xy^2 + a_{03}y^3,\\ \frac{dy}{dt} &= -2x^3 - 3a_{30}x^2y + b_{12}xy^2 + \frac{1}{2}(b_{12} + a_{30}^2)a_{30}y^3.\end{aligned} \tag{3.16}$$

Furthermore, system (3.16) can be changed into

$$\begin{aligned}\frac{d\xi}{dt} &= \frac{1}{8}\eta\big[8 + 4(2a_{21} - 3a_{30}^2)\xi^2 + (8a_{03} - 2a_{30}^2a_{21} + 2a_{30}^2b_{12} + 5a_{30}^4)\eta^2\big],\\ \frac{d\eta}{dt} &= \frac{1}{2}\xi\big[-(4 - 3a_{30}^2)\xi^2 + 2b_{12}\eta^2\big],\end{aligned} \tag{3.17}$$

by $\xi = x + \frac{1}{2}a_{30}y,\ \eta = y$. The system (3.17) is invariant under the change $(\xi,\eta) \to (-\xi,-\eta)$, implying that the origin of system (3.16) is a center.

When the condition (3.13) is satisfied, the system (3.2) becomes

$$\begin{aligned}\frac{dx}{dt} &= y + a_{30}x^3 + a_{21}x^2y - 3b_{03}xy^2 + a_{03}y^3 - \lambda y(x^2+y^2)^2,\\ \frac{dy}{dt} &= -2x^3 - 3a_{30}x^2y - a_{21}xy^2 + b_{03}y^3 + \lambda x(x^2+y^2)^2,\end{aligned} \tag{3.18}$$

which is a Hamiltonian system, so the origin is a center.

Finally, when the condition (3.14) holds, the system (3.2) can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= y + a_{30}x^3 + 2x^2y + 5a_{30}xy^2 - 6y^3 + 2(4 + a_{30}^2)y(x^2+y^2)^2,\\ \frac{dy}{dt} &= -2x^3 - 3a_{30}x^2y + 6xy^2 + a_{30}y^3 - 2(4 + a_{30}^2)x(x^2+y^2)^2.\end{aligned} \tag{3.19}$$

By a tedious computation, the following integrating factor

$$M = \frac{1}{1 + 8x^2 + 8a_{30}xy - 8y^2 + 4(4 + a_{30}^2)(x^2+y^2)^2}, \tag{3.20}$$

is obtained for the system (3.19), which means that the origin is a center. □

3.2. Bifurcation of limit cycles at the origin of the system (3.2)

Solving $R_2 = 0$ for b_{12} yields $b_{12} = \tilde{b}_{12}$, where

$$\begin{aligned}\tilde{b}_{12} = \frac{1}{2m_6}&(5604544 + 3340448a_{21} + 8154352a_{30}^2 + 938304a_{21}^2 \\ &+ 3837520a_{30}^2a_{21} + 51968a_{21}^3 + 4187684a_{30}^4 + 398224a_{30}^2a_{21}^2 \\ &+ 908714a_{30}^4a_{21} + 669053a_{30}^6).\end{aligned} \tag{3.21}$$

According to the proofs of Theorem 3.1 and Lemma 3.1, we have

Theorem 3.4. *The origin of the system* (3.2) *is an* 8*-th weak focus if and only if the following conditions hold:*

$$\begin{aligned}&a_{30} \neq 0, \quad b_{21} = -3a_{30}, \quad a_{12} = \frac{1}{10}(36 + 13a_{21} - 2b_{12})a_{30}, \\ &b_{03} = -\frac{1}{10}(12 + a_{21} - 4b_{12})a_{30}, \quad \lambda = -\frac{2}{5}(12 + a_{21} + b_{12} + 5a_{30}^2), \\ &a_{03} = -\frac{1}{250}\big[2(574 - 21a_{21} + 39b_{12}) - 2(a_{21} + b_{12})(4a_{21} - b_{12}) \\ &\qquad\qquad + 35a_{30}^2(2 - 4a_{21} + b_{12})\big], \\ &b_{12} = \tilde{b}_{12}, \quad R_1 = 0.\end{aligned} \tag{3.22}$$

Remark 3.1. (i) Denote $\tilde{\lambda} = -\frac{2}{5}(12+a_{21}+\tilde{b}_{12}+5a_{30}^2)$, $g = m_6^3\tilde{\lambda}(a_{21}+\tilde{b}_{12})(4a_{21}-\tilde{b}_{12}-2)$. Then, g is a polynomial in a_{21} and a_{30}. Moreover, it is easy to compute $Res(g, R_1, a_{21})$ which is a polynomial in a_{30}^2 with positive coefficients, implying that $\lambda(a_{21}+b_{12})(4a_{21}-b_{12}-2) \neq 0$ when the conditions in (3.22) hold.

(ii) Because R_1 is a polynomial in a_{21} and a_{30}^2 with positive coefficients, according to Theorem *3.4*, the origin of the system (3.22) is not a *8*-th weak focus when $a_{21} \geq 0$.

(iii) There exist solutions for the parameters such that $R_1 = 0$, for example, when the following two conditions are satisfied:

$$a_{21} = -1, \; -112224 + 612200a_{30}^2 + 540696a_{30}^4 + 218414a_{30}^6 + 35497a_{30}^8 = 0. \tag{3.23}$$

Theorem 3.5. *When the origin of the system* (3.2) *is an* 8th*-order weak focus, by choosing proper parameter values satisfying* $0 < |\delta_1| \ll 1$ *and* $|\delta_2| \ll 1$*, the system* (3.2) *has exactly 8 small-amplitude limit cycles bifurcating in a sufficiently small neighborhood of the origin.*

Proof. When the condition (3.22) holds, we get

$$J = \frac{D(V_2, V_4, V_6, V_8, V_{10}, V_{12}, V_{14})}{D(b_{21}, a_{12}, b_{03}, \lambda, a_{03}, b_{12}, a_{21})}$$

$$= \frac{\partial V_2}{\partial b_{21}} \frac{\partial V_4}{\partial a_{12}} \frac{\partial V_6}{\partial b_{03}} \frac{\partial V_8}{\partial \lambda} \frac{\partial V_{10}}{\partial a_{03}} \begin{vmatrix} \frac{\partial V_{12}}{\partial b_{12}} & \frac{\partial V_{12}}{\partial a_{21}} \\ \frac{\partial V_{14}}{\partial b_{12}} & \frac{\partial V_{14}}{\partial a_{21}} \end{vmatrix} \quad (3.24)$$

$$= -\frac{32768 a_{30}^4 (a_{21}+b_{12})^4 \lambda^6}{1229137284375 m_6} G,$$

where

$$\begin{aligned} G = {} & 251737946112 + 12(27871575 a_{30}^{10} + 611007730 a_{30}^8 + 6092505120 a_{30}^6 \\ & + 23366640576 a_{30}^4 + 33986397952 a_{30}^2 + 13321032192) a_{21} \\ & + 4(37548225 a_{30}^8 + 889808240 a_{30}^6 + 8554415520 a_{30}^4 + 26852503808 a_{30}^2 \\ & + 26275464448) a_{21}^2 + 64(68325 a_{30}^6 + 4686530 a_{30}^4 + 52472160 a_{30}^2 \\ & + 97468576) a_{21}^3 + (337069425 a_{30}^{10} + 6551116970 a_{30}^8 + 62496457400 a_{30}^6 \\ & + 278932954944 a_{30}^4 + 588728471552 a_{30}^2 + 586211992064) a_{30}^2. \end{aligned}$$

Moreover, $\mathrm{Res}(G, R_1, a_{21})$ is a polynomial in a_{30}^2 with positive coefficients, so $J \neq 0$ when (3.22) holds. According to Theorem 3.4 and Theorem 10.3.8 in [25], system (3.2) has exactly 8 small-amplitude limit cycles in a sufficiently small neighborhood of the origin. □

4. Bifurcation of limit cycles at infinity of the system (3.2)

A very natural extension from the bifurcation of limit cycles in the system (3.2) is to study the systems with a Hopf singular point at infinity. It should be emphasized that the study of a limit cycle bifurcation from infinity is an important part of the so-called weakened Hilbert's 16th problem which explores the number and distribution of limit cycles in systems close to the integrable (i.e., having a center) ones. In this section, bifurcation of limit cycles from the infinity of system (3.2) will be considered. By means of the transformation: $u = x+iy,\ v = x-iy,\ \tau = it$, he system (3.2) can be transformed to its complex conjugate system,

$$\begin{cases} \dfrac{du}{d\tau} = \dfrac{1}{8\lambda}(-4u + v + A_{30}u^3 + A_{21}u^2v + A_{12}uv^2 d + A_{03}v^3 + 8\lambda u^3 v^2), \\ \dfrac{dv}{d\tau} = \dfrac{1}{8\lambda}(-4u + v + B_{30}v^3 + B_{21}v^2u + B_{12}vu^2 + B_{03}u^3 - 8\lambda u^2 v^3). \end{cases} \quad (4.1)$$

The system (4.1) can be further changed to

$$\begin{aligned} \frac{dz}{dT} = & \frac{z}{40\lambda}\{3\big[(a_{21} - b_{12} - a_{03} - 2) - i(a_{12} + b_{21} - a_{30} - b_{03})\big] z^7 w^3 \\ & + \big[(a_{21} + b_{12} + 11a_{03} - 22) + i(5a_{12} - 5b_{21} + 7a_{30} - 7b_{03})\big] z^6 w^4 \\ & - \big[5(a_{21} - b_{12} + 3a_{03} + 6) - i(a_{12} + b_{21} + 3a_{30} + 3b_{03})\big] z^5 w^5 \\ & - \big[(a_{21} + b_{12} - 9a_{03} + 18) + i(5a_{12} - 5b_{21} + 3a_{30} - 3b_{03})\big] z^4 w^6 \end{aligned}$$

$$
\begin{aligned}
&+2\big[(a_{21}-b_{12}-a_{03}-2)+i(a_{12}+b_{21}-a_{30}-b_{03})\big]z^3w^7\\
&+4u^9v^9(3u^2-5uv+2v^2)+40\lambda\big\},
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
\frac{dw}{dT}=&\frac{-w}{40\lambda}\big\{3\big[(a_{21}-b_{12}-a_{03}-2)+i(a_{12}+b_{21}-a_{30}-b_{03})\big]z^7w^3\\
&+\big[(a_{21}+b_{12}+11a_{03}-22)-i(5a_{12}-5b_{21}+7a_{30}-7b_{03})\big]z^6w^4\\
&-\big[5(a_{21}-b_{12}+3a_{03}+6)+i(a_{12}+b_{21}+3a_{30}+3b_{03})\big]z^5w^5\\
&-\big[(a_{21}+b_{12}-9a_{03}+18)-i(5a_{12}-5b_{21}+3a_{30}-3b_{03})\big]z^4w^6\\
&+2\big[(a_{21}-b_{12}-a_{03}-2)-i(a_{12}+b_{21}-a_{30}-b_{03})\big]z^3w^7\\
&+4u^9v^9(3v^2-5uv+2u^2)+40\lambda\big\},
\end{aligned}
$$

by the transformation: $u=\frac{z}{z^3w^3}, v=\frac{w}{z^3w^3}, T=z^{10}w^{10}\tau$.
In (4.1),

$$
\begin{aligned}
A_{30}=\overline{B_{30}}=&-(a_{21}+b_{12}-a_{03}+2)+(a_{12}-b_{21}-a_{30}+b_{03})i,\\
A_{03}=\overline{B_{03}}=&\quad(a_{21}-b_{12}-a_{03}-2)+(a_{12}+b_{21}-a_{30}-b_{03})i,\\
A_{21}=\overline{B_{21}}=&-(a_{21}-b_{12}+3a_{03}+6)-(a_{12}+b_{21}+3a_{30}+3b_{03})i,\\
A_{12}=\overline{B_{12}}=&\quad(a_{21}+b_{12}+3a_{03}-6)-(a_{12}-b_{21}+3a_{30}-3b_{03})i.
\end{aligned}
$$

The above results show that the infinity of the system (4.1) can be changed to the origin of the system (4.2), implying that the study of bifurcation of limit cycles at the infinity of the system (4.1) is equivalent to considering the bifurcation of limit cycles around the origin of the system (4.2). Simultaneous bifurcation of limit cycles from two nests of periodic orbits was considered in [14]. The problem of simultaneous bifurcation of limit cycles and critical periods for a system of polynomial differential equations in the plane was also studied in [29]. According to [17,34,24], the following results are self-evident.

The first 9 Liapunov constants of the system (4.2) can be obtained by a careful computation. The details are omitted here for simplicity. Furthermore, we have the following result.

Theorem 4.1. *The first* 8 *Liapunov constants of the system* (4.2) *are all zero if and only if one of the following two conditions holds:*

$$b_{21}=-3a_{30},\quad a_{21}=-b_{12},\quad a_{12}=-3b_{03}; \tag{4.3}$$

$$b_{21}=a_{30}=a_{12}=b_{03}=0. \tag{4.4}$$

Further, we have

Theorem 4.2. *If one of the two conditions in Theorem* 4.1 *holds, the origin of the system* (4.2) *is a center.*

Proof. If the condition (4.3) holds, then the system (4.2) can be simplified to

$$
\begin{aligned}
\frac{dz}{dT} =& \frac{z}{40\lambda}\big\{-3\big[(2b_{12}+a_{03}+2)-4i(b_{03}+a_{30})\big]z^7w^3 \\
&+11\big[(a_{03}-2)-2i(b_{03}-a_{30})\big]z^6w^4 \\
&+5(2b_{12}-3a_{03}-6)z^5w^5 \\
&+9\big[(a_{03}-2)+2i(b_{03}-a_{30})\big]z^4w^6 \\
&-2\big[(2b_{12}+a_{03}+2)+4i(b_{03}+a_{30})\big]z^3w^7 \\
&+4u^9v^9(3u^2-5uv+2v^2)+40\lambda\big\}, \\
\frac{dw}{dT} =& \frac{-w}{40\lambda}\big\{-3\big[(2b_{12}+a_{03}+2)+4i(b_{03}+a_{30})\big]z^7w^3 \\
&+11\big[(a_{03}-2)+2i(b_{03}-a_{30})\big]z^6w^4 \\
&+5(2b_{12}-3a_{03}-6)z^5w^5 \\
&+9\big[(a_{03}-2)-2i(b_{03}-a_{30})\big]z^4w^6 \\
&-2\big[(2b_{12}+a_{03}+2)-4i(b_{03}+a_{30})\big]z^3w^7 \\
&+4u^9v^9(3v^2-5uv+2u^2)+40\lambda\big\},
\end{aligned}
\tag{4.5}
$$

which has an integrating factor $\frac{1}{(zw)^{16}}$, indicating that the origin of the system (4.5) is a center.

When the condition (4.4) holds, then the system (4.2) can be rewritten as

$$
\begin{aligned}
\frac{dz}{dT} =& \frac{z}{40\lambda}\big\{3\big[(a_{21}-b_{12}-a_{03}-2)\big]z^7w^3+\big[(a_{21}+b_{12}+11a_{03}-22)\big]z^6w^4 \\
&-\big[5(a_{21}-b_{12}+3a_{03}+6)\big]z^5w^5-\big[(a_{21}+b_{12}-9a_{03}+18)\big]z^4w^6 \\
&+2\big[(a_{21}-b_{12}-a_{03}-2)\big]z^3w^7+4u^9v^9(3u^2-5uv+2v^2)+40\lambda\big\}, \\
\frac{dw}{dT} =& \frac{-w}{40\lambda}\big\{3\big[(a_{21}-b_{12}-a_{03}-2)\big]z^7w^3+\big[(a_{21}+b_{12}+11a_{03}-22)\big]z^6w^4 \\
&-\big[5(a_{21}-b_{12}+3a_{03}+6)\big]z^5w^5-\big[(a_{21}+b_{12}-9a_{03}+18)\big]z^4w^6 \\
&+2\big[(a_{21}-b_{12}-a_{03}-2)\big]z^3w^7+4u^9v^9(3v^2-5uv+2u^2)+40\lambda\big\}.
\end{aligned}
\tag{4.6}
$$

It is easy to verify that the extended symmetrical principle [27] is satisfied for system (4.6), and so the origin of the system is a center. □

Theorem 4.3. *When the origin of the system (4.2) is a 9-order weak focus, then by choosing proper parameter values, the system (4.2) has exactly 9 limit cycles bifurcating in a sufficiently small neighborhood of the origin. Namely, the system (3.2) has exactly 9 large-amplitude limit cycles at infinity.*

Furthermore, the following results can be easily acquired.

Theorem 4.4. *When the origin of the system* (3.2) *is an* 8*-order weak focus, the infinity of the system* (3.2) *is a* 1*-order weak focus, then by choosing proper parameters, the system* (4.2) *has exactly 7 limit cycles in a sufficiently small neighborhood of the origin and 1 large limit cycle at infinity. When the infinity of the system* (3.2) *is a* 9*-order weak focus, the origin of the system* (3.2) *is a* 1*-order weak focus, then by choosing proper parameters, the system* (4.2) *has exactly 8 large limit cycles at infinity and 1 limit cycle in a sufficiently small neighborhood of the origin.*

5. Conclusion

In this article, we have discussed the problem of computing the singular point values of Z_2-equivariant systems with a nilpotent singular point. We have developed a new computation method to avoid complex integration calculation with only algebraic calculations, which makes the computations more efficient. For a specific class of 5-th order Z_2-equivariant systems, we calculate the focal values at both the origin and infinity and discuss in detail the limit cycle bifurcation problem. The method developed in this paper can be applied to study other synchronized bifurcation problems at both the origin and infinity.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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