Communications in Nonlinear Science and Numerical Simulation 120 (2023) 107167

Contents lists available at ScienceDirect

Communications in Nonlinear Science and Numerical Simulation

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Complete classification on center of cubic planar systems symmetric with respect to a straight line

Feng Li [a], Yusen Wu [b], Pei Yu [c,*]

[a] School of Mathematics and Statistics, Linyi University, Linyi, Shandong, 276005, China
[b] School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang, Henan, 471023, China
[c] Department of Mathematics, Western University, London, Ontario, Canada, N6A 5B7

ARTICLE INFO

Article history:
Received 5 October 2022
Received in revised form 7 February 2023
Accepted 8 February 2023
Available online 11 February 2023

Keywords:
Center-focus problem
Bi-center
Bi-isochronous center
Periodic constant
Linearization

ABSTRACT

In this paper, we study bi-center and bi-isochronous center problems in cubic planar systems which are symmetric with respect to a straight line. These systems can be transformed to ones which are symmetric with respect to the y-axis and have two symmetric singular points at $(\pm 1, 0)$, which can be classified as elementary and nilpotent singular points. A complete classification is given on the centers, including nine conditions for elementary singular points and four conditions for nilpotent singular points. Moreover, seven bi-isochronous center conditions are obtained for the elementary singular points.
© 2023 Elsevier B.V. All rights reserved.

1. Introduction

Center and isochronous center problems, which are closely related to the 16th problem proposed by D. Hilbert in 1900, are far from being solved. As a classical problem, center problem has been considered by many mathematicians. Some special systems have been investigated, for example, the systems given in the form of

$$\begin{aligned} \frac{dx}{dt} &= y + P_m(x, y), \\ \frac{dy}{dt} &= -x + Q_m(x, y), \end{aligned} \tag{1.1}$$

where $P_m(x, y)$ and $Q_m(x, y)$ are homogeneous polynomials of degree m. The three cases $m = 3, 4, 5$ have been studied in [1–3], respectively. The characterization of the centers for cubic systems described by

$$\begin{aligned} \frac{dx}{dt} &= y + P_2(x, y) + P_3(x, y), \\ \frac{dy}{dt} &= -x + Q_2(x, y) + Q_3(x, y), \end{aligned} \tag{1.2}$$

is not well investigated. Only some special cases with degenerate singular points at infinity have been completely characterized [4]. The center problem for systems with nilpotent singular points is more complicated because the classical methods used for studying elementary singular points are not applicable. Especially, a nilpotent center may not be an

* Corresponding author.
E-mail addresses: lf0539@126.com (F. Li), wuyusen621@126.com (Y. Wu), pyu@uwo.ca (P. Yu).

https://doi.org/10.1016/j.cnsns.2023.107167
1007-5704/© 2023 Elsevier B.V. All rights reserved.

analytical center, which has no analytical first integral. This implies that the center problem for degenerate singular points is much more difficult. Partial results have been obtained for the systems with null linear part, see for instance [5].

For analytical Liénard systems whose linear part has a pair of purely imaginary eigenvalues, an effective method was developed to derive necessary and sufficient conditions for the existence of centers, see [6–8]. The isochronous center problem is equivalent to determining whether the system can be transformed to a linear system by a formal change of the state variables. Over the past three decades, this problem has also been intensively investigated by many researchers, for example, see [9,10]. Periodic constants can be used to determine isochronous centers, and several methods have been developed to compute periodic constants, see [11–15]. However, these methods are hard to be applied to a concrete system because of the computation complexity. As far as we know, there are only a few complete classifications about isochronous centers for some special systems, see for instance [11,12,14,16–20]. On the other hand, the isochronous center problem for the homogeneous polynomial systems was considered and solved thoroughly. In 1964, Loud [17] classified the isochronous centers of systems with homogeneous polynomials of degree two. The isochronous center problem for systems whose nonlinear parts are homogeneous polynomials of degree three were solved by Pleshkan [18]. The results for the isochronous centers of systems with homogeneous perturbations of degrees four and five can be found in [11,12]. Linearization of linear systems perturbed by 5th-degree homogeneous polynomials was studied in [20].

As a class of special systems, the time-reversible systems were also studied by many researchers, for instance, the time-reversible cubic vector fields were investigated in [21,22]. More recently, the linearizability conditions of time-reversible quartic systems with homogeneous nonlinearities have been obtained in [23]. For the complex Lotka–Volterra system, the linearizability problem was solved in [24]. Some other methods have been developed in recent years, for example, the time-angle method was proposed in [25], which can also be found in the book [26]. For some planar polynomial Hamiltonian systems, the isochronicity and linearizability were also studied in [27,28].

The reason for studying the time-reversible systems is due to the existence of symmetry, which plays an important role in the qualitative analysis. Breaking of symmetry destroys the underlying order of nature. Another important kind of symmetric systems is the Z_n-equivariant system, with many good results obtained about the center and isochronous center problems. Recently, a complete study on the bi-center problem for Z_2-equivariant cubic vector fields was given in [29], and the bi-center problem for some Z_2-equivariant quintic systems was studied in [30]. In 2017, the bi-isochronous center problem for cubic systems in Z_2-equivariant vector fields with real coefficients was considered in [31]. In 2020, the isochronous center problem for the Z_2-equivariant cubic vector fields with complex coefficients are completely solved [32]. The Z_2-equivariant cubic vector fields with nilpotent singular point, weak saddles or resonant saddles were studied in [32–35], while the Z_2-equivariant cubic polynomial Hamiltonian systems with bi-center were investigated in [36].

Besides above mentioned symmetries, other types of symmetry are also important, and some of such systems have been considered. For example, systems with the y-axis symmetry were studied in [37], and the planar cubic differential systems with symmetric centers were investigated in [38]. In this paper, we will study cubic systems with the y-axis symmetry, and focus on the bi-center and bi-isochronous center problems with the main attention paid to the elementary and nilpotent singular points. We will provide a complete classification on the centers, including nine conditions for elementary singular points and four conditions for nilpotent singular points. Moreover, we derive six bi-isochronous center conditions for the elementary singular points.

The rest of the paper is organized as follows. In the next section, we simplify cubic systems symmetric with respect to a straight line and derive the condition possessing two elementary singular points or nilpotent singular points. In Section 3, nine cases are classified for the center conditions of the cubic systems with two elementary singular points. Then, the periodic constants are computed at $(\pm 1, 0)$ of cubic systems, which are used to obtain six isochronous center conditions. In Section 4, four cases are classified for the center conditions of cubic systems with two nilpotent singular points. Finally, conclusion is drawn in Section 5.

2. Simplification of cubic systems symmetric with respect to a straight line

A system is symmetric with respect to a straight line if the phase portrait is symmetric with respect to the straight line, and the straight line is called the axis of symmetry. Especially, a system is called symmetric with respect to the y-axis if the system is invariant under the transformation $(x, y, t) \to (-x, y, t)$. In this context, we consider cubic systems which are symmetric with respect to a straight line, and assume that $P_0 = (x_0, y_0)$ is a singular point which is not on this line of symmetry. If P_0 is a center, and $P_1 = (x_1, y_1)$ is the image of P_0 under the symmetric transformation, then P_1 is also a center. Then, the points P_0 and P_1 are called bi-center.

To investigate the existence of bi-center or bi-isochronous center for the cubic systems which possesses symmetry with respect to a straight line, we first find the normal form of the system using the symmetry. We have the following result.

Theorem 2.1. *Consider planar polynomial cubic systems with*

(1) *a singular point associated with a pair of purely imaginary eigenvalues $\pm mi$; and*
(2) *a straight line as the axis of symmetry, which does not include the singular point in* (1)*.*

Then there exists an affine transformation of the coordinates such that the straight line as the axis of symmetry is taken into the y-axis and the cubic systems can be rewritten in the following form:

$$\begin{aligned}\frac{dx}{dt} &= \frac{-x}{2b_3}(-2a_1b_3 + 2a_1b_3x^2 + 4a_1^2y + m^2y - 2a_8b_3y^2),\\ \frac{dy}{dt} &= -b_3 + b_3x^2 + (2a_1 - b_7)y + b_7x^2y + b_5y^2 + b_9y^3.\end{aligned} \tag{2.1}$$

Proof. Let (x_0, y_0) be a singular point of a symmetric cubic system with respect to a straight line such that its Jacobian matrix has a pair of purely imaginary eigenvalues. Assume that the cubic system possesses a straight line r: $ax + by = c$ $(a,\ b,\ c \in R)$ as the axis of symmetry and (x_0, y_0) does not belong to r. Without loss of generality, we may assume $a \neq 0$ (otherwise we simply apply the change of the coordinates $(x, y) \to (y, x)$). According to the results in [37], the straight line r becomes the y-axis (i.e., $x = 0$) and the singular point (x_0, y_0) is moved to the point $(1, 0)$ by a transformation. So, we can always assume that $(1, 0)$ is a singular point. Thus, with the symmetry, the system is changed to

$$\begin{aligned}\frac{dx}{dt} &= -x(-a_1 + a_1x^2 - a_4y - a_8y^2),\\ \frac{dy}{dt} &= -b_3 + b_3x^2 + b_2y + b_7x^2y + b_5y^2 + b_9y^3,\end{aligned}$$

whose Jacobian matrix is given by

$$J_{(1,0)} = \begin{bmatrix} -2a_1 & a_4 \\ 2b_3 & b_2 + b_7 \end{bmatrix}.$$

It is easy to see that $b_3 \neq 0$. Otherwise, according to condition (1), $b_3 = 0$ yields that $b_2 + b_7 = 2a_1$, leading to

$$\det(J_{(1,0)}) = -2a_1(b_2 + b_7) = -4a_1^2,$$

which contradicts with the assumption, $\det(J_{(1,0)}) = m^2$. So we get

$$b_2 = 2a_1 - b_7, \quad a_4 = -\frac{(4a_1^2 + m^2)}{2b_3}$$

from $\mathrm{Tr}(J_{(1,0)}) = 0$ and $\det(J_{(1,0)}) = m^2$, namely, system is changed to the normal form (2.1). □

Similarly, we can prove the following theorem.

Theorem 2.2. *Consider planar polynomial cubic systems with*

(1) *a singular point associated with a double-zero eigenvalue; and*
(2) *a straight line as the axis of symmetry which does not include the singular point in* (1)*.*

Then, there exists an affine transformation of the coordinates such that the straight line is taken into the y-axis, and the cubic systems can be rewritten as

$$\begin{aligned}\frac{dx}{dt} &= -\frac{x}{b_3}(-a_1b_3 + a_1b_3x^2 + 2a_1^2y - a_8b_3y^2),\\ \frac{dy}{dt} &= -b_3 + b_3x^2 + (2a_1 - b_7)y + b_7x^2y + b_5y^2 + b_9y^3.\end{aligned} \tag{2.2}$$

In order to prove center conditions, we introduce the results obtained in [39], where the so-called Liénard-like systems with a degenerate singular point are investigated. The Liénard-like systems are described by

$$\begin{aligned}\frac{dx}{dt} &= y,\\ \frac{dy}{dt} &= p_0(x) + p_1(x)y + p_2(x)y^2,\end{aligned} \tag{2.3}$$

where

$$p_0(x) = -x^{2n-1} + \sum_{k=2n}^{\infty} a_kx^k, \quad n \geq 1,$$

$$p_1(x) = Ax^{n-1} + \sum_{k=n}^{\infty} b_kx^k,$$

$$p_2(x) = \sum_{k=0}^{\infty} c_kx^k.$$

The following result for system (2.3) is usually applied to prove center conditions.

Theorem 2.3 ([39]). *The origin of system (2.3) is a centre if and only if the equations,*

$$W_1(x) = W_1(y) \quad \text{and} \quad W_2(x) = W_2(y),$$

where

$$W_1(x) = \frac{p_0p_1p_2 - p_1p_0' + p_0p_1'}{p_1^3}, \quad W_2(x) = \frac{W_1'(x)p_0}{p_1^2},$$

have an analytical solution $y = \varphi(x)$ with $\varphi(0) = 0$ and $\varphi'(0) = -1$ in the neighborhood of $x = 0$.

It should be pointed out that the conclusion in Theorem 2.3 is also true for the case when $W_1(x)$ or $W_2(x)$ is a constant, which is not included in the result of [39].

3. Bi-center and bi-isochronous center of system (2.1)

For system (2.1), there are two elementary singular points $(\pm 1, 0)$ associated with a pair of purely imaginary eigenvalues $\pm mi$. We need only to study the center problem at the singular point $(1, 0)$ because of the symmetry. System (2.1) can be transformed into

$$\begin{aligned} \frac{dx}{dt} &= -\frac{x+1}{2b_3}(4a_1b_3x + 2a_1b_3x^2 + 4a_1^2y + m^2y - 2a_8b_3y^2), \\ \frac{dy}{dt} &= 2b_3x + b_3x^2 + 2a_1y + 2b_7xy + b_7x^2y + b_5y^2 + b_9y^3, \end{aligned} \tag{3.1}$$

by $x = \bar{x} + 1$, where we still use x for $\bar{x}$ for convenience. As a result, the singular point $(1, 0)$ of system (2.1) is shifted to the origin of system (3.1). Furthermore, under the transformation,

$$u = -\frac{y}{2b_3}, \quad v = \frac{x}{m} + \frac{a_1y}{mb_3}, \quad \tau = mt,$$

system (3.1) can be changed to

$$\begin{aligned} \frac{dx}{d\tau} &= (1 + 2a_1x)(-1 + 2b_7x)y + \frac{1}{2}m(-1 + 2b_7x)y^2 \\ &\quad + \frac{2}{m}x^2(2a_1b_7 + a_1^2(-1 + 2b_7x) + b_3(-b_5 + 2b_3b_9x)), \\ \frac{dy}{d\tau} &= x - \frac{8a_1(a_1^2(a_1 + b_7) + b_3^2(-a_8 + b_9))x^3}{m^2} - \frac{4a_1(a_1 + b_7)xy}{m} \\ &\quad + \frac{2x^2[2a_8b_3^2(1 + my) + a_1(2b_3b_5 - 4a_1b_7 + m^2 - 2a_1(3a_1 + 2b_7)my)]}{m^2} \\ &\quad + xy(m - 2a_1(3a_1 + b_7)y) - a_1y^2(2 + my). \end{aligned} \tag{3.2}$$

Now we are ready to derive the center and isochronous center conditions for system (3.2).

3.1. Bi-center conditions of system (2.1)

By the complex transformation, $z = x + iy$, $w = x - iy$, $\tau = iT$, system (3.2) can be changed to its complex concomitant system. Then, computing and analyzing the focus values at the origin of the resulting system with the formal series method developed in [26] yields the following result.

Theorem 3.1. *The first three focus values at the origin of system (3.2) are*

$$\begin{aligned} \mu_1 &= \frac{i}{m^3}\{4(2a_1^2 + b_3b_5 - a_1b_7)(-b_3(a_8b_3 + a_1b_5) + 2a_1^2b_7) \\ &\qquad + [(2a_1 - b_7)(-b_3b_5 + 2a_1b_7) - 3b_3^2b_9]m^2\}, \\ \mu_2 &= \frac{4i}{9m^7}(-10a_8b_3^2 - 10a_1b_3b_5 + 20a_1^2b_7 + 4a_1m^2 + b_7m^2)f_1f_2, \\ \mu_3 &= \frac{8i}{225m^5}(a_1 - b_7)(4a_1 + b_7)(2a_1 + 3b_7)f_1f_2, \end{aligned}$$

where

$$\begin{aligned} f_1 =& \ 8a_1^3 + 4a_1b_3b_5 - 4a_1^2b_7 + 2a_1m^2 - b_7m^2, \\ f_2 =& -4a_1^2a_8b_3^2 - 4a_1^3b_3b_5 - 2a_8b_3^3b_5 - 2a_1b_3^2b_5^2 + 8a_1^4b_7 \\ &+ 2a_1a_8b_3^2b_7 + 6a_1^2b_3b_5b_7 - 4a_1^3b_7^2 - a_1b_3b_5m^2 \\ &+ 2a_1^2b_7m^2 - b_3b_5b_7m^2 + 2a_1b_7^2m^2. \end{aligned}$$

The following two theorems directly follow from Theorem 3.1.

Theorem 3.2. *All the first three focus values at the origin of system (3.2) vanish if and only if one of the following nine conditions holds:*

$$\begin{aligned} &C_1: \ b_9 = -\frac{b_5b_7}{b_3}, \quad a_8 = -\frac{b_7(2b_3b_5 - 4b_7^2 - m^2)}{2b_3^2}, \quad a_1 = b_7; \\ &C_2: \ b_9 = \frac{b_7(2b_3b_5 + b_7^2)}{4b_3^2}, \quad a_8 = \frac{b_7(2b_3b_5 + b_7^2 - m^2)}{8b_3^2}, \quad a_1 = -\frac{b_7}{4}; \\ &C_3: \ b_9 = -\frac{2b_7(2b_3b_5 + 4b_7^2)}{b_3^2}, \quad a_8 = \frac{b_7(3b_3b_5 + 9b_7^2 - m^2)}{2b_3^2}, \quad a_1 = -\frac{3b_7}{2}; \\ &C_4: \ b_9 = \frac{b_7(b_3b_5 - 2a_1b_7)}{b_3^2}, \\ &\qquad a_8 = -\frac{(b_3b_5 - 2a_1b_7)(4a_1^3 + 2a_1b_3b_5 + 2a_1^2b_7 + a_1m^2 + b_7m^2)}{2b_3^2(2a_1^2 + b_3b_5 - a_1b_7)}; \\ &C_5: \ b_9 = -\frac{b_7^3}{b_3^2}, \quad b_5 = -\frac{3b_7^2}{b_3}, \quad a_1 = -b_7; \\ &C_6: \ b_9 = a_1 = b_5 = 0; \\ &C_7: \ b_9 = 0, \quad b_5 = -\frac{b_7^2}{b_3}, \quad a_1 = -\frac{b_7}{2}; \\ &C_8: \ b_9 = \frac{a_8(2a_1 - b_7)}{3a_1}, \quad b_5 = -\frac{(2a_1 - b_7)(4a_1^2 + m^2)}{4a_1b_3}; \\ &C_9: \ b_9 = -\frac{4a_8b_3b_5}{3m^2}, \quad a_1 = b_7 = 0. \end{aligned}$$

Theorem 3.2 gives the necessary conditions under which the origin of system (3.2) is a center. Next, we use Theorem 2.3 to prove that these conditions are also sufficient.

Theorem 3.3. *The origin of system (3.2) is a center if and only if one of the nine conditions in Theorem 3.2 holds.*

Proof. When the condition C_1 in Theorem 3.2 holds, system (3.2) becomes

$$\begin{aligned} \frac{dx}{d\tau} =& \ \frac{1}{2m}\Big\{-4b_3b_5x^2 + 8b_3^2b_9x^3 - 2\,my \\ &\quad + (2b_7x + my)[-my + 2b_7x(1 + 2b_7x + my)]\Big\}, \\ \frac{dy}{d\tau} =& \ x(1+2b_7x)^2 - 2b_7(1+4b_7x)y^2 + my(x + 2b_7x^2 - b_7y^2), \\ &- \frac{4b_7x(b_3b_5x + b_7(2 + 3b_7x))y}{m}. \end{aligned} \tag{3.3}$$

When $b_7 \neq 0$, system (3.3) has an integral factor

$$I_1 = g_1^{\frac{4b_7^2+m^2-4b_3b_5}{4b_7^2+m^2}}\, g_2^{\frac{4b_7^2+m^2-4b_3b_5}{2(4b_7^2+m^2)}},$$

in the neighborhood of the origin, where

$$\begin{aligned} g_1 &= 1 + 2b_7x + my, \\ g_2 &= -\frac{m^2(1+2b_7x)^2}{b_7^2} + 8m(1+2b_7x)y + 4m^2y^2. \end{aligned}$$

When $b_7=0$, system (3.3) becomes

$$\frac{dx}{d\tau}=-\frac{1}{2m}(4b_3b_5x^2+2\,my+m^2y^2),\quad \frac{dy}{d\tau}=x(1+my),$$

which is time-reversible, implying that the origin is a center.

When the condition C_2 in Theorem 3.2 holds, system (3.2) is reduced to

$$\begin{aligned}\frac{du}{dt}&=\frac{u+1}{4b_3^2}(2b_3^2b_7u-b_3b_7^2v-4b_3m^2v+2b_3b_5b_7v^2+b_7^3v^2),\\ \frac{dv}{dt}&=\frac{1}{4b_3^2}(4b_3^3u-2b_3^2b_7v+4b_3^2b_7uv+4b_3^2b_5v^2+2b_3b_5b_7v^3+b_7^3v^3),\end{aligned}$$

by the transformation $u=x^2-1$, $v=y$, and can be further changed to

$$\begin{aligned}\frac{dx_1}{d\tau}&=\frac{1}{4b_3^2m}(4b_3^2b_5x_1^2+2b_3b_7^2x_1^2+2b_3b_5b_7x_1^3+b_7^3x_1^3\\&\qquad+4b_3^2\,my_1+4b_3b_7\,mx_1y_1),\\ \frac{dy_1}{d\tau}&=-\frac{1}{4b_3^2m}(4b_3^2\,mx_1+2b_3b_7\,mx_1^2+b_3b_7^2x_1y_1+4b_3m^2x_1y_1\\&\qquad-2b_3b_5b_7x_1^2y_1-b_7^3x_1^2y_1-2b_3b_7\,my_1^2).\end{aligned}\tag{3.4}$$

by using $x_1=v$, $y_1=\frac{b_3}{m}u-\frac{b_7}{2m}v$, $\tau=mt$. Moreover, with the transformation,

$$X=x_1,\quad Y=\frac{1}{4b_3^2m}(4b_3^2b_5x_1^2+2b_3b_7^2x_1^2+2b_3b_5b_7x_1^3+b_7^3x_1^3+4b_3^2\,my_1+4b_3b_7\,mx_1y_1),$$

system (3.4) can be changed to the following Liénard-like system,

$$\frac{dX}{d\tau}=Y,\quad \frac{dY}{d\tau}=p_0(X)+p_1(X)Y+p_2(X)Y^2,\tag{3.5}$$

where

$$\begin{aligned}p_0(X)&=-\frac{X(2b_3+b_7X)}{32b_3^4m^2(b_3+b_7X)}(2b_3^2+2b_3b_7X-2b_3b_5X^2-b_7^2X^2)\\&\quad\times(8b_3^2m^2+8b_3b_7m^2X-2b_3b_5b_7^2X^2-b_7^4X^2),\\ p_1(X)&=\frac{X}{4b_3^2\,m(b_3+b_7X)}(8b_3^3b_5+23b_3^2b_7^2-4b_3^2m^2+26b_3^2b_5b_7X+116b_3b_7^3X\\&\qquad-4b_3b_7m^2X+22b_3b_5b_7^2X^2+95b_7^4X^2),\\ p_3(X)&=\frac{3b_7}{2(b_3+b_7X)}.\end{aligned}$$

Let

$$W_1(X)=\frac{p_0(X)p_1(X)p_2(X)-p_1(X)p_0'(X)+p_0(X)p_1'(X)}{p_1(X)^3},\quad W_2(X)=\frac{W_1'(X)p_0(X)}{p_1(X)^2}.$$

It is easy to verify that $W_1(X)-W_1(Y)=0$ and $W_2(X)-W_2(Y)=0$ have a solution $b_3X+b_3Y+b_7XY=0$. According to Theorem 2.3, the origin of system (3.2) is a center.

When the condition C_3 in Theorem 3.2 holds, system (3.2) is reduced to

$$\begin{aligned}\frac{du}{d\tau}&=\frac{u+1}{b_3^2}(3b_3^2b_7u-9b_3b_7^2v-b_3m^2v+3b_3b_5b_7v^2+9b_7^3v^2-b_7m^2v^2),\\ \frac{dv}{d\tau}&=\frac{1}{b_3^2}(b_3^3u-3b_3^2b_7v+b_3^2b_7uv+b_3^2b_5v^2+2b_3b_5b_7v^3+8b_7^3v^3),\end{aligned}$$

by the transformation $u = x^2 - 1$, $v = y$, and can be further changed to

$$\begin{aligned}\frac{dx_1}{d\tau} &= \frac{1}{b_3^2 m}(b_3^2 b_5 x_1^2 + 3b_3 b_7^2 x_1^2 + 2b_3 b_5 b_7 x_1^3 + 8b_7^3 x_1^3 \\ &\quad + b_3^2\, my_1 + b_3 b_7\, mx_1 y_1), \\ \frac{dy_1}{d\tau} &= -\frac{1}{b_3^2 m^2}(b_3^2\, mx_1 + 4b_3 b_7 m^2 x_1^2 - 3b_3 b_5 b_7^2 x_1^3 - 3b_7^4 x_1^3 \\ &\quad + 3b_7^2 m^2 x_1^3 - 6b_3 b_7^2\, mx_1 y_1 + b_3 m^3 x_1 y_1 - 3b_3 b_5 b_7 x_1^2 y_1 \\ &\quad - 9b_7^3\, mx_1^2 y_1 + b_7 m^3 x_1^2 y_1 - 3b_3 b_7 m^2 y_1^2),\end{aligned} \tag{3.6}$$

by using $x_1 = v$, $y_1 = \frac{b_3}{m}u - \frac{3b_7}{m}v$, $\tau = mt$. System (3.6) can be changed to the Liénard-like system (3.5) with

$$\begin{aligned}p_0(X) &= -\frac{X(b_3 + 2b_7 X)}{b_3^4 m^2 (b_3 + b_7 X)}(b_3^2 + 2b_3 b_7 X - b_3 b_5 X^2 - 4b_7^2 X^2) \\ &\quad \times (b_3^2 m^2 + 2b_3 b_7 m^2 X + 3b_3 b_5 b_7^2 X^2 + 15b_7^4 X^2 + b_7^2 m^2 X^2), \\ p_1(X) &= \frac{X}{b_3^2\, m(b_3 + b_7 X)}(2b_3^3 b_5 + 12b_3^2 b_7^2 - b_3^2 m^2 + 4b_3^2 b_5 b_7 X + 24b_3 b_7^3 X \\ &\quad - 2b_3 b_7 m^2 X - 5b_3 b_5 b_7^2 X^2 - 23b_7^4 X^2 - b_7^2 m^2 X^2), \\ p_3(X) &= \frac{4b_7}{2(b_3 + b_7 X)},\end{aligned}$$

by the transformation,

$$\begin{aligned}X &= x_1, \\ Y &= \frac{1}{b_3^2 m}(b_3^2 b_5 x_1^2 + 3b_3 b_7^2 x_1^2 + 2b_3 b_5 b_7 x_1^3 + 8b_7^3 x_1^3 + b_3^2\, my_1 + b_3 b_7\, mx_1 y_1).\end{aligned}$$

Note that $W_1(X) - W_1(Y)$ and $W_2(X) - W_2(Y)$ have a common factor $(X - Y)(b_3 X + b_3 Y + 2b_7 XY)$. According to Theorem 2.3, the origin of system (3.2) is a center.

When the condition C_4 in Theorem 3.2 holds, system (3.2) has an integrating factor,

$$I_2 = x^{1+\frac{-4b_3 b_5 + 4a_1(-2a_1 + b_7)}{m^2}}(b_3 + b_7 y)^{\frac{-8a_1^3 + 4a_1^2 b_7 + b_7 m^2 - 2a_1(2b_3 b_5 + m^2)}{b_7 m^2}}$$

if $b_7 \neq 0$. In fact, for any real b_7, system (3.2) can be changed to

$$\begin{aligned}\frac{dx_1}{d\tau} &= \frac{1}{b_3^2 m}(b_3 + b_7 x_1)(b_3 b_5 x_1^2 - 2a_1 b_7 x_1^2 + b_3\, my_1), \\ \frac{dy_1}{d\tau} &= -\frac{b_3^2 m^2 (2a_1^2 + b_3 b_5 - a_1 b_7) x_1 + P(x_1, y_1)}{b_3^2 m^2 (2a_1^2 + b_3 b_5 - a_1 b_7)},\end{aligned} \tag{3.7}$$

by using $x_1 = y$, $y_1 = \frac{b_3}{m}(x^2 - 1) - \frac{2a_1}{m}y$, $\tau = mt$, where

$$\begin{aligned}P(x_1, y_1) &= (-4a_1^3 b_3 m^2 - a_1 b_3^2 b_5 m^2 + b_3^2 b_5 b_7 m^2 - 2a_1 b_3 b_7^2 m^2) x_1^2 \\ &\quad + (16a_1^5 b_7 - 8a_1^4 b_3 b_5 - 4a_1^2 b_3^2 b_5^2 + 8a_1^3 b_3 b_5 b_7 - 2a_1 b_3^2 b_5^2 b_7 + 6a_1^2 b_3 b_5 b_7^2 \\ &\quad - 4a_1^3 b_7^3 - 2a_1^2 b_3 b_5 m^2 + 4a_1^3 b_7 m^2 - 2a_1 b_3 b_5 b_7 m^2 + 4a_1^2 b_7^2 m^2) x_1^3 \\ &\quad + (-8a_1^4 b_3 m - 4a_1^2 b_3^2 b_5\, m - 2a_1 b_3^2 b_5 b_7 m + 2a_1^2 b_3 b_7^2\, m + 2a_1^2 b_3 m^3 \\ &\quad + b_3^2 b_5 m^3 - a_1 b_3 b_7 m^3) x_1 y_1 + (4a_1^3 b_3 b_5\, m + 2a_1 b_3^2 b_5^2 m - 8a_1^4 b_7 m \\ &\quad - 6a_1^2 b_3 b_5 b_7 m + 4a_1^3 b_7^2 m + a_1 b_3 b_5 m^3 - 2a_1^2 b_7 m^3 + b_3 b_5 b_7 m^3 \\ &\quad - 2a_1 b_7^2 m^3) x_1^2 y_1 + (4a_1^3 b_3 m^2 + 2a_1 b_3^2 b_5 m^2 - 2a_1^2 b_3 b_7 m^2) y_1^2.\end{aligned}$$

Then, by the transformation,

$$\begin{aligned}X &= x_1, \\ Y &= \frac{1}{b_3^2 m}(b_3 + b_7 x_1)(b_3 b_5 x_1^2 - 2a_1 b_7 x_1^2 + b_3\, my_1),\end{aligned}$$

system (3.7) can be changed to the Liénard-like system (3.5), where

$$
\begin{aligned}
p_0(X) = &\frac{X(b_3+2b_7X)}{b_3^4(2a_1^2+b_3b_5-a_1b_7)}(-b_3^2+2b_3a_1X+b_3b_5X^2-2a_1b_7X^2)\\
&\times(2a_1^2b_3+b_3^2b_5-a_1b_3b_7+a_1b_3b_5X-2a_1^2b_7X+b_3b_5b_7X-2a_1b_7^2X),\\
p_1(X) = &\frac{X(4a_1^2+2b_3b_5-2a_1b_7-m^2)}{b_3^2m(2a_1^2+b_3b_5-a_1b_7)}(2a_1^2b_3+b_3^2b_5-a_1b_3b_7+a_1b_3b_5X\\
&-2a_1^2b_7X+b_3b_5b_7X-2a_1b_7^2X),\\
p_3(X) = &\frac{2a_1-b_7}{2(b_3+b_7X)}.
\end{aligned}
$$

If $4a_1^2+2b_3b_5-2a_1b_7-m^2=0$, the Liénard-like system is symmetric with the X-axis. Otherwise, after a tedious computation, we obtain

$$
W_1(X) = -\frac{2(2a_1^2+b_3b_5-a_1b_7)m^2}{(4a_1^2+2b_3b_5-2a_1b_7-m^2)^2},
$$

which is a constant, implying that the origin is a center.

When the conditions C_5, C_6 and C_7 in Theorem 3.2 hold, system (3.2) can be reduced to

$$
\begin{cases}
\dfrac{dx}{dt} = \dfrac{1}{2b_3}xy(-m^2+2a_8b_3y),\\
\dfrac{dy}{dt} = (x^2-1)(b_3+b_7y),
\end{cases}
\quad \text{for } C_5,
$$
$$
\begin{cases}
\dfrac{dx}{dt} = \dfrac{1}{2b_3}x(-2b_3b_7+2b_3b_7x^2-4b_7^2y-m^2y+2a_8b_3y^2),\\
\dfrac{dy}{dt} = \dfrac{1}{b_3^2}(-b_3+b_3x-b_7y)(b_3+b_7y)(b_3+b_3x+b_7y).
\end{cases}
\quad \text{for } C_6,
$$
$$
\begin{cases}
\dfrac{dx}{dt} = \dfrac{x}{2b_3}(-b_3b_7+b_3b_7x^2-b_7^2y-m^2y+2a_8b_3y^2),\\
\dfrac{dy}{dt} = \dfrac{1}{b_3}(-b_3+b_3x^2-b_7y)(b_3+b_7y),
\end{cases}
\quad \text{for } C_7,
$$

which admit the following inverse integrating factors:

$$
I_5 = x(b_3+b_7y), \quad I_6 = x(b_3+b_7y)^3 \quad \text{and} \quad I_7 = x(b_3+b_7y)^2,
$$

for C_5, C_6 and C_7, respectively, indicating that the origin of system (3.2) is a center under each of these three conditions.

When the condition C_8 in Theorem 3.2 holds, system (3.2) can be changed to

$$
\begin{aligned}
\frac{dx_1}{d\tau} = &-\frac{1}{12a_1b_3m}(24a_1^3x_1^2+12a_1^2b_7x_1^2+6a_1m^2x_1^2-3b_7m^2x_1^2\\
&-8a_1a_8b_3x_1^3+4a_8b_3b_7x_1^3-12a_1b_3\,my_1-12a_1b_7\,mx_1y_1),\\
\frac{dy_1}{d\tau} = &\frac{-1}{6b_3m^2}(6b_3m^2x_1+24a_1^3x_1^2-12a_8b_3^2x_1^2+12a_1^2b_7x_1^2-6a_1m^2x_1^2\\
&-3b_7m^2x_1^2+16a_1a_8b_3x_1^3+4a_8b_3b_7x_1^3-24a_1^2mx_1y_1\\
&-12a_1b_7\,mx_1y_1+6m^3x_1y_1-12a_8b_3\,mx_1^2y_1+12a_1m^2y_1^2).
\end{aligned}
\tag{3.8}
$$

by using $x_1=y$, $y_1=\frac{b_3}{m}(x^2-1)-\frac{b_7}{m}y$, $\tau=mt$. Applying the same method with the transformation,

$$
\begin{aligned}
X = &\,x_1,\\
Y = &-\frac{1}{12a_1b_3m}(24a_1^3x_1^2+12a_1^2b_7x_1^2+6a_1m^2x_1^2-3b_7m^2x_1^2\\
&-8a_1a_8b_3x_1^3+4a_8b_3b_7x_1^3-12a_1b_3\,my_1-12a_1b_7\,mx_1y_1),
\end{aligned}
$$

we can change system (3.8) to the Liénard-like system (3.5) with

$$\begin{aligned}
p_0(X) = {} & \frac{X}{72a_1b_3^2m^2(b_3+b_7X)}(-6b_3m^2-24a_1^3X+12a_8b_3^2X-12a_1^2b_7X \\
& -6a_1m^2X-3b_7m^2X+8a_1a_8b_3X^2+8a_8b_3b_7X^2)(12a_1b_3^2-24a_1^2b_3X \\
& +12a_1b_3b_7X+24a_1^3X^2-12a_1^2b_7X^2+6a_1m^2X^2-3b_7m^2X^2 \\
& -8a_1a_8b_3X^3+4a_8b_3b_7X^3), \\
p_1(X) = {} & \frac{(-4a_1+b_7)X}{12a_1b_3m(b_3+b_7X)}(6b_3m^2+24a_1^3X-12a_8b_3^2X+12a_1^2b_7X \\
& +6a_1m^2X+3b_7m^2X-8a_1a_8b_3X^2-8a_8b_3b_7X^2), \\
p_3(X) = {} & \frac{2a_1-b_7}{b_3+b_7X}.
\end{aligned}$$

If $b_7=4a_1$, the Liénard-like system is symmetric with the X-axis. Otherwise, a simple computation shows that

$$W_1(X) = -\frac{2a_1(2a_1-b_7)}{(-4a_1+b_7)^2},$$

which is a constant, implying that the origin of system (3.2) is a center according to Theorem 2.3.

When the condition C_9 in Theorem 3.2 holds, system (3.2) is reduced to

$$\begin{aligned}
\frac{dx}{dt} &= \frac{xy}{2b_3}(-m^2+2a_8b_3y), \\
\frac{dy}{dt} &= \frac{1}{3m^2}(-3b_3m^2+3b_3m^2x^2+3b_5m^2y^2-4a_8b_3b_5y^3),
\end{aligned}$$

which admits an integrating factor $I_8 = x^{\frac{4b_3b_5-m^2}{m^2}}$, showing that the origin of system (3.2) is a center.

The proof for Theorem 3.3 is complete. □

3.2. Bi-isochronous center conditions of system (2.1)

For each case listed in Theorem 3.2, we compute and analyse the periodic constants at the origin of system (3.2) to obtain the following Lemmas.

Lemma 3.1. *If C_1 in Theorem 3.2 holds, then system (3.2) has an isochronous center at the origin if and only if one of the following conditions holds:*

$$\begin{aligned}
L_1: &\quad a_1=0,\quad a_8=0,\quad b_5=-\frac{m^2}{b_3},\quad b_7=0,\quad b_9=0; \\
L_2: &\quad a_1=0,\quad a_8=0,\quad b_5=-\frac{m^2}{4b_3},\quad b_7=0,\quad b_9=0.
\end{aligned}$$

Proof. When the condition C_1 holds, the first two periodic constants of system (3.2) can be obtained as

$$\begin{aligned}
T_1 &= -\frac{4b_3^2b_5^2-28b_3b_5b_7^2+40b_7^4+5b_3b_5m^2+14b_7^2m^2+m^4}{3m^2}, \\
T_2 &= \frac{6b_7^2(-20b_3b_5b_7^2+36b_7^4+5b_3b_5m^2+13b_7^2m^2+m^4)}{m^2}.
\end{aligned}$$

It is obvious that $T_2=0$ yields a solution $b_7=0$ which in turn leads to $a_1=a_8=b_9=0$ due to the condition C_1. Then, for $b_7=0$, the equation $T_1=0$ gives two solutions: $b_5=-\frac{m^2}{b_3}$ and $b_5=-\frac{m^2}{4b_3}$, yielding the conditions L_1 and L_2.

If $b_7\neq 0$, then eliminating b_7 from the two equations: $T_1=T_2=0$ gives a solution,

$$b_7^2 = \frac{b_3b_5(36b_3b_5-5m^2)-m^4}{52b_3b_5+4m^2},$$

and a resultant equation,

$$b_3b_5(64b_3^2b_5^2+189b_3b_5m^2+21m^4)=0.$$

It is easy to show that the resultant equation yields solutions for b_3 and b_5 such that $b_7^2<0$.

When the conditions L_1 and L_2 hold, system (3.2) can be rewritten as

$$\begin{cases} \dfrac{dx}{d\tau} = \dfrac{1}{2}(-2y + 4mx^2 - my^2), \\ \dfrac{dy}{d\tau} = x(1 + my), \end{cases} \quad \text{for } L_1, \qquad \begin{cases} \dfrac{dx}{d\tau} = \dfrac{1}{2}(-2y + mx^2 - my^2), \\ \dfrac{dy}{d\tau} = x(1 + my), \end{cases} \quad \text{for } L_2,$$

which admit the transversal commuting systems,

$$\begin{aligned} \frac{dx}{d\tau} &= -x(-1 + m^2x^2 - 2\,my), \\ \frac{dy}{d\tau} &= -\frac{1}{2}y(-2 + 2m^2x^2 - 3\,my + m^2y^2), \end{aligned}$$

for L_1 and

$$\begin{aligned} \frac{dx}{d\tau} &= x(1 + my), \\ \frac{dy}{d\tau} &= \frac{1}{2}(-mx^2 + 2y + my^2), \end{aligned}$$

for L_2, respectively. This implies that the origin of system (3.2) is an isochronous center according to Corollary 5.1 in [40]. □

Lemma 3.2. *If the condition C_2 in Theorem 3.2 holds, then system (3.2) has an isochronous center at the origin if and only if either L_2 or the following condition L_3 is satisfied:*

$$L_3:\ a_1 = -\frac{b_7}{4},\ a_8 = \frac{b_7(11b_7^2 + 6m^2)}{4b_3^2},\ b_5 = -\frac{3b_7^2 + 4m^2}{4b_3},\ b_9 = \frac{b_7(3b_7^2 + 4m^2)}{4b_3^2}.$$

Proof. When the condition C_2 holds, the first two periodic constants of system (3.2) are given by

$$\begin{aligned} T_1 &= -\frac{(4b_3b_5 + 3b_7^2 + 4m^2)(16b_3b_5 + 9b_7^2 + 4m^2)}{48m^2}, \\ T_2 &= \frac{b_7^2(b_7^2 + 4m^2)(4b_3b_5 + 3b_7^2 + 4m^2)}{384m^2}. \end{aligned}$$

It is easy to see that the solution $b_7 = 0$ yields the condition L_2. If $b_7 \neq 0$, then b_5 is easily derived from the common factor of T_1 and T_2: $4b_3b_5 + 3b_7^2 + 4m^2 = 0$, and then the condition C_1 leads to the expressions of a_1, a_8 and b_9. This gives the condition L_3.

If the condition L_3 holds, system (3.2) can be brought into

$$\begin{aligned} \frac{dx}{dT} &= \frac{1}{8b_3^2m}(2b_3b_7^2x^2 + 8b_3m^2x^2 + b_7^3x^3 + 4b_7m^2x^3 \\ &\qquad - 8b_3^2\,my - 8b_3b_7\,mxy), \\ \frac{dy}{dT} &= \frac{1}{8b_3^2m}(8b_3^2\,mx + 4b_3b_7\,mx^2 + 2b_3b_7^2xy + 8b_3m^2xy \\ &\qquad + b_7^3x^2y + 4b_7m^2x^2y - 4b_3b_7\,my^2), \end{aligned} \tag{3.9}$$

which has a transversal commuting system,

$$\begin{aligned} \frac{dx}{dT} &= \frac{1}{8b_3^2m}(8b_3^2\,mx + 4b_3b_7\,mx^2 + 2b_3b_7^2xy + 8b_3m^2xy + b_7^3x^2y \\ &\qquad + 4b_7m^2x^2y - 4b_3b_7\,my^2), \\ \frac{dy}{dT} &= \frac{y(8b_3^2\,m + 8b_3b_7\,mx + 2b_3b_7^2y + 8b_3m^2y + b_7^3xy + 4b_7m^2xy)}{8b_3^2m}. \end{aligned} \tag{3.10}$$

This shows that the origin of system (3.10) is an isochronous center according to Corollary 5.1 in [40]. □

If the condition C_3 in Theorem 3.2 holds, we get the same condition as that for the condition C_1.

Lemma 3.3. *If the condition C_4 in Theorem 3.2 holds, then system (3.2) has an isochronous center at the origin if and only if one of the following conditions holds:*

$$L_4:\quad a_1=-b_7,\quad a_8=b_9=-\frac{b_7(4b_7^2+m^2)}{4b_3^2},\quad b_5=-\frac{12b_7^2+m^2}{4b_3m};$$
$$L_5:\quad a_1=-b_7,\quad a_8=b_9=-\frac{b_7(b_7^2+m^2)}{b_3^2},\quad b_5=-\frac{12b_7^2+m^2}{4b_3m}.$$

Proof. When the condition C_4 holds, the first three periodic constants of system (3.2) are obtained as

$$T_1=-\frac{1}{3(2a_1^2+b_3b_5-a_1b_7)^2m^2}f_3,$$
$$T_2=\frac{(a_1+b_7)}{8(2a_1^2+b_3b_5-a_1b_7)^4}f_4,$$
$$T_3=-\frac{(a_1+b_7)}{23040(2a_1^2+b_3b_5-a_1b_7)^6m^2}f_5,$$

where

$$\begin{aligned}f_3=&\,4(2a_1^2+b_3b_5-a_1b_7)^4+(32a_1^6-48a_1^5b_7+3a_1b_3b_5b_7(b_3b_5-15b_7^2)\\&+b_3^2b_5^2(5b_3b_5+12b_7^2)+a_1^4(48b_3b_5+76b_7^2)+a_1^3(-96b_3b_5b_7+64b_7^3)\\&+a_1^2(36b_3^2b_5^2-54b_3b_5b_7^2+43b_7^4))m^2+(2a_1^2+b_3b_5-a_1b_7)^2m^4.\end{aligned}$$

The lengthy expressions f_4 and f_5 are omitted here for brevity. Similarly, we can prove that the three equations, $T_1=T_2=T_3=0$, yield the conditions L_4 and L_5.

When the condition L_4 holds, by the complex transformation $z=x+iy$, $w=x-iy$, $\tau=iT$, system (3.2) can be changed to its complex concomitant system,

$$\begin{aligned}\frac{dz}{dT}&=-\frac{1}{2}z(-2-ib_7m^2w^2+2b_7^2w^2+imz+4b_7z-ib_7\,mz^2-2b_7^2z^2),\\\frac{dw}{dT}&=-\frac{1}{2}w(2+imw-4b_7w-ib_7\,mw^2+2b_7^2w^2-ib_7\,mz^2-2b_7^2z^2),\end{aligned}\tag{3.11}$$

which has a linearizability transformation L_3 constructed by the simple integral curves, as shown in Table 1.

When the condition L_5 holds, by the same complex transformation for proving L_4, system (3.2) can be changed to its complex concomitant system,

$$\begin{aligned}\frac{dz}{dT}=&\,\frac{1}{8}z(-3imw^2+8z-6imwz-8b_7^2w^2z+10ib_7\,mw^2z-16b_7z^2\\&-7imz^2+12ib_7\,mwz^2+8b_7^2z^3+10ib_7\,mz^3),\\\frac{dw}{dT}=&\,\frac{1}{8}w(-8w+16b_7w^2-7imw^2-8b_7^2w^3+10ib_7\,mw^3\\&-6i\,mwz+12ib_7\,mw^2z-3imz^2+8b_7^2wz^2+10ib_7\,mwz^2),\end{aligned}\tag{3.12}$$

which also has a linearizability transformation L_5 constructed by the simple integral curves, as given in Table 1.

When the conditions C_5, C_6 and C_7 in Theorem 3.2 hold, there do not exist more isochronous center conditions. When the condition C_9 in Theorem 3.2 holds, we get the same condition as that for the condition C_1. Finally, we consider the condition C_8.

Lemma 3.4. *If the condition C_8 in Theorem 3.2 holds, then system (3.2) has an isochronous center at the origin if and only if one of the following conditions holds:*

$$L_6:\quad a_1=-\frac{\sqrt{2}m}{4},\quad a_8=b_9=0,\quad b_5=-\frac{3m^2}{b_3},\quad b_7=\frac{3\sqrt{2}m}{2};$$
$$L_7:\quad a_1=\frac{\sqrt{2}m}{4},\quad a_8=b_9=0,\quad b_5=-\frac{3m^2}{b_3},\quad b_7=-\frac{3\sqrt{2}m}{2};$$
$$L_8:\quad b_7=-2a_1,\quad a_8=b_9=0,\quad b_5=-\frac{4a_1^2+m^2}{b_3}.$$

Table 1
Linearizability transformations for systems (3.11) and (3.12).

Linearizability transformation	Integral curves
L_4: $\xi = -zh_1^{-\frac{4ib_7}{m}} h_2 h_3^{\frac{-m+2b_7 i}{m}}$, $\eta = wh_1^{\frac{4ib_7}{m}} h_2^{-1} h_4^{\frac{-2b_7 i}{m}}$.	$h_1 = -1 + b_7(w+z)$, $h_2 = -2 - imw + 2b_7(w+z)$, $h_3 = (-1+b_7 w)(2 - 2b_7 w + imw) + (4b_7 + im)(1+w)z - b_7(2b_7 + im)z^2$, $h_4 = -8 + 4imz + 4b_7(4 + im(w-z))(w+z) - 8b_7^2(w+z)^2 - mw(4i + mz)$.
L_5: $\xi = h_5^{-2ib_7} h_6^{-2+2b_7 i} f_7$, $\eta = h_5^{1+ib_7} f_6^{-1} h_7^{\frac{1+b_7 i}{2}}$.	$h_5 = -1 + b_7\ m(w+z)$, $h_6 = mw^2 + 8iz - 2\ mwz - 8ib_7\ mwz + mz^2 - 8ib_7\ mz^2$, $h_7 = 2 + 4imw - 4b_7\ mw - 3m^2w^2 - 4ib_7 m^2 w^2 + 2b_7^2 m^2 w^2 - 4imz - 4b_7\ mz - 2m^2 wz + 4b_7^2 m^2 wz - 3m^2 z^2 + 4ib_7 m^2 z^2 + 2b_7^2 m^2 z^2$.

Proof. When the condition C_8 holds, the first three periodic constants of system (3.2) are given by

$$
\begin{aligned}
T_1 &= -\frac{1}{12a_1m^4} f_6,\\
T_2 &= \frac{(a_1+b_7)}{8(2a_1^2+b_3b_5-a_1b_7)^4} f_4,\\
T_3 &= -\frac{(a_1+b_7)}{23040(2a_1^2+b_3b_5-a_1b_7)^6 m^2} f_5,
\end{aligned}
$$

where

$$
\begin{aligned}
f_6 = {} & 160(2a_1^4 - a_1a_8b_3^2 + a_1^3b_7)^2 + 48a_1^2(6a_1^4 - 2a_1a_8b_3^2\\
& + 3a_1^3b_7 + a_8b_3^2b_7)m^2 + 6a_1^2(4a_1^2 - b_7^2)m^4 - (a_1-b_7)(2a_1+b_7)m^6.
\end{aligned}
$$

The lengthy expressions f_7 and f_8 are not given here for brevity. Similarly, we can use the equations $T_1 = T_2 = T_3 = 0$ to obtain the conditions L_6, L_7 and L_8.

When $a_8 = 0$, system (3.8) can be simplified to

$$
\begin{aligned}
\frac{dx_1}{d\tau} &= -\frac{1}{4a_1b_3m}(8a_1^3x_1^2 + 4a_1^2b_7x_1^2 + 2a_1m^2x_1^2 - b_7m^2x_1^2\\
&\qquad - 4a_1b_3\ my_1 - 4a_1b_7\ mx_1y_1),\\
\frac{dy_1}{d\tau} &= -\frac{1}{2b_3m^2}(2b_3m^2x_1 + 8a_1^3x_1^2 + 4a_1^2b_7x_1^2 - 2a_1m^2x_1^2\\
&\qquad - b_7m^2x_1^2 - 8a_1^2mx_1y_1 - 4a_1b_7\ mx_1y_1 + 2m^3x_1y_1 + 4a_1m^2y_1^2),
\end{aligned}
$$

which can be further changed to

$$
\begin{aligned}
\frac{du}{d\tau} &= -v + \frac{2a_1-b_7}{4a_1}u^2 + \frac{a_1(2a_1+b_7)}{m^2}v^2,\\
\frac{dv}{d\tau} &= u(1+v).
\end{aligned}
\tag{3.13}
$$

by the transformation $u = -\frac{mx_1}{b_3} - \frac{2a_1y_1}{b_3}$, $v = -\frac{2a_1x_1}{b_3} + \frac{my_1}{b_3}$. It is well known that the origin of a quadratic system is an isochronous center if and only if the system can be brought into one in the form of

$$
\begin{aligned}
\frac{du}{d\tau} &= -v + Au^2 + Bv^2,\\
\frac{dv}{d\tau} &= u(1+v).
\end{aligned}
$$

where $(A, B) \in \{(1, 0),\ (\frac{1}{2}, -\frac{1}{2}),\ (\frac{1}{4}, 0),\ (2, -\frac{1}{2})\}$, see [40]. When the condition L_6 or L_7 holds, the coefficients of u^2 and v^2 in system (3.13) are 2 and $-\frac{1}{2}$, respectively. When the condition L_8 holds, the coefficients of u^2 and v^2 in system (3.13) are 1 and 0, respectively. So the origin of system (3.13) is an isochronous center. □

It is seen that the condition L_1 can be included in condition L_8. Summarizing the above results, we have the following theorem.

Theorem 3.4. *The origin of system (3.2) is an isochronous center if and only if one of the following seven conditions holds:*

$$
\begin{aligned}
&L_2: \quad a_1=0, \quad a_8=0, \quad b_5=-\frac{m^2}{4b_3}, \quad b_7=0, \quad b_9=0;\\
&L_3: \quad a_1=-\frac{b_7}{4}, \; a_8=\frac{b_7(11b_7^2+6m^2)}{4b_3^2}, \; b_5=-\frac{3b_7^2+4m^2}{4b_3}, \; b_9=\frac{b_7(3b_7^2+4m^2)}{4b_3^2};\\
&L_4: \quad a_1=-\frac{b_7}{m}, \quad a_8=b_9=-\frac{b_7(4b_7^2+m^2)}{4b_3^2}, \quad b_5=-\frac{12b_7^2+m^2}{4b_3m};\\
&L_5: \quad a_1=-\frac{b_7}{m}, \quad a_8=b_9=-\frac{b_7(b_7^2+m^2)}{b_3^2}, \quad b_5=-\frac{12b_7^2+m^2}{4b_3m};\\
&L_6: \quad a_1=-\frac{\sqrt{2}m}{4}, \quad a_8=b_9=0, \quad b_5=-\frac{3m^2}{b_3}, \quad b_7=\frac{3\sqrt{2}m}{2};\\
&L_7: \quad a_1=\frac{\sqrt{2}m}{4}, \quad a_8=b_9=0, \quad b_5=-\frac{3m^2}{b_3}, \quad b_7=-\frac{3\sqrt{2}m}{2};\\
&L_8: \quad b_7=-2a_1, \quad a_8=b_9=0, \quad b_5=-\frac{4a_1^2+m^2}{b_3}.
\end{aligned}
$$

4. Bi-center conditions of system (2.2)

With a proper linear transformation, planar autonomous analytic systems with a nilpotent critical point can always be given in the form of

$$
\begin{aligned}
\frac{dx}{dt}&=\Phi(x,y)=y+\sum_{k+j=2}^{\infty}a_{kj}x^ky^j,\\
\frac{dy}{dt}&=\Psi(x,y)=\sum_{k+j=2}^{\infty}b_{kj}x^ky^j,
\end{aligned}
\tag{4.1}
$$

where $\Phi(x,y)$, $\Psi(x,y)$ are analytic in the neighborhood of the origin.

The results given in [41] show that the origin of system (4.1) is a monodromic critical point if and only if the following conditions hold:

$$
\begin{aligned}
&\Psi(x,f(x)=\alpha x^{2n-1}+o(x^{2n-1})), \quad \alpha\neq 0,\\
&\left[\frac{\partial\Phi}{\partial x}+\frac{\partial\Psi}{\partial y}\right]_{y=f(x)}=\beta x^{n-1}+o(x^{n-1}),\\
&\beta^2+4n\alpha<0,
\end{aligned}
\tag{4.2}
$$

where n is a positive integer.

For system (2.2), there are two nilpotent singular points $(\pm1,0)$ with a double-zero eigenvalue. We need only to study the center problem at the singular point $(1,0)$ due to the symmetry. System (2.2) can be transformed into

$$
\begin{aligned}
\frac{dx}{dt}&=-\frac{x+1}{b_3}(2a_1b_3x+a_1b_3x^2+2a_1^2y-a_8b_3y^2),\\
\frac{dy}{dt}&=2b_3x+b_3x^2+2a_1y+2b_7xy+b_7x^2y+b_5y^2+b_9y^3,
\end{aligned}
\tag{4.3}
$$

by $x=\bar{x}+1$, where we still use x for $\bar{x}$ for convenience. The singular point $(1,0)$ of system (2.2) is translated to the origin of system (4.3). It is noted that when $a_1=0$, by the transformation, $u=y$, $v=2b_3x$, system (4.3) becomes

$$
\begin{aligned}
\frac{dx}{dt}&=\frac{4b_3^2b_5x^2+4b_3^2b_9x^3+4b_3^2y+4b_3b_7xy+b_3y^2+b_7xy^2}{4b_3^2},\\
\frac{dy}{dt}&=a_8x^2(2b_3+y).
\end{aligned}
\tag{4.4}
$$

It is easy to verify that the origin of system (4.4) is not a monodromic singular point. Therefore, we assume $a_1 \neq 0$ in system (4.3). Then, by the transformation, $u = b_3 x$, $v = -2a_1 b_3 x - 2a_1^2 y$, system (4.3) can be changed to

$$\begin{aligned}
\frac{dx}{dt} &= -\frac{(b_3+x)(4a_1^5x^2 - 4a_1^2a_8b_3^2x^2 - 4a_1^4b_3y - 4a_1a_8b_3^2xy - a_8b_3^2y^2)}{4a_1^4b_3^2},\\
\frac{dy}{dt} &= \frac{1}{4a_1^4b_3^2}\big(-8a_1^3a_8b_3^3x^2 - 8a_1^4b_3^2b_5x^2 + 16a_1^5b_3b_7x^2 + 8a_1^6x^3\\
&\quad - 8a_1^3a_8b_3^2x^3 + 8a_1^5b_7x^3 + 8a_1^3b_3^2b_9x^3 - 8a_1^5b_3xy\\
&\quad - 8a_1^2a_8b_3^3xy - 8a_1^3b_3^2b_5xy + 8a_1^4b_3b_7xy - 8a_1^2a_8b_3^2x^2y\\
&\quad + 4a_1^4b_7x^2y + 12a_1^2b_3^2b_9x^2y - 2a_1a_8b_3^3y^2 - 2a_1^2b_3^2b_5y^2\\
&\quad - 2a_1a_8b_3^2xy^2 + 6a_1b_3^2b_9xy^2 + b_3^2b_9y^3\big).
\end{aligned} \tag{4.5}$$

Now we are ready to derive the center conditions for system (4.5). Using the results in [41], we obtain

$$\begin{aligned}
\alpha_2 &= a_8b_3 - a_1(-b_3b_5 + 2a_1b_7),\\
\alpha_3 &= a_1^3(-b_3b_5b_7 + 2a_1b_7^2 + b_3^2b_9) < 0,\\
\alpha_4 &= \frac{8a_1(a_1+b_7)(b_3b_5 - 2a_1b_7)(a_1^2 + b_3b_5 - 2a_1b_7)}{b_3},\\
\alpha_5 &= -2a_1(a_1+3b_7)(-b_3b_5 + 2a_1b_7),\\
\beta_1 &= -\frac{2(2a_1^2 + b_3b_5 - a_1b_7)}{a_1b_3}.
\end{aligned}$$

The origin of system (4.5) is a third-multiple monodromic singular point if $\alpha_2 = 0$, $\alpha_3 < 0$ and $\beta_1^2 + 8\alpha_3 < 0$. When $\alpha_3 = 0$, namely, $b_9 = \frac{b_7(b_3b_5 - 2a_1b_7)}{b_3^2}$, the origin of system (4.5) is at most a fourth-multiple singular point when $\alpha_4 \neq 0$. The origin is a singular point with multiplicity four because of the symmetry. So the origin is a monodromic singular point of system (4.5) with multiplicity three.

Due to the complexity in the monodromic condition of the nilpotent singular point, we compute the quasi-focus values before discussing the monodromic condition. The first two quasi-focus values at the origin of system (4.5) are

$$\begin{aligned}
\mu_1 &= -\frac{(-b_3b_5b_7 + 2a_1b_7^2 + b_3^2b_9)}{15a_1^3b_3^4}(16a_1^3 + 18a_1b_3b_5 - 24a_1^2b_7\\
&\quad - 3b_3b_5b_7 + 8a_1b_7^2 + 15b_3^2b_9),\\
\mu_2 &= -\frac{8(-b_3b_5b_7 + 2a_1b_7^2 + b_3^2b_9)}{2625a_1^5b_3^6}(a_1 - b_7)(4a_1 + b_7)(2a_1 + 3b_7)\\
&\quad \times (2a_1^2 + b_3b_5 - a_1b_7).
\end{aligned}$$

Theorem 4.1. *The first two quasi-focal values at the origin of system (4.5) vanish if and only if one of the following four conditions holds:*

$$\begin{aligned}
&\mathrm{NC}_1:\ b_9 = -\frac{b_5b_7}{b_3},\quad a_1 = b_7;\\
&\mathrm{NC}_2:\ b_9 = \frac{b_7(2b_3b_5 + b_7^2)}{4b_3^2},\quad a_1 = -\frac{b_7}{4};\\
&\mathrm{NC}_3:\ b_9 = \frac{2b_7(2b_3b_5 + 4b_7^2)}{b_3^2},\quad a_1 = -\frac{3b_7}{2};\\
&\mathrm{NC}_4:\ b_9 = \frac{a_1(4a_1^2 - b_7^2)}{b_3^2},\quad b_5 = -\frac{a_1(a_1 - b_7)}{b_3}.
\end{aligned}$$

Theorem 4.1 implies that the four conditions together with $\alpha_3 < 0$ and $\beta_1^2 + 8\alpha_3 < 0$ are necessary for the origin of system (4.5) to be a center. Next, we prove that these conditions are also sufficient.

Theorem 4.2. *The origin of system (4.5) is a center if and only if one of the four conditions in 4.1 together with $\alpha_3 < 0$ and $\beta_1^2 + 8\alpha_3 < 0$ hold.*

Proof. When the condition NC_1 in Theorem 4.1 holds, system (4.5) admits an inverse integrating factor,

$$I_7 = (b_3 + x)^{1 - \frac{b_3b_5}{b_7^2}}\, y^{\frac{1}{2} - \frac{b_3b_5}{b_7^2}}\, (4b_3b_7 + 4b_7x + y)^{\frac{1}{2} - \frac{b_3b_5}{b_7^2}}.$$

When the condition NC_2 in Theorem 4.1 holds, system (2.2) can be changed to

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{1}{4b_3^2}(4b_3^2b_5x_1^2 + 2b_3b_7^2x_1^2 + 2b_3b_5b_7x_1^3 + b_7^3x_1^3 + 4b_3^2y_1 + 4b_3b_7x_1y_1),\\ \frac{dy_1}{dt} &= -\frac{b_7y_1}{4b_3^2}(b_3b_7x_1 - 2b_3b_5x_1^2 - b_7^2x_1^2 - 2b_3y_1),\end{aligned} \tag{4.6}$$

by using $x_1 = y$, $y_1 = b_3(x^2-1) - \frac{b_7}{2}y$. System (4.6) admits an inverse integrating factor,

$$I_8 = y_1^{-3+\frac{8b_3b_5}{b_7^2}}(2b_3 + b_7x_1 + 2y_1)^{7+\frac{8b_3b_5}{b_7^2}}.$$

When the condition NC_3 in Theorem 4.1 holds, system (2.2) can be changed to

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{1}{b_3^2}(b_3^2b_5x_1^2 + 3b_3b_7^2x_1^2 + 2b_3b_5b_7x_1^3 + 8b_7^3x_1^3 + b_3^2y_1 + b_3b_7x_1y_1),\\ \frac{dy_1}{dt} &= \frac{3b_7}{4b_3^2}(b_3b_5b_7x_1^3 + b_7^3x_1^3 + 2b_3b_7x_1y_1 + b_3b_5x_1^2y_1 + 3b_7^2x_1^2y_1 + b_3y_1^2).\end{aligned} \tag{4.7}$$

by using $x_1 = y$, $y_1 = b_3(x^2-1) - 3b_7y$. Moreover, by the transformation,

$$\begin{aligned}X &= x_1,\\ Y &= \frac{1}{b_3^2}(b_3^2b_5x_1^2 + 3b_3b_7^2x_1^2 + 2b_3b_5b_7x_1^3 + 8b_7^3x_1^3 + b_3^2y_1 + b_3b_7x_1y_1),\end{aligned}$$

system (4.7) can be changed to the Liénard-like system (3.5), where

$$\begin{aligned}p_0(X) &= \frac{3b_7^2(b_3b_5 + 5b_7^2)X^3(b_3 + 2b_7X)}{b_3^4(b_3 + b_7X)}(b_3^22b_3b_7X - b_3b_5X^2 - 4b_7^2X^2),\\ p_1(X) &= \frac{X}{b_3^2(b_3 + b_7X)}(2b_3^3b_5 + 12b_3^2b_7^2 + 4b_3^2b_5b_7X + 24b_3b_7^3X\\ &\qquad - 5b_3b_5b_7^2X^2 - 23b_7^4X^2),\\ p_3(X) &= \frac{4b_7}{2(b_3 + b_7X)}.\end{aligned}$$

Since $W_1(X) - W_1(Y)$ and $W_2(X) - W_2(Y)$ have a common factor $(X-Y)(b_3X + b_3Y + 2b_7XY)$, the origin of system (4.7) is a center according to Theorem 2.3.

When the condition NC_4 in Theorem 4.1 holds, system (2.2) can be changed to

$$\begin{aligned}\frac{dx_1}{dt} &= \frac{1}{3b_3^2}(- 6a_1^2b_3x_1^2 - 3a_1b_3b_7x_1^2 + 4a_1^3x_1^3 - a_1b_7^2x_1^3\\ &\quad + 3b_3^2y_1 + 3b_3b_7x_1y_1),\\ \frac{dy_1}{dt} &= \frac{2a_1}{3b_3^2}(- 8a_1^3x_1^3 - 6a_1^2b_7x_1^3 - a_1b_7^2x_1^3 + 6a_1b_3x_1y_1\\ &\quad + 3b_3b_7x_1y_1 + 6a_1^2x_1^2y_1 + 3a_1b_7x_1^2y_1 - 3b_3y_1^2),\end{aligned} \tag{4.8}$$

by using $x_1 = y$, $y_1 = b_3(x^2-1) + 2a_1y$. Moreover, system (4.8) can be changed to the Liénard-like system (3.5) by the transformation,

$$\begin{aligned}X &= x_1,\\ Y &= \frac{1}{3b_3^2}(-6a_1^2b_3x_1^2 - 3a_1b_3b_7x_1^2 + 4a_1^3x_1^3 - a_1b_7^2x_1^3 + 3b_3^2y_1 + 3b_3b_7x_1y_1),\end{aligned}$$

where

$$\begin{aligned}p_0(X) &= -\frac{4a_1^2(a_1 + b_7)(2a_1 + b_7)X^3(b_3 + 2b_7X)}{9b_3^4(b_3 + b_7X)}(-3b_3^3 + 6a_1b_3^2X\\ &\quad - 3b_3^2b_7X - 6a_1^2b_3X^2 + 3a_1b_3b_7X^2 + 4a_1^3X^3 - a_1b_7^2X^3),\\ p_1(X) &= \frac{2a_1(4a_1 - b_7)(a_1 + b_7)(2a_1 + b_7)X^3}{3b_3^2(b_3 + b_7X)},\\ p_3(X) &= -\frac{2a_1 - b_7}{2(b_3 + b_7X)}.\end{aligned}$$

If $b_7 = 4a_1$, the Liénard-like system is symmetric with the X-axis. Otherwise, a simple computation shows that $W_1(X) = \frac{2a_1(2a_1-b_7)}{(4a_1-b_7)^2}$ is a constant, implying that the origin of system (4.8) is a center according to Theorem 2.3. □

5. Conclusion

In this paper, we have investigated the bi-center and bi-isochronous center problems in cubic planar systems which are symmetric with respect to a straight line. We first apply a transformation to move the symmetric line on the y-axis with two symmetric singular points at $(\pm 1, 0)$, which are classified as elementary and nilpotent singular points. A complete classification is provided, with nine conditions for elementary singular points and four conditions for nilpotent singular points. Moreover, six bi-isochronous center conditions are obtained for the elementary singular points.

It should be pointed out that for the nilpotent singular points, the classical method is first to give the monodromic condition, and then to compute the quasi-focus values in order to solve the center problem. However, in this paper we first compute the quasi-focus values, and then obtain the center conditions. In fact, when $\alpha_2 = 0$ and $\alpha_3 > 0$, the origin of system (4.5) is a nilpotent saddle point with multiplicity three. The conditions in Theorem 4.1 are also integrability conditions of nilpotent saddle point, which will be considered in future work.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Feng Li reports financial support was provided by The National Natural Science Foundation of China. Pei Yu reports financial support was provided by The Natural Sciences and Engineering Research Council of Canada

Data availability

No data was used for the research described in the article.

Acknowledgments

This research was partially supported by the National Natural Science Foundation of China, No. 12071198 (F. Li), and the Natural Sciences and Engineering Research Council of Canada, No. R2686A02 (P. Yu).

References

[1] Chavarriga J. Integrable systems in the plan with a center type linear part. Appl Math 1994;22:285–309.
[2] Chavarriga J, Giné J. Integrability of a linear center perturbed by fourth degree homogeneous polynomial. Publ Mat 1996;40:21–39.
[3] Chavarriga J, Giné J. Integrability of a linear center perturbed by fifth degree homogeneous polynomial. Publ Mat 1997;41:335–56.
[4] Chavarriga J, Giné J. Integrability of cubic systems with degenerate infinity. Differ Equ Dyn Syst 1998;6:425–38.
[5] Chavarriga J, Giacomini H, Giné J, Llibre J. Local analytic integrability for nilpotent centers. Ergodic Theory Dynam Systems 2003;23:417–28.
[6] Cima A, Gasull A, Manosa F. Cyclicity of a family of vector fields. J Math Anal Appl 1995;196:921–37.
[7] Christopher CJ, Lloyd NG, Pearson JM. On a Cherkas method for centre conditions. Nonlinear World 1995;2:459–69.
[8] Christopher CJ, Lloyd NG. Small-amplitude limit cycles in polynomial Liénard systems. Nonlinear Differential Equations Appl 1996;3:183–90.
[9] Giné J. Isochronous foci for analytic differential systems. Int J Bifurcation Chaos 2003;13(6):1617–23.
[10] Giné J, Grau M. Characterization of isochronous foci for planar analytic differential systems. Proc R Soc Edinb Sect A 2005;135(5):985–98.
[11] Chavarriga J, Giné J, García IA. Isochronous centers of a linear center perturbed by fourth degree homogeneous polynomial. Bull Sci Math 1999;123:77–96.
[12] Chavarriga J, Giné J, García IA. Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomial. J Comput Appl Math 2001;126:351–68.
[13] Liu YR, Li JB. Theory of values of singular point in complex autonomous differential systems. Sci China Ser A 1990;33(1):10–24.
[14] Lin YP, Li JB. The normal form of a planar autonomous system and critical points of the period of closed orbits. Acta Math Sinica 1991;34:490–501, (in Chinese).
[15] Liu YR, Huang WT. A new method to determine isochronous center conditions for polynomial differential systems. Bull Sci Math 2003;127(2):133–48.
[16] Lukashevich NA. The isochronism of a center of certain systems of differential equations. Diff Nye Uravn 1965;1:295–302.
[17] Loud WS. Behavior of the period of solutions of certain plane autonomous systems near centers. Contrib Differ Equ 1964;3:21–36.
[18] Pleshkan I. A new method of investigating the isochronicity of a system of two differential equations. Differ Equ 1969;5:796–802.
[19] Romanovski VG, Robnik M. The centre and isochronicity problems for some cubic systems. J Phys A Math Gen 2001;34(47):10267–92.
[20] Romanovski VG, Chen XW, Hu ZP. Linearizability of linear systems perturbed by fifth degree homogeneous polynomials. J Phys A 2007;40(22):5905–19.
[21] Chavarriga J, Giné J, Garcia I. Isochronicity into a family of time-reversible cubic vector fields. Appl Math Comput 2001;121:129–45.
[22] Cairó L, Giné J, Llibre J. A class of reversible cubic systems with an isochronous center. Comput Math Appl 1999;38:39–53.
[23] Chen XW, Romanovski VG, Zhang W. Linearizability conditions of time-reversible quartic systems having homogeneous nonlinearities. Nonlinear Anal 2008;69(5):1525–39.
[24] Dukaric M, Giné J. Integrability of Lotka–Volterra planar complex cubic systems. Int J Bifurcation Chaos 2016;26(1):1650002.
[25] Liu YR, Li JB. Periodic constants and time-angle of isochronous centers for complex analytic systems. Int J Bifurcation Chaos 2006;16(12):3747–57.
[26] Liu YR, Li JB, Huang WT. Classical problems in planar vector fields. Beijing: Science Press; 2010.
[27] Llibre J, Romanovski VG. Isochronicity and linearizability of planar polynomial Hamiltonian systems. J Differ Equ 2015;259(5):1649–62.

[28] Mereu A, Llibre J, Braun F. Isochronicity for trivial quintic and septic planar polynomial Hamiltonian systems. Discrete Contin Dyn Syst 2016;36(10):5245–55.
[29] Liu YR, Li JB. Complete study on a bi-center problem for the Z_2-equivariant cubic vector fields. Acta Math Sinica 2011;27(7):1379–94.
[30] Romanovski VG, Fernandes W, Oliveira R. Bi-center problem for some classes of Z_2-equivariant systems. J Comput Appl Math 2017;320:61–75.
[31] Du CX, Y R. Liu Isochronicity for a Z_2-equivariant cubic system. Nonlinear Dyn 2017;87(2):1235–52.
[32] Li F, Liu YR, Liu YY, Yu P. Complex isochronous centers and linearization transformations for cubic Z_2-equivariant planar systems. J Differ Equ 2020;268:3819–47.
[33] Li F, Liu YR, Tian YY, Yu P. Integrability and linearizability of cubic Z_2 systems with non-resonant singular points. J Differ Equ 2020;269:9026–492.
[34] Li F, Liu YR, Liu YY, Yu P. Bi-center problem and bifurcation of limit cycles from nilpotent singular points in Z_2-equivariant cubic vector fields. J Differ Equ 2018;265:4965–92.
[35] Li F, Liu YY, Yu P, Wang JL. Complex integrability and linearizability of cubic Z_2-equivariant systems with two 1: q resonant singular points. J Differ Equ 2021;300:786–813.
[36] Chen T, Li S, Llibre J. Z_2-Equivariant linear type bi-center cubic polynomial Hamiltonnian vector fields. J Differ Equ 2020;269:832–61.
[37] Fernandes W, Valerio VP, Tempesta P. Isochronicity of bi-centers for symmetric quartic differential systems. Discrete Contin Dyn Syst Ser B 2022;27(7):3991–4006.
[38] Dukaric M, Fernandes W, Oliveira R. Symmetric centers on planar cubic differential systems. Nonlinear Anal 2020;197:111868.
[39] Gasull A, Torregrosa J. Center problem for several differential equations via Cherkas method. J Math Anal Appl 1998;228:322–43.
[40] Sabatini M, Chavarriga J. A survey of isochronous centers. Qual Theory Dyn Syst 1999;1(1):1–70.
[41] Amelbkin VV, Lukasevnky NA, Sadovski AP. Nonlinear oscillations in second order systems. Minsk, BGY Lenin. B. I. Press; 1992, (in Russian).