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# Chaos control and synchronization for a special generalized Lorenz canonical system – The SM system

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**Abstract**

This paper presents some simple feedback control laws to study global stabilization and global synchronization for a special chaotic system described in the generalized Lorenz canonical form (GLCF) when $\tau = -1$ (which, for convenience, we call Shimizu–Morioka system, or simply SM system). For an arbitrarily given equilibrium point, a simple feedback controller is designed to globally, exponentially stabilize the system, and reach globally exponent synchronization for two such systems. Based on the system's coefficients and the structure of the system, simple feedback control laws and corresponding Lyapunov functions are constructed. Because all conditions are obtained explicitly in terms of algebraic expressions, they are easy to be implemented and applied to real problems. Numerical simulation results are presented to verify the theoretical predictions.


## 1. Introduction

The discovery of the Lorenz chaotic attractor [1] has led to a new era in the study of chaos and its applications. Since the 1960s of the last century, based on the Lorenz system, many researchers from different disciplines such as mathematics and physics extensively investigated the property and applications of chaotic systems.

For a quite long period, due to the high sensitivity of a chaotic system to its initial condition, people thought that chaos was not controllable, and two same type of chaotic systems could not be synchronized. However, the OGY method [2] developed in the 1990s, and in particular the concept of the synchronization proposed by Pccora and Garrol [3] in 1990, have completely changed the situation. This has attracted more researchers to study chaos control and synchronization.

One goal of chaos control is: for a given unstable or locally stable equilibrium point, to design a feedback control law such that the equilibrium point becomes globally, asymptotically stable or even globally, exponentially stable. Up to now, a number of efficient methods have been proposed mainly for the Lorenz system, Chen system and Lü system [2,4–13], but very little has been achieved for global and exponential stability.

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Although many results have been obtained for the study of the synchronization of two chaotic systems [1,3,11,12,14–24] most of them were focused on the Lorenz system, Chua circuit, Chen system and Lü system. The results obtained for global synchronization are less than that of local synchronization, and that for globally exponential synchronization are even lesser.

State feedback control is perhaps the most generally and universally applicable method. However, to design the possibly simplest feedback control law requires good experience and skills. Generally, when the vector field of a chaotic system satisfies global Lipschitz condition (such as Chua circuit), or a system has a globally attractive set (such as the Lorenz system), the global and exponential stability can always be obtained by using a linear feedback control law. When a chaotic system does not belong to the above two types of chaotic systems (or has not been proved to have a globally attractive set), linear feedback controls cannot be used to reach global and asymptotic stability, and global synchronization. In this case, nonlinear feedback controls must be considered. To design the possibly simplest feedback control law, which does not change or only partially changes the structure of the original system, experience and skills are needed. Just as the construction of Lyapunov functions, there are no general rules to follow. The researchers have to develop their own methodologies to solve the problem.

We have thoroughly studied the general methods of chaos control and chaos synchronization for the six types of existing Rössler systems, which are relatively more difficult in the study of chaos control and synchronization [25,26]. These methods are general and can be easily extended to consider other chaotic systems.

Recently, Čelikovský and Chen [27] have used a canonical form to utilize different chaotic systems, including the Lorenz system, Chen system, and Lü system. This canonical form is called generalized Lorenz canonical form (GLCF), given by

$$\begin{pmatrix} \frac{dy_1}{dt'} \\ \frac{dy_2}{dt'} \\ \frac{dy_3}{dt'} \end{pmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + (1 \quad -1 \quad 0) \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{bmatrix}, \tag{1}$$

where $\lambda_1 > 0$, $\lambda_{2,3} < 0$, and $\tau \in R$.

It has been shown in [27] that with any $\tau \neq -1$, system (1) is state-equivalent to the following system:

$$\begin{pmatrix} \frac{dx_1}{dt'} \\ \frac{dx_2}{dt'} \\ \frac{dx_3}{dt'} \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{sign}(\tau+1) \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{2}$$

where

$$\begin{aligned} a_{11} &= \lambda_1 + \frac{\lambda_2 - \lambda_1}{\tau + 1}, \\ a_{12} &= -\frac{\lambda_2 - \lambda_1}{\tau + 1}, \\ a_{21} &= \lambda_1 - \lambda_2 + \frac{\lambda_2 - \lambda_1}{\tau + 1}, \\ a_{22} &= \lambda_2 - \frac{\lambda_2 - \lambda_1}{\tau + 1}, \end{aligned} \tag{3}$$

via the following linear transformation of coordinates:

$$\begin{aligned} x_1 &= \sqrt{|\tau + 1|}(y_1 - y_2), \\ x_2 &= \sqrt{|\tau + 1|}(y_1 + \tau y_2), \\ x_3 &= |\tau + 1| y_3. \end{aligned} \tag{4}$$

As pointed out in [27], the GLCF with $\tau > -1$ is equivalent to the generalized Lorenz systems (GLS), while the GLCF with $\tau < -1$ is equivalent to the hyperbolic-type generalized Lorenz systems (HGLS). It is easy to see that the case of $\tau = -1$ is well defined in (2), but cannot go through transformations of (3) and (4). Yet, this case is not equivalent to the GLS, nor to the HGLS, it in fact stands for the transform between the hyperbolic and non-hyperbolic cases. Since the two cases have qualitatively different structures in their nonlinear parts, they cannot be continuously changed from one to another. Actually, this special case corresponds to two models reported in the literature: one is the Shimizu–Morioka model developed in the 1970s of the last century [28], while the other is a model studied recently [29]. It has been shown in [27] that these two models are state-equivalent. Thus we call such a model as SM system. The classification of the GLCF is shown in Table 1.

Since chaos control and chaos synchronization for the GLCF system (2) with $\tau \neq -1$ have been extensively studied (e.g., see [5–9,14,15]), in this paper we will particularly consider chaos control and chaos synchronization for the CLCF when $\tau = -1$, i.e., for the SM system. A series of constructive and simple algebraic methods are obtained.

## 2. The SM system, the definitions of chaos control and synchronization, and the relative lemmas

The original Shimizu–Morioka model is described by the following equations [28]:

$$\begin{aligned} \dot{x} &\equiv \frac{\mathrm{d}x}{\mathrm{d}t} = y, \\ \dot{y} &\equiv \frac{\mathrm{d}y}{\mathrm{d}t} = x(1-z) - \beta y, \\ \dot{z} &\equiv \frac{\mathrm{d}z}{\mathrm{d}t} = -\alpha z + x^2, \end{aligned} \tag{5}$$

which can be transformed to the GLCF (1) with $\tau = -1$ via the following transformations:

$$\begin{aligned} x &= (y_1 - y_2)\sqrt{\frac{\lambda_1 - \lambda_2}{(-\lambda_1\lambda_2)^{3/2}}}, \\ y &= (\lambda_1 y_1 - \lambda_2 y_2)\sqrt{\frac{\lambda_1 - \lambda_2}{(-\lambda_1\lambda_2)^{5/2}}}, \\ z &= y_3\left(\frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2}\right), \\ t' &= t\sqrt{-\lambda_1\lambda_2}, \end{aligned}$$

and $\alpha = \frac{\lambda_3}{\sqrt{-\lambda_1\lambda_2}}$, $\beta = -\frac{\lambda_1+\lambda_2}{\sqrt{-\lambda_1\lambda_2}}$.

The model studied recently by Liu et al. [29] is given by

$$\begin{aligned} \dot{\hat{x}} &= a(\hat{y} - \hat{x}), \\ \dot{\hat{y}} &= b\hat{x} - k\hat{x}\hat{z}, \\ \dot{\hat{z}} &= -c\hat{z} + h\hat{x}^2, \end{aligned} \tag{6}$$

where $a$, $b$, $c$, $k$ and $h$ are positive numbers. For example: $a = 10$, $b = 40$, $c = 2.5$, $k = 1$ and $h = 4$.

To reduce the number of the coefficients, introduce the following transformations:

$$x = \hat{x}\sqrt{hk}, \quad y = \hat{y}\sqrt{hk}, \quad z = k\hat{z},$$

into (6) to obtain

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= bx - xz, \\ \dot{z} &= -cz + x^2. \end{aligned} \tag{7}$$

Table 1
Classificaion of the GLCF and their equivalents [27]

| GLCF | Equivalent systems |
|---|---|
| $\tau \in (-\infty, -1)$ | HGLS |
| $\tau = -1$ | Shimizu–Morika model or Liu–Liu–Liu–Liu model |
| $\tau = (-1,0)$ | GLS with $a_{12}a_{21} < 0$, Chen system |
| $\tau = 0$ | GLS with $a_{12}a_{21} = 0$, Lü system |
| $\tau \in (0,\infty)$ | GLS with $a_{12}a_{21} > 0$, Lorenz system |

It has been shown in [27] that system (7) is also state-equivalent to the GLCF (1) via linear transformations. This indicates that the model (6) is state-equivalent to the Shimizu–Morioka model (5) via a linear transformation of coordinates and a constant time scaling. Therefore, in the remaining of the paper, for convenience, we will call this model as SM system.

Based on the SM system (5), in this paper we will present a detailed study on chaos control and chaos synchronization. With the aid of the general method developed in [25,26], we can construct the possibly simplest linear feedback or nonlinear feedback control law and the corresponding Lyapunov function to achieve the algebraic conditions for chaos control and chaos synchronization.

First note that the equilibrium points of (5) are:

$$E_1 = (0,0,0), \quad E_2 = (\sqrt{\alpha},0,1), \quad E_3 = (-\sqrt{\alpha},0,1),$$

where $\alpha = 0.03$, $\beta = 0.3$ are positive constants. Let $(x^*, y^*, z^*)$ represent any one of the equilibrium points $E_1$, $E_2$, $E_3$, and further let

$$\bar{x} = x - x^*, \quad \bar{y} = y - y^*, \quad \bar{z} = z - z^*.$$

Suppose the feedback control law is given by

$$u_1(\bar{x},\bar{y},\bar{z}), \qquad u_2(\bar{x},\bar{y},\bar{z}), \qquad u_3(\bar{x},\bar{y},\bar{z}),$$

where $u_i$'s are linear or nonlinear functions which satisfy $u_i(0,0,0) = 0$, $i = 1, 2, 3$. Then, adding the above feedback controls to system (5) yields

$$\begin{aligned} \dot{\bar{x}} &= \bar{y} - u_1(\bar{x},\bar{y},\bar{z}), \\ \dot{\bar{y}} &= \bar{x} - xz + x^*z^* - \beta\bar{y} - u_2(\bar{x},\bar{y},\bar{z}), \\ \dot{\bar{z}} &= -\alpha\bar{z} + x^2 - x^{*2} - u_3(\bar{x},\bar{y},\bar{z}). \end{aligned} \tag{8}$$

We will study how to select the simplest linear feedback controls $u_1$, $u_2$, $u_3$ or the possibly simplest control law of the combination of linear and nonlinear feedbacks, such that the zero solution of (8) is globally, exponentially stable, and thus to achieve the global and exponential stability of an arbitrarily equilibrium point.

To further study the global and exponential synchronization of two such systems, consider system (5) as a drive system (transmitter):

$$\begin{aligned} \dot{x}_d &= y_d, \\ \dot{y}_d &= x_d - x_d z_d - \beta y_d, \\ \dot{z}_d &= -\alpha z_d + x_d^2, \end{aligned} \tag{9}$$

where the subscript d indicates the ‘‘drive’’. The corresponding receiving system is:

$$\begin{aligned} \dot{x}_r &= y_r - u_1(e_x,e_y,e_z), \\ \dot{y}_r &= x_r - x_r z_r - \beta y_r - u_2(e_x,e_y,e_z), \\ \dot{z}_r &= -\alpha z_r + x_r^2 - u_3(e_x,e_y,e_z), \end{aligned} \tag{10}$$

where r represents the ‘‘receive’’, and $e_x = x_d - x_r$, $e_y = y_d - y_r$, $e_z = z_d - z_r$ denote the errors. The error system is given by

$$\begin{aligned} \dot{e}_x &= e_y + u_1(e_x,e_y,e_z), \\ \dot{e}_y &= e_x - x_d z_d + x_r z_r - \beta e_y + u_2(e_x,e_y,e_z), \\ \dot{e}_z &= -\alpha e_z + x_d^2 - x_r^2 + u_3(e_x,e_y,e_z). \end{aligned} \tag{11}$$

Next, we will show how to choose the possibly simplest linear or nonlinear feedback control laws $u_1$, $u_2$, $u_3$ such that the zero solution of (11) is globally, exponentially stable, and thus the two systems (9) and (10) are globally, exponentially synchronized.

**Definition 1.** For any arbitrary given initial condition $x_d(t_0)$, $y_d(t_0)$, $z_d(t_0) \in R^3$ of the drive system and the corresponding initial condition $x_r(t_0)$, $y_r(t_0)$, $z_r(t_0)$ of the receiving system, the zero solution of (11) satisfies the following inequality:

$$e_x^2(t) + e_y^2(t) + e_z^2(t) \leqslant k(e(t_0))e^{-\alpha(t-t_0)}, \tag{12}$$

where $k(e(t_0))$ is a constant depending on $e(t_0)$, and $\alpha > 0$ is a positive constant independent of $e(x_0)$. Then, the zero solution of (11) is globally, exponentially stable, and the two systems (9) and (10) are globally, exponentially synchronized.

For convenience in the following analysis, we present a lemma on global and exponential stability about the zero solution of (11).

**Lemma 1.** *For a system given in the form of* (11), *if there exists a positive definite quadratic form*:

$$V = (\,e_x \quad e_y \quad e_z\,)P(\,e_x \quad e_y \quad e_z\,)^{\mathrm{T}},$$

*where* $P = P^{\mathrm{T}}$ *is a positive definite quadratic matrix, such that the derivative of V w.r.t. time t along the trajectory of* (8) *satisfies*

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(1)} \leqslant (\,e_x \quad e_y \quad e_z\,)Q(\,e_x \quad e_y \quad e_z\,)^{\mathrm{T}},$$

*where* $Q = Q^{\mathrm{T}}$ *is negative definite, then the zero solution of* (11) *is globally, exponentially stable, and so systems* (9) *and* (10) *are globally, exponentially synchronized. Further, we have the following estimation*:

$$e_x^2(t) + e_y^2(t) + e_z^2(t) \leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}[e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)]\mathrm{e}^{\frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}(t-t_0)}. \tag{13}$$

*where* $\lambda_{\min}(P)$ *and* $\lambda_{\max}(P)$ *are the minimum and maximum eigenvalues of P, respectively, while* $\lambda_{\max}(Q)$ *is the maximum eigenvalue of Q.*

**Proof.** Due to the symmetry of $P$, we have

$$\begin{aligned}\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(3)} &\leqslant (\,e_x(t) \quad e_y(t) \quad e_z(t)\,)Q(\,e_x(t) \quad e_y(t) \quad e_z(t)\,)^{\mathrm{T}} \leqslant \lambda_{\max}(Q)[e_x^2(t) + e_y^2(t) + e_z^2(t)] \\ &\leqslant \frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}(\,e_x(t) \quad e_y(t) \quad e_z(t)\,)P(\,e_x(t) \quad e_y(t) \quad e_z(t)\,)^{\mathrm{T}} = \frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}V.\end{aligned}$$

Thus,

$$\lambda_{\min}(P)[e_x^2(t) + e_y^2(t) + e_z^2(t)] \leqslant V(t) \leqslant V(t_0)\,\mathrm{e}^{\frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}(t-t_0)} \leqslant \lambda_{\max}(P)[e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)]\mathrm{e}^{\frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}(t-t_0)},$$

and finally we obtain

$$e_x^2(t) + e_y^2(t) + e_z^2(t) \leqslant \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}[e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)]\mathrm{e}^{\frac{\lambda_{\max}(Q)}{\lambda_{\max}(P)}(t-t_0)}. \qquad \square \tag{14}$$

**Remark.**

(1) Because the study of globally asymptotic stability (globally exponential stability) requires that the constructed positive definite Lyapunov function must be radially unbounded. Therefore, in the following, all the constructed Lyapunov functions are positive definite, quadratic forms, which are certainly radially unbounded. Thus, we shall not repeat the property of radially unbounded in the rest of the paper.
(2) Although Lemma 1 is based on system (11), it is applicable to system (8), since this is a special case for $e_x = \bar{x} = x - x^*$, $e_y = \bar{y} = y - y^*$, $e_z = \bar{z} = z - z^*$.

**Lemma 2.** *If with a proper feedback control law, system* (8) *or* (11) *becomes a linear system, then the necessary and sufficient condition for the zero solution of the corresponding linear system being globally, exponentially stable is that the coefficient matrix of the linear system is a Hurwitz matrix.*

This is a standard result (e.g., see [13]) and thus the proof is omitted. The following result is also well known.

**Lemma 3.** *The necessary and sufficient conditions for the cubic-degree polynomial* $\lambda^3 + P\lambda^2 + Q\lambda + R$ *with real coefficients being a Hurwitz polynomial are*: $P > 0$ *and* $PQ > R > 0$ (*see* [13]).

## 3. Globally exponential stabilization of the SM system

In this section, we consider the global, exponential stabilization of an arbitrary equilibrium point of system (5), $E_1$, $E_2$ or $E_3$ denoted by $X^* = (x^*, y^*, z^*)$. First we define the global, exponential stabilization.

**Definition 2.** When a proper feedback control law, given by $u_1$, $u_2$, $u_3$, is chosen such that the zero solution of system (8) becomes globally, exponentially stable, then it is said that $X^* = (x^*, y^*, z^*)$ is globally, exponentially stabilized.

**Theorem 1.** *Under the following control law*: $u_1 = 0$, $u_2 = -x\bar{z} - \delta_1\bar{x}(\delta_1 \geqslant 0)$, $u_3 = x\bar{x}$, *when* $\delta_1 < -1$, *any equilibrium point of system* (8), $E_1$, $E_2$, *or* $E_3$, *is globally, exponentially stabilized. When* $\delta_1 < 0$, $E_2$ *or* $E_3$ *are globally, exponentially stabilized.*

**Proof.** Under the given control law, system (8) becomes:

$$\begin{aligned} \dot{\bar{x}} &= \bar{y}, \\ \dot{\bar{y}} &= \bar{x} - \beta\bar{y} - z^*\bar{x} + \delta_1\bar{x}, \\ \dot{\bar{z}} &= x^*\bar{x} - \alpha\bar{z}. \end{aligned} \tag{15}$$

Since $x^*$, $y^*$ and $z^*$ are constants, system (15) is a linear system with constant coefficients. Let

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ (1+\delta_1 - z^*) & -\beta & 0 \\ x^* & 0 & -\alpha \end{bmatrix}.$$

Then

$$\begin{aligned} \det(\lambda E_3 - A_1) &= \begin{vmatrix} \lambda & -1 & 0 \\ -(1+\delta_1 - z^*) & \lambda+\beta & 0 \\ -x^* & 0 & \lambda+\alpha \end{vmatrix} = \lambda(\lambda+\beta)(\lambda+\alpha) - (1+\delta_1 - z^*)(\lambda+\alpha) \\ &= (\lambda+\alpha)[\lambda^2 + \beta\lambda - (1+\delta_1 - z^*)], \end{aligned} \tag{16}$$

where $\alpha > 0$, $\beta > 0$. When $\delta_1 < 0$ and $z^* = 1$, $1 + \delta_1 - z^* = \delta_1 < 0$. $A_1$ is a Hurwitz matrix. When $\delta_1 < -1$ and $z^* = 0$, $1 + \delta_1 - z^* = 1 + \delta_1 < 0$. $A_1$ is still a Hurwitz matrix. According to Lemma 2, when $\delta_1 < -1$, any equilibrium point of system (8), $E_1$, $E_2$, or $E_3$, is globally, exponentially stabilized. When $\delta_1 < 0$, $E_2$ or $E_3$ are globally, exponentially stabilized. □

**Remark.** Because the control law is constructed according to a given equilibrium point, i.e., the control law depends on the choice of the equilibrium point ($x^*$, $y^*$, $z^*$). Once the equilibrium point is determined, the control law can only be designed for the specific equilibrium point. However, the necessary condition of the global, asymptotic stability requires the uniqueness of the equilibrium point, so under the control law, all other equilibrium points of system (5) disappear and thus chaos is vanished. With this regard, it has shown the importance of the special dynamics and asymptotic behavior of the global, exponential stability of the designated equilibrium point.

**Theorem 2.** *For system* (8), *when the control law is chosen as*: $u_1 = k_1\bar{x}$, $u_2 = -z\bar{x}$, $u_3 = \bar{x}x$, *assuming* $k_2\alpha + \alpha\beta + k_1\beta - 1 > 0$, $(k_2 + \beta + \alpha)(k_2\alpha + \alpha\beta + k_2\beta - 1) > \alpha\beta k_1 - \alpha + x^* > 0$, *then the zero solution of* (8), *i.e., an arbitrary equilibrium point of system* (8), ($x^*$, $y^*$, $z^*$), *is globally, exponentially stabilized.*

**Proof.** Under the given control law of Theorem 2, system (8) can be written as:

$$\begin{aligned} \dot{\bar{x}} &= -k_1\bar{x} + \bar{y}, \\ \dot{\bar{y}} &= \bar{x} - \beta\bar{y} - x^*\bar{z}, \\ \dot{\bar{z}} &= x^*\bar{x} - \alpha\bar{z}. \end{aligned} \tag{17}$$

The coefficient matrix of linear system (17) is

$$A_2 = \begin{bmatrix} -k_1 & 1 & 0 \\ 1 & -\beta & -x^* \\ x^* & 0 & -\alpha \end{bmatrix}.$$

Thus,

$$\det(\lambda E_3 - A_2) = \begin{vmatrix} \lambda + k_1 & -1 & 0 \\ -1 & \lambda + \beta & x^* \\ -x^* & 0 & \lambda + \alpha \end{vmatrix} = (\lambda + k_1)(\lambda + \beta)(\lambda + \alpha) + x^{*2} - (\lambda + \alpha) = \lambda^3 + (k_1 + \beta + \alpha)\lambda^2$$
$$+ (k_1\alpha + \alpha\beta + k_1\beta)\lambda + \alpha\beta k_1 - \lambda - \alpha + x^{*2} = \lambda^3 + (k_1 + \beta + \alpha)\lambda^2$$
$$+ (k_1\alpha + \alpha\beta + k_1\beta - 1)\lambda - \alpha + x^{*2} + \alpha\beta k_1. \tag{18}$$

According to Lemma 3, the necessary and sufficient conditions for the cubic-degree polynomial (18) having a Hurwitz matrix are

$$\begin{cases} k_1\alpha + \alpha\beta + k_1\beta - 1 > 0, \\ (k_1 + \beta + \alpha)(k_1\alpha + \alpha\beta + k_1\beta - 1) > \alpha\beta k_1 \geqslant 2\beta k_1 - \alpha + x^{*2} > 0, \end{cases}$$

which shows that the zero solution of (18), i.e., any equilibrium point, $E_1$, $E_2$ or $E_3$, is globally, exponentially stable. Obviously, when

$$\begin{cases} k_1\alpha + \alpha\beta + k_1\beta - 1 > 0, \\ (k_1 + \beta + \alpha)(k_1\alpha + \alpha\beta + k_1\beta - 1) > 2\beta k_1 - \alpha > 0, \end{cases}$$

equilibrium point $E_1 = (0,0,0)$ is globally, exponentially stabilized. □

**Theorem 3.** *For system (8), if the control law is chosen as*: $u_1 = 0$, $u_2 = -z\bar{x} + h_1\bar{y} + \delta_2\bar{z}$, $u_3 = \bar{x}x - \delta_2\bar{x}$, *where*

$$\delta_2 \begin{cases} > 0, & \text{when } x^* > 0, \\ < 0, & \text{when } x^* < 0, \end{cases}$$

*when* $(\alpha + \beta + h_1)[\alpha(\beta + h_1) - 1] > 2\delta_2 x^* + \delta_2^2 + x^{*2} - \alpha > 0$, *any equilibrium point*, $E_1$, $E_2$ *or* $E_3$, *is globally*, *exponentially stabilized*; *when* $(\alpha + \beta + h_1)[\alpha(\beta + h_1) - 1] > \delta_2^2 - \alpha > 0$, *equilibrium point* $E_1 = (0,0,0)$ *is globally*, *exponentially stabilized.*

**Proof.** Under the given feedback control law, system (8) can be written as

$$\begin{aligned} \dot{\bar{x}} &= \bar{y}, \\ \dot{\bar{y}} &= \bar{x} - (\beta + h_1)\bar{y} - (x^* + \delta_2)\bar{z}, \\ \dot{\bar{z}} &= (x^* + \delta_2)\bar{x} - \alpha\bar{z}. \end{aligned} \tag{19}$$

The coefficient matrix of (19) is

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\beta - h_1 & -x^* - \delta_2 \\ x^* + \delta_2 & 0 & -\alpha \end{bmatrix}.$$

The corresponding characteristic polynomial is

$$\det(\lambda E_3 - A_3) = \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda + \beta + h_1 & x^* + \delta_2 \\ -x^* - \delta_2 & 0 & \lambda + \alpha \end{vmatrix} = \lambda(\lambda + \beta + h_1)(\lambda + \alpha) + (x^* + \delta_2)^2 - (\lambda + \alpha)$$
$$= \lambda^3 + (\beta + h_1 + \alpha)\lambda^2 + [\alpha(\beta + h_1) - 1]\lambda + \delta_2^2 + 2x^*\delta_2 + x^{*2} - \alpha. \tag{20}$$

By Lemma 3, the condition $(\alpha + \beta + h_1)[\alpha(\beta + h_1) - 1] > 2\delta_2 x^* + \delta_2^2 + x^{*2} - \alpha > 0$ guarantees that $A_3$ is a Hurwitz matrix. Therefore, all the equilibrium points of system (8), i.e., $E_1$, $E_2$ and $E_3$ can be globally, exponentially stabilized. When $(\alpha + \beta + h_1)[\alpha(\beta + h_1) - 1] > \delta_2^2 - \alpha > 0$, $E_1 = (0,0,0)$ is globally, exponentially stable. □

**Theorem 4.** *If in Eq. (8) choose the following control law*: $u_1 = k_2\bar{x}$, $u_2 = -\bar{x}\bar{z}$, $u_3 = -\bar{x}x$, *then when the positive number* $k_2$ *satisfies* $(\alpha + \beta + k_2)(\alpha\beta + \beta k_2 + \alpha k_2) > (2\beta k_2 + \alpha)$, $E_2$ *or* $E_3$ *can be globally*, *exponentially stabilized*; *when* $k_2 > \frac{1}{\beta}$, $E_1$ *can be globally*, *exponentially stabilized.*

**Proof.** For the control law given in this theorem, system (8) can be rewritten as

$$\begin{aligned}\dot{\bar{x}} &= -k_2\bar{x} + \bar{y},\\ \dot{\bar{y}} &= \bar{x} - \beta\bar{y} - x^*\bar{z} - z^*\bar{x},\\ \dot{\bar{z}} &= x^*\bar{x} - \alpha\bar{z}.\end{aligned} \tag{21}$$

The coefficient matrix of system (21) is

$$A_4 = \begin{bmatrix} -k_2 & 1 & 0 \\ 1 - z^* & -\beta & -x^* \\ x^* & 0 & -\alpha \end{bmatrix}.$$

When $z^* = 1$ and $x^* = \pm\sqrt{\alpha}$,

$$A'_4 = \begin{bmatrix} -k_2 & 1 & 0 \\ 0 & -\beta & \mp\sqrt{\alpha} \\ \pm\sqrt{\alpha} & 0 & -\alpha \end{bmatrix}.$$

the corresponding characteristic polynomial is

$$\det(\lambda E_3 - A'_4) = \begin{vmatrix} \lambda + k_2 & -1 & 0 \\ 0 & \lambda + \beta & \pm\sqrt{\alpha} \\ \mp\sqrt{\alpha} & 0 & \lambda + \alpha \end{vmatrix} = (\lambda + k_2)(\lambda + \beta)(\lambda + \alpha) + \alpha$$

$$= \lambda^3 + (\alpha + \beta + k_2)\lambda^2 + (\alpha\beta + k_2\beta + \alpha k_2)\lambda + \alpha\beta k_2 + \alpha.$$

When $(\alpha + \beta + k_2)(\alpha\beta + k_2\beta + \alpha k_2) > \alpha\beta k_2 + \alpha$, according to Lemma 3, $E_2$ or $E_3$ is globally, exponentially stable. When $x^* = y^* = z^* = 0$,

$$A''_4 = \begin{bmatrix} -k_1 & 1 & 0 \\ 1 & -\beta & 0 \\ 0 & 0 & -\alpha \end{bmatrix}.$$

Obviously, if and only if $k_1 > \frac{1}{\beta}$, $A''_4$ is a Hurwitz matrix. Thus, if and only if $k_1 > \frac{1}{\beta}$, $E_1$ is globally, exponentially stable. □

**Theorem 5.** *Under the following control law*: $u_1 = x\bar{z} + k_3\bar{x}$, $u_2 = 0$, $u_3 = -x\bar{y}$, *when* $k_3 > \max[\frac{(1+\beta)}{4\beta}, \frac{1}{\beta}]$, *any equilibrium point of system* (8), $E_1$, $E_2$ *or* $E_3$ *can be globally, exponentially stabilized*; *when* $k_3 > \frac{1}{\beta}$, *equilibrium point* $E_1$ *can be globally, exponentially stabilized.*

**Proof.** Under the given control law, system (8) can be written as:

$$\begin{aligned}\dot{\bar{x}} &= -k_3\bar{x} + \bar{y} - x\bar{z},\\ \dot{\bar{y}} &= (1 - z^*)\bar{x} - \beta\bar{y} - x\bar{z},\\ \dot{\bar{z}} &= x\bar{x} + x^*\bar{x} + x\bar{y} - \alpha\bar{z}.\end{aligned} \tag{22}$$

For the above system, construct the quadratic, positive definite Lyapunov function,

$$V = \bar{x}^2 + \bar{y}^2 + \bar{z}^2. \tag{23}$$

Let $P = \mathrm{diag}(1,1,1) = I_3$. Then, $\lambda_{\max}(P) = \lambda_{\min}(P) = 1$. Evaluating the $\frac{dV}{dt}$ along the trajectory of system (22) results in

$$\left.\frac{dV}{dt}\right|_{(22)} = 2\bar{x}\dot{\bar{x}} + 2\bar{y}\dot{\bar{y}} + 2\bar{z}\dot{\bar{z}} = -2k_3\bar{x}^2 + 2\bar{x}\bar{y} - 2x\bar{x}\bar{z} + 2(1 - z^*)\bar{x}\bar{y} - 2\beta\bar{y}^2 - 2x\bar{y}\bar{z} + 2x\bar{x}\bar{z} + 2x^*\bar{x}\bar{z} + 2x\bar{y}\bar{z} - 2\alpha\bar{z}^2$$

$$= \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2k_3 & 2 - z^* & x^* \\ [0.5ex]2 - z^* & -2\beta & 0 \\ x^* & 0 & -2\alpha \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \stackrel{\mathrm{def}}{=} (\bar{x} \quad \bar{y} \quad \bar{z})Q_1(\bar{x} \quad \bar{y} \quad \bar{z})^{\mathrm{T}}.$$

Thus, for $E_1$, i.e., for $x^* = y^* = z^* = 0$, $Q_1$ becomes

$$Q'_1 = \begin{bmatrix} -2k_3 & 2 & 0 \\ 2 & -2\beta & 0 \\ 0 & 0 & -2\alpha \end{bmatrix}.$$

Hence when $k_1 > \frac{1}{\beta}$, $Q_1'$ is a symmetric negative definite matrix.

Next, for $(x^*, y^*, z^*) \neq 0$, i.e., for $E_2$ and $E_3$, $Q_1$ becomes

$$Q_1'' = \begin{bmatrix} -2k_3 & 1 & \pm\sqrt{\alpha} \\ 1 & -2\beta & 0 \\ \pm\sqrt{\alpha} & 0 & -2\alpha \end{bmatrix}.$$

Obviously, when $-8\alpha\beta k_3 + 2\alpha\beta + 2\alpha < 0$, i.e., $k_3 > \max[\frac{(1+\beta)}{4\beta}, \frac{1}{\beta}]$, $Q_1''$ is a symmetric negative definite matrix.

By Lemma 1, for $E_1$, we have the following estimation:

$$\bar{x}^2(t) + \bar{y}^2(t) + \bar{z}^2(t) \leqslant [\bar{x}^2(t_0) + \bar{y}^2(t_0) + \bar{z}^2(t_0)]e^{\lambda_{\max}(Q_1')(t-t_0)};$$

while for $E_2$ or $E_3$, the estimation is:

$$\bar{x}^2(t) + \bar{y}^2(t) + \bar{z}^2(t) \leqslant [\bar{x}^2(t_0) + \bar{y}^2(t_0) + \bar{z}^2(t_0)]e^{\lambda_{\max}(Q_1'')(t-t_0)}. \qquad \square$$

**Theorem 6.** *If the following control law: $u_1 = k_4\bar{x} - z^*\bar{y} + x\bar{z}$, $u_2 = 0$, $u_3 = -x\bar{y}$ is chosen for Eq. (8), then when $k_4 > \frac{1}{\beta}$, $E_1 = (0,0,0)$ is globally, exponentially stabilized; and when $k_4 > \max[\frac{\beta+4\alpha}{4\beta}, \frac{1}{\beta}]$, $E_2$ or $E_3$ can be globally, exponentially stabilized.*

**Proof.** Under the given control law, system (8) becomes

$$\begin{aligned} \dot{\bar{x}} &= k_4\bar{x} + z^*\bar{y} - x\bar{z} + \bar{y}, \\ \dot{\bar{y}} &= \bar{x} - \beta\bar{y} - z^*\bar{x} - x\bar{z}, \\ \dot{\bar{z}} &= (x + x^*)\bar{x} - \alpha\bar{z} + x\bar{y}. \end{aligned} \tag{24}$$

Taking the same positive definite and radially unbounded Lyapunov function (23) yields

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(24)} &= 2\bar{x}\dot{\bar{x}} + 2\bar{y}\dot{\bar{y}} + 2\bar{z}\dot{\bar{z}} \\ &= -2k_4\bar{x}^2 + 2z^*\overline{xy} - 2x\overline{xz} + 2\overline{xy} + 2\overline{xy} - 2\beta\bar{y}^2 - 2z^*\overline{xy} - 2x\overline{yz} + 2x\overline{xz} + 2x^*\overline{xz} - 2\alpha\bar{z}^2 + 2x\overline{yz} \\ &= \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2k_4 & 2 & x^* \\ 2 & -2\beta & 0 \\ x^* & 0 & -2\alpha \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \stackrel{\text{def}}{=} (\bar{x}\ \bar{y}\ \bar{z})\ Q_2\ (\bar{x}\ \bar{y}\ \bar{z})^{\mathrm{T}}. \end{aligned} \tag{25}$$

For $E_1$, $Q_2$ becomes

$$Q_2' = \begin{bmatrix} -2k_4 & 2 & 0 \\ 2 & -2\beta & 0 \\ 0 & 0 & -2\alpha \end{bmatrix}.$$

Hence, in this case, when $k_4 > \frac{1}{\beta}$, $Q_2'$ is negative definite. We have the following estimation:

$$\bar{x}^2(t) + \bar{y}^2(t) + \bar{z}^2(t) \leqslant [\bar{x}^2(t_0) + \bar{y}^2(t_0) + \bar{z}^2(t_0)]e^{\lambda_{\max}(Q_2')(t-t_0)}.$$

For $E_2$ and $E_3$, $Q_2$ becomes

$$Q_2'' = \begin{bmatrix} -2k_4 & 2 & x^* \\ 2 & -2\beta & 0 \\ x^* & 0 & -2\alpha \end{bmatrix}.$$

Hence, when $k_4 > \max[\frac{\beta+4\alpha}{4\beta}, \frac{1}{\beta}]$, $Q_2''$ is negative definite. We have the following estimation:

$$\bar{x}^2(t) + \bar{y}^2(t) + \bar{z}^2(t) \leqslant [\bar{x}^2(t_0) + \bar{y}^2(t_0) + \bar{z}^2(t_0)]e^{\lambda_{\max}(Q_2'')(t-t_0)},$$

indicating that the conclusion of Theorem 5 holds. $\square$

## 4. Globally exponential synchronization of two SM chaotic systems

In this section, we shall consider the global and exponential stability of the zero solution of (11), i.e., the global, exponential synchronization between systems (9) and (10).

**Theorem 7.** *In Eq. (10), if we take any one of the following control laws:*

(1) $u_1 = -\hat{k}_1 e_x - x_d e_z - x_r e_z + z_r e_y,\ u_2 = 0,\ u_3 = x_d e_y;$
(2) $u_1 = -\hat{k}_1 e_x - x_d e_z - x_r e_z + z_r e_y,\ u_2 = x_d e_z,\ u_3 = 0;$
(3) $u_1 = -\hat{k}_1 e_x + z_r e_y,\ u_2 = 0,\ u_3 = -(x_d + x_r)\ e_x + x_d e_y;$

*where $\hat{k}_1 > \frac{1}{\beta}$, the zero solution of (11) is globally, exponentially stable and so systems (9) and (10) are globally, exponentially synchronized.*

**Proof.** With the designed control law (1) of Theorem 7, system (11) becomes:

$$\begin{aligned}\dot{e}_x &= -\hat{k}_1 e_x - (x_d e_z + x_r e_z) + z_r e_y + e_y,\\ \dot{e}_y &= e_x - x_d e_z - z_r e_x - \beta e_y,\\ \dot{e}_z &= (x_d + x_r) e_x + x_d e_y - \alpha e_z.\end{aligned} \tag{26}$$

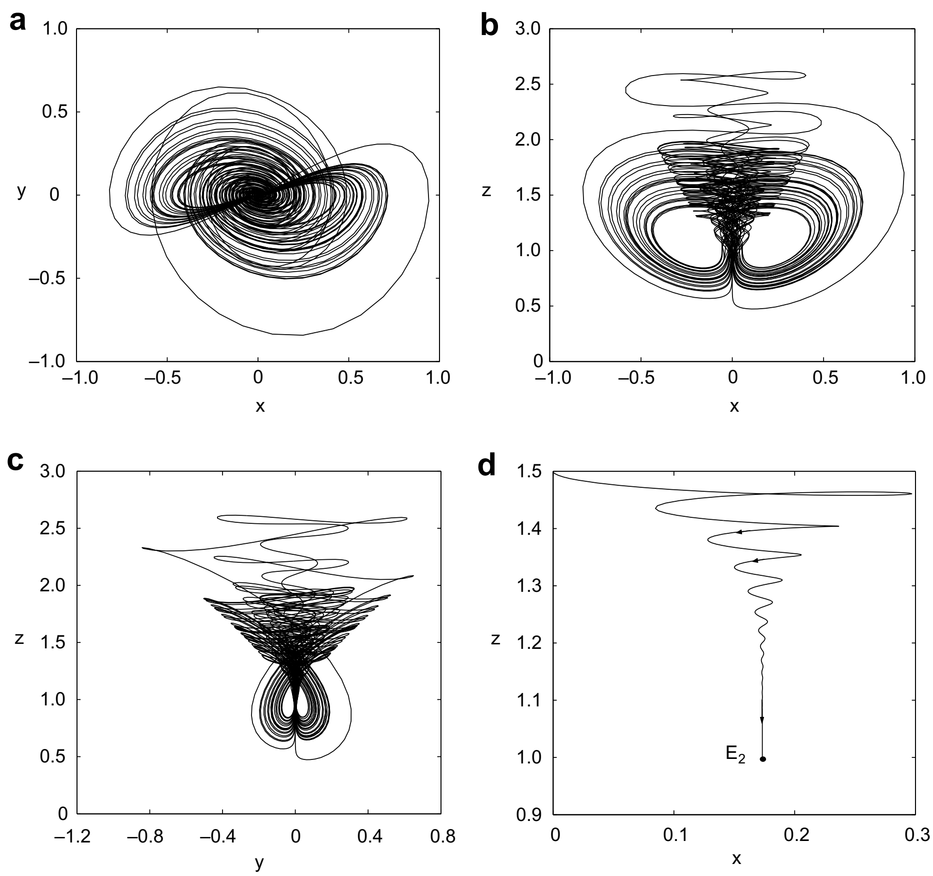


Fig. 1. Trajectories of system (5) with the initial condition, $x(0) = 2.88 \times 10^{-5}$, $y(0) = 0.3 \times 10^{-6}$, $z(0) = 1.5$ for chaotic attractor without control: (a) projected on the $x$–$y$ plane, (b) projected on the $x$–$z$ plane, (c) projected on the $y$–$z$ plane; and (d) with the control law given in Theorem 1 for $\delta_1 = -2$, globally convergent to the equilibrium point, $E_2$: (0.1732,0,1).

Under the designed control law (2) of Theorem 7, system (11) becomes:

$$\begin{aligned}\dot{e}_x &= -\hat{k}_1 e_x + e_y - (x_\mathrm{d} e_z + x_\mathrm{r} e_z) + z_\mathrm{r} e_y,\\ \dot{e}_y &= e_x - z_\mathrm{r} e_x - \beta e_y,\\ \dot{e}_z &= (x_\mathrm{d} + x_\mathrm{r}) e_x - \alpha e_z.\end{aligned} \tag{27}$$

By the designed control law (3) of Theorem 7, system (11) becomes:

$$\begin{aligned}\dot{e}_x &= -\hat{k}_1 e_x + e_y + z_\mathrm{r} e_y,\\ \dot{e}_y &= e_x - x_\mathrm{d} e_z - z_\mathrm{r} e_x - \beta e_y,\\ \dot{e}_z &= x_\mathrm{d} e_y - \alpha e_z.\end{aligned} \tag{28}$$

By constructing the positive definite and radially unbounded Lyapunov function,

$$V = e_x^2 + e_y^2 + e_z^2,$$

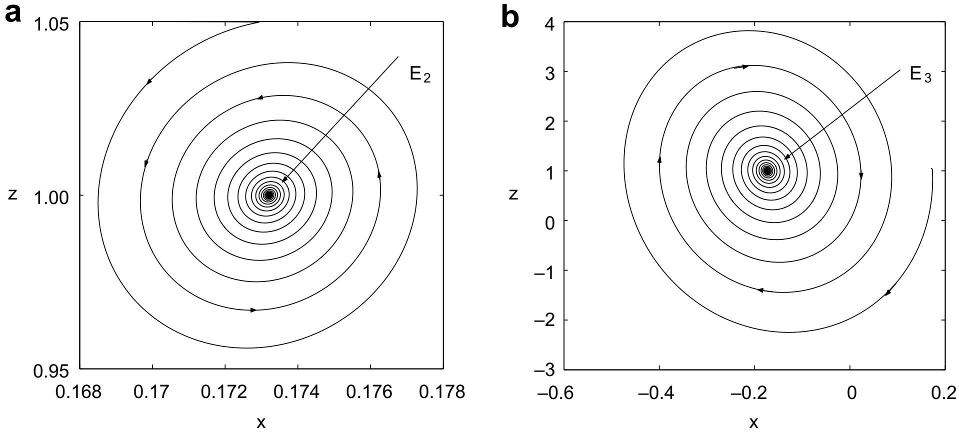


Fig. 2. Trajectories of system (5) with the control law given in Theorem 3 using the initial condition, $x(0) = 0.17$, $y(0) = 0.3$, $z(0) = 1.05$: (a) convergent to the equilibrium point, $E_2$: (0.1732,0,1) when $h_1 = 100$ and $\delta_2 = 2$; and (b) convergent to the equilibrium point, $E_3$: (−0.1732,0,1) when $h_1 = 100$ and $\delta_2 = -2$.

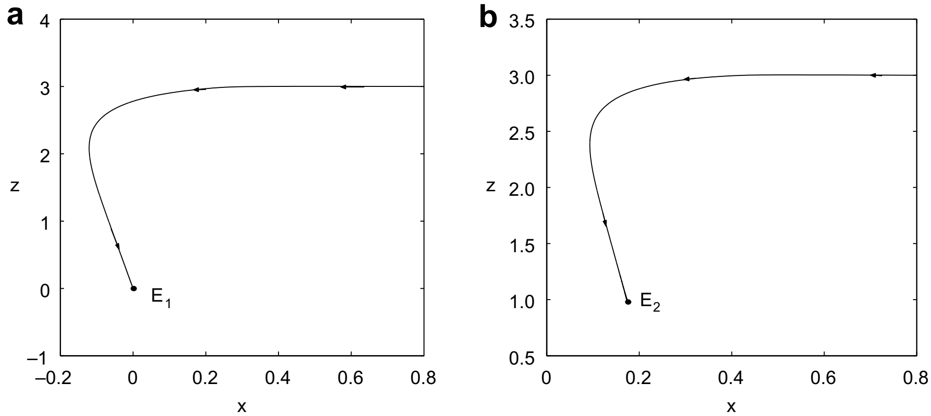


Fig. 3. Trajectories of system (5) with the control law given in Theorem 4 using the initial condition, $x(0) = 0.8$, $y(0) = 3$, $z(0) = 3$: (a) convergent to the equilibrium point, $E_1$: (0,0,0) when $k_2 = 10$; and (b) convergent to the equilibrium point, $E_2$: (0.1732,0, 1) when $k_2 = 10$.

we obtain

$$
\begin{aligned}
\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(26)} &= 2e_x\dot{e}_x + 2e_y\dot{e}_y + 2e_z\dot{e}_z \\
&= 2e_x[-\hat{k}_1 e_x - (x_\mathrm{d} e_z + x_\mathrm{r} e_z) + z_\mathrm{r} e_y + e_y] + 2e_y[e_x - x_\mathrm{d} e_z - z_\mathrm{r} e_x - \beta e_y] + 2e_z[(x_\mathrm{d} + x_\mathrm{r})e_x + x_\mathrm{d} e_y - \alpha e_z] \\
&= -2\hat{k}_1 e_x^2 - 2x_\mathrm{d} e_x e_z - 2x_\mathrm{r} e_x e_z + 2z_\mathrm{r} e_x e_y + 2e_x e_y + 2e_x e_y - 2x_\mathrm{d} e_y e_z - 2z_\mathrm{r} e_x e_y - 2\beta e_y^2 + 2x_\mathrm{d} e_x e_z \\
&\quad + 2x_\mathrm{r} e_x e_z + 2x_\mathrm{d} e_y e_z - 2\alpha e_z^2 \\
&= \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}^\mathrm{T} \begin{bmatrix} -2\hat{k}_1 & 2 & 0 \\ 2 & -2\beta & 0 \\ 0 & 0 & -2\alpha \end{bmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \stackrel{\mathrm{def}}{=} (\, e_x \quad e_y \quad e_z \,) Q_3 (\, e_x \quad e_y \quad e_z \,)^\mathrm{T}.
\end{aligned} \tag{29}
$$

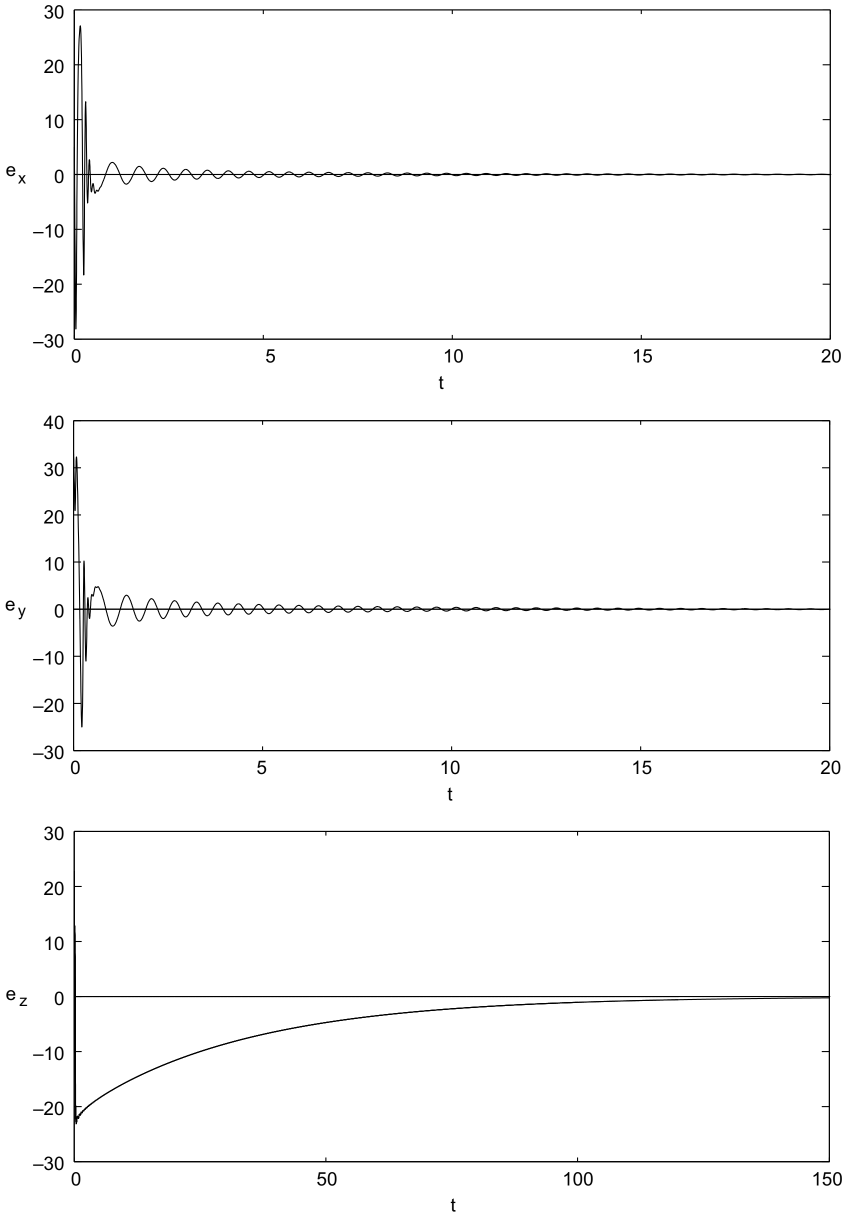


Fig. 4. Error history of system (11) with the control law given in Theorem 7, (1) using the initial conditions, $x_\mathrm{d}(0) = 10$, $y_\mathrm{d}(0) = 20$, $z_\mathrm{d}(0) = 8$ and $x_\mathrm{r}(0) = 5$, $y_\mathrm{r}(0) = -13$, $z_\mathrm{r}(0) = -10$; when $\hat{k}_1 = 4$.

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(27)} &= 2e_x\dot{e}_x + 2e_y\dot{e}_y + 2e_z\dot{e}_z \\ &= 2e_x[-\hat{k}_1e_x + e_y - (x_de_z + x_re_z) + z_re_y] + 2e_y[e_x - z_re_x - \beta e_y] + 2e_z[(x_d + x_r)e_x - \alpha e_z] \\ &= -2\hat{k}_1e_x^2 + 2e_xe_y - 2x_de_xe_z - 2x_re_xe_z + 2z_re_xe_y + 2e_xe_y - 2z_re_xe_y - 2\beta e_y^2 \\ &\quad + 2x_de_xe_z + 2x_re_xe_z - 2\alpha e_z^2 \stackrel{\text{def}}{=} (e_x \quad e_y \quad e_z)Q_3(e_x \quad e_y \quad e_z)^{\mathrm{T}}.\end{aligned} \tag{30}$$

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(28)} &= 2e_x\dot{e}_x + 2e_y\dot{e}_y + 2e_z\dot{e}_z = 2e_x[-\hat{k}_1e_x + z_re_y + e_y] + 2e_y[e_x - x_de_z - z_re_x - \beta e_y] \\ &\quad + 2e_z[x_de_y - \alpha e_z] = -2\hat{k}_1e_x^2 + 2z_re_xe_y + 2e_xe_y + 2e_xe_y - 2x_de_ye_z - 2z_re_xe_y - 2\beta e_y^2 \\ &\quad + 2x_de_ye_z - 2\alpha e_z^2 \stackrel{\text{def}}{=} (e_x \quad e_y \quad e_z)Q_3(e_x \quad e_y \quad e_z)^{\mathrm{T}}.\end{aligned} \tag{31}$$

Here, $Q_3$ is symmetric. Since $\hat{k}_1 > \frac{1}{\beta}$, $Q_3$ is negative definite. Consequently, by Lemma 1 we have

$$e_x^2(t) + e_y^2(t) + e_z^2(t) \leqslant [e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)]e^{\lambda_{\max}(Q_3)(t-t_0)}. \tag{32}$$

The zero solution of system (11) is globally, exponentially stabilized and so systems (9) and (10) are globally exponentially synchronized. □

**Theorem 8.** *In Eq.* (10), *if any one of the following control laws were taken*:

(1) $u_1 = -\hat{k}_1e_x - (x_d + x_r)e_z$, $u_2 = x_de_z + z_re_x$, $u_3 = 0$;
(2) $u_1 = -\hat{k}_1e_x - x_de_z - z_re_x + z_re_y$, $u_2 = 0$, $u_3 = x_de_y$;
(3) $u_1 = -\hat{k}_1e_x + z_de_y - x_de_z - x_re_z$, $u_2 = x_re_z$, $u_3 = 0$;
(4) $u_1 = -\hat{k}_1e_x + z_re_y$, $u_2 = 0$, $u_3 = -(x_d + x_r)\ e_x + x_d\ e_y$;

where $\hat{k}_1 > \frac{1}{\beta}$, *the zero solution of Eq.* (11) *is globally*, *exponentially stable and so systems* (9) *and* (10) *are globally*, *exponentially synchronized with the same estimation Eq.* (32).

**Proof.** With the control laws given in Theorem 8, system (11) can be written as

$$\begin{aligned}\dot{e}_x &= -\hat{k}_1e_x + e_y - (x_d + x_r)e_z, \\ \dot{e}_y &= e_x - x_dz_d + x_rz_r - \beta e_y + x_de_z + z_re_x, \\ \dot{e}_z &= (x_d + x_r)e_x - \alpha e_z,\end{aligned} \tag{33}$$

$$\begin{aligned}\dot{e}_x &= -\hat{k}_1e_x + e_y - x_de_z - x_re_z + z_re_y, \\ \dot{e}_y &= e_x - x_de_z - z_re_x - \beta e_y, \\ \dot{e}_z &= (x_d + x_r)e_x + x_de_y - \alpha e_z.\end{aligned} \tag{34}$$

$$\begin{aligned}\dot{e}_x &= -\hat{k}_1e_x + e_y + z_de_y - x_de_z - x_re_z, \\ \dot{e}_y &= e_x - x_re_z - z_de_x - \beta e_y + x_re_z, \\ \dot{e}_z &= (x_d + x_r)e_x - \alpha e_z,\end{aligned} \tag{35}$$

and

$$\begin{aligned}\dot{e}_x &= -\hat{k}_1e_x + e_y + z_re_y, \\ \dot{e}_y &= e_x - x_de_z - z_re_x - \beta e_y, \\ \dot{e}_z &= x_de_y - \alpha e_z,\end{aligned} \tag{36}$$

respectively. We still construct the positive definite and radially unbounded Lyapunov function.

$$V = e_x^2 + e_y^2 + e_z^2.$$

Similar to the proof of Theorem 7, we have

$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{()(33)-(36)} = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}^{\mathrm{T}} \begin{bmatrix} -2\hat{k}_1 & 2 & 0 \\ 2 & -2\beta & 0 \\ 0 & 0 & -2\alpha \end{bmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \stackrel{\mathrm{def}}{=} (\, e_x \quad e_y \quad e_z \,) Q_4 (\, e_x \quad e_y \quad e_z \,)^{\mathrm{T}},$$

where $\hat{k}_1 > \frac{1}{\beta}$, which leads to the same estimation given in the proof of Theorem 7:

$$e_x^2(t) + e_y^2(t) + e_z^2(t) \leqslant [e_x^2(t_0) + e_y^2(t_0) + e_z^2(t_0)]\mathrm{e}^{\lambda_{\max}(Q_4)(t-t_0)}.$$

Thus zero solution of system (11) is globally, exponentially stabilized and so systems (9) and (10) are globally exponentially synchronized. □

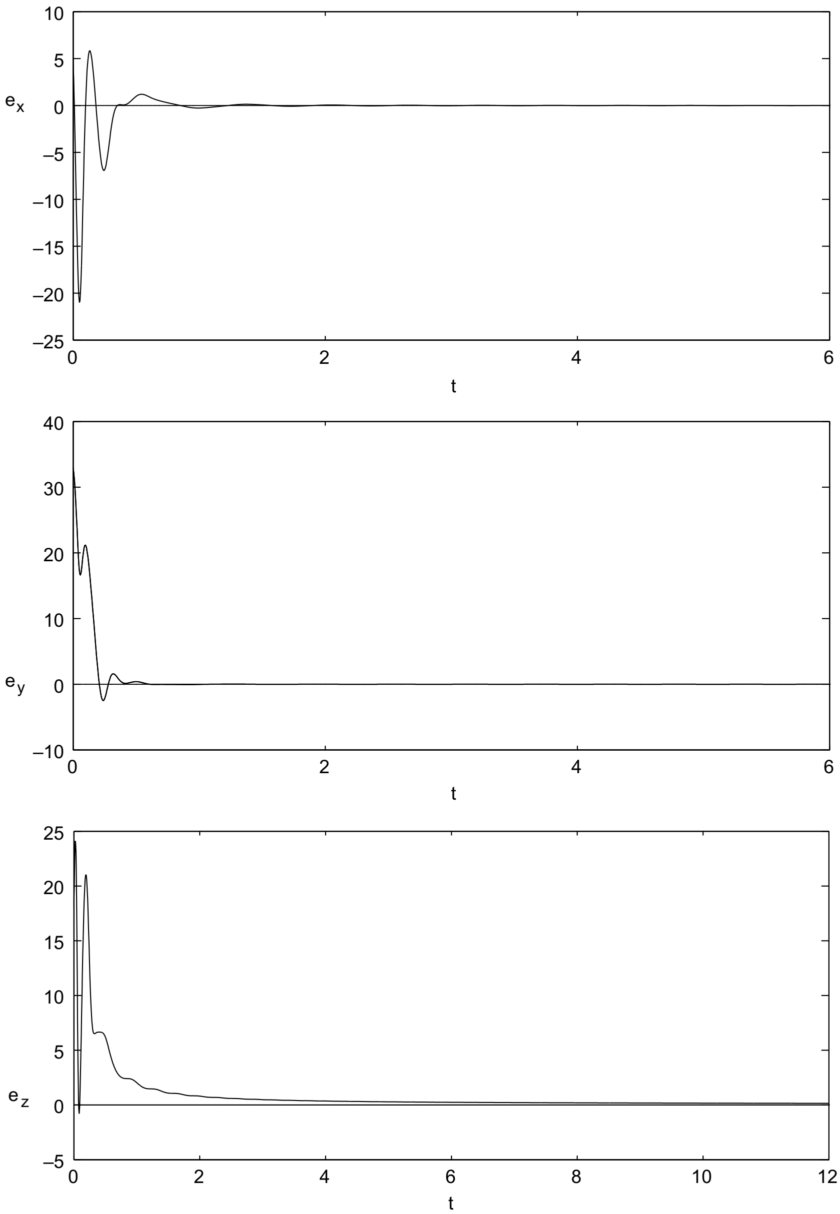


Fig. 5. Error history of system (11) with the control law given in Theorem 8, (2) using the initial conditions, $x_{\mathrm{d}}(0) = 10$, $y_{\mathrm{d}}(0) = 20$, $z_{\mathrm{d}}(0) = 8$ and $x_{\mathrm{r}}(0) = 5$, $y_{\mathrm{r}}(0) = -13$, $z_{\mathrm{r}}(0) = -10$; when $\hat{k}_1 = 4$.

## 5. Numerical simulation examples

In this section, we use a 4th-order Runge-Kutta method to simulate the stabilization of equilibrium points by using system (5) with the control law given in Eq. (8). We also present numerical results for the synchronization between systems (9) and (10) under the feedback control. The system parameters take the values: $\alpha = 0.03$ and $\beta = 0.3$.

For the stabilization problem, we show three examples using the control laws given in Theorems 1, 3 and 4, respectively, since the remaining three cases are similar to Theorem 4. The initial conditions are chosen as

$$x(0) = 2.88 \times 10^{-5}, \quad y(0) = 3 \times 10^{-7}, \quad z(0) = 1.5, \tag{37}$$

$$x(0) = 0.17, \quad y(0) = 0.3, \quad z(0) = 1.05 \tag{38}$$

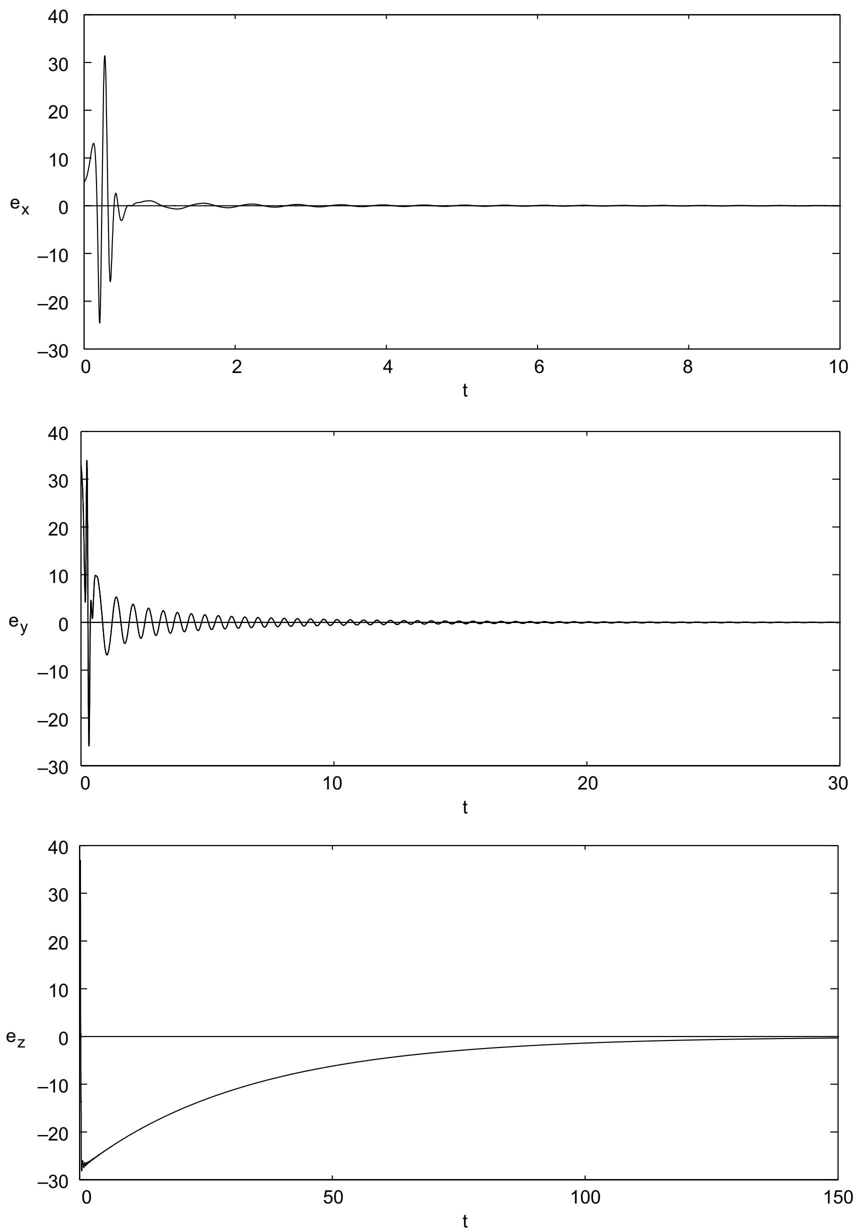


Fig. 6. Error history of system (11) with the control law given in Theorem 8, (3) using the initial conditions, $x_d(0) = 10$, $y_d(0) = 20$, $z_d(0) = 8$ and $x_r(0) = 5$, $y_r(0) = -13$, $z_r(0) = -10$; when $\hat{k}_1 = 4$.

and

$$x(0) = 0.8, \quad y(0) = 3 \quad z(0) = 3, \tag{39}$$

respectively.

Without the feedback control, the system exhibits chaotic motion, as shown in Figs. 1a–c. The strange attractor is similar to Chen's attractor [16]. When the control law given in Theorem 1, with $\delta_1 = -2$, is applied to system (5) and using $E_2$ as the designed equilibrium point, the trajectory of the controlled system globally, exponentially converges to $E_2$, as depicted by Fig. 1d.

When the control law given in Theorem 3 is applied to system (5), with $h_1 = 100$ and $\delta_2 = 2$, and using $E_2$ as the designed equilibrium point, it is shown (see Fig. 2a) that the trajectory converges to $E_2$; while for $h_1 = 100$ and

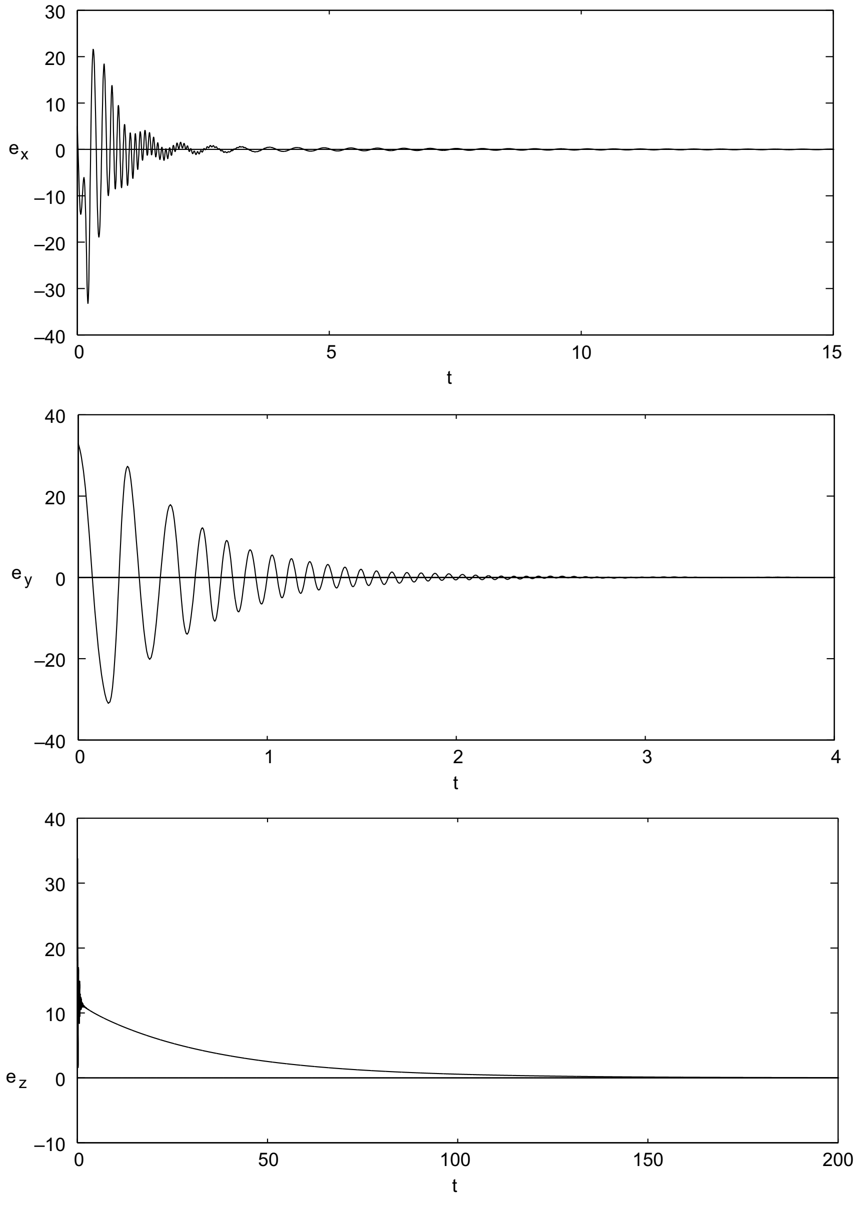


Fig. 7. Error history of system (11) with the control law given in Theorem 8, (4) using the initial conditions, $x_d(0) = 10$, $y_d(0) = 20$, $z_d(0) = 8$ and $x_r(0) = 5$, $y_r(0) = -13$, $z_r(0) = -10$; when $\hat{k}_1 = 4$.

$\delta_2 = -2$, and using $E_3$ as the designed equilibrium point it can be seen from Fig. 2b that the trajectory converges to $E_3$. Similarly, applying the control law given in Theorem 4 to system (5), we obtain the convergence to $E_1$ or $E_2$ by taking $k_2 = 10$, as depicted in Fig. 3.

For the synchronization problem, we present four examples by employing the four control laws given in Theorems 7 and 8, respectively. The initial conditions for the driving system are:

$$x_r(0) = 10, \quad y_r(0) = 20, \quad z_r(0) = 8, \tag{40}$$

while that for the receiving system are:

$$x_r(0) = 5, \quad y_r(0) = -13, \quad z_r(0) = -10, \tag{41}$$

which are quite different from that of the driving system. The values of parameters $\hat{k}_1$ is chosen the same for all the four cases: $\hat{k}_1 = 4$. The time histories for the error signals $e_x(t)$, $e_y(t)$ and $e_z(t)$ obtained for the four cases are displayed in Figs. 4–7. All the four cases show the convergence of the errors to zero exponentially as expected, but with irregular vibrating motions during the transient period.

## 6. Conclusion

In this paper, we have studied in detail the chaos control and chaos synchronization of the SM (Shimizu–Morioka) chaotic system. For a given equilibrium point, we designed various linear and nonlinear feedback control laws to globally, exponentially stabilize the equilibrium point. We also designed a variety of possibly simplest feedback control laws for two SM chaotic systems such that the zero solution of the error system is globally, exponentially stable, and thus the two SM chaotic systems are globally, exponentially synchronized.

Because all the control laws obtained in this paper are designed according to the properties of the given system, and thus all the conditions are simple, constructional, and algebraic. Since all conditions are not abstract or describing the existence theory, it is very convenient to apply the method and conclusion presented in this paper to in practical problems. The methods can be easily generalized to consider other systems to obtain similar results.

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