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Centers and isochronous centers of a class of quasi-analytic switching systems

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Abstract In this paper, we study the integrability and linearization of a class of quadratic quasi-analytic switching systems. We improve an existing method to compute the focus values and periodic constants of quasi-analytic switching systems. In particular, with our method, we demonstrate that the dynamical behaviors of quasi-analytic switching systems are more complex than those of continuous quasi-analytic systems, by showing the existence of six and seven limit cycles in the neighborhood of the origin and infinity, respectively, in a quadratic quasi-analytic switching system. Moreover, explicit conditions are obtained for classifying the centers and isochronous centers of the system.

Keywords quasi-analytic switching systems, Lyapunov constant, limit cycle, center, isochronous center

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1 Introduction

The problem of characterizing the centers and isochronous centers of dynamical systems has attracted the attention of many researchers. So far, regarding the family of polynomial differential systems, a complete classification of the centers and isochronous centers has only been solved for quadratic polynomial systems, or simply quadratic systems. Quadratic systems having a center were classified by Dulac [5], Kapteyn [10, 11], Bautin [2], Żołądek [35] and Yu and Han [34], while quadratic systems having an isochronous center were characterized by Loud [25]. Centers of the cubic systems with homogeneous nonlinearities were studied in [24, 32], and the isochronous centers for such cubic systems were further investigated by Pleshkan [29]. However, it is still far away from obtaining a complete classification of the centers and isochronous centers for polynomial differential systems of degree three, and it is extremely difficult to study these problems when the degree of the systems is increased. For example, consider the following systems:

$$\dot{z} = (\lambda + \mathrm{i})z + (z\bar{z})^{\frac{d-5}{2}}(Az^{4+j}\bar{z}^{1-j} + Bz^3\bar{z}^2 + Cz^{2-j}\bar{z}^{3+j} + D\bar{z}^5), \quad d = 2m + 1 \geqslant 5, \tag{1.1}$$

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$$\dot{z} = \mathrm{i}z + (z\bar{z})^{\frac{d-4}{2}}(Az^3\bar{z} + Bz^2\bar{z}^2 + C\bar{z}^4), \quad d = 2m \geqslant 4, \tag{1.2}$$

$$\dot{z} = (\lambda + \mathrm{i})z + (z\bar{z})^{\frac{d-3}{2}}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3), \quad d = 2m + 1 \geqslant 3, \tag{1.3}$$

$$\dot{z} = (\lambda + \mathrm{i})z + (z\bar{z})^{\frac{d-2}{2}}(Az^2 + Bz\bar{z} + C\bar{z}^2), \quad d = 2m \geqslant 2, \tag{1.4}$$

which have been investigated by Llibre and Valls [20–23], and the conditions on the centers and isochronous centers were obtained. However, in these articles, the parameter d was restricted to making the systems be polynomial systems.

Recently, switching systems have been widely used in modelling many practical problems in science and engineering. A theory suggests that switching systems can be considered as a uniform limit of continuous systems, and that the global dynamics of continuous models may be approximated by switching systems. In fact, the richness of dynamical behavior found in switching systems covers almost all the phenomena discussed in general continuous systems, such as limit cycles, homoclinic and heteroclinic orbits, strange attractors. For example, Leine and Nijmeijer [13], and Zou et al. [36] considered non-smooth Hopf bifurcation in switching systems. Bifurcation of limit cycles from the centers of discontinuous quadratic systems was studied by Chen and Du [3]. Limit cycles in a class of continuous and discontinuous cubic polynomial differential systems were investigated by Llibre et al. [18]. Bifurcation of limit cycles in discontinuous quadratic differential systems with two zones was considered in [19]. The Melnikov function method has also been extended to study homoclinic bifurcation of non-smooth systems (see [4, 12]). In addition, some general efficient methods have also been developed to study non-smooth systems. Among these methods, normal form computation for impact oscillators was given in [6], and a general methodology for reducing multidimensional flows to low-dimensional maps in piecewise nonlinear oscillators was proposed in [28]. The center and isochronous center conditions for switching systems associated with elementary singular points were discussed in [14].

More recently, quasi-analytic systems have also been widely used in modelling many practical problems. By "quasi-analytic", we mean that the system may be analytic for some parameters but not for some other parameters. For example, an axis-symmetric quasi-analytical model was developed in order to simulate the behavior of a remote field eddy current (RFEC) system during its operation (see [26]). A quasi-analytical model for scattering infrared near-field microscopy has been designed for predicting and analyzing signals on layered samples (see [9]). A simple quasi-analytical model was developed in [27] to study the response of ice-sheets to climate. On the other hand, a general type of quasi-analytic systems, described by

$$\begin{aligned} \dot{x} &= \delta x - y + \sum_{k=2}^{\infty}(x^2 + y^2)^{\frac{(k-1)(\lambda-1)}{2}} X_k(x, y), \\ \dot{y} &= x + \delta y + \sum_{k=2}^{\infty}(x^2 + y^2)^{\frac{(k-1)(\lambda-1)}{2}} Y_k(x, y), \end{aligned} \tag{1.5}$$

where

$$X_k(x, y) = \sum_{\alpha+\beta=k} A_{\alpha\beta}x^{\alpha}y^{\beta}, \quad Y_k(x, y) = \sum_{\alpha+\beta=k} B_{\alpha\beta}x^{\alpha}y^{\beta},$$

has been studied by Liu [15] and Liu et al. [17]. As the special cases, quadratic quasi-analytic systems have been studied in [16] and cubic quasi-analytic systems in [33]. In particular, generalized focal values and bifurcation of limit cycles for quadratic quasi-analytic systems were discussed in [17]. Here, a quadratic quasi-analytic system is defined by taking $k = 2$ only in (1.5). Similarly, cubic quasi-analytic systems can be defined.

Similar to quasi-analytic continuous systems, in this paper, we propose to study the center and isochronous center conditions for the following class of discontinuous planar systems:

$$\begin{aligned} \dot{x} &= \delta x - y + \sum_{k=2}^{\infty}(x^2 + y^2)^{\frac{(k-1)(\lambda-1)}{2}} F_k^+(x, y), \\ \dot{y} &= x + \delta y + \sum_{k=2}^{\infty}(x^2 + y^2)^{\frac{(k-1)(\lambda-1)}{2}} G_k^+(x, y), \end{aligned} \quad y > 0, \tag{1.6a}$$

$$\begin{aligned}\dot{x} &= \delta x - y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} F_k^-(x,y),\\ \dot{y} &= x + \delta y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} G_k^-(x,y),\end{aligned} \qquad y<0, \tag{1.6b}$$

for $(x,y)\in\mathbb{R}^2$, where the two subsystems (1.6a) and (1.6b) describe dynamics on the upper and lower half planes, called the upper and lower systems, respectively. For $\lambda>0$ (<0), the linear terms in (1.6) are the lowest (highest) order terms in these functions. Hence, when $\lambda>0$, the origin of (1.6) is a center or a focus. When $\lambda<0$, System (1.6) has no real singular point in the equator of Poincaré compactification, but the point at infinity is a center or a focus. Therefore, it is necessary to determine, for $\lambda\neq 0$, whether or not the origin (when $\lambda>0$) or infinity (when $\lambda<0$) is a center (or a weak focus). As a continuous work of [14, 15, 17], we generalize the study to consider the center and isochronous center conditions of quasi-analytic switching systems and hope to promote the research in this direction.

As compared with bifurcations, the center and isochronous center problems of switching systems have not received much attention. They should be considered carefully for switching systems because they are closely related to the conditions on integrability and linearization. Freire [7] discussed the center problem in a piecewise linear system. But it is much harder to solve the center or isochronous center problem for nonlinear or piecewise linear systems because the classical methods for computing Lyapunov constants and periodic constants are no longer applicable. Thus, new techniques are needed to develop.

In this paper, we study quasi-analytic switching systems mainly from two aspects. First of all, we modify and improve existing methods to compute Lyapunov constants and periodic constants for quasi-analytic switching systems. Secondly, as an application, we study a quadratic quasi-analytic switching system and derive its center and isochronous center conditions.

The rest of the paper is organized as follows. In Section 2, we present a method to compute the return map of System (1.6). As an application, a class of quadratic quasi-analytic switching systems is studied in Section 3, and the center and isochronous center are classified by using our method. Finally, our conclusion is drawn in Section 4.

2 Lyapunov constants of the quasi-analytic switching system (1.6)

Under the transformation of the polar coordinates

$$x = r^{\frac{1}{\lambda}}\cos\theta, \quad y = r^{\frac{1}{\lambda}}\sin\theta, \tag{2.1}$$

(1.5) becomes

$$\dot{r} = \lambda r\left(\delta + \sum_{k=1}^{\infty}\varphi_{k+2}(\theta) r^k\right), \quad \dot{\theta} = 1 + \sum_{k=1}^{\infty}\psi_{k+2}(\theta) r^k, \quad r\geqslant 0, \tag{2.2}$$

where $\varphi_k(\theta)$ and $\psi_k(\theta)$ are polynomial functions in $\cos\theta$ and $\sin\theta$, given in the form of

$$\begin{aligned}\varphi_k(\theta) &= \cos\theta X_{k-1}(\cos\theta,\sin\theta) + \sin\theta Y_{k-1}(\cos\theta,\sin\theta),\\ \psi_k(\theta) &= \cos\theta Y_{k-1}(\cos\theta,\sin\theta) - \sin\theta X_{k-1}(\cos\theta,\sin\theta).\end{aligned}$$

Then, it follows from (2.2) that

$$\frac{dr}{d\theta} = \lambda r\frac{\delta + \Sigma_{k=1}^{\infty}\varphi_{k+2}(\theta)r^k}{1+\Sigma_{k=1}^{\infty}\psi_{k+2}(\theta)r^k}. \tag{2.3}$$

Obviously, the polar coordinate form of the quasi-analytic system (1.5) differs from that of analytic systems by only a constant factor λ. It is easy to see that (2.3) is a special case of the following equation:

$$\frac{dr}{d\theta} = r\sum_{k=1}^{\infty}R_k(\theta)r^k, \quad r\geqslant 0. \tag{2.4}$$

By the method of small parameters of Poincaré, the general solution of (2.3) can be expressed as [1], i.e.,

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k,$$

where $v_1(0) = 1$, $v_k(0) = 0$, $\forall\, k \geqslant 2$. Now, substituting the above solution $r = \tilde{r}(\theta, h)$ into (2.4) yields

$$\begin{aligned} v_1'(\theta) &= R_0(\theta) v_1(\theta), \\ v_2'(\theta) &= R_0(\theta) v_2(\theta) + R_1(\theta) v_1(\theta)^2, \\ &\vdots \\ v_m'(\theta) &= R_0(\theta)\Omega_{1,m}(\theta) + R_1(\theta)\Omega_{2,m}(\theta) + \cdots + R_{m-1}(\theta)\Omega_{m,m}(\theta). \end{aligned} \tag{2.5}$$

Thus, we may solve $v_k(\theta)$ one by one to obtain

$$\begin{aligned} v_1(\theta) &= \mathrm{e}^{\oint_0^{\vartheta} R_0(\varphi) d\varphi}, \\ v_2(\theta) &= 2 v_1(\theta) \oint_0^{\vartheta} R_1(\varphi) v_1(\varphi) d\varphi, \\ &\vdots \\ v_m(\theta) &= v_1(\theta) \oint_0^{\vartheta} \frac{R_1(\varphi)\Omega_{2,m}(\varphi) + \cdots + R_{m-1}(\varphi)\Omega_{m,m}(\varphi)}{v_1(\varphi)} d\varphi. \end{aligned} \tag{2.6}$$

Note that $R_0(\theta) = \lambda\delta$ for (2.2). Furthermore, we define the successive function as

$$\Delta(h) = \tilde{r}(2\pi, h) - h,$$

and thus the critical point being a center must satisfy $\Delta(h) = 0$, namely,

$$\tilde{r}(2\pi, h) = h.$$

Many methods have been developed to compute the successive function $\Delta(h)$ (see, e.g., [17]).

From the second equation of (2.2), we can also obtain

$$t = T(\theta, h) = \int_0^{\theta} \frac{d\vartheta}{1 + \sum_{k=1}^{\infty} \psi_{k+2}(\theta) \tilde{r}(\theta, h)^k}, \tag{2.7}$$

which implies that the critical point being an isochronous center should satisfy

$$\tilde{r}(2\pi, h) = h \quad \text{and} \quad T(2\pi, h) = 2\pi.$$

However, the classical methods and formulas cannot be directly applied to a non-analytic switching system due to discontinuity. We need to modify the existing methods to resolve this problem. Similar to the return map defined for analytic switching systems (see [8]), the approach used in [8, Lemma 2.1] can be extended to define the return maps for the quasi-analytic switching system (1.6). The basic idea is briefly illustrated as follows (see Figure 1). First of all, we define the positive half-return map of the upper phase of (1.6a). Then, by a transformation $y \to -y$, the lower half phase could be transformed into the upper phase, as shown in Figure 2. Furthermore, using a time reverse changing, the computation of this transformed half-return map of the lower phase is replaced by computing the positive half-return map of the following system:

$$\begin{aligned} \dot{x} &= \delta x - y - \sum_{k=2}^{\infty} (x^2 + y^2)^{\frac{(k-1)(\lambda-1)}{2}} F_k^-(x, -y), \\ \dot{y} &= x + \delta y + \sum_{k=2}^{\infty} (x^2 + y^2)^{\frac{(k-1)(\lambda-1)}{2}} G_k^-(x, -y), \end{aligned} \qquad y > 0, \tag{2.8}$$

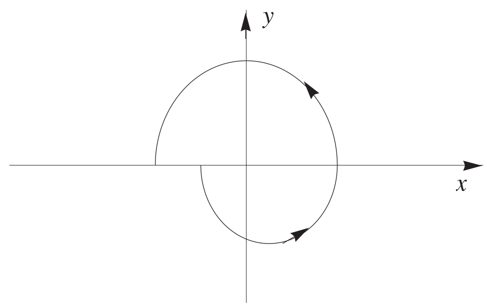

Figure 1 Half-return maps for (1.6a) and (1.6b)

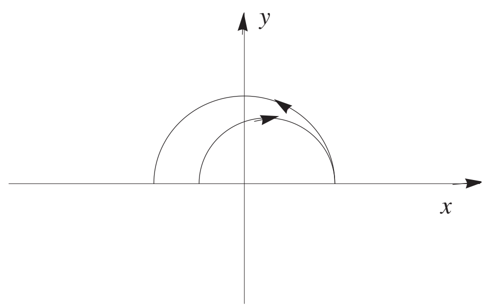

Figure 2 The lower-half plane changed to the upper-half plane

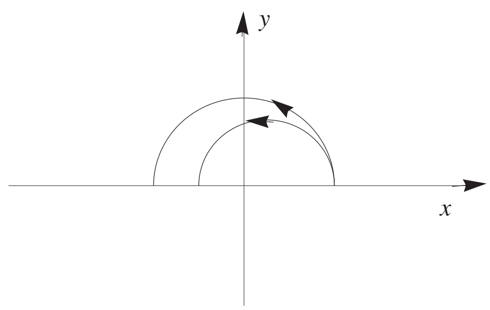

Figure 3 Vector fields of (1.6a) and (2.8)

which is shown in Figure 3. Therefore, we only need to compute the two positive half-return maps for (1.6a) and (2.8).

By defining the successive functions for (1.6a) and (2.8), respectively, as

$$\Delta_1(h) = \tilde{r}_1(\pi, h) - h \quad \text{and} \quad \Delta_2(h) = \tilde{r}_2(\pi, h) - h,$$

then we obtain the successive function for the switching system (1.6), defined as

$$\Delta(h) = \Delta_1(h) - \Delta_2(h) = \tilde{r}_1(\pi, h) - \tilde{r}_2(\pi, h). \tag{2.9}$$

Similarly, the period constants for (1.6a) and (2.8) can be defined as

$$T_1(\theta, h) = \int_0^\pi \frac{d\vartheta}{1 + \sum_{k=1}^\infty \psi_{2+k}(\theta)\tilde{r_1}^k(\vartheta, h)},$$
$$T_2(\theta, h) = \int_0^\pi \frac{d\vartheta}{1 + \sum_{k=1}^\infty \psi_{2+k}(\theta)\tilde{r_2}^k(\vartheta, h)},$$

which in turn yield the period function for the switching system (1.6) in the form of

$$T = T_1(\pi, h) + T_2(\pi, h) = 2\pi + \sum_{k=1}^\infty T_k h^k. \tag{2.10}$$

In particular, if the equations describing the lower half plane are given by

$$\begin{aligned} \dot{x} &= \delta x - y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} F_k^-(x,y) = -y, \\ \dot{y} &= x + \delta y + \sum_{k=2}^{\infty}(x^2+y^2)^{\frac{(k-1)(\lambda-1)}{2}} G_k^-(x,y) = x, \end{aligned} \tag{2.11}$$

then we only need to compute $\Delta_1(h)$ and $T_1(\theta, h)$.

Based on the above results, we can define the focus values and periodic constants for the quasi-analytic switching system (1.6).

Definition 2.1. $\Delta(h)$ can be written as

$$\Delta(h) = \sum_{1}^{n}[u_1(\pi) - v_1(\pi)]h^k = \sum_{k=1}^{\infty} V_k h^k,$$

where V_k is called the k-th-order focus value at the origin (or infinity) of the quasi-analytic switching system (1.6).

Definition 2.2. $T(h)$ can be expressed as

$$T(h) = T_1(\pi, h) + T_2(\pi, h) = 2\pi + \sum_{k=1}^{\infty} T_k h^k,$$

where T_k is called the k-th periodic constant at the origin (or infinity) of the quasi-analytic switching system (1.6).

Having defined V_k and T_k, we now describe the steps in computing them.

(1) Introduce the transformations: $y \to -y$ and $t \to -t$ for the lower half plane.

(2) By means of the transformation of polar coordinates,

$$x = r^{\frac{1}{\lambda}} \cos\theta, \quad y = r^{\frac{1}{\lambda}} \sin\theta,$$

in (1.6a) and (2.8), write the solutions for (2.8) and (2.10) as

$$r_1 = \tilde{r}_1(\theta, h) = \sum_{k=1}^{\infty} u_k(\theta) h^k \quad \text{and} \quad r_2 = \tilde{r}_2(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k,$$

respectively, satisfying $u_1(0) = v_1(0) = 1$, $u_k(0) = v_k(0) = 0$, $\forall\, k \geqslant 2$.

(3) Solve $u_k(\theta)$ and $v_k(\theta)$.

(4) Compute the successive function for the switching system by the formula,

$$\Delta(h) = \Delta_1(h) - \Delta_2(h) = \tilde{r}_1(\pi, h) - \tilde{r}_2(\pi, h).$$

(5) Compute the periodic constants for the switching system by the formula,

$$T = T_1(\pi, h) + T_2(\pi, h).$$

Obviously, the symmetry principle for continuous systems is no longer applicable for switching systems. We need to redefine symmetry for switching systems in order to derive the center conditions of switching systems.

Definition 2.3. If both (1.6a) and (1.6b) are symmetric with respect to the y-axis, then (1.6) is said to be symmetric with respect to the y-axis. Furthermore, if the vector fields of (1.6a) and (1.6b) satisfy

$$F_k^+(x,y) = -F_k^-(x,-y) \quad \text{and} \quad G_k^+(x,y) = G_k^-(x,-y),$$

then (1.6) is said to be symmetric with respect to the x-axis.

So obviously, if (1.6) is symmetric with respect to the x-axis or the y-axis, then the origin of System (1.6) is a center.

3 A quadratic quasi-analytic switching system

In this section, we consider a quadratic quasi-analytic switching system to demonstrate the application of the formulae and results obtained in the previous section. We will use our method to determine the center conditions and isochronous center conditions of the system we will consider, given by

$$\begin{aligned}&\left.\begin{aligned}\dot{x} &= \delta x - y + (x^2+y^2)^{\frac{(\lambda-1)}{2}}(a_{20}x^2 + a_{11}xy + a_{02}y^2),\\ \dot{y} &= x + \delta y + (x^2+y^2)^{\frac{(\lambda-1)}{2}}(b_{20}x^2 + b_{11}xy + b_{02}y^2),\end{aligned}\right\} \quad y>0,\\ &\left.\begin{aligned}\dot{x} &= \delta x - y,\\ \dot{y} &= x + \delta y,\end{aligned}\right\} \quad y<0.\end{aligned} \tag{3.1}$$

The case $\lambda = 1$ (a polynomial system) has been studied in [8], which becomes a special quadratic switching system. It is shown in [8] that the highest order of focus values for this special system is 5, and 5 small-amplitude limit cycles are obtained. We want to extend the study to the case $\lambda \neq 1$. However, when the lower system is not in a simple form, even for general quadratic switching systems, it is very difficult to determine the center conditions and isochronous center conditions. Thus, in this paper we focus on the study of (3.1) for $\lambda \neq 1$.

3.1 Center conditions and limit cycles for System (3.1)

We first study the center conditions and bifurcation of limit cycles in (3.1). It has been recently noticed that Tian and Yu [31] studied a quadratic switching Bautin system and obtained 10 small-amplitude limit cycles. Here, we want to show that (3.1) with $\lambda \neq 1$ can bifurcate 7 limit cycles around the origin, two more than those of the system with $\lambda = 1$.

In order to consider the center and isochronous center conditions, and determine the number of limit cycles bifurcating in the small neighborhood of the origin (or infinity), we need to compute the Lyapunov constants and periodic constants. With the aid of a computer algebra system—Mathematica, we obtain the following Lyapunov constants of (3.1).

The first three Lyapunov constants at the origin are given by

$$\begin{aligned}L_0 &= 2\pi\delta,\\ L_1 &= -\frac{2}{3}(a_{11} + 2b_{02} + b_{20})\lambda,\\ L_2 &= -\frac{\pi}{8}[b_{20}(a_{20}+a_{02}) + (2a_{20}+b_{11})(b_{20}+b_{02})]\lambda.\end{aligned} \tag{3.2}$$

For higher Lyapunov constants, we have two cases.

Case (A) $b_{20} \neq 0$. For this case, L_3 is given by

$$L_3 = -\frac{2}{105}\lambda\{[6a_{20}b_{02}(2a_{20}+b_{11}) + b_{20}(3a_{20}^2 + 4b_{02}^2)](\lambda+6) + 14b_{20}[3a_{20}(2a_{20}+b_{11}) + 2b_{20}b_{02}]\}.$$

Then, there are three sub-cases in computing L_i, $i \geqslant 4$.

Case (A1) $a_{20}[7b_{20} + b_{02}(\lambda+6)] \neq 0$, for which we have

$$\begin{aligned}L_4 = {}& \frac{a_{20}b_{20}\lambda\pi}{1536[7b_{20}+b_{02}(\lambda+6)]^2}[4b_{20}+b_{02}(\lambda+3)][12b_{20}+b_{02}(\lambda+9)]\\ &\times\{3a_{20}^2(\lambda+6)^2 - 28b_{20}[7b_{20}+b_{02}(\lambda+6)]\}.\end{aligned}$$

(a) If $b_{20} = -\frac{1}{4}b_{02}(\lambda+3)$, then

$$\begin{aligned}L_5 &= -\frac{b_{02}\lambda(\lambda+3)}{3243240(\lambda-1)^2}f_1,\\ L_6 &= -\frac{\pi b_{02}\lambda^3(\lambda+3)^2}{5308416a_{20}(\lambda-1)^3}f_2,\\ L_7 &= -\frac{b_{02}\lambda(\lambda+3)}{6788231049600a_{20}^2(\lambda-1)^4}f_3,\end{aligned}$$

where

$$\begin{aligned} f_1 &= (3868a_{20}^4 - 2909a_{20}^2b_{02}^2 - 1085b_{02}^4)\lambda^5 + (34320a_{20}^4 - 5962a_{20}^2b_{02}^2 - 1085b_{02}^4)\lambda^4 + 2(41274a_{20}^4 \\ &\quad + 3162a_{20}^2b_{02}^2 + 3563b_{02}^4)\lambda^3 - 6b_{02}^2(2937a_{20}^2 + 259b_{02}^2)\lambda^2 + 243b_{02}^2(83a_{20}^2 - 35b_{02}^2)\lambda + 5103b_{02}^4, \\ f_2 &= [(4a_{20}^2 - 7b_{02}^2)\lambda^2 + 2(24a_{20}^2 - 7b_{02}^2)\lambda + 3(48a_{20}^2 + 7b_{02}^2)] \\ &\quad \times [(196a_{20}^4 + 25a_{20}^2b_{02}^2 + 4b_{02}^4)\lambda^2 - 2b_{02}^2(53a_{20}^2 + 4b_{02}^2)\lambda + 81a_{20}^2b_{02}^2 + 4b_{02}^4], \\ f_3 &= 56b_{02}^8(\lambda-1)^4(\lambda+3)(1027505\lambda^4 + 426036\lambda^3 - 1498554\lambda^2 + 551124\lambda + 1476225) \\ &\quad + 128a_{20}^8\lambda^5(9756302\lambda^4 + 203092731\lambda^3 + 1398704409\lambda^2 + 3748721013\lambda + 2969005833) \\ &\quad - 4a_{20}^6b_{02}^2\lambda^3(\lambda-1)(488213167\lambda^5 + 3344141799\lambda^4 + 596192742\lambda^3 \\ &\quad - 12028807314\lambda^2 + 57678458859\lambda + 102228640299) \\ &\quad + a_{20}^2b_{02}^6(\lambda-1)^3(24084923\lambda^6 - 1850127480\lambda^5 - 6861739149\lambda^4 \\ &\quad + 1283638536\lambda^3 + 11199899889\lambda^2 - 5507145936\lambda - 10721822175) \\ &\quad - 3a_{20}^4b_{02}^4(\lambda-1)^2\lambda(192786817\lambda^6 + 2736149472\lambda^5 + 4596112377\lambda^4 \\ &\quad - 12782671128\lambda^3 - 13724853597\lambda^2 - 2481265224\lambda - 22052498589). \end{aligned}$$

(b) If $b_{20} = -\frac{1}{12}b_{02}(\lambda+9)$, then

$$\begin{aligned} L_5 &= -\frac{b_{02}\lambda(\lambda+9)}{4169880(5\lambda+9)^3}\tilde{f}_1, \\ L_6 &= -\frac{\pi b_{02}\lambda^3(\lambda+9)}{143327232a_{20}(5\lambda+9)^4}\tilde{f}_2, \\ L_7 &= -\frac{b_{02}\lambda(\lambda+9)}{183282238339200a_{20}^2(5\lambda+9)^5}\tilde{f}_3, \end{aligned}$$

where

$$\begin{aligned} \tilde{f}_1 &= (43092a_{20}^4 + 52305a_{20}^2b_{02}^2 + 8675b_{02}^4)\lambda^6 + (386532a_{20}^4 + 597639a_{20}^2b_{02}^2 + 86980b_{02}^4)\lambda^5 \\ &\quad + 9(45900a_{20}^4 + 56538a_{20}^2b_{02}^2 - 8377b_{02}^4)\lambda^4 - 162(5202a_{20}^4 + 68961a_{20}^2b_{02}^2 + 11720b_{02}^4)\lambda^3 \\ &\quad - 729b_{02}^2(47283a_{20}^2 + 6887b_{02}^2)\lambda^2 - 177147b_{02}^2(159a_{20}^2 + 28b_{02}^2)\lambda - 1594323b_{02}^4, \\ \tilde{f}_2 &= [(108a_{20}^2 + 35b_{02}^2)\lambda^2 + 54(24a_{20}^2 + 7b_{02}^2)\lambda + 3888a_{20}^2 + 567b_{02}^2] \\ &\quad \times [(11772a_{20}^4 + 15255a_{20}^2b_{02}^2 + 2300b_{02}^4)\lambda^4 + 24(1035a_{20}^4 + 2256a_{20}^2b_{02}^2 + 220b_{02}^4)\lambda^3 \\ &\quad - 18(2034a_{20}^4 + 13479a_{20}^2b_{02}^2 + 1036b_{02}^4)\lambda^2 - 2592b_{02}^2(417a_{20}^2 + 25b_{02}^2)\lambda - 12393b_{02}^2(81a_{20}^2 + 4b_{02}^2)], \\ \tilde{f}_3 &= 279936a_{20}^8(\lambda-1)\lambda^5(38218052\lambda^4 + 790049163\lambda^3 + 5940016749\lambda^2 + 19219084137\lambda + 22987749699) \\ &\quad + 56b_{02}^8(\lambda+9)(5\lambda+9)^4(4362337\lambda^5 + 301281\lambda^4 - 20129886\lambda^3 - 60022134\lambda^2 - 143311923\lambda \\ &\quad - 39858075) + 324a_{20}^6b_{02}^2\lambda^3(5\lambda+9)(12357727123\lambda^6 + 228212204300\lambda^5 + 1303142758287\lambda^4 \\ &\quad + 688198402656\lambda^3 - 17536030592607\lambda^2 - 57374137182060\lambda - 55078300245699) \\ &\quad + 3a_{20}^2b_{02}^6(5\lambda+9)^3(5875349809\lambda^7 + 75871528713\lambda^6 + 173810927169\lambda^5 - 698576296239\lambda^4 \\ &\quad - 3526727842629\lambda^3 - 8234719793325\lambda^2 - 10205502265773\lambda - 2605402788525) \\ &\quad + 27a_{20}^4b_{02}^4\lambda(5\lambda+9)^2(15822388201\lambda^7 + 255144264153\lambda^6 + 1105548667713\lambda^5 - 1017555100407\lambda^4 \\ &\quad - 18518298666957\lambda^3 - 51274399423149\lambda^2 - 74254559009373\lambda - 56058065936181). \end{aligned}$$

(c) If $a_{20}^2 = \frac{28b_{20}[7b_{20}+b_{02}(\lambda+6)]}{3(\lambda+6)^2}$, the following hold:

$$\begin{aligned} L_5 &= \frac{128b_{20}^5}{715(\lambda+6)^4}\lambda(\lambda-1)(2\lambda-9)(5\lambda+16), \\ L_6 &= 0, \\ L_7 &= -\frac{128b_{20}^7}{692835(\lambda+6)^6}\lambda(\lambda-1)(2\lambda-9)(1587\lambda^3 - 82636\lambda^2 - 527988\lambda - 838080). \end{aligned}$$

Case (A2) $a_{20}=0$, under which L_3 is reduced to

$$L_3=-\frac{8b_{20}b_{02}}{105}\lambda[7b_{20}+b_{02}(\lambda+6)],$$

and higher Lyapunov constants are

$$\begin{aligned}
L_4 &= -\frac{\pi b_{02}^3 b_{11}}{3136}\lambda(\lambda-1)(5\lambda+9),\\
L_5 &= -\frac{128b_{02}^3}{108153045(\lambda+6)}\lambda(\lambda-1)(2\lambda-9)[9b_{02}^2(5\lambda+16)(\lambda+6)^2+49b_{11}^2(27\lambda+50)],\\
L_6 &= -\frac{\pi b_{02}^3 b_{11}}{59006976(\lambda+6)}\lambda(\lambda-1)[3b_{02}^2(\lambda+6)^2(562\lambda^2+5085\lambda+13365)\\
&\quad -b_{11}^2(60074\lambda^2-6615\lambda-235935)],\\
L_7 &= \frac{128b_{02}^3}{231084662834025(\lambda+6)^3}\lambda(\lambda-1)[405b_{02}^4(2\lambda-9)(\lambda+6)^4(1587\lambda^3-82636\lambda^2-527988\lambda\\
&\quad -838080)+441b_{02}^2b_{11}^2(\lambda+6)^2(2699488\lambda^4+1664883\lambda^3-12695418\lambda^2+64285812\lambda\\
&\quad +193185000)+b_{11}^4(17594187058\lambda^4+24560280388\lambda^3-109958232048\lambda^2-10609846128\lambda\\
&\quad +343997992800)].
\end{aligned}$$

Case (A3) $b_{20}=-\frac{1}{7}b_{02}(6+\lambda)$, for which L_3 becomes

$$L_3=\frac{2a_{20}^2b_{02}}{245}\,\lambda(\lambda+6)^2,$$

and higher Lyapunov constants are given by

$$\begin{aligned}
L_4 &= -\frac{\pi\, b_{02}\lambda(\lambda-1)(9+5\lambda)}{43904(6+\lambda)}[a_{20}b_{02}^2\lambda^2+2b_{20}^2(20a_{20}+7b_{11})\lambda\\
&\quad +12a_{20}(17b_{02}^2-49a_{20}^2-49a_{20}b_{11})+21b_{11}(4b_{02}^2-7a_{20}b_{11})],\\
L_5 &= \frac{18a_{20}^4b_{02}}{105105}\,\lambda(2\lambda+9)(7\lambda+12)(\lambda+2)(\lambda+6),\\
L_6 &= \frac{\pi a_{20}b_{02}^3\lambda(\lambda+6)}{4956585984}[2(34496a_{20}^2+15375b_{02}^2)\lambda^4+3(29792a_{20}^2+165045b_{02}^2)\lambda^3\\
&\quad -18(279104a_{20}^2+37917b_{02}^2)\lambda^2-27(130144a_{20}^2+103149b_{02}^2)\lambda+1620(1568a_{20}^2+891b_{02}^2)],\\
L_7 &= \frac{2a_{20}^2b_{02}}{245}\lambda(\lambda+6)^2.
\end{aligned}$$

Case (B) $b_{20}=0$. For this case, we have

$$\begin{aligned}
L_1 &= -\frac{2(a_{11}+2b_{02})}{3}\lambda,\\
L_2 &= -\frac{\pi b_{02}(2a_{20}+b_{11})}{8}\lambda,\\
L_3 &= -\frac{2b_{02}(2a_{20}+b_{11})}{315}(8a_{02}-9b_{11})\,\lambda(\lambda+6),\\
L_4 &= -\frac{\pi b_{02}(2a_{20}+b_{11})}{9216}(28a_{02}^2+36b_{02}^2-36a_{02}b_{11}+27b_{11}^2)\lambda(\lambda+3)(\lambda+9),\\
L_5 &= \frac{b_{02}(2a_{20}+b_{11})}{1216215}[27b_{11}(27a_{02}^2+120b_{02}^2-60a_{02}b_{11}+35b_{11}^2)-128a_{02}(10a_{02}^2+27b_{02}^2)]\\
&\quad \times\lambda(\lambda+2)(\lambda+12)(2\lambda+9),\\
L_6 &= -\frac{\pi b_{02}(2a_{20}+b_{11})}{28311552}[16(143a_{02}^4+594a_{02}^2b_{02}^2+243b_{02}^4)-288a_{02}b_{11}(11a_{02}^2+45b_{02}^2)\\
&\quad +1080b_{11}^2(3a_{02}^2+7b_{02}^2-45b_{11}^3(56a_{02}-27b_{11})]\\
&\quad \times\lambda(\lambda+3)(\lambda+6)(\lambda+15)(2\lambda+3).
\end{aligned}$$

Note that in the above computations, $L_{k-1} = 0$, $k = 1, 2, \dots, 6$ have been used in computing L_k.

Now, by carefully analyzing the above Lyapunov constants, we obtain the following result.

Theorem 3.1. *For* (3.1)*, maximal six small-amplitude limit cycles can bifurcate from the origin and maximal seven small-amplitude limit cycles can exist in the neighborhood of infinity. Moreover, the first seven Lyapunov constants at the origin* (*or infinity*) *of* (3.1) *vanish if and only if one of the following conditions is satisfied*:

(i) $\delta = a_{11} = b_{20} = b_{02} = 0$;

(ii) $\delta = b_{20} = a_{11} + 2b_{02} = b_{11} + 2a_{20} = 0$;

(iii) $\delta = a_{20} = b_{02} = a_{11} + b_{20} = b_{11} + a_{02} = 0$;

(iv) $\delta = \lambda - 1 = a_{02} = a_{20} = b_{02} + b_{20} = a_{11} - b_{20} = 0$;

(v)

$$\begin{cases} \delta = \lambda - 1 = 2a_{11}b_{20} + 3a_{20}^2 - 2b_{20}^2 = 2b_{11} + 5a_{20} = 0, \\ 8a_{02}b_{20}^2 + a_{20}(8b_{20}^2 - 3a_{20}^2) = 4b_{02}b_{20} - 3a_{20}^2 + 4b_{20}^2 = 0; \end{cases}$$

(vi)

$$\begin{cases} \delta = \lambda - \dfrac{9}{2} = 12a_{11}b_{20} + 27a_{20}^2 - 4b_{20}^2 = 4b_{11} + 11a_{20} = 0, \\ 32a_{02}b_{20}^2 + 3a_{20}(8b_{20}^2 - 9a_{20}^2) = 24b_{02}b_{20} - 27a_{20}^2 + 16b_{20}^2 = 0. \end{cases}$$

Proof. First of all, note that $\delta = 0$ is a necessary condition for all cases in order to get limit cycles bifurcating from the origin (or infinity), under which $L_0 = 0$.

We start from Case (B) in which $b_{20} = 0$. It is easy to see that for this case all L_i, $i = 2, 3, \dots, 6$ contain a same factor $b_{02}(2a_{20} + b_{11})$. Thus, $L_i = 0$, $i = 2, 3, \dots 6$, if $b_{02}(2a_{20} + b_{11}) = 0$, implying that the maximal number of limit cycles that can be obtained is two. When $b_{20} = b_{02} = 0$, $L_2 = 0$ and $L_1 = 0$ yield one solution: $a_{11} = 0$, which gives the condition (i). If $b_{20} = 2a_{20} + b_{11} = 0$, then $L_2 = 0$, and $L_1 = 0$ requires $a_{11} + 2b_{02} = 0$, which yields the condition (ii).

Case (A3) in which $b_{20} = -\frac{1}{7}b_{02}(6 + \lambda)$ is simple since it is assumed that $b_{20} \neq 0$ and so $b_{02} \neq 0$. It is also assumed that $a_{20} \neq 0$ for this case, yielding $L_3 \neq 0$, implying that three limit cycles can be obtained since one can choose appropriate values of a_{11} and b_{11} to set $L_1 = L_2 = 0$.

Next, consider Case (A2) in which $a_{20} = 0$. Maximal 5 limit cycles may be obtained by choosing $b_{02} \neq 0$, and setting $b_{11} = 0$ (so $L_4 = 0$), $b_{20} = 0$ (so $L_2 = L_3 = 0$), and $a_{11} + 2b_{02} = 0$ (so $L_1 = 0$). If $b_{02} = 0$, then $L_3 = L_4 = \cdots = L_7 = 0$. Furthermore, setting $L_1 = L_2 = 0$ we obtain $a_{11} + b_{20} = b_{11} + a_{02} = 0$, which is the condition (iii). Another possibility for this case is to set $\lambda = 1$, given $L_4 = L_5 = L_6 = L_7 = 0$. Furthermore, letting $L_1 = L_2 = L_3 = 0$ yields $a_{02} = b_{02} + b_{20} = a_{11} - b_{20} = 0$, leading to the condition (iv).

For Case (A1)(c), it is noted that $\lambda = 1$ or $\lambda = \frac{9}{2}$ yields $L_5 = L_7 = 0$ (L_6 is already equal to zero). $L_1 = 0$ gives $b_{02} = -\frac{1}{2}(a_{11} + b_{20})$ which is substituted into

$$a_{20}^2 = \frac{28b_{20}[7b_{20} + b_{02}(\lambda + 6)]}{3(\lambda + 6)^2}$$

to obtain $2a_{11}b_{20} + 3a_{20}^2 - 2b_{20}^2 = 0$ for $\lambda = 1$, and $12a_{11}b_{20} + 27a_{20}^2 - 4b_{20}^2 = 0$ for $\lambda = \frac{9}{2}$. Then, for the condition (v) for which $\lambda = 1$, using $a_{20}^2 = \frac{4}{3}b_{20}(b_{20} + b_{02})$ to simplify $L_3 = 0$ yields $2b_{11} + 5a_{20} = 0$, and again using the expression of a_{20}^2 as well as $b_{11} = -\frac{5}{2}a_{20}$ to simplify $L_2 = 0$ we obtain $8a_{02} + a_{20}(8b_{20}^2 - 3a_{20}^2) = 0$. Finally, using $a_{11} = \frac{2b_{20}^2 - 3a_{20}^2}{2b_{20}}$ to simplify $L_1 = 0$ results in $4b_{02}b_{20} - 3a_{20}^2 + 4b_{20}^2 = 0$. Summarizing the above results leads to the condition (v). Following the same procedure, we can obtain the condition (vi).

Now we come to Cases (A1)(a) and (A1)(b). First note that for these two cases, $b_{02} \neq 0$ since it is assumed $b_{20} \neq 0$. Therefore, instead of considering the equation $L_5 = L_6 = L_7 = 0$, we consider the polynomial equations $f_1 = f_2 = f_3 = 0$ for Case (A1)(a) and $\tilde{f}_1 = \tilde{f}_2 = \tilde{f}_3 = 0$ for Case (A1)(b). In order to obtain the maximal number of limit cycles, we need to find the conditions such that $L_i = 0$,

$i = 0, 1, \ldots, k-1$, but $L_k \neq 0$. Actually, setting $L_i = 0$, $i = 0, 1, 2, 3, 4$, we obtain two sets of solutions

$$\begin{aligned} \delta &= 0, \\ b_{20} &= -\frac{b_{02}}{4}(\lambda+3), \\ a_{11} &= \frac{b_{02}}{4}(\lambda-5), \\ a_{02} &= \frac{a_{20}^2\lambda - b_{02}^2(\lambda-1)}{6a_{20}}, \\ b_{11} &= -\frac{a_{20}^2(\lambda^2+21\lambda+6) - b_{02}^2(\lambda-1)(\lambda+3)}{6a_{20}(\lambda-1)} \end{aligned} \tag{3.3}$$

for Case (A1)(a), and

$$\begin{aligned} \delta &= 0, \\ b_{20} &= -\frac{b_{02}}{12}(\lambda+9), \\ a_{11} &= \frac{b_{02}}{12}(\lambda-15), \\ a_{02} &= -\frac{9a_{20}^2\lambda(\lambda+13) + b_{02}^2(\lambda-3)(5\lambda+9)}{18a_{20}(5\lambda+9)}, \\ b_{11} &= \frac{a_{20}^2(9\lambda^2-45\lambda+162) + b_{02}^2(\lambda+9)(5\lambda+9)}{18a_{20}(5\lambda+9)} \end{aligned} \tag{3.4}$$

for Case (A1)(b).

Note that the functions f_1, f_2, f_3 and $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$ are homogeneous polynomials in a_{20}^2 and b_{02}^2. So we may introduce $b_{02}^2 = k\,a_{20}^2$ $(k > 0)$ into these polynomials. Note that f_2 and $\tilde{f}_2$ have two factors, and one is linear in k and one is quadratic in k, given as follows:

$$\begin{aligned} f_2 &= -a_{20}^6 f_{2a} f_{2b} \\ &= -a_{20}^6[7(\lambda-1)(\lambda+3)k - 4(\lambda+6)^2][4(\lambda-1)^2k^2 + (\lambda-1)(25\lambda-81)k + 196\lambda^2]. \end{aligned} \tag{3.5}$$

We first solve $f_{2a} = 0$ to obtain $k = \frac{4(\lambda+6)^2}{7(\lambda-1)(\lambda+3)}$, which is then substituted into f_1 and f_3 to yield

$$\begin{aligned} f_1 &= -\frac{1296a_{20}^4}{7}(\lambda-1)(2\lambda-9)(5\lambda+16)(\lambda+3)^2, \\ f_3 &= -\frac{699840a_{20}^8}{49}(\lambda-1)^2(2\lambda-9)(\lambda+3)^3C_3, \end{aligned}$$

where

$$C_3 = (1587\lambda^3 - 82636\lambda^2 - 527988\lambda - 838080). \tag{3.6}$$

Since $(\lambda-1)(\lambda+3) \neq 0$, the only solution satisfying $f_1 = 0$ and $f_3 \neq 0$ is $\lambda = -\frac{16}{5}$. When $\lambda = \frac{9}{2}$, $L_i = 0$, $i = 0, 1, \ldots, 7$, but it is easy to verify that this is a special case of (vi).

Moreover, for the solution $\lambda = -\frac{16}{5}$,

$$\det(J) = \det\left[\frac{\partial(L_5, L_6)}{\partial(k, \lambda)}\right] = \frac{8384103915264}{78125}\,a_{20}^{10} \neq 0 \quad \text{for} \quad a_{20} \neq 0,$$

implying that seven limit cycles can bifurcate in the small neighborhood of infinity.

For the second factor f_{2b}, we eliminate k from the two equations $f_1 = f_{2b} = 0$ to obtain the solution for k,

$$k = -\frac{4\lambda^2(6337\lambda^2 + 27872\lambda + 27783)}{(\lambda-1)(1721\lambda^3 - 2945\lambda^2 - 38097\lambda - 45927)}$$

and a resultant equation

$$\mathrm{R}_{12} = \lambda(\lambda+1)(\lambda-1)(397378\lambda^4 + 3696797\lambda^3 + 12760835\lambda^2 + 19311435\lambda + 10762227) = 0,$$

which has three real solutions: $\lambda = -2.59473685\cdots, -1.58363608\cdots, -1,$ but none of them yields $k > 0$. Hence, there are no solutions from $f_{2b} = 0$ to generate seven limit cycles.

When the conditions in (3.4) are satisfied, similarly, we can use $b_{02}^2 = \tilde{k}\, a_{20}^2$ to find that

$$\begin{aligned}\tilde{f}_2 &= a_{20}^6 \tilde{f}_{2a}\tilde{f}_{2b}\\ &= a_{20}^6[7(\lambda+9)(5\lambda+9)\tilde{k} + 108(\lambda+6)^2][4(5\lambda+9)^2(23\lambda^2 - 30\lambda - 153)\tilde{k}^2\\ &\quad + 9(5\lambda L + 9)(339\lambda^3 + 593\lambda^2 - 6459\lambda - 12393)\tilde{k} + 108\lambda^2(\lambda-1)(109\lambda+339)]. \end{aligned} \tag{3.7}$$

Solving $\tilde{f}_{2a} = 0$ to obtain $\tilde{k} = -\frac{108(\lambda+6)^2}{7(\lambda+9)(5\lambda+9)}$, and then substituting it into $\tilde{f}_1$ and $\tilde{f}_3$ yields

$$\begin{aligned}\tilde{f}_1 &= -\frac{34992a_{20}^4}{49}(\lambda-1)(2\lambda-9)(5\lambda+16)(\lambda+9)^2,\\ \tilde{f}_3 &= -\frac{17061120a_{20}^8}{49}(\lambda-1)^2(2\lambda-9)(5\lambda+9)^2(\lambda+9)^3\, C_3,\end{aligned}$$

where C_3 is given in (3.6). Since $(5\lambda+9)(\lambda+9) \neq 0$, the only solution satisfying $f_1 = 0$ and $f_3 \neq 0$ is $\lambda = -\frac{16}{5}$. $\lambda = 1$ and $\lambda = \frac{9}{2}$ are not the solutions since they yield $\tilde{k} < 0$. For the solution $\lambda = -\frac{16}{5}$, we have

$$\det(J) = \det\left[\frac{\partial(L_5, L_6)}{\partial(\tilde{k}, \lambda)}\right] = -\frac{304277047914997479168}{78125} a_{20}^{10} \neq 0 \quad \text{for} \quad a_{20} \neq 0,$$

implying that seven limit cycles can bifurcate in the small neighborhood of infinity.

For the second factor $\tilde{f}_{2b}$, similarly we eliminate $\tilde{k}$ from the two equations $\tilde{f}_1 = 0$ and $\tilde{f}_{2b} = 0$ to obtain the solution for $\tilde{k}$,

$$\tilde{k} = -\frac{36\lambda^2(\lambda-1)(223\lambda^4 + 8120\lambda^3 - 127950\lambda^2 - 778680\lambda - 741393)}{(5\lambda+9)/(6419\lambda^6 + 31564\lambda^5 - 1363197\lambda^4 - 4806072\lambda^3 + 20436705\lambda^2 + 88888428\lambda + 81310473)}$$

and a resultant equation

$$\begin{aligned}\tilde{\mathrm{R}}_{12} = \lambda(\lambda-1)(\lambda-9)(5\lambda+9)&(8397602\lambda^9 + 84616511\lambda^8 - 1494342124\lambda^7 - 16706405616\lambda^6\\ &+ 16325397720\lambda^5 + 375950483190\lambda^4 - 756410507892\lambda^3 - 10202463072792\lambda^2\\ &- 22941003813786\lambda - 15945065065773),\end{aligned}$$

which has four real solutions: $\lambda = -12.38286360\cdots, -8.54108488\cdots, 1, 9, 13.13951523\cdots$, but all of them yield $\tilde{k} \leqslant 0$. Thus, there are no solutions from $\tilde{f}_{2b} = 0$ to give seven limit cycles.

Summarizing the above results obtained for Cases (A1)(a) and (A1)(b), we conclude that there exist two sets of infinite solutions such that (3.1) can have seven limit cycles bifurcating in the small neighborhood of infinity.

Although we cannot obtain seven limit cycles around the origin of (3.1), we may find an infinite number of solutions for six limit cycles which bifurcate in the small neighborhood of the origin, which is still better than the five limit cycles obtained in [8]. To find the solutions, it needs L_i, $i = 0, 1, \ldots, 5$, but $L_6 \neq 0$. Thus, we only need to solve $f_1 = 0$ (or $\tilde{f}_1 = 0$). Let $b_{02}^2 = ka_{20}^2$, and $f_1 = 0$ becomes $f_1 = -a_{20}^2[A_2k^2 + A_1k + A_0]$, where

$$\begin{aligned}A_2 &= 7(\lambda-1)^2(\lambda+3)(155\lambda^2 - 243),\\ A_1 &= \lambda(\lambda-1)(2909\lambda^3 + 8871\lambda^2 + 2547\lambda + 20169),\\ A_0 &= -4\lambda^3(967\lambda^2 + 8580\lambda + 20637).\end{aligned}$$

We want to find the solutions satisfying $k > 0$ and $\lambda > 0$ ($\lambda \neq 1$). It is easy to show that $f_1 = 0$ has a unique positive solution for k when $\lambda \geqslant 9\sqrt{3/155}$, and does not have the solutions when $1 < \lambda < 9\sqrt{3/155}$. When $\lambda \in (0,1)$, $A_i < 0$, $i = 0, 1, 2$, and

$$\begin{aligned}\Delta &= A_1^2 - 4A_2A_0\\ &= -81\lambda^2(1-\lambda)^3(311721\lambda^5 + 3409519\lambda^4 + 14178654\lambda^3 + 25596434\lambda^2 + 14511609\lambda - 5022081),\end{aligned}$$

which is positive for $\lambda \in (0,\ 0.23512585)$, leading to that $f_1 = 0$ has two solutions for k and each λ chosen from this interval. Hence, there exists an infinite number of solutions for k satisfying $f_1 = 0$ when

$$\lambda \in (0,\ 0.23512585) \cup (9\sqrt{3/155},\ \infty).$$

These solutions do not yield $f_2 = 0$, since in the above we have already shown that $f_1 = f_2 = 0$ does not have the solutions satisfying $k > 0$ and $\lambda > 0$. This indicates that for Case (A1)(a) there exist an infinite number of solutions for the existence of six limit cycles around the origin of (3.1). Similarly, for Case (A1)(b), we can prove that $f_1 = 0$ has two positive solutions for k when $\lambda \in (0,\ 1)$ and one positive solution when $\lambda \in (1,\ 5.132341426)$, implying that for Case (A1)(b) there also exist an infinite number of solutions for the existence of six limit cycles around the origin of (3.1).

The proof is completed. □

Note that the conditions (i)–(vi) given in Theorem 3.1 yield $L_i = 0$, $i = 0, 1, \ldots, 7$, implying that they are necessary conditions for the origin (or infinity) of (3.1) to be a center. In the following, we will show that these conditions are also sufficient for the origin (or infinity) of (3.1) to be a center. We have the following theorem.

Theorem 3.2. *The conditions* (i)–(vi) *given in Theorem* 3.1 *are necessary and sufficient for the origin* (*or infinity*) *of* (3.1) *to be a center.*

Proof. The *necessity* has been shown in the proof of Theorem 3.1. Hence, we only need to prove the *sufficiency*. First note that for all the six cases, the lower-half plane is same (as $\delta = 0$), described by

$$\dot{x} = -y, \quad \dot{y} = x, \quad y < 0.$$

This system has a first integral $H_0(x, y) = x^2 + y^2$, which is an even function of x (i.e., symmetric with the y-axis). Thus, in the following, for each case we only list the equations for the upper-half plane.

When the condition (i) holds, the equations for the upper-half plane of (3.1) become

$$\begin{aligned}\dot{x} &= -y + (a_{20}x^2 + a_{02}y^2)(x^2+y^2)^{\frac{(\lambda-1)}{2}},\\ \dot{y} &= x + b_{11}xy(x^2+y^2)^{\frac{(\lambda-1)}{2}},\end{aligned} \qquad y > 0, \tag{3.8}$$

which is symmetric with the y-axis, so the origin (or infinity) is a center.

When the condition (ii) is satisfied, the equations for the upper-half plane of (3.1) can be rewritten as

$$\begin{aligned}\dot{x} &= -y + (a_{20}x^2 - 2b_{02}xy + a_{02}y^2)(x^2+y^2)^{\frac{(\lambda-1)}{2}},\\ \dot{y} &= x - (2a_{20}x - b_{02}y)y\,(x^2+y^2)^{\frac{(\lambda-1)}{2}},\end{aligned} \qquad y > 0, \tag{3.9}$$

which has an integrating factor $\frac{2}{3}\lambda(x^2+y^2)^{\frac{(1-\lambda)}{2}}$. Then, (3.9) becomes

$$\begin{aligned}\dot{x} &= -\frac{2}{3}\lambda y(x^2+y^2)^{\frac{(1-\lambda)}{2}} + \frac{2}{3}\lambda(a_{20}x^2 - 2b_{02}xy + a_{02}y^2),\\ \dot{y} &= \frac{2}{3}\lambda x(x^2+y^2)^{\frac{(1-\lambda)}{2}} - \frac{2}{3}\lambda(2a_{20}x - b_{02}y)y,\end{aligned}$$

which has a first integral,

$$H_1(x, y) = 6\lambda(x^2+y^2)^{\frac{(3-\lambda)}{2}} + 2\lambda(\lambda-3)\,y[3(a_{20}x - b_{02}y)x + a_{02}y^2].$$

It is seen that H_1 is an even function of x when $y = 0$, so the origin (or infinity) of (3.1) is a center (see, e.g., [14, Theorem 2.2]).

When the condition (iii) holds, the equations for the upper-half plane of (3.1) can be rewritten as

$$\begin{aligned} \dot{x} &= -y - (b_{20}x - a_{02}y)y\,(x^2+y^2)^{\frac{(\lambda-1)}{2}}, \\ \dot{y} &= x + (b_{20}x - a_{02}y)x\,(x^2+y^2)^{\frac{(\lambda-1)}{2}}, \end{aligned} \quad y > 0. \tag{3.10}$$

It is easy to see that (3.10) has a first integral,

$$H_2(x,y) = (x^2+y^2)^{\frac{\lambda}{3}},$$

which is an even function of x, so the origin (or infinity) is a center.

When the condition (iv) is satisfied, the equations for the upper-half plane of (3.1) become

$$\begin{aligned} \dot{x} &= -y + b_{20}xy, \\ \dot{y} &= x + (b_{20}x^2 + b_{11}xy - b_{20}y^2), \end{aligned} \quad y > 0. \tag{3.11}$$

It can be shown that (3.11) has a first integral,

$$H_3(x,y) = (b_{20}x - 1)[b_{20}x + 1 + 12(b_{11}-\gamma)y]^{\alpha}[b_{20}x + 1 + 12(b_{11}+\gamma)y]^{(1-\alpha)},$$

where

$$\alpha = \frac{4b_{20}^2}{\gamma(\gamma+b_{11})}, \quad \gamma = \sqrt{b_{11}^2 + 8b_{20}^2}.$$

$H_3(x,y)$ is an even function of x when $y = 0$ because $H_3(x,0) = b_{20}^2x^2 - 1$, so the origin is a center (see [14]).

When the condition (v) holds, the upper-half plane has a first integral

$$H_4(x,y) = (2b_{20}x - a_{20}y - 2)^2[4(b_{20}x+1)^2 - (4a_{20} + 12a_{20}b_{20}x)y + (3a_{20}^2 - 8b_{20}^2)y^2],$$

which is an even function of x when $y = 0$ since $H_4(x,0) = 16(1 - b_{20}^2x^2)^2$. So the origin is a center (see [14]).

Finally, when the condition (vi) is satisfied, the equations for the upper-half plane of (3.1) become

$$\begin{aligned} \dot{x} &= -y + \frac{1}{96b_{20}^2}(x^2+y^2)^{\frac{7}{4}}[96a_{20}b_{20}^2x^2 - 8b_{20}(27a_{20}^2 - 4b_{20}^2)xy + 9a_{20}(9a_{20}^2 - 8b_{20}^2)y^2], \\ \dot{y} &= x + \frac{1}{24b_{20}}(x^2+y^2)^{\frac{7}{4}}[24b_{20}^2x^2 - 66a_{20}b_{20}xy + (27a_{20}^2 - 16b_{20}^2)y^2], \end{aligned} \tag{3.12}$$

which has a first integral

$$\begin{aligned} H_5(x,y) = {} & \frac{9a_{20}^2}{64b_{20}^2(9a_{20}^2 + 16b_{20}^2)(x^2+y^2)^{\frac{3}{2}}} \\ & \times \{4096b_{20}^4 + (4b_{20}x - 3a_{20}y)^4[8b_{20}^2(x^2-y^2) - 3a_{20}(8b_{20}x - 3a_{20}y)y](x^2+y^2)^{\frac{3}{2}} \\ & + 128b_{20}^2y(4b_{20}x - 3a_{20}y)(24a_{20}b_{20}x - 9a_{20}^2y + 16b_{20}^2y)(x^2+y^2)^{\frac{3}{4}}\}. \end{aligned}$$

Since

$$H_5(x,0) = -\frac{288a_{20}^2b_{20}^2(2 + b_{20}^2x^6|x|^3)}{(9a_{20}^2 + 16b_{20}^2)|x|^3},$$

$H_5(x,y)$ is an even function of x when $y = 0$, and hence the origin is a center (see [14]). □

Combining the results in Theorems 3.1 and 3.2, we have the following theorem.

Theorem 3.3. *For* (3.1), *the highest order of focus value is* 7.

3.2 Isochronous centers of (3.1)

Having established the center conditions in the previous section for (3.1), we now discuss the isochronous center conditions for this system.

First, by a direct computation, we can show that under any of the conditions (ii)–(vi), no isochronous center can exist because these conditions cannot lead to all periodic constants vanishing. When the condition (i) holds, the periodic constants are obtained as

$$\begin{aligned}
\tau_1 &= \frac{2}{3}\,(a_{20} + 2a_{02} - b_{11}),\\
\tau_2 &= \frac{\pi}{16} a_{02}^2 [2a_{20}\lambda + a_{02}(3\lambda + 5)],\\
\tau_3 &= -\frac{2a_{02}^3}{105}(\lambda^2 - 1),\\
\tau_4 &= \frac{3\,\pi a_{02}^4}{1024}(\lambda^2 - 1)(\lambda + 3),\\
\tau_5 &= -\frac{2a_{02}^5}{1001}(\lambda^2 - 1)(14\lambda^2 + 91\lambda + 139),\\
\tau_6 &= \frac{3\pi a_{02}^6}{4587521024}(\lambda^2 - 1)(1190\lambda^2 + 8817\lambda + 13898),\\
&\ \vdots
\end{aligned}$$

Therefore, we have the following theorem.

Theorem 3.4. *The origin* (*or infinity*) *of* (3.1) *is an isochronous center if and only if one of the following conditions holds*:

(I) $a_{11} = b_{02} = b_{20} = a_{02} = b_{11} - a_{20} = 0$;

(II) $\lambda - 1 = a_{11} = b_{02} = b_{20} = b_{11} + 2a_{02} = a_{20} + 4a_{02} = 0$;

(III) $\lambda + 1 = a_{11} = b_{02} = b_{20} = b_{11} - 3a_{02} = a_{20} - a_{02} = 0$.

Proof. The *necessity* can be easily proved by setting the periodic constants $\tau_i = 0$, $i = 1, 2, \ldots, 6$. To prove the *sufficiency*, we consider three systems under three conditions (I)–(III). First, consider the condition (I). When this condition holds, the equations for the upper-half plane of (3.1) can be written as

$$\begin{aligned}
\dot{x} &= -y + a_{20}x^2(x^2 + y^2)^{\frac{(\lambda-1)}{2}},\\
\dot{y} &= x + a_{20}xy(x^2 + y^2)^{\frac{(\lambda-1)}{2}}.
\end{aligned} \tag{3.13}$$

A simple calculation gives $\frac{d\theta}{dt} = 1$, so the origin (or infinity) of (3.1) is an isochronous center.

When the condition (II) is satisfied, (3.1) becomes

$$\begin{aligned}
&\left.\begin{aligned}\dot{x} &= -y - a_{02}(4x^2 - y^2),\\ \dot{y} &= x - 2a_{02}xy,\end{aligned}\right\} \quad y > 0,\\
&\left.\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x,\end{aligned}\right\} \quad y < 0.
\end{aligned} \tag{3.14}$$

This system has a transversal system

$$\begin{aligned}
&\left.\begin{aligned}\dot{x} &= x - 4a_{02}\,xy\,(1 - a_{02}y),\\ \dot{y} &= y - a_{02}\,y^2(3 - 2a_{02}y),\end{aligned}\right\} \quad y > 0,\\
&\left.\begin{aligned}\dot{x} &= x,\\ \dot{y} &= y,\end{aligned}\right\} \quad y < 0.
\end{aligned} \tag{3.15}$$

So by [30, Theorem 2.1], the origin of (3.1) is an isochronous center.

When the condition (III) is satisfied, introducing the transformation

$$u = x(x^2+y^2)^{-\frac{2}{3}}, \quad v = y(x^2+y^2)^{-\frac{2}{3}},$$

into (3.1) yields

$$\begin{aligned} &\left.\begin{aligned}\dot{u} &= -v - \frac{1}{3}\,a_{02}(u^4+10u^2v^2-3v^4),\\ \dot{v} &= u + \frac{1}{3}\,a_{02}(5u^2-7v^2)\,uv,\end{aligned}\right\} \quad v>0,\\ &\left.\begin{aligned}\dot{u} &= -v,\\ \dot{v} &= u,\end{aligned}\right\} \quad v<0.\end{aligned} \tag{3.16}$$

The upper-half plane has an inverse integrating factor

$$V(u,v) = (u^2+v^2)^2 f_7^{\frac{5}{6}},$$

from which a first integral can be obtained as

$$H(u,v) = \frac{u^2+v^2}{f_7^{\frac{1}{6}} + 16a_{02}(u^2+v^2)\int u f_7^{-\frac{5}{6}} du},$$

where

$$f_7 = 9[1+2a_{02}\,v(u^2-v^2)+a_{02}^2v^2(u^2+v^2)^2].$$

Thus, this system has a transversal system

$$\begin{aligned} &\left.\begin{aligned}\dot{u} &= u(3-9a_{02}^2v^3)\frac{u^2+v^2}{f_7^{\frac{1}{6}}H(u,v)},\\ \dot{v} &= v(3+9a_{02}u^2v-3a_{02}v^3)\frac{u^2+v^2}{f_7^{\frac{1}{6}}H(u,v)},\end{aligned}\right\} \quad v>0,\\ &\left.\begin{aligned}\dot{u} &= u,\\ \dot{v} &= v,\end{aligned}\right\} \quad v<0,\end{aligned} \tag{3.17}$$

implying that infinity of (3.1) is an isochronous center. □

4 Conclusion

In this paper, quasi-analytic switching systems have been considered. A modified and improved method for computing the return maps of quasi-analytic switching systems is presented. In particular, a quadratic quasi-analytic switching system is investigated using this method. The center and isochronous center conditions are explicitly derived. Compared with the special case $\lambda = 1$ for which five limit cycles are obtained around the origin (see [8]), we have shown that there exist an infinite number of solutions of $\lambda > 0$, $\lambda \neq 1$ for the existence of six limit cycles around the origin, and two solutions for the existence of seven limit cycles for $\lambda = -\frac{16}{5}$, which bifurcate in the small neighborhood of infinity of the system. This shows that the dynamics of quasi-analytic switching systems is more complex.

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