International Journal of Bifurcation and Chaos, Vol. 26, No. 2 (2016) 1650026 (14 pages)

DOI: 10.1142/S0218127416500267

Analytic Integrability of Two Lopsided Systems

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Received January 29, 2015; Revised July 14, 2015

In this paper, we present two classes of lopsided systems and discuss their analytic integrability. The analytic integrable conditions are obtained by using the method of inverse integrating factor and theory of rotated vector field. For the first class of systems, we show that there are $n+4$ small-amplitude limit cycles enclosing the origin of the systems for $n \geq 2$, and ten limit cycles for $n = 1$. For the second class of systems, we prove that there exist $n+4$ small-amplitude limit cycles around the origin of the systems for $n \geq 2$, and nine limit cycles for $n = 1$.

Keywords: Nilpotent Poincaré systems; analytic integrability; Lyapunov constant; rotated vector field.

1. Introduction

Integrability is one of the most important and difficult problems in studying ordinary differential systems. To explain the problem, consider a planar analytic differential system, described by

$$\begin{aligned} \dot{u} &= -v + U(u,v), \\ \dot{v} &= u + V(u,v), \end{aligned} \tag{1}$$

where dot indicates differentiation with respect to time t, U and V are real analytic functions whose series expansions in a neighborhood of the origin start at least from second-order terms. By the Poincaré–Lyapunov theorem, system (1) has a center at the origin if and only if there exists a first integral, given in the form of

$$\phi(u,v) = u^2 + v^2 + \sum_{k+j=3}^{\infty} \phi_{kj} u^k v^j, \tag{2}$$

where the series converges in a neighborhood of the origin. Determining whether the origin of system (1) is a center or focus is called a *center problem*. Another important problem in the study of system (1) is the existence of analytical first integral in a *small* neighborhood of the origin of system (1). If there exists such an analytical first integral, the origin of system (1) is a center, in particular, called an analytic center, see [Algaba *et al.*, 2012].

It is well known that it is difficult to distinguish focus from center when the singular point is degenerate. Much research has been done in this direction. For example, analytic systems having a

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nilpotent singular point at the origin were studied by Andreev [1958] in order to obtain their local phase portraits. However, Andreev's results do not distinguish focus from center. Takens [1974] provided a normal form for nilpotent center of foci. Later, Moussu [1982] found the C^∞ normal form for analytic nilpotent centers. Further, Berthier and Moussu [1994] studied the reversibility of nilpotent centers. Teixeria and Yang [2001] analyzed the relationship between reversibility and the center-focus problem, expressed in a convenient normal form, and studied the reversibility of certain types of polynomial vector fields. Han *et al.* considered polynomial Hamiltonian systems with a nilpotent singular point, and they obtained necessary and sufficient conditions for quadratic and cubic Hamiltonian systems with a nilpotent singular point which may be a center, a cusp or a saddle, see [Han *et al.*, 2010]. In particular, the local analytic integrability for nilpotent centers was investigated in [Chavarriga *et al.*, 2003], for the differential systems in the form of

$$\dot{x} = y + P_3(x, y),$$
$$\dot{y} = Q_3(x, y),$$

which has a local analytic first integral, where P_3 and Q_3 represent homogeneous polynomials of degree three. For third-order nilpotent singular points of a planar dynamical system, the analytic center problem was solved by using the integrating factor method, see for example [Liu *et al.*, 2013].

The Kukles system, as a well-known example, has been investigated intensively on the existence of its limit cycles as well as its integrability. For the following Kukles system,

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3\\ &\quad + a_5x^2y + a_6xy^2 + a_7y^3,\end{aligned}$$

the conditions under which the origin of the system is a center have been examined in [Christopher & Lloyd, 1990; Jin & Wang, 1990; Lloyd & Pearson, 1990, 1992; Rousseau *et al.*, 1995; Wu *et al.*, 1999; Zang *et al.*, 2008]. More details about the Kukles system can be found in [Pearson & Lloyd, 2010]. The so-called extended Kukles system,

$$\begin{aligned}\dot{x} &= y(1 + kx),\\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3\\ &\quad + a_5x^2y + a_6xy^2 + a_7y^3,\end{aligned}$$

has also been considered to obtain the center conditions [Hill *et al.*, 2007a, 2007b]. Recently, center problem for some more generalized Kukles type systems have been studied [Rabanal, 2014; Grin & Schneider, 2013; Llibre & Mereu, 2011]. A type $(n, 4)$ $(3 \le n \le 27)$ of Liénard systems was investigated and the lower bound of the maximal number of limit cycles for this type of system was obtained [Yang & Liang, 2015]. Center conditions of a class of nilpotent-Poincaré system were obtained in [Li & Wu, 2014] by using the method of inverse integrating factor and theory of rotated vector field.

Research on Hilbert's 16th problem usually proceeds by the investigation on specific classes of polynomial systems. In recent years, much effort on the research has been devoted to investigate various systems such as Poincaré system, Abel equation, lopsided system and so on. The Kukles system is perhaps the earliest example of lopsided systems which can be written in the form of

$$\dot{x} = -y, \quad \dot{y} = x + P(x, y),$$

or of

$$\dot{x} = -y + P(x, y), \quad \dot{y} = x.$$

Since then, lopsided systems have drawn more and more attention to researchers. Lopsided quartic and quintic polynomial vector fields have been studied and center conditions were obtained [Salih & Pons, 2002; Pons, 2002]. Furthermore, Gine [2002] proved that there is exactly one isochronous system for lopsided quartic system, and the origin cannot be an isochronous center for lopsided quintic system. For 7-degree polynomial lopsided systems, Soriano and Salih [2002] showed that the origin is a center if and only if the system is time-reversible and if it is not, no more than seven local limit cycles can bifurcate from the origin under certain conditions. However, when the origin is a degenerate singular point, there are fewer results in the literature because it is difficult to compute the Lyapunov constants. The cubic lopsided system with a nilpotent singular point has been investigated intensively. For example, Alvarez and Gasull [2006] proved that three limit cycles can bifurcate from a nilpotent singular point of the following system:

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= a_1x^2 + a_2xy + a_3y^2 + a_4x^3\\ &\quad + a_5x^2y + a_6xy^2 + a_7y^3,\end{aligned} \tag{3}$$

via an analysis based on normal forms. Then, Liu and Li [2009] showed that with a small perturbation to the linear terms of system (3), the system can exhibit four small-amplitude limit cycles. Bifurcation of limit cycles and center conditions for the following two families of lopsided systems with nilpotent singularities,

$$\dot{x} = -y + P_4(x, y),$$
$$\dot{y} = -2x^3$$

and

$$\dot{x} = -y + P_5(x, y),$$
$$\dot{y} = -2x^3,$$

have been considered by Li *et al.* [2013], where $P_4(x, y)$ and $P_5(x, y)$ represent homogeneous polynomials in x and y of degrees four and five, respectively. Their results show that it is more difficult to distinguish focus from center when the singular point is degenerate. As far as analytic center of lopsided systems is concerned, it is more challenging to distinguish it from focus. So, in this paper, we shall discuss analytic center conditions and bifurcation of limit cycles for two classes of lopsided systems with a cubic-order nilpotent singular point, given by

$$\begin{aligned}\dot{x} &= y + H_3(x, y) + H_{2n+3}(x, y),\\ \dot{y} &= -2x^3\end{aligned} \tag{4}$$

and

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -2x^3 + H_3(x, y) + H_{2n+3}(x, y),\end{aligned} \tag{5}$$

where $H_k(x, y)$ represent a kth-degree homogeneous polynomial in x and y.

The main goal of this paper is to apply the method of integrating factor and theory of rotating vector fields to study analytic integrability conditions and to find the conditions for analytic centers. This work is a continuation of that for the Kukles system with a degenerate singular point. In the next section, we present some known results which are necessary for proving the main result. We derive the analytic center conditions for the centers of systems (4) and (5) in Secs. 3 and 4, respectively. Finally, conclusion is drawn in Sec. 5.

2. Preliminary Results

In this section, we present some relative notions and results taken from [Liu & Li, 2010a, 2010b], which will be used in the following sections. A system whose origin is a cubic-order monodromic singular point can be written as

$$\begin{aligned}\dot{x} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij} x^i y^j = X(x, y),\\ \dot{y} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij} x^i y^j = Y(x, y).\end{aligned} \tag{6}$$

Theorem 1. *For any positive integer s and a given number sequence $\{c_{0\beta}\}$, $\beta \geq 3$, a formal series can be constructed successively in terms of the coefficients $c_{\alpha\beta}$ $(\alpha \neq 0)$ as*

$$M(x, y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta} x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x, y), \tag{7}$$

satisfying

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=3}^{\infty} \omega_m(s, \mu) x^m, \tag{8}$$

where $M_k(x, y)$ is a kth-degree homogeneous polynomial in x and y, satisfying $s\mu = 0$ for all k.

Theorem 2. *For $\alpha \geq 1$, $\alpha + \beta \geq 3$ in (7) and (8), $c_{\alpha\beta}$ can be uniquely determined by the recursive formula,*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}). \tag{9}$$

For $m \geq 1$, $\omega_m(s, \mu)$ can be uniquely determined by the recursive formulae:

$$\omega_m(s, \mu) = A_{m,0} + B_{m,0}, \tag{10}$$

$$\lambda_m = \frac{\omega_{2m+4}(s, \mu)}{2m - 4s - 1} \tag{11}$$

where

$$\begin{aligned}A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [k - (s+1)(\alpha - k + 1)] \times a_{kj} c_{\alpha-k+1,\beta-j},\\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} [j - (s+1)(\beta - j + 1)] \times b_{kj} c_{\alpha-k,\beta-j+1}.\end{aligned} \tag{12}$$

Theorem 3. *The origin of system (6) is an analytic center if and only if the origin of system (6) is a center of ∞-class, namely, the origin of system (6) is a center for any natural number s.*

3. Analytic Centers of System (4)

Now, we discuss the analytic centers of system (4) in two cases.

3.1. *Case 1*: $n = 1$

For this case, system (4) can be written as

$$\begin{aligned} \dot{x} &= y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ &\quad + a_{50}x^5 + a_{41}x^4y + a_{32}x^3y^2 + a_{23}x^2y^3 \\ &\quad + a_{14}xy^4 + a_{05}y^5, \\ \dot{y} &= -2x^3. \end{aligned} \tag{13}$$

According to Theorem 1, we can find a formal series $M(x, y) = x^4 + y^2 + o((x^2 + y^2)^2)$ for system (13), such that Eq. (8) holds. Applying the recursive formulae in Theorem 2 to system (13), with the help of Mathematica, we obtain (a Mathematica code for computing the coefficients ω_m is given in Appendix of [Li *et al.*, 2015])

$$\begin{aligned} &\omega_3 = \omega_4 = \omega_5 = 0, \quad \omega_6 = (4s - 1)a_{30}, \quad \omega_7 = 3(a + 1)c_{03}, \\ &\omega_8 = -\frac{1}{5}(4s - 3)(2a_{12} + 5a_{50}), \quad \omega_9 = 0, \quad \omega_{10} = -\frac{1}{7}(4s - 5)(2a_{32} + 3a_{21}a_{50}), \\ &\omega_{11} = \frac{15}{4}(s + 1)c_{05}, \quad \omega_{12} = -\frac{1}{45}(4s - 7)(12a_{14} + 30a_{03}a_{50} + 5a_{41}a_{50}), \quad \omega_{13} = 0, \\ &\omega_{14} = -\frac{3a_{50}}{77}(4s - 9)(6a_{23} + a_{21}a_{41} - 10a_{50}^2), \quad \omega_{15} = \frac{35}{8}(s + 1)c_{07}, \\ &\omega_{16} = -\frac{a_{50}}{117}(4s - 11)(60a_{05} + 10a_{03}a_{41} + a_{41}^2 - 3a_{21}a_{50}^2), \quad \omega_{17} = 0, \\ &\omega_{18} = \frac{a_{50}}{1155}(4s - 13)(2a_{21}a_{41}^2 + 300a_{03}a_{50}^2 + 9a_{21}^2a_{50}^2 + 100a_{41}a_{50}^2), \quad \omega_{19} = \frac{315}{64}(s + 1)c_{09}, \\ &\omega_{20} = -\frac{a_{50}}{895050}(4s - 15)(28a_{21}a_{41}^4 + 252a_{21}^2a_{41}^2a_{50}^2 + 800a_{41}^3a_{50}^2 + 567a_{21}^3a_{50}^4 + 3600a_{21}a_{41}a_{50}^4 + 4500a_{50}^6), \\ &\omega_{21} = 0, \\ &\omega_{22} = -\frac{4a_{50}}{235125}(4s - 17)(4a_{21}^2a_{41}^4 + 36a_{21}^3a_{41}^2a_{50}^2 + 100a_{21}a_{41}^3a_{50}^2 + 81a_{21}^4a_{50}^4 + 450a_{21}^2a_{41}a_{50}^4 - 125a_{41}^2a_{50}^4), \\ &\omega_{23} = \frac{693}{128}(s + 1)c_{011}, \quad \omega_{24} = \frac{a_{50}}{42089726250000(s + 1)}f_1, \end{aligned} \tag{14}$$

where

$$\begin{aligned} f_1 = &-15174868212a_{21}^6a_{41}^4 - 84454927200a_{21}^4a_{41}^5 + 22768748000a_{21}^2a_{41}^6 - 136573813908a_{21}^7a_{41}^2a_{50}^2 \\ &- 1193662008000a_{21}^5a_{41}^3a_{50}^2 - 2087643726000a_{21}^3a_{41}^4a_{50}^2 + 651216400000a_{21}a_{41}^5a_{50}^2 \\ &- 307291081293a_{21}^8a_{50}^4 - 3661266760200a_{21}^6a_{41}a_{50}^4 - 9313504335000a_{21}^4a_{41}^2a_{50}^4 \\ &+ 5946721200000a_{21}^2a_{41}^3a_{50}^4 + 23826000000a_{41}^4a_{50}^4 - 17785962180a_{21}^6a_{41}^4s \\ &- 98929404000a_{21}^4a_{41}^5s + 26934900000a_{21}^2a_{41}^6s - 160073659620a_{21}^7a_{41}^2a_{50}^2s \\ &- 1398534984000a_{21}^5a_{41}^3a_{50}^2s - 2442981870000a_{21}^3a_{41}^4a_{50}^2s + 18810000000a_{41}^4a_{50}^4s \\ &+ 770106000000a_{21}a_{41}^5a_{50}^2s - 360165734145a_{21}^8a_{50}^4s + 6998670000000a_{21}^2a_{41}^3a_{50}^4s \end{aligned}$$

$$- 4290086997000a_{21}^{6}a_{41}a_{50}^{4}s - 10903637205000a_{21}^{4}a_{41}^{2}a_{50}^{4}s + 4967473392a_{21}^{6}a_{41}^{4}s^{2}$$
$$+ 27628675200a_{21}^{4}a_{41}^{5}s^{2} - 7529168000a_{21}^{2}a_{41}^{6}s^{2} + 44707260528a_{21}^{7}a_{41}^{2}a_{50}^{2}s^{2}$$
$$+ 390585888000a_{21}^{5}a_{41}^{3}a_{50}^{2}s^{2} + 682203816000a_{21}^{3}a_{41}^{4}a_{50}^{2}s^{2} - 215262400000a_{21}a_{41}^{5}a_{50}^{2}s^{2}$$
$$+ 100591336188a_{21}^{8}a_{50}^{4}s^{2} + 1198155823200a_{21}^{6}a_{41}a_{50}^{4}s^{2} + 3044973060000a_{21}^{4}a_{41}^{2}a_{50}^{4}s^{2}$$
$$- 1955419200000a_{21}^{2}a_{41}^{3}a_{50}^{4}s^{2} - 5016000000a_{41}^{4}a_{50}^{4}s^{2}.$$

Based on (11) and (14), it is easy to find the first ten quasi-Lyapunov constants of system (13).

Theorem 4. *The first ten quasi-Lyapunov constants evaluated at the origin of system (13) are given by*

$$\begin{aligned}
\lambda_1 &= a_{30},\\
\lambda_2 &= \frac{1}{5}(2a_{12} + 5a_{50}),\\
\lambda_3 &= \frac{1}{7}(2a_{32} + 3a_{21}a_{50}),\\
\lambda_4 &= \frac{1}{45}(12a_{14} + 30a_{03}a_{50} + 5a_{41}a_{50}),\\
\lambda_5 &= \frac{3\,a_{50}}{77}(6a_{23} + a_{21}a_{41} - 10a_{50}^{2}),\\
\lambda_6 &= -\frac{a_{50}}{117}(60a_{05} + 10a_{03}a_{41} + a_{41}^{2} - 3a_{21}a_{50}^{2}),\\
\lambda_7 &= -\frac{a_{50}}{1155}(2a_{21}a_{41}^{2} + 300a_{03}a_{50}^{2}\\
&\quad + 9a_{21}^{2}a_{50}^{2} + 100a_{41}a_{50}^{2}),\\
\lambda_8 &= -\frac{a_{50}}{895050}(28a_{21}a_{41}^{4} + 252a_{21}^{2}a_{41}^{2}a_{50}^{2}\\
&\quad + 800a_{41}^{3}a_{50}^{2} + 567a_{21}^{3}a_{50}^{4}\\
&\quad + 3600a_{21}a_{41}a_{50}^{4} + 4500a_{50}^{6}),\\
\lambda_9 &= -\frac{4a_{50}}{235125}(4a_{21}^{2}a_{41}^{4} + 36a_{21}^{3}a_{41}^{2}a_{50}^{2}\\
&\quad + 100a_{21}a_{41}^{3}a_{50}^{2} + 81a_{21}^{4}a_{50}^{4}\\
&\quad + 450a_{21}^{2}a_{41}a_{50}^{4} - 125a_{41}^{2}a_{50}^{4}),\\
\lambda_{10} &= -\frac{a_{50}}{42089726250000(s+1)(4s-19)}f_1,
\end{aligned}\tag{15}$$

where $\lambda_{k-1} = 0$ for $k = 2, \ldots, 10$ have been used in the computation of λ_k.

It follows from Theorem 4 that the following assertion holds.

Proposition 1. *For $n = 1$, the origin of system (13) is an analytic center if and only if the following conditions are satisfied:*

$$a_{30} = a_{12} = a_{32} = a_{14} = a_{50} = 0. \tag{16}$$

Proof. By setting $\lambda_1 = \lambda_2 = \cdots = \lambda_{10} = 0$, it is easy to get the conditions in (16). Assume $a_{50} \neq 0$, and denote

$$\begin{aligned}
f_2 &= 28a_{21}a_{41}^{4} + 252a_{21}^{2}a_{41}^{2}a_{50}^{2} + 800a_{41}^{3}a_{50}^{2}\\
&\quad + 567a_{21}^{3}a_{50}^{4} + 3600a_{21}a_{41}a_{50}^{4} + 4500a_{50}^{6},\\
f_3 &= 4a_{21}^{2}a_{41}^{4} + 36a_{21}^{3}a_{41}^{2}a_{50}^{2} + 100a_{21}a_{41}^{3}a_{50}^{2}\\
&\quad + 81a_{21}^{4}a_{50}^{4} + 450a_{21}^{2}a_{41}a_{50}^{4} - 125a_{41}^{2}a_{50}^{4}.
\end{aligned}\tag{17}$$

Then, we have

$$\begin{aligned}
R_1 &= \mathrm{Resultant}[f_2, f_3, a_{21}]\\
&= 252226880859375a_{50}^{28}(37a_{41}^{6} + 36000a_{41}^{3}a_{50}^{4} + 864000a_{50}^{8}),\\
R_2 &= \mathrm{Resultant}[f_2, f_1, a_{21}]\\
&= -750785873641864353168750000000000000000a_{50}^{44}(-879390304066912a_{41}^{12}\\
&\quad + 4798354710699940353360a_{41}^{9}a_{50}^{4} + 494455332558037158426660a_{41}^{6}a_{50}^{8}
\end{aligned}$$

$$
\begin{aligned}
&+ 14560575327441724719285 00a_{41}^{3}a_{50}^{12} + 6498810664995012399669375a_{50}^{16}\\
&- 2759277767198304a_{41}^{12}s + 16929770682572631 6960a_{41}^{9}a_{50}^{4}s\\
&+ 1734707435937166329417 00a_{41}^{6}a_{50}^{8}s + 5108374765584631369552500a_{41}^{3}a_{50}^{12}s\\
&+ 2285112445546857058184 0625a_{50}^{16}s - 2080601609429376a_{41}^{12}s^{2}\\
&+ 151804543373289707520a_{41}^{9}a_{50}^{4}s^{2} + 1544224232625002916388 20a_{41}^{6}a_{50}^{8}s^{2}\\
&+ 4547533910484627929569500a_{41}^{3}a_{50}^{12}s^{2} + 20400917084512169654885625a_{50}^{16}s^{2}\\
&+ 610343850576768a_{41}^{12}s^{3} - 33200851039501326720a_{41}^{9}a_{50}^{4}s^{3}\\
&- 342149472446615260119 00a_{41}^{6}a_{50}^{8}s^{3} - 1007536381850376947992500a_{41}^{3}a_{50}^{12}s^{3}\\
&- 4496729306788344912103125a_{50}^{16}s^{3} + 573878672057856a_{41}^{12}s^{4}\\
&- 4987607699325806592 0a_{41}^{9}a_{50}^{4}s^{4} - 504246200112021820111 20a_{41}^{6}a_{50}^{8}s^{4}\\
&- 1484973484158264557562000a_{41}^{3}a_{50}^{12}s^{4} - 6678213700042142946607500a_{50}^{16}s^{4}\\
&- 217264690435584a_{41}^{12}s^{5} + 18276082133674805760a_{41}^{9}a_{50}^{4}s^{5}\\
&+ 1849690602964280495520 0a_{41}^{6}a_{50}^{8}s^{5} + 5447200235062263224400 00a_{41}^{3}a_{50}^{12}s^{5}\\
&+ 2448661869754899507450000a_{50}^{16}s^{5} + 19914634381312a_{41}^{12}s^{6}\\
&- 1701986627481384960a_{41}^{9}a_{50}^{4}s^{6} - 1721646399680525295360a_{41}^{6}a_{50}^{8}s^{6}\\
&- 507012990157076679360 00a_{41}^{3}a_{50}^{12}s^{6} + 2279637270657990507600 00a_{50}^{16}s^{6}).
\end{aligned}
$$

With the aid of Mathematica, we obtain for $\forall\, s \in Z^{+}$,

$$
\begin{aligned}
G_1 &= \mathrm{Resultant}[R_1, R_2, a_{41}]\\
&= -182848672642886912449902102931881129741668701171875a_{50}^{96}(1+s)^{6}(-19+4s)^{6}\\
&\quad \times (122421605949432884774972492589507679 57 + 5718719099641891112424347359750198554 0s\\
&\quad + 8421005783784110519044481758755994470 2s^{2} + 2205334187859295741442697387622502658 0s^{3}\\
&\quad - 3474644745036108705758192186363144052 3s^{4} - 7190180552428847800895138514692327280s^{5}\\
&\quad + 8952012886140489676856041653019558112s^{6} - 1982180847477328550724618213150339840s^{7}\\
&\quad + 138354459536790840295491820367594752s^{8})^{3} \neq 0.
\end{aligned}
$$

So there are no solutions for the set of equations, $f_1 = f_2 = f_3 = 0$, implying that there do not exist other analytic center conditions for system (13) if $a_{50} \neq 0$.

Under the conditions in (16), system (13) becomes

$$
\begin{aligned}
\dot{x} &= y + a_{21}x^{2}y + a_{03}y^{3} + a_{41}x^{4}y\\
&\quad + a_{23}x^{2}y^{3} + a_{05}y^{5},\\
\dot{y} &= -2x^{3}.
\end{aligned}
\tag{18}
$$

Obviously, system (18) is symmetric with the y-axis. According to Theorem 11 in [Liu *et al.*, 2013], the origin is an analytic center of system (13). ■

Proposition 1 implies that

Theorem 5. *The necessary and sufficient conditions for the origin of system (13) being an analytic center are determined from the vanishing of the first ten quasi-Lyapunov constants, that is, the conditions given in Proposition 1 are satisfied.*

When the cubic-order nilpotent singular point, $O(0,0)$, is a tenth-order weak focus, it is easy to show that the perturbed system of (13), given by

$$\begin{aligned}\dot{x} &= \delta x + y + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 \\ &\quad + a_{03}y^3 + a_{50}x^5 + a_{41}x^4y + a_{32}x^3y^2 \\ &\quad + a_{23}x^2y^3 + a_{14}xy^4 + a_{05}y^5, \\ \dot{y} &= \delta y - 2x^3,\end{aligned} \tag{19}$$

can generate ten limit cycles enclosing an elementary node at the origin of system (21). We omit the details of the proof for brevity.

It follows from the above statement and Theorem 2.2 in [Liu & Li, 2010b], we have the following result. The detailed proof can be seen in [Li *et al.*, 2015].

Theorem 6. *If the origin of system* (*19*) *is a tenth-order weak focus, then within a small neighborhood of the origin, for* $0 < \delta \ll 1$, *perturbing the coefficients of system* (*19*) *can yield ten small-amplitude limit cycles bifurcating from the elementary node* $O(0,0)$.

3.2. *Case 2*: $n \geq 2$

For this case, system (4) can be written as

$$\begin{aligned}\dot{x} &= y + x(a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ &\quad + a_{2n+3,0}x^{2n+3} + a_{2n+2,1}x^{2n+2}y \\ &\quad + a_{2n+1,2}x^{2n+1}y^2 + \cdots + a_{1,2n+2}xy^{2n+2} \\ &\quad + a_{0,2n+3}y^{2n+3}) \\ &\equiv X_1(x,y), \\ \dot{y} &= -2x^3.\end{aligned} \tag{20}$$

Theorem 7. *For* $n \geq 2$, *the origin of system* (*20*) *is at most an* $(n+4)$*th-order weak focus. If the origin of system* (*20*) *is an* $(n+4)$*th-order weak focus, then within a small neighborhood of the origin, perturbing the coefficients of system* (*20*) *can yield* $n+4$ *small-amplitude limit cycles around the elementary node* $O(0,0)$.

Proof. For a nilpotent system, in order to study the dynamical behavior in the neighborhood of the origin, we could consider y and x^2 to be of infinitesimal equivalence in the neighborhood of the origin, see [Liu & Li, 2010b]. Construct a comparison system for system (20),

$$\begin{aligned}\dot{x} &= y + x(a_{21}x^2y + a_{03}y^3 + a_{2n+2,1}x^{2n+2}y \\ &\quad + \cdots + a_{1,2n+2}xy^{2n+2}) \\ &\equiv X_2(x,y), \\ \dot{y} &= -2x^3,\end{aligned} \tag{21}$$

which shows that the system is symmetric with the x-axis, and so the origin $O(0,0)$ is a center.

Next, we compute the determinant of systems (20) and (21) to obtain

$$\begin{aligned}J_1 &= \det\begin{bmatrix} X_1(x,y) & -2x^3 \\ X_2(x,y) & -2x^3 \end{bmatrix} \\ &= -2x^4(a_{30}x^2 + a_{12}y^2 + a_{2n+3,0}x^{2n+2} \\ &\quad + a_{2n+1,2}x^{2n}y^2 + \cdots + a_{3,2n}x^2y^{2n} \\ &\quad + a_{1,2n+2}y^{2n+2}).\end{aligned}$$

By treating the y and x^2 as infinitesimal equivalence in the neighborhood of the origin, we have

$$\begin{aligned}J_1 &= -2x^4(a_{30}x^2 + a_{12}x^4 + a_{2n+3,0}x^{2n+2} \\ &\quad + a_{2n+1,2}x^{2n+4} + \cdots + a_{3,2n}x^{4n+2} \\ &\quad + a_{1,2n+2}x^{4n+4}),\end{aligned} \tag{22}$$

which implies that $a_{30}, a_{12}, a_{2n+3,0}, a_{2n+1,2}, \ldots, a_{3,2n}, a_{1,2n+2}$ could be taken as the focus values of system (19). So for $n \geq 2$, the origin of system (20) is at most an $(n+4)$th-order weak focus. According to Theorem 4.1.5 in [Liu & Li, 2010a, 2010b], within a small neighborhood of the origin, perturbing the coefficients of system (20) can yield $n+4$ small-amplitude limit cycles around the elementary node $O(0,0)$. ∎

Furthermore, we have the following result.

Theorem 8. *For* $n \geq 2$, *the origin of system* (*20*) *is an analytic center if and only if*

$$\begin{aligned}a_{30} &= a_{12} = a_{2n+3,0} = a_{2n+1,2} \\ &= \cdots = a_{3,2n} = a_{1,2n+2} = 0.\end{aligned} \tag{23}$$

The proof can be found in [Li *et al.*, 2015].

4. Analytic Centers of System (5)

Now we turn to discuss the analytic center conditions for system (5). It also has two cases.

4.1. *Case A*: $n = 1$

For this case, system (5) can be written as

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -2x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3\\ &\quad + a_{50}x^5 + a_{41}x^4y + a_{32}x^3y^2\\ &\quad + a_{23}x^2y^3 + a_{14}xy^4 + a_{05}y^5,\end{aligned} \tag{24}$$

for which we can find a formal series $M(x,y) = x^4 + y^2 + o((x^2+y^2)^2)$ according to Theorem 1, provided that (8) holds. Carrying out calculations with help of Mathematica and applying the recursive formulae in Theorem 2 to system (24), we obtain

$$\begin{aligned}&\omega_3 = \omega_4 = \omega_5 = 0, \quad \omega_6 = -\frac{1}{3}(4s-1)a_{21},\\ &\omega_7 = 3(s+1)c_{03},\\ &\omega_8 = -\frac{1}{5}(4s-3)(6a_{03}+a_{41}), \quad \omega_9 = 0,\\ &\omega_{10} = -\frac{1}{7}(4s-5)(2a_{03}a_{12} - 2a_{23} + 3a_{03}a_{50}),\\ &\omega_{11} = \frac{15}{4}(s+1)c_{05},\\ &\omega_{12} = \frac{1}{30}(4s-5)(40a_{05} - 4a_{03}a_{32}\\ &\qquad - 2a_{03}a_{12}a_{50} - 5a_{03}a_{50}^2),\\ &\omega_{13} = 0,\\ &\omega_{14} = \frac{a_{031}}{154}(4s-9)(48a_{03}^2 - 40a_{14}\\ &\qquad + 12a_{12}a_{32} + 6a_{12}^2a_{50} + 12a_{32}a_{50}\\ &\qquad + 21a_{12}a_{50}^2 + 18a_{50}^3).\end{aligned}$$

Then, for $a_{12} + 2a_{50} \neq 0$,

$$\begin{aligned}&\omega_{15} = \frac{35}{8}(s+1)c_{07},\\ &\omega_{16} = \frac{a_{03}}{520}(4s-11)(64a_{03}^2a_{12} + 16a_{32}^2 + 128a_{03}^2a_{50} + 16a_{12}a_{32}a_{50}\\ &\qquad + 4a_{12}^2a_{50}^2 + 32a_{32}a_{50}^2 + 20a_{12}a_{50}^3 + 23a_{50}^4),\\ &\omega_{17} = 0,\\ &\omega_{18} = -\frac{a_{03}}{61600(a_{12}+2a_{50})}(4s-13)(4a_{32} + 2a_{12}a_{50} + 5a_{50}^2)(112a_{12}^2a_{32} - 432a_{32}^2\\ &\qquad + 56a_{12}^3a_{50} - 96a_{12}a_{32}a_{50} + 200a_{12}^2a_{50}^2 - 640a_{32}a_{50}^2 + 120a_{12}a_{50}^3 - 85a_{50}^4),\\ &\omega_{19} = \frac{315}{64}(1+s)c_{09},\\ &\omega_{20} = -\frac{a_{03}}{40840800(a_{12}+2a_{50})^2}(4s-15)(4a_{32} + 2a_{12}a_{50} + 5a_{50}^2)(14372996a_{12}^4a_{32}\\ &\qquad - 63894256a_{12}^2a_{32}^2 + 34076160a_{32}^3 + 7186498a_{12}^5a_{50} - 10734116a_{12}^3a_{32}a_{50}\\ &\qquad - 12772032a_{12}a_{32}^2a_{50} + 28572751a_{12}^4a_{50}^2 - 99036264a_{12}^2a_{32}a_{50}^2\\ &\qquad + 45544768a_{32}^2a_{50}^2 + 26958196a_{12}^3a_{50}^3 - 39087216a_{12}a_{32}a_{50}^3),\\ &\omega_{21} = 0,\\ &\omega_{22} = \frac{a_{03}}{11639628000(a_{12}+2a_{50})^2}(4a_{32} + 2a_{12}a_{50} + 5a_{50}^2)f_4\end{aligned} \tag{25}$$

and for $a_{12} + 2a_{50} = 0$,

$$\begin{aligned}&\omega_{16} = \frac{a_{03}}{520}(4s-11)(-4a_{32} + a_{50}^2)(4a_{32} + a_{50}^2),\\ &\omega_{17} = 0.\end{aligned}$$

If in addition $a_{32} = \frac{a_{50}^2}{4}$, then

$$\omega_{18} = \frac{9a_{03}}{7700}(4s-13)(4a_{32}+a_{50}^2)(24a_{03}^2+a_{50}^3), \quad \omega_{19} = \frac{315}{64}(s+1)c_{09},$$

$$\omega_{20} = -\frac{7a_{03}}{13260}(4s-15)a_{50}^6, \quad \omega_{21} = 0, \quad \omega_{22} = -\frac{a_{03}a_{50}^7}{13856700(1+s)}(5391-205861s+66718s^2)$$

and if $a_{32} = -\frac{a_{50}^2}{4}$, we have

$$\omega_{18} = 0, \quad \omega_{19} = \frac{315}{64}(s+1)c_{09}, \quad \omega_{20} = \frac{2a_{03}}{5525}(4s-15)(16a_{03}^2+a_{50}^3)(27a_{03}^2+2a_{50}^3),$$

$$\omega_{21} = 0, \quad \omega_{22} = -\frac{4a_{03}a_{50}}{1154725(1+s)}(16a_{03}^2+a_{50}^3)(27a_{03}^2+2a_{50}^3).$$

Here,

$$\begin{aligned}
f_4 = {} & -175151884140096a_{12}^5a_{32} + 900479057104608a_{12}^3a_{32}^2 - 870691837997952a_{12}a_{32}^3 \\
& - 87575942070048a_{12}^6a_{50} + 191710117401504a_{12}^4a_{32}a_{50} + 108696440458488a_{12}^2a_{32}^2a_{50} \\
& - 135343772601984a_{32}^3a_{50} - 348204560750520a_{12}^5a_{50}^2 + 1449441463187484a_{12}^3a_{32}a_{50}^2 \\
& - 1227387674864544a_{12}a_{32}^2a_{50}^2 - 328545834076764a_{12}^4a_{50}^3 + 705733010157654a_{12}^2a_{32}a_{50}^3 \\
& - 180965555611680a_{32}^2a_{50}^3 - 196530256579516a_{12}^5a_{32}s + 956545497558256a_{12}^3a_{32}^2s \\
& - 775708121404800a_{12}a_{32}^3s - 98265128289758a_{12}^6a_{50}s + 188198718433828a_{12}^4a_{32}a_{50}s \\
& - 92031891970176a_{32}^3a_{50}s - 390699835897045a_{12}^5a_{50}^2s + 1519172249991516a_{12}^3a_{32}a_{50}^2s \\
& - 1080068393470816a_{12}a_{32}^2a_{50}^2s - 368634374118786a_{12}^4a_{50}^3s + 690504693129726a_{12}^2a_{32}a_{50}^3s \\
& - 123060065895200a_{32}^2a_{50}^3s + 80500800862640a_{12}^5a_{32}s^2 - 396831061224512a_{12}^3a_{32}^2s^2 \\
& + 336523546570752a_{12}a_{32}^3s^2 + 40250400431320a_{12}^6a_{50}s^2 - 79597335692936a_{12}^4a_{32}a_{50}s^2 \\
& - 56511998888512a_{12}^2a_{32}^2a_{50}s^2 + 43311880631808a_{32}^3a_{50}s^2 + 160035098537960a_{12}^5a_{50}^2s^2 \\
& - 632261525950008a_{12}^3a_{32}a_{50}^2s^2 + 470151299998208a_{12}a_{32}^2a_{50}^2s^2 + 150997488382038a_{12}^4a_{50}^3s^2 \\
& + 142715605813496a_{12}^2a_{32}^2a_{50}s - 292287689531688a_{12}^2a_{32}a_{50}^3s^2 + 57905489716480a_{32}^2a_{50}^3s^2.
\end{aligned}$$

Based on (11) and (25), it is easy to find the first nine quasi-Lyapunov constants of system (24).

Theorem 9. *The first nine quasi-Lyapunov constants evaluated at the origin of system* (24) *are given by*

$$\lambda_1 = -\frac{1}{3}a_{21}, \quad \lambda_2 = -\frac{1}{5}(6a_{03}+a_{41}), \quad \lambda_3 = -\frac{1}{7}(2a_{03}a_{12}-2a_{23}+3a_{03}a_{50}),$$

$$\lambda_4 = \frac{1}{30}(40a_{05}-4a_{03}a_{32}-2a_{03}a_{12}a_{50}-5a_{03}a_{50}^2),$$

$$\lambda_5 = \frac{a_{031}}{154}(48a_{03}^2-40a_{14}+12a_{12}a_{32}+6a_{12}^2a_{50}+12a_{32}a_{50}+21a_{12}a_{50}^2+18a_{50}^3).$$

Then, for $a_{12}+2a_{50} \neq 0$,

$$\lambda_6 = \frac{a_{03}}{520}(64a_{03}^2a_{12}+16a_{32}^2+128a_{03}^2a_{50}+16a_{12}a_{32}a_{50}+4a_{12}^2a_{50}^2+32a_{32}a_{50}^2+20a_{12}a_{50}^3+23a_{50}^4),$$

$$\lambda_7 = -\frac{a_{03}}{61600(a_{12}+2a_{50})}(4a_{32}+2a_{12}a_{50}+5a_{50}^2)(112a_{12}^2a_{32}-432a_{32}^2+56a_{12}^3a_{50} - 96a_{12}a_{32}a_{50}+200a_{12}^2a_{50}^2-640a_{32}a_{50}^2+120a_{12}a_{50}^3-85a_{50}^4),$$

$$\lambda_8 = -\frac{a_{03}}{40840800(a_{12}+2a_{50})^2}(4a_{32}+2a_{12}a_{50}+5a_{50}^2)(14372996a_{12}^4a_{32}-63894256a_{12}^2a_{32}^2 + 34076160a_{32}^3+7186498a_{12}^5a_{50}-10734116a_{12}^3a_{32}a_{50}-12772032a_{12}a_{32}^2a_{50}+28572751a_{12}^4a_{50}^2 - 99036264a_{12}^2a_{32}a_{50}^2+45544768a_{32}^2a_{50}^2+26958196a_{12}^3a_{50}^3-39087216a_{12}a_{32}a_{50}^3),$$

$$\lambda_9 = \frac{a_{03}}{11639628000(a_{12}+2a_{50})^2}(4a_{32}+2a_{12}a_{50}+5a_{50}^2)f_4;$$

while for $a_{12}+2a_{50}=0$,

$$\lambda_6 = \frac{a_{03}}{520}(-4a_{32}+a_{50}^2)(4a_{32}+a_{50}^2)$$

and in addition if $a_{32}=\frac{a_{50}^2}{4}$,

$$\lambda_7 = \frac{9a_{03}}{7700}(4a_{32}+a_{50}^2)(24a_{03}^2+a_{50}^3),$$

$$\lambda_8 = -\frac{7a_{03}}{13260}a_{50}^6,$$

$$\lambda_9 = -\frac{a_{03}a_{50}^7}{13856700(1+s)} \times (5391-205861s+66718s^2);$$

if $a_{32}=-\frac{a_{50}^2}{4}$,

$$\lambda_7 = 0,$$

$$\lambda_8 = \frac{2a_{03}}{5525}(16a_{03}^2+a_{50}^3)(27a_{03}^2+2a_{50}^3),$$

$$\lambda_9 = -\frac{4\,a_{03}a_{50}}{1154725(1+s)}(16a_{03}^2+a_{50}^3) \times (27a_{03}^2+2a_{50}^3),$$

where $\lambda_{k-1}=0$ *for* $k=2,\ldots,9$ *have been used in computing* λ_k.

Furthermore, the following result can be easily obtained.

Proposition 2. *For* $n=1$, *the origin of system (24) is an analytic center if and only if one of the following conditions holds*:

$$a_{21}=a_{03}=a_{41}=a_{23}=a_{05}=0; \tag{26}$$

$$a_{21}=a_{14}=a_{05}=0,\quad a_{41}=-6a_{03},$$
$$a_{23}=\frac{1}{2}(2a_{12}+3a_{50})a_{03},\quad a_{03}^2=-\frac{1}{16}a_{50}^3; \tag{27}$$

$$a_{21}=a_{05}=0,\quad a_{41}=-6a_{03},$$
$$a_{12}=-2a_{50},\quad a_{23}=-\frac{1}{2}a_{03}a_{50},$$
$$a_{32}=\frac{1}{4}a_{50}^2,\quad a_{03}^2=-\frac{2}{27}a_{50}^3, \tag{28}$$
$$a_{14}=-\frac{1}{72}a_{50}^3.$$

Proof. It is easy to get the conditions (26)–(28) by setting $\lambda_1=\lambda_2=\cdots=\lambda_9=0$. When $a_{50}\neq 0$, let

$$f_5 = 112a_{12}^2a_{32}-432a_{32}^2+56a_{12}^3a_{50}-96a_{12}a_{32}a_{50}+200a_{12}^2a_{50}^2-640a_{32}a_{50}^2+120a_{12}a_{50}^3-85a_{50}^4,$$

$$f_6 = 14372996a_{12}^4a_{32}-63894256a_{12}^2a_{32}^2+34076160a_{32}^3+7186498a_{12}^5a_{50}-10734116a_{12}^3a_{32}a_{50} - 12772032a_{12}a_{32}^2a_{50}+28572751a_{12}^4a_{50}^2-99036264a_{12}^2a_{32}a_{50}^2+45544768a_{32}^2a_{50}^2 + 26958196a_{12}^3a_{50}^3-39087216a_{12}a_{32}a_{50}^3.$$

Then, we obtain

$$R_3 = \mathrm{Resultant}[f_4,f_5,a_{32}] = -200206652313600a_{50}^3(a_{12}+2a_{50})^4(72a_{12}^5+652a_{12}^4a_{50}+2694a_{12}^3a_{50}^2 + 6043a_{12}^2a_{50}^3+7092a_{12}a_{50}^4+3463a_{50}^5),$$

$$\begin{aligned}
R_4 &= \text{Resultant}[f_4, f_6, a_{32}]\\
&= -289298612593152000a_{50}^3(a_{12}+2a_{50})^4(278734084992a_{12}^7 + 3588989330016a_{12}^6a_{50}\\
&\quad + 21026493958464a_{12}^5a_{50}^2 + 71854303647672a_{12}^4a_{50}^3 + 152324731255716a_{12}^3a_{50}^4\\
&\quad + 197669760539760a_{12}^2a_{50}^5 + 144211704399495a_{12}a_{50}^6 + 45502270176438a_{50}^7\\
&\quad + 1651136278704a_{12}^7s + 18838630495800a_{12}^6a_{50}s + 98153819816532a_{12}^5a_{50}^2s\\
&\quad + 294846821571666a_{12}^4a_{50}^3s + 534120897148782a_{12}^3a_{50}^4s + 565346904516561a_{12}^2a_{50}^5s\\
&\quad - 310952885702031a_{12}a_{50}^6s + 62534145621954a_{50}^7s + 1859553268056a_{12}^7s^2\\
&\quad + 17577034105268a_{12}^6a_{50}s^2 + 76054378966082a_{12}^5a_{50}^2s^2 + 181916843290105a_{12}^4a_{50}^3s^2\\
&\quad + 238094684972342a_{12}^3a_{50}^4s^2 + 147267056136244a_{12}^2a_{50}^5s^2 + 19507558179230a_{12}a_{50}^6s^2\\
&\quad - 6987410240074a_{50}^7s^2 - 1714957345728a_{12}^7s^3 - 17270801713568a_{12}^6a_{50}s^3\\
&\quad - 80278393721648a_{12}^5a_{50}^2s^3 - 212296048411048a_{12}^4a_{50}^3s^3 - 328607615124608a_{12}^3a_{50}^4s^3\\
&\quad - 285216632755972a_{12}^2a_{50}^5s^3 - 121914622605938a_{12}a_{50}^6s^3 - 19568176640258a_{50}^7s^3\\
&\quad + 301352777088a_{12}^7s^4 + 3072037299008a_{12}^6a_{50}s^4 + 14465726297408a_{12}^5a_{50}^2s^4\\
&\quad + 38896515312256a_{12}^4a_{50}^3s^4 + 61684793716376a_{12}^3a_{50}^4s^4 + 55602984609100a_{12}^2a_{50}^5s^4\\
&\quad + 25319000517368a_{12}a_{50}^6s^4 + 4451109045332a_{50}^7s^4).
\end{aligned}$$

Further, with the aid of Mathematica, we obtain for $\forall\, s \in Z^+$

$$\begin{aligned}
G_2 &= \text{Resultant}[R_3, R_4, a_{12}]\\
&= 30984189289342953910272000a_{50}^{35}(1+s)^5(-17+4s)^5\\
&\quad \times (-12328775079356225692983907585995321 6\\
&\quad - 33902812452688795016920021129342044624s\\
&\quad - 180855325034978657368019444342423080236s^2\\
&\quad - 106706695920461596165900474148839286557 5s^3\\
&\quad - 332834343796244437534076209999289147211 0s^4\\
&\quad - 477319695465556239084800555585494692145 9s^5\\
&\quad + 142415408037847599167272364364103357143 20s^6\\
&\quad - 107320884674960964670635027958153054751 20s^7\\
&\quad + 372143624839985736429555836313184066803 2s^8\\
&\quad - 625676462230475741935920059840273863168s^9\\
&\quad + 41454785818979861302571809000901003264s^{10})\\
&\neq 0.
\end{aligned}$$

The above calculations indicate that the equations $f_4 = f_5 = f_6 = 0$ do not have real solutions, namely, there do not exist other analytic center conditions for system (24) if $a_{50} \neq 0$.

When the conditions in (26) hold, system (24) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -2x^3 + a_{12}xy^2 + a_{50}x^5 \\ &\quad + a_{32}x^3y^2 + a_{14}xy^4. \end{aligned} \tag{29}$$

Obviously, this system is symmetric with the y-axis, implying that the origin of system (29) is an analytic center due to Theorem 11 in [Liu *et al.*, 2013].

When the conditions in (27) are satisfied, system (24) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{1}{4}(-8x^3 + 4a_{50}x^5 - 24a_{03}x^4y + 4a_{12}xy^2 \\ &\quad - 2a_{12}a_{50}x^3y^2 - 5a_{50}^2x^3y^2 + 4a_{03}y^3 \\ &\quad + 4a_{03}a_{12}x^2y^3 + 6a_{03}a_{50}x^2y^3). \end{aligned} \tag{30}$$

Introducing the transformation,

$$x = x, \quad y = \frac{(-2 + a_{50}x^2)z}{2(-2 + a_{50}x^2 + a_{03}xz)}$$

and time scaling,

$$T = \frac{2(-2 + a_{50}x^2)^3 t}{-2 + a_{50}x^2 - 2a_{03}xy},$$

into system (30) yields

$$\begin{aligned} \frac{dx}{dT} &= z(a_{50}x^2 - 2)^2, \\ \frac{dz}{dT} &= -\frac{1}{4}x(128x^2 - 192a_{50}x^4 + 96a_{50}^2x^6 \\ &\quad - 16a_{50}^3x^8 - 16a_{12}z^2 + 24a_{12}a_{50}x^2z^2 \\ &\quad + 20a_{50}^2x^2z^2 - 96a_{03}^2x^4z^2 - 12a_{12}a_{50}^2x^4z^2 \\ &\quad - 20a_{50}^3x^4z^2 + 48a_{03}^2a_{50}x^6z^2 + 2a_{12}a_{50}^3x^6z^2 \\ &\quad + 5a_{50}^4x^6z^2 + 32a_{03}^3x^5z^3 + 2a_{03}a_{50}^3x^5z^3), \end{aligned} \tag{31}$$

which is symmetric with the z-axis because $a_{03}^2 = -\frac{a_{50}^3}{16}$. Thus, according to Theorem 11 in [Liu *et al.*, 2013], the origin of system (30) is an analytic center.

Similarly, when the conditions in (28) hold, system (24) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \frac{1}{72}(-144x^3 + 72a_{50}x^5 - 432a_{03}x^4y \\ &\quad - 144a_{50}xy^2 - 18a_{50}^2x^3y^2 + 72a_{03}y^3 \\ &\quad - 36a_{03}a_{50}x^2y^3 - a_{50}^3xy^4), \end{aligned} \tag{32}$$

for which there exists an analytic integrating factor,

$$u(x,y) = \frac{e^{-\frac{3}{8}a_{50}^2x^4}}{\left(1 - \frac{1}{2}a_{50}x^2 + \frac{3}{4}a_{50}xy\right)^4},$$

indicating that the origin of system (32) is an analytic center. ■

Therefore, Proposition 2 implies the following result.

Theorem 10. *The necessary and sufficient conditions for the origin of system (24) being an analytic center are given by the vanishing of the first nine quasi-Lyapunov constants, that is, one of the conditions in Proposition 2 is satisfied.*

Similarly, when the cubic-order nilpotent singular point $O(0,0)$ is a ninth-order weak focus, it is easy to prove that the perturbed system of (24), given by

$$\begin{aligned} \dot{x} &= \delta x + y, \\ \dot{y} &= \delta y - 2x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ &\quad + a_{50}x^5 + a_{41}x^4y + a_{32}x^3y^2 + a_{23}x^2y^3 \\ &\quad + a_{14}xy^4 + a_{05}y^5, \end{aligned} \tag{33}$$

can generate nine limit cycles enclosing an elementary node at the origin. The proof is similar to that for Theorem 9 and thus omitted for brevity.

Based on the above statement and Theorem 2.2 in [Liu & Li, 2010b] we have the following theorem.

Theorem 11. *If the origin of system (33) is a ninth-order weak focus, then within a small neighborhood of the origin, for $0 < \delta \ll 1$, system (33) can yield nine small-amplitude limit cycles around the elementary node $O(0,0)$.*

The proof of Theorem 11 can be seen in [Li *et al.*, 2015].

4.2. *Case B*: $n \geq 2$

For this case, system (5) can be written as

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -2x^3 + (a_{21}x^2y + a_{12}xy^2 + a_{03}y^3\\ &\quad + a_{2n+3,0}x^{2n+3} + a_{2n+2,1}x^{2n+2}y\\ &\quad + a_{2n+1,2}x^{2n+1}y^2 + \cdots + a_{1,2n+2}xy^{2n+2}\\ &\quad + a_{0,2n+3}y^{2n+3})\\ &\equiv Y_1(x,y).\end{aligned} \tag{34}$$

Theorem 12. *For $n \geq 2$, the origin of system (34) is at most an $(n+4)$th-order weak focus. If the origin of system (34) is an $(n+4)$th-order weak focus, then within a small neighborhood of the origin of its perturbed system, perturbing the coefficients of system (34) can yield $n+4$ small-amplitude limit cycles enclosing the elementary node $O(0,0)$.*

Proof. The proof is similar to that for Theorem 7. We construct a comparison system for system (34),

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= -2x^3 + x(a_{12}xy^2 + a_{2n+3,0}x^{2n+3}\\ &\quad + \cdots + a_{2,2n+1}x^2y^{2n+1})\\ &\equiv Y_2(x,y).\end{aligned} \tag{35}$$

It is easy to see that system (35) is symmetric with the x-axis, and so $O(0,0)$ is a center.

Next, we compute the determinant of systems (34) and (35), yielding

$$\begin{aligned}J_2 &= \det\begin{bmatrix} y & Y_1(x,y)\\ y & Y_2(x,y)\end{bmatrix}\\ &= a_{21}x^2y^2 + a_{03}y^4 + a_{2n+2,1}x^{2n+2}y^2\\ &\quad + a_{2n,3}x^{2n}y^4 + \cdots + a_{2,2n+1}x^2y^{2n+2}\\ &\quad + a_{0,2n+3}y^{2n+4}.\end{aligned}$$

Similarly, we take the y and x^2 as infinitesimal equivalence in the neighborhood of the origin in order to study the dynamical behavior of (34) around the origin. So, J_2 becomes

$$\begin{aligned}J_2 &= x^4(a_{21}x^2 + a_{03}x^4 + a_{2n+2,1}x^{2n+2}\\ &\quad + a_{2n,3}x^{2n+2} + \cdots + a_{2,2n+1}x^{4n+2}\\ &\quad + a_{0,2n+3}x^{4n+4}),\end{aligned} \tag{36}$$

implying that $a_{21}, a_{03}, a_{2n+2,1}, a_{2n,3}, \ldots, a_{2,2n+1}$, $a_{0,2n+3}$ could be considered as the focal values of the system. Therefore, for $n \geq 2$, the origin of system (34) is at most an $(n+4)$th-order weak focus. According to Theorem 4.1.5 in [Liu & Li, 2010a, 2010b], within a small neighborhood of the origin, one can perturb the coefficients of system (34) to obtain $n+4$ small-amplitude limit cycles around the elementary node $O(0,0)$. ■

Moreover, we have a similar theorem for this case.

Theorem 13. *For $n \geq 2$, the origin of system (34) is an analytic center if and only if*

$$\begin{aligned}a_{21} &= a_{03} = a_{2n+2,1} = a_{2n,3}\\ &= \cdots = a_{2,2n+1} = a_{0,2n+3} = 0.\end{aligned} \tag{37}$$

For more details about the proof, see [Li *et al.*, 2015].

5. Conclusion

In this paper, two classes of lopsided systems have been studied on their analytic integrable conditions and bifurcation of limit cycles. We have obtained some analytic integrability conditions for each class of the systems for case $n = 1$. By using certain transformations or integrating factors, we have proved that all conditions are sufficient and necessary. For case $n \geq 2$, we have constructed different comparison systems for each class of the systems and shown that $n+4$ limit cycles may bifurcate from the origin of each system. In addition, conditions for the origin being an analytic center are obtained simultaneously.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (NSFC Nos. 11201211, 11371373), the Applied Mathematics Enhancement Program of Linyi University and the Natural Science and Engineering Research Council of Canada (NSERC No. R2686A02).

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