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Center and isochronous center conditions for switching systems associated with elementary singular points

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ABSTRACT

In this paper, an existing method is modified for computing the focal values and period constants of switching systems associated with elementary singular points. In particular, a quadratic switching system is considered to illustrate the computational efficiency of this method. Further, with this method, a cubic switching system is constructed to show existence of 15 limit cycles, which is the best result so far obtained for cubic switching systems.

1. Introduction

As one of the most important bifurcation phenomena, Hopf bifurcation plays an important role in the study of nonlinear dynamical systems. Many results on Hopf bifurcation for continuous systems have been obtained, especially for planar differential systems, see for example [1–3]. As far as the maximal number of small-amplitude limit cycles, bifurcating from an elementary center or focus, is concerned, the best known result is $M(2) = 3$, obtained by Bautin in 1952 [4]. Here, $M(n)$ denotes the maximal number of small-amplitude limit cycles around a singular point with n being the degree of polynomials in the vector field. For $n = 3$, a number of results have been obtained. Around an elemental focus, James and Lloyd [5] considered a special class of cubic systems to obtain 8 limit cycles in 1991, and the systems were reinvestigated couple of years later by Ning et al. [6] to find another solution of 8 limit cycles. Yu and Corless [7] constructed a cubic system and combined symbolic and numerical computations to show 9 limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [8]. Another cubic system was also recently constructed by Lloyd and Pearson [9] to show 9 limit cycles with purely symbolic computation. Recently, Yu and Tian [10] have shown that there can exist 12 limit cycles around an elementary center in a planar cubic-degree polynomial system. This is the best result obtained so far for cubic polynomial systems with all limit cycles around a single singular point. For $n \geq 4$, there are very few results, for example, Huang gave an example of a quartic system with 8 limit cycles bifurcating from a fine focus [11].

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However, in modeling practical physical and engineering problems, there exist many problems which involve discontinuous or non-smooth functions, see for instance [12] and [13], and the references therein. Such examples include relay feedback systems in control theory [14,15], switching circuits in power electronics [16], impact and dry frictions in mechanical engineering [17,18], etc. In recent years, study of switching systems associated with Hopf bifurcation has attracted many researchers. Leine and Nijmeijer [19], and Zou et al. [20] considered non-smooth Hopf bifurcation. Freire et al. [21] discussed the focus-center limit cycle bifurcation in a symmetric three-dimensional, piecewise linear system. For homoclinic bifurcation, the Melnikov function method has been extended to study non-smooth systems [22,23]. General effective methods have also been developed to investigate non-smooth systems. For example, normal form computation for impact oscillators was given in [24], and a general methodology for reducing multidimensional flows to low dimensional maps in piecewise nonlinear oscillators was proposed in [25]. Due to complexity in non-smooth systems, such systems can exhibit not only all types of bifurcations that occur in smooth systems, but also complicated nonstandard bifurcation phenomena that are unique in non-smooth ones, such as grazing [26,27], sliding effects [17], border collision [28], etc. There are many articles in the literature, devoted to study various nonstandard bifurcations for non-smooth systems; see, for example, [17,18,26–29] and the references therein.

Recently, Chen and Du constructed a quadratic switching system to obtain 9 limit cycles [30]. Llibre et al. studied the maximum number of limit cycles that bifurcate from the periodic orbits of isochronous centers in switching cubic polynomial differential systems [31] and in switching quadratic polynomial differential systems [32]. These examples show that there exist more limit cycles in switching systems than in continuous systems, and the dynamics of these systems are more complex.

In this paper, the switching planar system, described by the following ordinary differential equations:

$$\begin{cases}\dfrac{dx}{dt}=-y+F^{+}(x,y),\\[2mm] \dfrac{dy}{dt}=x+G^{+}(x,y),\end{cases}\quad (y>0),\qquad \begin{cases}\dfrac{dx}{dt}=-y+F^{-}(x,y),\\[2mm] \dfrac{dy}{dt}=x+G^{-}(x,y),\end{cases}\quad (y<0), \tag{1.1}$$

will be used to investigate bifurcation of limit cycles. In particular, a method for computing the Lyapunov constants of system (1.1) is present in Section 2, and then an approach for computing the period constants of system (1.1) is given in Section 3. Then, in Section 4 a quadratic switching system, as an example, is given to illustrate the computation efficiency of our methods; and further in the same section we construct a cubic switching system to show that system (1.1) can exhibit at least 15 limit cycles, which is a new best result for such systems. Finally, conclusion is drawn in Section 5.

2. Computation of Lyapunov constants of system (1.1)

In this section, we present a method for computing the Lyapunov constants of the switching system (1.1). First, we introduce some basic formulas of computing Lyapunov constants and period constants. The classical method to solve center problems is based on computing Lyapunov constants, with the procedure described as follows.

The general differential system,

$$\begin{aligned}\frac{dx}{dt}&=\delta x-y+\sum_{k=2}^{n}X_k(x,y)\equiv X(x,y),\\ \frac{dy}{dt}&=x+\delta y+\sum_{k=2}^{n}Y_k(x,y)\equiv Y(x,y),\end{aligned} \tag{2.1}$$

under the polar coordinates transformation,

$$x=r\cos\theta,\quad y=r\sin\theta, \tag{2.2}$$

can be rewritten as

$$\begin{aligned}\frac{dr}{dt}&=r\left(\delta+\sum_{k=2}^{n}\varphi_{k+2}(\theta)r^k\right),\\ \frac{d\theta}{dt}&=1+\sum_{k=2}^{n}\psi_{k+2}(\theta)r^k,\end{aligned} \tag{2.3}$$

where $\varphi_k(\theta)$, $\psi_k(\theta)$ are polynomials of $\cos\theta$ and $\sin\theta$, given by

$$\begin{aligned}\varphi_k(\theta)&=\cos\theta X_{k-1}(\cos\theta,\sin\theta)+\sin\theta Y_{k-1}(\cos\theta,\sin\theta),\\ \psi_k(\theta)&=\cos\theta Y_{k-1}(\cos\theta,\sin\theta)-\sin\theta X_{k-1}(\cos\theta,\sin\theta).\end{aligned}$$

From Eq. (2.3) we have

$$\frac{dr}{d\theta}=\frac{r(\delta+\sum_{k=2}^{n}\varphi_{k+2}(\theta)r^k)}{1+\sum_{k=2}^{n}\psi_{k+2}(\theta)r^k}, \tag{2.4}$$

whose expansion around $r=0$ can be expressed in the form of

$$\frac{dr}{d\theta}=r\sum_{k=1}^{\infty}R_k(\theta)r^k. \tag{2.5}$$

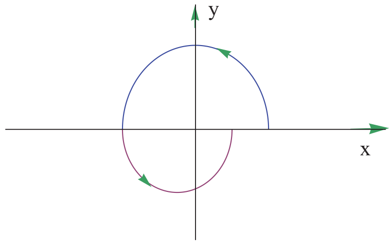

Fig. 1. The successive function $\Delta(h)$.

By the method of small parameters of Poincaré, the general solution of (2.5) can be obtained as

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k,$$

where $v_1(0) = 1, v_k(0) = 0, \forall k \geq 2$. Now, substituting the above solution $r = \tilde{r}(\theta, h)$ into (2.5) yields

$$\begin{aligned} v_1'(\theta) &= R_0(\theta) v_1(\theta), \\ v_2'(\theta) &= R_0(\theta) v_2(\theta) + R_1(\theta) v_1^2(\theta), \\ &\vdots \\ v_m'(\theta) &= R_0(\theta)\Omega_{1,m}(\theta) + R_1(\theta)\Omega_{2,m}(\theta) + \cdots + R_{m-1}(\theta)\Omega_{m,m}(\theta). \end{aligned} \tag{2.6}$$

Thus, we may solve $v_k(\theta)$ one by one, yielding

$$\begin{aligned} v_1(\theta) &= e^{\oint_0^{\vartheta} R_0(\varphi) d\varphi}, \\ v_2(\theta) &= 2 v_1(\theta) \oint_0^{\vartheta} R_1(\varphi) v_1(\varphi) d\varphi, \\ &\vdots \\ v_m(\theta) &= v_1(\theta) \oint_0^{\vartheta} \frac{R_1(\varphi)\Omega_{2,m}(\varphi) + \cdots + R_{m-1}(\varphi)\Omega_{m,m}(\varphi)}{v_1(\varphi)} d\varphi. \end{aligned} \tag{2.7}$$

Furthermore, we define the successive function as

$$\Delta(h) = \tilde{r}(2\pi, h) - h, \tag{2.8}$$

as shown in Fig. 1, which in turn gives the condition to define a center, as

$$r(2\pi, h) = h. \tag{2.9}$$

Many classical methods have been developed for computing the successive function of continuous systems, see, for example [33].

From the second equation of (2.3), we can also obtain

$$t = T(\theta, h) = \int_0^{\theta} \frac{d\varphi}{1 + \sum_{k=2}^{n} \psi_{k+2}(\varphi) r^k(\varphi, h)}, \tag{2.10}$$

which shows that the condition, corresponding to an isochronous center, is given by

$$r(2\pi, h) = h, \quad T(2\pi, h) = 2\pi. \tag{2.11}$$

However, unfortunately, the classical methods and formulas cannot be directly applied to a switching system due to discontinuity. We need to develop new methods to overcome this difficulty.

Note that the polar coordinates expression for (1.1) can be written as

$$\begin{aligned} &(R^+(r,\theta), 1 + \Theta^+(r,\theta)), \quad \theta \in [0, \pi], \\ &(R^-(r,\theta), 1 + \Theta^-(r,\theta)), \quad \theta \in [\pi, 2\pi]. \end{aligned} \tag{2.12}$$

Also note that although a return map cannot be simply defined for (1.1) like that for continuous systems, we may follow the approach presented in [34] to define half-return maps, but the method of computing of the return map based on a suitable decomposition of certain one-forms associated with the expression of the system in polar coordinates in [34] is difficult to understand and complex. So we modified the method to compute the return map near the singular point.

By Lemma 2.1 in [34], we may equivalently compute the positive half-return map for

$$\begin{aligned} \frac{dx}{dt} &= -y + F^+(x,y) \equiv X^+(x,y), \\ \frac{dy}{dt} &= x + G^+(x,y) \equiv Y^+(x,y), \end{aligned} \quad (y > 0) \tag{2.13}$$

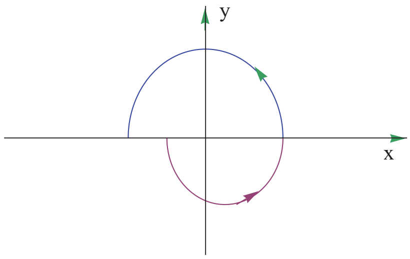

Fig. 2. Positive half-return maps.

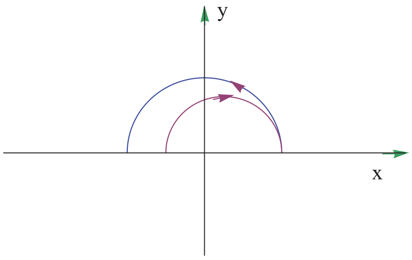

Fig. 3. Lower half plane changed to upper half plane.

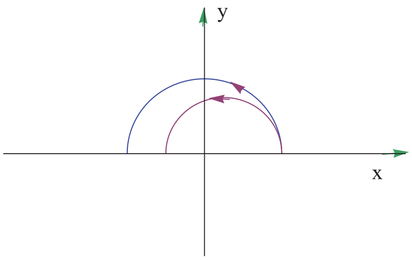

Fig. 4. Vector fields of system (2.13) and (2.14).

and that for

$$\begin{aligned} \frac{dx}{dt} &= -y - F^-(x, -y) \equiv X^-(x, y), \\ \frac{dy}{dt} &= x + G^-(x, -y) \equiv Y^-(x, y), \end{aligned} \quad (y > 0). \tag{2.14}$$

The idea to obtain the above system (2.14) is illustrated in Figs. 2 and 3. We first redefine the positive half-return maps for the upper and lower half planes, defined by (1.1), as shown in Fig. 2. Then, we introduce the transformation $y \to -y$ to change the lower half phase into upper phase, as shown in Fig. 3.

Further, with a time reversing $t \to -t$, the computation of the half-return map of the lower plane in (1.1) is equivalent to the computation of the positive half-return map of (2.14), as depicted in Fig. 4.

The above discussion indicates that we only need to compute the positive half-return maps for the systems (2.13) and (2.14). Suppose

$$r_1 = \tilde{r_1}(\theta, h) = \sum_{k=1}^{\infty} u_k(\theta) h^k$$

and

$$r_2 = \tilde{r_2}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k$$

are the solutions of system (2.13) and (2.14), respectively, satisfying $u_1(0) = v_1(0) = 1, u_k(0) = v_k(0) = 0, \forall k \geq 2$. We can then define the following successive functions:

$$\Delta_1(h) = \tilde{r_1}(\pi, h) - h$$

and

$$\Delta_2(h) = \tilde{r_2}(\pi, h) - h,$$

for systems (2.13) and (2.14), respectively. Then, the successive function for the switching system (1.1) can be defined as

$$\Delta(h) = \Delta_1(h) - \Delta_2(h) = \tilde{r_1}(\pi, h) - \tilde{r_2}(\pi, h). \tag{2.15}$$

Definition 2.1. Define

$$\Delta(h) = \sum_{k=1}^{n} (u_k(\pi) - v_k(\pi))h^k = \sum_{k=1}^{n} V_k h^k, \tag{2.16}$$

where V_k is called the kth-order focal value of the switching system (1.1).

Obviously, the symmetry principle for continuous systems cannot be used to prove the center conditions of switching systems. We need to redefine symmetry of switching systems in order to derive the center conditions for switching systems.

Definition 2.2. If the functions on the right-hand side of systems (1.1) satisfy the following conditions:

$$F^+(x,y) = F^+(-x,y), \quad G^+(x,y) = -G^+(-x,y),$$
$$F^-(x,y) = F^-(-x,y), \quad G^-(x,y) = -G^-(-x,y),$$

then system (1.1) is said to be symmetric with the y-axis. If the functions on the right-hand side of systems (1.1) satisfy

$$F^+(x,y) = -F^-(x,-y), \quad G^+(x,y) = G^-(x,-y),$$

then system (1.1) is said to be symmetric with the x-axis.

With the above definitions, we have the following result.

Theorem 2.1. *If system (1.1) is symmetric with the x-axis or the y-axis, then the origin of system (1.1) is a center.*

Proof. When system (1.1) is symmetric with the y-axis, we have

$$F^+(x,y) = F^+(-x,y), \quad G^+(x,y) = G^+(-x,y),$$
$$F^-(x,y) = F^-(-x,y), \quad G^-(x,y) = G^-(-x,y).$$

Because the upper half plane and lower half plane are symmetry, the origins are the centers of the upper half plane and lower half plane, then the origin of system (1.1) is a center because of its symmetry.

When (1.1) is symmetric with the x-axis, namely,

$$F^+(x,y) = -F^-(x,-y), \quad G^+(x,y) = G^-(x,-y),$$

we may use the transformation, $y \to -y, \quad t \to -t$, to transfer system (2.14) into (2.13). So the origin of (1.1) is a center. □

To end this section, we present an example to demonstrate that the origin of the switching system is not a center even if both the upper half plane and lower half plane have analytic first integrals.

Example 2.1. Consider the following system:

$$\begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x + 3x^2 + 2x^3, \end{cases} (y > 0), \qquad \begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x, \end{cases} (y < 0). \tag{2.17}$$

Obviously, the upper half plane has a first integral,

$$H(x,y) = x^2 + y^2 + 2x^3 + x^4;$$

and the lower half plane has a first integral,

$$H(x,y) = x^2 + y^2.$$

However, the origin of system (2.12) is not a center, as illustrated in Fig. 5.

This example implies the following result.

Theorem 2.2. *If the upper half plane and lower half plane of system (1.1) have analytic first integrals, $H_1(x,y)$ and $H_2(x,y)$, respectively, then the origin of system (1.1) is a center if and only if $H_1(x,y)$ and $H_2(x,y)$ satisfy one of the following conditions:*

(1) *$H_1(x_i, 0) = h$, and $H_2(x_1, 0) = H_2(x_2, 0)$ for any real values of h, x_1, x_2;*
(2) *$H_2(x_i, 0) = h$, and $H_1(x_1, 0) = H_1(x_2, 0)$ for any real values of h, x_1, x_2.*

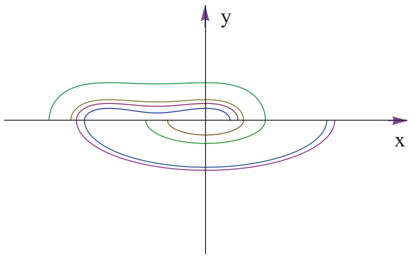

Fig. 5. The vector field of system (2.12).

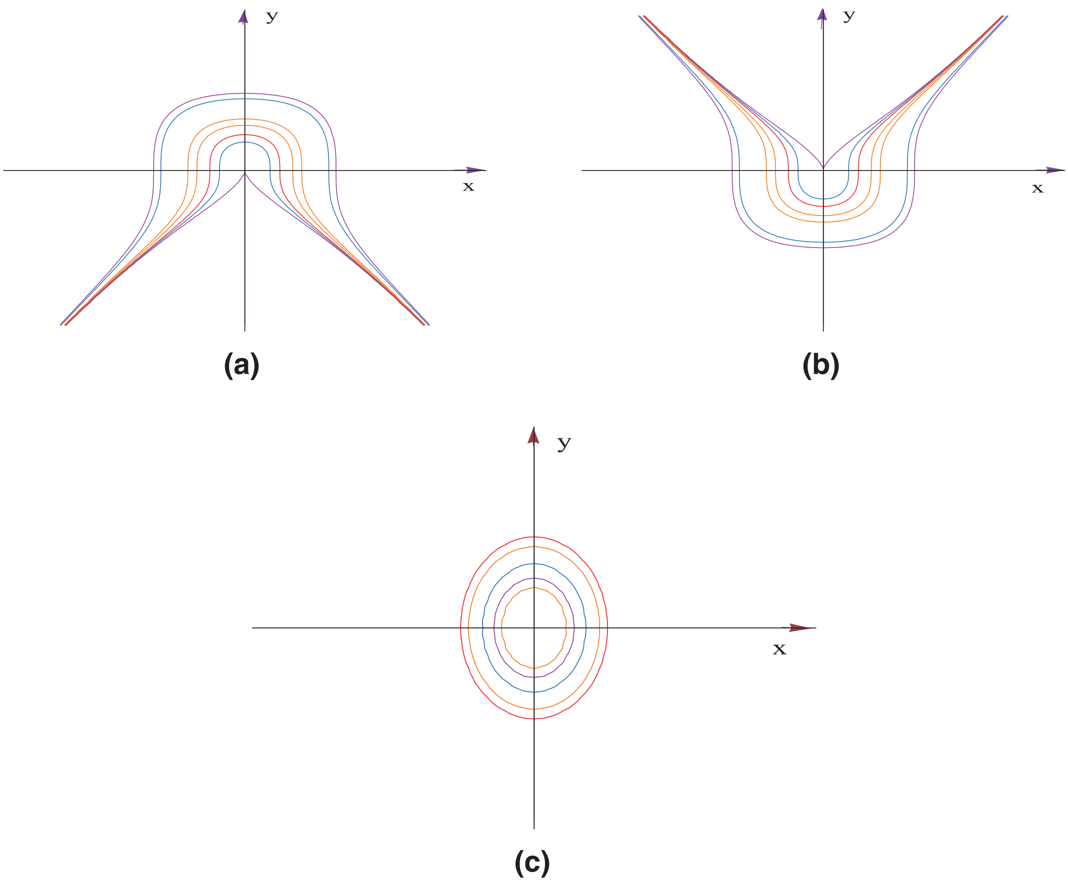

Fig. 6. (a) The phase portrait of the upper half place of (2.18); (b) the phase portrait of the lower half place of (2.18); and (c) the phase portrait of (2.18).

In particular, when $H_1(x, 0)$ ($H_2(x, 0)$) is symmetric with the y-axis, the origin of system (1.1) is a center if and only if $H_2(x, 0)$ ($H_1(x, 0)$) is also symmetric with the y-axis.

Conversely, the origin of the switching system may be a center even if neither of the upper half plane or the lower half plane has the origin as its center, as illustrated by the following example.

Example 2.2. Consider the following system:

$$\begin{cases} \dfrac{dx}{dt} = 3y^2, \\ \dfrac{dy}{dt} = -2x - 4x^3, \end{cases} \quad (y > 0), \qquad \begin{cases} \dfrac{dx}{dt} = -3y^2, \\ \dfrac{dy}{dt} = -2x - 4x^3, \end{cases} \quad (y < 0). \tag{2.18}$$

The phase portraits near the origin of the upper half plane and the lower half plane are shown in Fig. 6(a) and (b), respectively, indicating that they are not centers. But the origin of the switching system is a center, as depicted in Fig. 6(c).

3. Computation of period constants of system (1.1)

Similarly, the half period functions for system (1.1) can be defined as

$$\begin{aligned} T_1(\pi,h) &= \int_0^{\pi} \frac{d\vartheta}{1+\sum_{k=2}^{n}\psi_{k+2}(\vartheta)r_1^k(\vartheta,h))}, \\ T_2(\pi,h) &= \int_0^{\pi} \frac{d\vartheta}{1+\sum_{k=2}^{n}\psi_{k+2}(\vartheta)r_2^k(\vartheta,h))}. \end{aligned} \tag{3.1}$$

Then, the period function for the switching system can be defined as

$$T = T_1(\pi,h) + T_2(\pi,h). \tag{3.2}$$

Definition 3.1. The period function of the switching system is defined by

$$T(h) = T_1(\pi,h) + T_2(\pi,h) = 2\pi + \sum_{k=1}^{n} T_k h^k, \tag{3.3}$$

where T_k is called the kth period constant of the switching system.

The following example illustrates the difference between the period functions of continuous systems and switching systems.

Example 3.1. Consider the switching system,

$$\begin{cases} \dfrac{dx}{dt} = -y(1+2x+4x^2), \\ \dfrac{dy}{dt} = x + x^2 - y^2 - 4xy^2, \end{cases} (y>0), \qquad \begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x, \end{cases} (y<0). \tag{3.4}$$

It is easy to see that the origin is an isochronous center for both the upper half and lower half planes, but it is not an isochronous center of the switching system.

Based on the results obtained in the previous section and this section, we summarize the main steps in computing the focal values and period constants of the switching system as follows.

1. Introduce the transformation: $y \to -y, t \to -t$ into the lower half plane.
2. Use the formulas in (2.7) to solve $u_k(\theta), v_k(\theta)$.
3. Use the formula (2.15) to compute the successive function $\Delta(h)$ of the switching system.
4. When the origin is a center, use the formula (3.2) to compute the period constants T of the switching system.

Remark 3.1. Especially, if the system for the lower half plane is defined by

$$\begin{aligned} \frac{dx}{dt} &= -y - F^-(x,-y) = -y, \\ \frac{dy}{dt} &= x + G^-(x,-y) = x, \end{aligned} \tag{3.5}$$

then one only needs to compute $\Delta_1(h)$ and $T_1(\theta,h)$.

4. Applications

In this section, we apply the results obtained in the previous sections to consider two examples. In particular, the second example shows that a simple cubic switching system can have at least 15 limit cycles.

4.1. Example 1

The first example has been studied in [34]. We reinvestigate this example here to illustrate the computation efficiency of our methods, by computing the Lyapunov constants and period constants, to find the center conditions and isochronous center conditions of the system. The system equations are given by

$$\begin{cases} \dfrac{dx}{dt} = -y + (a_{20}x^2 + a_{11}xy + a_{02}y^2), \\ \dfrac{dy}{dt} = x + (b_{20}x^2 + b_{11}xy + b_{02}y^2), \end{cases} (y>0), \qquad \begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x, \end{cases} (y<0). \tag{4.1}$$

4.1.1. Center conditions and bifurcation of limit cycles for (4.1)

First, we consider the center conditions and the number of bifurcating limit cycles. With the aid of symbolic computation, we obtain the following result.

Theorem 4.1. *For system (4.1), the first five Lyapunov constants at the origin are given by*

$$\lambda_1 = \frac{2}{3}(a_{11} + 2b_{02} + b_{20}),$$
$$\lambda_2 = \frac{\pi}{16}\left(2a_{11}a_{20} + a_{11}b_{11} - 2a_{02}b_{20} - 4a_{20}b_{20} - b_{11}b_{20}\right),$$

with two cases:

(I) $b_{20} \neq 0$.

$$\lambda_3 = \frac{2}{15}\left(6a_{11}a_{20}^2 + 3a_{11}a_{20}b_{11} - a_{11}^2 b_{20} - 9a_{20}^2 b_{20} - 3a_{20}b_{11}b_{20} + b_{20}^3\right),$$

which has two sub-cases:

Sub-case (Ia). If $a_{20}(a_{11} - b_{20}) \neq 0$, *we have*

$$\lambda_4 = \frac{a_{20}b_{20}\pi}{192(a_{11} - b_{20})}(5a_{11} - 7b_{20})\left(3a_{20}^2 + 2a_{11}b_{20} - 2b_{20}^2\right),$$
$$\lambda_5 = \frac{128b_{20}^5}{21875(a_{11} - b_{20})^2}\left(3a_{20}^2 + 2a_{11}b_{20} - 2b_{20}^2\right).$$

Sub-case (Ib). If $a_{20} = 0$, *we obtain*

$$\lambda_4 = \frac{2}{15}b_{20}(-a_{11} + b_{20})(a_{11} + b_{20}),$$
$$\lambda_5 = -\frac{\pi}{192}b_{11}(-a_{11} + b_{20})(a_{11} + b_{20})(-5a_{11} + 7b_{20}).$$

(II) $b_{20} = 0$.

$$\lambda_2 = \frac{\pi}{16}a_{11}(2a_{20} + b_{11}),$$
$$\lambda_3 = \frac{2}{45}a_{11}(2a_{20} + b_{11})(4a_{02} + 5a_{20} - 2b_{11}),$$
$$\lambda_4 = \frac{\pi}{2304}a_{11}(2a_{20} + b_{11})\left(140a_{02}^2 + 45a_{11}^2 + 260a_{02}a_{20} + 176a_{20}^2 - 50a_{02}b_{11} - 130a_{20}b_{11} + 26b_{11}^2\right),$$
$$\lambda_5 = \frac{2}{14175}a_{11}(2a_{20} + b_{11})\left(1600a_{02}^3 + 1080a_{02}a_{11}^2 + 3840a_{02}^2a_{20} + 1269a_{11}^2a_{20} + 3936a_{02}a_{20}^2 + 1874a_{20}^3 - 240a_{02}^2b_{11} - 378a_{11}^2b_{11} - 1626a_{02}a_{20}b_{11} - 1968a_{20}^2b_{11} + 228a_{02}b_{11}^2 + 726a_{20}b_{11}^2 - 92b_{11}^3\right).$$

Note that in computing the above expressions λ_k, $k = 2, \cdots, 5$, $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0$ *have been used.*

The following proposition follows directly from Theorem 4.1.

Proposition 4.1. *The first five Lyapunov constants evaluated at the origin of system (4.1) become zero if and only if one of the following conditions is satisfied:*

$$b_{02} = 0, \quad a_{02} = -b_{11}, \quad a_{20} = 0, \quad a_{11} = -b_{20}; \tag{4.2}$$

$$b_{02} = -\frac{1}{2}a_{11}, \quad b_{20} = 0, \quad 2a_{20} + b_{11} = 0; \tag{4.3}$$

$$b_{02} = -b_{20}, \quad a_{02} = a_{20} = 0, \quad a_{11} = b_{20}; \tag{4.4}$$

$$\begin{aligned} &2a_{11}b_{20} + 3a_{20}^2 - 2a_{20}^2 = 2b_{11} + 5a_{20} = 0, \\ &a_{02}b_{20}^2 + a_{20}(8b_{20}^2 - 3a_{20}^2) = 4a_{02}b_{20} - 3b_{20}^2 + 4a_{20}^2 = 0; \end{aligned} \tag{4.5}$$

$$a_{11} = b_{02} = b_{20} = 0. \tag{4.6}$$

They are also the center conditions of system (4.1).

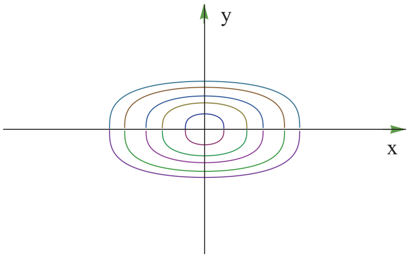

Fig. 7. Phase portrait of system (4.1) when the conditions in (4.3) are satisfied with $b_{11} = 2, b_{02} = 1, a_{02} = 3$.

Proof. It has been shown in [34] that when the conditions in (4.2)–(4.5) hold, system (4.1) has respectively the following first integrals:

$$\begin{aligned} H_1(x,y) &= x^2 + y^2, \\ H_2(x,y) &= \frac{1}{2}(x^2+y^2) + \frac{b_{11}}{2}x^2y + b_{02}xy^2 - \frac{a_{02}}{3}y^3, \\ H_3(x,y) &= (b_{20}x-1)[b_{20}x+12(b_{11}-\gamma)y+1]^{\alpha}[b_{20}x+12(b_{11}+\gamma)y+1]^{(1-\alpha)}, \\ H_4(x,y) &= (-2b_{20}x + a_{20}y + 2)^2 \\ &\quad \times \left[4(b_{20}x+1)^2 - (4a_{20} + 12a_{20}b_{20}x)y + (3a_{20}^2 - 8b_{20}^2)y^2\right], \end{aligned}$$

where

$$\alpha = \left(4b_{20}^2\right) \Big/ (\gamma(\gamma + b_{11})), \gamma = \sqrt{b_{11}^2 + 8b_{20}^2}.$$

Obviously, all $H_i(x,0), i = 1,2,3,4$, are symmetric with the y-axis. Thus, the origin of system (4.1) is a center. For example, when the conditions in (4.3) hold and $b_{11} = 2, b_{02} = 1, a_{02} = 3$, the system has a first integral,

$$H_2(x,y) = \frac{1}{2}(x^2+y^2) + x^2y + xy^2 - y^3,$$

and the phase portrait of (4.1) is shown in Fig. 7.

When the conditions in (4.6) hold, system (4.1) becomes

$$\begin{cases} \dfrac{dx}{dt} = -y + (a_{20}x^2 + a_{02}y^2), \\ \dfrac{dy}{dt} = x + b_{11}xy, \end{cases} (y > 0), \quad \begin{cases} \dfrac{dx}{dt} = -y, \\ \dfrac{dy}{dt} = x, \end{cases} (y < 0). \tag{4.7}$$

Obviously, the system is symmetric with the y-axis, and so the origin is a center of system (4.1). □

As far as limit cycles are concerned, it follows from Theorem 4.1 that at most 5 limit cycles can bifurcate from the origin of system (4.1). Next, to obtain 5 limit cycles, we consider the perturbed system of (4.1), given as follows:

$$\begin{cases} \dfrac{dx}{dt} = \delta_1 x - y + (a_{20}x^2 + a_{11}xy + a_{02}y^2), \\ \dfrac{dy}{dt} = x + \delta_1 y + (b_{20}x^2 + b_{11}xy + b_{02}y^2), \end{cases} (y > 0), \quad \begin{cases} \dfrac{dx}{dt} = \delta_2 x - y, \\ \dfrac{dy}{dt} = x + \delta_2 y, \end{cases} (y < 0), \tag{4.8}$$

for which we have the following theorem.

Theorem 4.2. *If the origin of system (4.8) is a 5th-order weak focus, then for $0 < \delta_1, \delta_2 \ll 1$, 5 small-amplitude limit cycles can bifurcate from the origin of the perturbed system (4.8).*

Proof. When the origin of system (4.8) is a 5th-order weak focus, the following conditions:

$$\begin{aligned} b_{02} &= -\frac{1}{2}(a_{11} + b_{20}), \\ a_{02} &= \frac{2a_{11}a_{20} + a_{11}b_{11} - 4a_{20}b_{20} - b_{11}b_{20}}{2b_{20}}, \\ b_{11} &= -\frac{-6a_{11}a_{20}^2 + a_{11}^2b_{20} + 9a_{20}^2b_{20} - b_{20}^3}{3a_{20}(a_{11} - b_{20})}, \\ a_{11} &= \frac{7}{5}b_{20}, \\ &a_{20}b_{20}(a_{11} - b_{20})\left(3a_{20}^2 + 2a_{11}b_{20} - 2b_{20}^2\right) \neq 0, \end{aligned}$$

should be satisfied, which result in

$$\frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{\partial(b_{02}, a_{02}, b_{11}, a_{11})} = \frac{a_{20}^2 b_{20}^2 (15a_{20}^2 + 4b_{20}^2)\pi^2}{2880} \neq 0,$$

implying that 5 small-amplitude limit cycles can bifurcate from the origin of the perturbed system (4.8). □

4.1.2. Isochronous centers of system (4.1)

Next, we discuss isochronous center conditions of system (4.1). By simple computations, we can show that when one set of the conditions in (4.2), (4.4) or (4.5) is satisfied, the period constants of system (4.1) cannot be zero simultaneously, so there do not exist isochronous center conditions for these three cases.

When the conditions in (4.3) hold, the period constants are obtained as

$$\begin{aligned}
\tau_1 &= \frac{2}{3}(2a_{02} + 3a_{20}),\\
\tau_2 &= \frac{9}{64}\left(a_{11}^2 + 5a_{20}^2\right)\pi,\\
\tau_3 &= -2a_{20}^3,\\
\tau_4 &= \frac{27}{2048}\left(7a_{11}^4 + 78a_{11}^2a_{20}^2 + 183a_{20}^4\right)\pi,\\
\tau_5 &= -50944a_{20}^3\left(a_{11}^2 + 6a_{20}^2\right),\\
&\ \vdots
\end{aligned}$$

Note that $\tau_2 = 0$ implies that $a_{11} = a_{20} = 0$, and so all $\tau_i\,(i \geq 3)$ equal zero.

When the conditions in (4.6) are satisfied, the period constants are given by

$$\begin{aligned}
\tau_1 &= \frac{2}{3}(2a_{02} + a_{20} - b_{11}),\\
\tau_2 &= \frac{1}{8}a_{02}(a_{20} + 4a_{02})\pi,\\
\tau_3 &= \frac{8}{15}a_{02}(a_{20} + 4a_{02})(a_{20} + a_{02}),\\
\tau_4 &= \frac{\pi}{256}a_{02}(a_{20} + 4a_{02})\left(84a_{02}^2 + 97a_{02}a_{20} + 48a_{20}^2\right),\\
\tau_5 &= \frac{16}{1575}a_{02}(a_{20} + 4a_{02})(a_{20} + a_{02})\left(198a_{02}^2 + 127a_{02}a_{20} + 60a_{20}^2\right),\\
&\ \vdots
\end{aligned}$$

Note for this case that $\tau_i\,(i \geq 3)$ contains a common factor $a_{02}(a_{20} + 4a_{02})$.

Therefore, it is obvious that the origin is an isochronous center if and only if the first two period constants vanish, and we obtain the following result.

Theorem 4.3. *The origin of system (4.1) is an isochronous center if and only if one of the following conditions holds:*

(a) $a_{20} = a_{11} = a_{02} = b_{20} = b_{11} = b_{02} = 0$;
(b) $a_{11} = b_{02} = b_{20} = a_{02} = 0, b_{11} = a_{20}$;
(c) $a_{11} = b_{02} = b_{20} = 0, b_{11} = -2a_{02}, a_{20} = -4a_{02}$.

Proof. When the conditions in (a) are satisfied, the equations for the upper plane of system (4.1) can be rewritten as

$$\frac{dx}{dt} = -y,\qquad \frac{dy}{dt} = x,$$

which shows that the origin of system (4.1) is an isochronous center.

When the conditions in (b) hold, the equations for the upper plane of system (4.1) become

$$\begin{aligned}
\frac{dx}{dt} &= -y + a_{20}x^2,\\
\frac{dy}{dt} &= x + a_{20}xy,
\end{aligned} \tag{4.9}$$

A simple calculation yields

$$\frac{d\theta}{dt} = 1,$$

showing that the conclusion is true.

When the conditions in (c) are satisfied, the equations for the upper phase of system (4.1) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= -y - 4a_{02}x^2 + a_{02}y^2, \\ \frac{dy}{dt} &= x - 2a_{02}xy, \end{aligned} \tag{4.10}$$

which has a transversal system

$$\begin{cases} \dfrac{dx}{dt} = x - 4a_{02}xy + 4a_{02}^2xy^2, \\ \dfrac{dy}{dt} = y - 3a_{02}y^2 + 2a_{02}^2y^3, \end{cases} (y > 0), \quad \begin{cases} \dfrac{dx}{dt} = x, \\ \dfrac{dy}{dt} = y, \end{cases} (y < 0), \tag{4.11}$$

indicating that the origin of system (4.1) is an isochronous center. □

4.2. Example 2

As we know, the second part of Hilbert's 16th problem is to find the maximal number and relative locations of the limit cycles bifurcating in polynomial systems of degree n. This problem is far from being completely solved. Let $H(n)$ denote this number, called Hilbert number. For $n = 2$, Shi [35] and Chen and Wang [36] independently constructed concrete examples to show existence of 4 limit cycles more than 30 years ago, but whether $H(2) = 4$ is still open. For $n = 3$, Li and Huang first proved $H(3) \geq 11$ [37]. Li and Liu [38], and Liu et al. [39] respectively found more concrete cubic systems which have 11 limit cycles with the same distribution. Later, Han et al. [40,41] used the method of stability-changing in a homoclinic loop to give more cubic systems which have 11 limit cycles with two different distributions. Further, Yu and Han [42–44] proved $H(3) \geq 12$ by studying Hopf bifurcation in a Z_2-symmetric cubic system. Later, this Z_2-symmetric system was reinvestigated by Li et al. [45] to prove existence of one more limit cycle, namely, $H(3) \geq 13$. Liu and Li obtained a sufficient condition for existence of these 13 limit cycles [33,46], with the distribution of one large limit cycle bifurcating from the equator, which surrounds 12 small limit cycles bifurcating from two symmetric foci. Around the same time, Li et al. obtained another example of 13 limit cycles by perturbing a Hamiltonian system [46], and was confirmed later by using a computation method [47].

In this section, motivated by the work of Liu et al. [46], we will consider a cubic switching system, and apply our method and formulas to show that there can exist at least 15 limit cycles in this system. The system to be considered is given by

$$\begin{aligned} &\begin{cases} \dfrac{dx}{dt} = -y + 3x^2y - 2a_2xy^2 - 2a_3y^3, \\ \dfrac{dy}{dt} = x - x^3 - 2a_5xy^2 - 2a_6y^3, \end{cases} (y > 0), \\ &\begin{cases} \dfrac{dx}{dt} = -y + 3x^2y - 2b_2xy^2 - 2b_3y^3, \\ \dfrac{dy}{dt} = x - x^3 - 2b_5xy^2 - 2b_6y^3, \end{cases} (y < 0). \end{aligned} \tag{4.12}$$

It is easy to see from system (4.12) that both the upper and lower half planes of this system are Z_2 equivalent. This system has three singular points at $(0, 0)$ and $(\pm 1, 0)$, so we may only need to study the singular points $(0, 0)$ and $(1, 0)$ since the upper and lower half planes of this system are Z_2 equivalent.

First, consider the singular point $(1, 0)$. By using the translation: $u = x - 1$, $v = y$, we can transfer the singular point $(1, 0)$ of system (4.12) to the origin of the following system:

$$\begin{aligned} &\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv + a_2(1+u)v^2 + a_3v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + a_5(1+u)v^2 + a_6v^3, \end{cases} (v > 0), \\ &\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv + b_2(1+u)v^2 + b_3v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + b_5(1+u)v^2 + b_6v^3, \end{cases} (v < 0). \end{aligned} \tag{4.13}$$

In the following, we will use our method to compute the focal values and obtain the necessary and sufficient conditions for the origin of system (4.13) being a center.

4.2.1. Center conditions of system (4.13)

By using a computer algebra system, the Lyapunov constants associated with the singular point $(\pm 1, 0)$ of system (4.12) are obtained, as given in the following theorem.

Theorem 4.4. *For system (4.13), the first seven Lyapunov constants at the origin are given by*

$$\lambda_1 = \frac{4}{3}(a_5 - b_5),$$
$$\lambda_2 = \frac{1}{8}(-2a_2 + 3a_6 - 2b_2 + 2a_2b_5 + 2b_2b_5 + 3b_6)\pi,$$
$$\lambda_3 = -\frac{4}{45}\Big(9a_3 - 8a_2b_2 - 8b_2^2 - 9b_3 - 6a_3b_5 + 8a_2b_2b_5 + 8b_2^2b_5 + 6b_3b_5 + 12a_2b_6 + 12b_2b_6\Big).$$

(i) When $b_5 \neq \frac{3}{2}$,

$$\lambda_4 = \frac{(a_2+b_2)}{288(-3+2b_5)}(216 + 30a_2b_2 + 90b_2^2 + 270b_3 - 126b_5 - 50a_2b_2b_5 - 110b_2^2b_5 - 360b_3b_5 - 120b_5^2 + 20a_2b_2b_5^2 + 20b_2^2b_5^2 + 120b_3b_5^2 + 72b_5^3 - 45a_2b_6 + 45b_2b_6 + 30a_2b_5b_6 - 150b_2b_5b_6 - 270b_6^2)\pi,$$

$$\lambda_5 = \frac{32(a_2+b_2)^2}{14175}(27a_2 - 135b_2 - 20a_2^2b_2 - 40a_2b_2^2 - 20b_2^3 + 36a_2b_5 + 144b_2b_5 + 20a_2^2b_2b_5 + 40a_2b_2^2b_5 + 20b_2^3b_5 - 36a_2b_5^2 - 36b_2b_5^2 + 162b_6 + 30a_2^2b_6 + 60a_2b_2b_6 + 30b_2^2b_6 - 108b_5b_6);$$

(ia) If $b_5 \neq \frac{1}{18}(27 + 5(a_2+b_2)^2)$,

$$\lambda_6 = \frac{(a_2+b_2)}{92160(27 + 5a_2^2 + 10a_2b_2 + 5b_2^2 - 18b_5)^2}(-3+2b_5)(1+2b_5)f_1\pi,$$
$$\lambda_7 = \frac{8(a_2+b_2)^3(a_2-b_2)^3(-3+2b_5)(1+2b_5)(36+5(a_2+b_2)^2)}{567(12 + 35a_2^2 + 70a_2b_2 + 35b_2^2)(27 + 5a_2^2 + 10a_2b_2 + 5b_2^2 - 18b_5)^3}f_2;$$

(ib) If $b_5 = \frac{1}{18}(27 + 5(a_2+b_2)^2)$,

$$\lambda_5 = -\frac{32}{25515}(a_2-b_2)(a_2+b_2)^3\Big(36 + 5a_2^2 + 10a_2b_2 + 5b_2^2\Big),$$
$$\lambda_6 = \frac{1}{933120b_2}(-15309b_2^2 - 59805b_2^4 - 173400b_2^6 + 14000b_2^8 - 91854b_2b_6 - 51030b_2^3b_6 + 340200b_2^5b_6 - 137781b_6^2 + 229635b_2^2b_6^2)\pi,$$
$$\lambda_7 = \frac{4096}{14467005(-3+5b_2^2)}b_2^3(567 - 6b_2^2 + 35b_2^4)(9b_2 + 20b_2^3 + 27b_6);$$

(ii) When $b_5 = \frac{3}{2}$,

$$\lambda_3 = -\frac{16}{45}(a_2+b_2)(b_2+3b_6),$$
$$\lambda_4 = -\frac{5}{64}(a_3-b_3)(b_2+3b_6)\pi,$$
$$\lambda_5 = \lambda_6 = 0,$$
$$\lambda_7 = -\frac{759285}{16384}a_2^3a_3^2b_2^2b_6\pi.$$

In the above expressions of λ_k, *we have used* $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0$, *for* $k = 2, \cdots, 7$. *Here,*

$$f_1 = -69984 - 366120a_2^2 - 40200a_2^4 + 875a_2^6 - 187920a_2b_2 - 312000a_2^3b_2 - 26250a_2^5b_2 - 366120b_2^2 - 543600a_2^2b_2^2 - 112875a_2^4b_2^2 - 312000a_2b_2^3 - 171500a_2^3b_2^3 - 40200b_2^4 - 112875a_2^2b_2^4 - 26250a_2b_2^5 + 875b_2^6 + 93312b_5 + 289440a_2^2b_5 + 50400a_2^4b_5 + 578880a_2b_2b_5 + 201600a_2^3b_2b_5 + 289440b_2^2b_5 + 302400a_2^2b_2^2b_5 + 201600a_2b_2^3b_5 + 50400b_2^4b_5 - 31104b_5^2 - 90720a_2^2b_5^2 - 181440a_2b_2b_5^2 - 90720b_2^2b_5^2,$$
$$f_2 = 9072 - 24a_2^2 + 35a_2^4 - 48a_2b_2 + 140a_2^3b_2 - 24b_2^2 + 210a_2^2b_2^2 + 140a_2b_2^3 + 35b_2^4.$$

Next, we discuss the center conditions of system (4.13). From Theorem 4.4 we obtain the following result.

Proposition 4.2. *The first seven Lyapunov constants at the origin of system (4.13) become zero if and only if one of the following conditions is satisfied:*

$$a_5 = b_5, \quad a_3 = b_3, \quad a_6 = -b_6, \quad a_2 = -b_2; \tag{4.14}$$

$$a_5 = b_5 = a_3 = b_3 = -\frac{1}{2}, \quad a_6 = a_2, \quad b_6 = b_2; \tag{4.15}$$

$$a_5 = b_5 = \frac{3}{2}, \quad a_6 = a_2 = 0, \quad b_2 = -3b_6; \tag{4.16}$$

$$a_5 = b_5 = \frac{3}{2}, \quad a_6 = -\frac{a_2}{3}, \quad b_2 = -3b_6, \quad a_3 = 0. \tag{4.17}$$

They are also the conditions for the origin of system (4.13) being a center.

Proof. When the conditions in (4.14) hold, system (4.13) becomes

$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv - b_2(1+u)v^2 + b_3v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + b_5(1+u)v^2 - b_6v^3, \end{cases} \quad (v > 0),$$
$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv + b_2(1+u)v^2 + b_3v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + b_5(1+u)v^2 + b_6v^3, \end{cases} \quad (v < 0), \tag{4.18}$$

showing that the system is symmetric with the u-axis.

When the conditions in (4.15) are satisfied, system (4.13) can be rewritten as

$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv + a_2(1+u)v^2 - \dfrac{1}{2}v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 - \dfrac{1}{2}(1+u)v^2 + a_2v^3, \end{cases} \quad (v > 0),$$
$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv + b_2(1+u)v^2 - \dfrac{1}{2}v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 - \dfrac{1}{2}(1+u)v^2 + b_2v^3, \end{cases} \quad (v < 0). \tag{4.19}$$

The upper half plane has an analytic first integral,

$$H_1(u,v) = \frac{1}{4(1+2u+u^2+v^2)}\{1 + 2a_2v + 2a_2uv + 2v^2 - 2a_2(1+2u+u^2+v^2)\arctan\frac{v}{1+u} + (1+2u+u^2+v^2)\log[(1+u)^2+v^2]\},$$

and the lower half plane has an analytic first integral,

$$H_2(u,v) = \frac{1}{4(1+2u+u^2+v^2)}\{1 + 2b_2v + 2b_2uv + 2v^2 - 2b_2(1+2u+u^2+v^2)\arctan\frac{v}{1+u} + (1+2u+u^2+v^2)\log[(1+u)^2+v^2]\},$$

showing that $H_1(u,0) = H_2(u,0)$. So the origin of system (4.13) is a center.

When the conditions in (4.16) are satisfied, system (4.13) can be rewritten as

$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv + a_3v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + \dfrac{3}{2}(1+u)v^2, \end{cases} \quad (v > 0),$$
$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv - 3b_6(1+u)v^2 + b_3v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + \dfrac{3}{2}(1+u)v^2 + b_6v^3, \end{cases} \quad (v < 0). \tag{4.20}$$

The upper half plane of this system has an analytic first integral,

$$H_3(u,v) = u^2 + v^2 + \frac{1}{2}u^3 + \frac{1}{8}u^4 - \frac{1}{4}a_3v^4 + \frac{3}{2}uv^2 + \frac{3}{4}u^2v^2,$$

and the lower half plane has an analytic first integral,

$$H_4(u,v) = u^2 + v^2 + \frac{1}{2}u^3 + \frac{1}{8}u^4 + b_6v^3 + b_6uv^3 - \frac{1}{4}b_3v^4 + \frac{3}{2}uv^2 + \frac{3}{4}u^2v^2,$$

indicating that $H_3(u,0) = H_4(u,0)$, and so the origin of system (4.13) is a center.

When the conditions in (4.17) hold, system (4.13) becomes

$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv + a_2(1+u)v^2, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + \dfrac{3}{2}(1+u)v^2 - \dfrac{a_2}{3}v^3, \end{cases} \quad (v > 0),$$
$$\begin{cases} \dfrac{du}{dt} = -v - \dfrac{3}{2}(2+u)uv - 3b_6(1+u)v^2 + b_3v^3, \\ \dfrac{dv}{dt} = u + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + \dfrac{3}{2}(1+u)v^2 + b_6v^3. \end{cases} \quad (v < 0). \tag{4.21}$$

The upper half plane of this system has an analytic first integral,

$$H_5(u,v) = u^2 + v^2 + \frac{1}{2}u^3 + \frac{1}{8}u^4 + \frac{3}{2}uv^2 + \frac{3}{4}u^2v^2 - \frac{1}{3}a_2(1+u)v^3,$$

and the lower half plane has an analytic first integral,

$$H_6(u,v) = u^2 + v^2 + \frac{1}{2}u^3 + \frac{1}{8}u^4 - \frac{1}{4}b_3v^4 + b_6(1+u)v^3 + \frac{3}{2}uv^2 + \frac{3}{4}u^2v^2.$$

So $H_5(u,0) = H_6(u,0)$, implying that the origin of system (4.13) is also a center. □

4.2.2. Bifurcation of limit cycles in system (4.12)

In this section, we study bifurcation of limit cycles in system (4.12). First, we consider the limit cycles bifurcating from the symmetric singular points $(\pm 1, 0)$. From Theorem 4.4 we obtain the following result.

Theorem 4.5. *The singular point* $(1,0)$ *or* $(-1,0)$ *is a 7th-order weak focus of (4.12) if and only if*

$$\begin{aligned}
a_5 &= b_5, \\
a_6 &= \frac{-27b_2+5a_2(27+4(a_2+b_2)^2)-4(9b_2+a_2(36+5(a_2+b_2)^2))b_5+36(a_2+b_2)b_5^2}{6(27+5(a_2+b_2)^2-18b_5)}, \\
a_3 &= -\frac{1}{20(27+5(a_2+b_2)^2-18b_5)^2}\Big\{1000a_2^3b_2 + 324(3-2b_5)^2(4+3b_5) \\
&\quad - 125a_2^4(1+6b_5) + 25b_2^4(19+18b_5) \\
&\quad - 720b_2^2\left(-6+b_5+6b_5^2\right) + 30a_2^2\left[36-168b_5+b_2^2(95+90b_5)\right] \\
&\quad + 40a_2b_2\left[b_2^2(55+60b_5)+9(33+4(5-3b_5)b_5)\right]\Big\}, \\
b_3 &= -\frac{1}{20(27+5(a_2+b_2)^2-18b_5)^2}\Big\{360b_2^2(3-14b_5) + 324(3-2b_5)^2(4+3b_5) \\
&\quad - 125b_2^4(1+6b_5) + 200a_2^3b_2(11+12b_5) + 25a_2^4(19+18b_5) \\
&\quad + 40a_2b_2\left[25b_2^2+9(33+4(5-3b_5)b_5)\right] \\
&\quad + 30a_2^2\left[b_2^2(95+90b_5) - 24\left(-6+b_5+6b_5^2\right)\right]\Big\}, \\
b_6 &= \frac{1}{6(27+5(a_2+b_2)^2-18b_5)}\Big\{135b_2+20a_2^2b_2(1-b_5) + 4b_2\left[5b_2^2(1-b_5)\right. \\
&\quad \left.- 9(4-b_5)b_5\right] + a_2\left[40b_2^2(1-b_5) - 9(3-2b_5)(1+2b_5)\right]\Big\},
\end{aligned}$$

with $[b_5 - \frac{1}{18}(27+5(a_2+b_2)^2)](b_5 - \frac{3}{2})(a_2+b_2)(2b_5+1)(6b_5-13) \neq 0$.

Proof. When $[b_5 - \frac{1}{18}(27+5(a_2+b_2)^2)](b_5 - \frac{3}{2})(a_2+b_2)(2b_5+1)(6b_5-13) \neq 0$, solving $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$ ($\lambda_i, i = 1, 2, \ldots, 5$, given in Theorem 4.4) yields the solutions given in Theorem 4.5. Moreover, it is easy to verify that there is no real solutions for $f_2 = 0$, which implies that $(0,0)$ is a 7th-order weak focus of (4.13), so $(1,0)$ and $(-1,0)$ are 7th-order weak focuses of (4.12). □

Further, we have the following theorem.

Theorem 4.6. *When* $(0,0)$ *is a 7th-order weak focus points, seven small-amplitude limit cycles can bifurcate from the origin* $(0,0)$ *of the perturbed system of (4.13):*

$$\begin{cases} \dfrac{du}{dt} = \delta_1 u - v - \dfrac{3}{2}(2+u)uv + a_2(1+u)v^2 + a_3v^3, \\ \dfrac{dv}{dt} = u + \delta_1 v + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + a_5(1+u)v^2 + a_6v^3, \end{cases} \quad (v>0),$$
$$\begin{cases} \dfrac{du}{dt} = \delta_2 u - v - \dfrac{3}{2}(2+u)uv + b_2(1+u)v^2 + b_3v^3, \\ \dfrac{dv}{dt} = u + \delta_2 v + \dfrac{3}{2}u^2 + \dfrac{1}{2}u^3 + b_5(1+u)v^2 + b_6v^3, \end{cases} \quad (v<0), \tag{4.22}$$

where $0 < \delta_1, \delta_2 \ll 1$.

Proof. When the origin of is a 7th-order weak focus, we find that

$$\det \begin{bmatrix} \frac{d\lambda_1}{da_5} & \frac{d\lambda_1}{da_6} & \frac{d\lambda_1}{da_3} & \frac{d\lambda_1}{db_3} & \frac{d\lambda_1}{db_6} & \frac{d\lambda_1}{db_5} \\ \frac{d\lambda_2}{da_5} & \frac{d\lambda_2}{da_6} & \frac{d\lambda_2}{da_3} & \frac{d\lambda_2}{db_3} & \frac{d\lambda_2}{db_6} & \frac{d\lambda_2}{db_5} \\ \frac{d\lambda_3}{da_5} & \frac{d\lambda_3}{da_6} & \frac{d\lambda_3}{da_3} & \frac{d\lambda_3}{db_3} & \frac{d\lambda_3}{db_6} & \frac{d\lambda_3}{db_5} \\ \frac{d\lambda_4}{da_5} & \frac{d\lambda_4}{da_6} & \frac{d\lambda_4}{da_3} & \frac{d\lambda_4}{db_3} & \frac{d\lambda_4}{db_6} & \frac{d\lambda_4}{db_5} \\ \frac{d\lambda_5}{da_5} & \frac{d\lambda_5}{da_6} & \frac{d\lambda_5}{da_3} & \frac{d\lambda_5}{db_3} & \frac{d\lambda_5}{db_6} & \frac{d\lambda_5}{db_5} \\ \frac{d\lambda_6}{da_5} & \frac{d\lambda_6}{da_6} & \frac{d\lambda_6}{da_3} & \frac{d\lambda_6}{db_3} & \frac{d\lambda_6}{db_6} & \frac{d\lambda_6}{db_5} \end{bmatrix}$$

$$= \det \begin{bmatrix} \frac{d\lambda_1}{da_5} & \frac{d\lambda_1}{da_6} & \frac{d\lambda_1}{da_3} & \frac{d\lambda_1}{db_3} & \frac{d\lambda_1}{db_6} \\ 0 & \frac{d\lambda_2}{da_6} & \frac{d\lambda_2}{da_3} & \frac{d\lambda_2}{db_3} & \frac{d\lambda_2}{db_6} \\ 0 & 0 & \frac{d\lambda_3}{da_3} & \frac{d\lambda_3}{db_3} & \frac{d\lambda_3}{db_6} \\ 0 & 0 & 0 & \frac{d\lambda_4}{db_3} & \frac{d\lambda_4}{db_6} \\ 0 & 0 & 0 & 0 & \frac{d\lambda_5}{db_6} \end{bmatrix} \times \frac{d\lambda_6}{db_5}$$

$$= \frac{8(a_2+b_2)^2(27+5a_2^2+10a_2b_2+5b_2^2-18b_5)(-3+2b_5)^2\pi^2}{42525} f_3,$$

where

$$f_3 = \frac{35(a_2-b_2)^2\left(-12+5a_2^2+10a_2b_2+5b_2^2\right)}{8\left(12+35a_2^2+70a_2b_2+35b_2^2\right)}\left(36+5a_2^2+10a_2b_2+5b_2^2\right)$$
$$\times\big(-93312-434160a_2^2-40200a_2^4+875a_2^6-324000a_2b_2-312000a_2^3b_2$$
$$-26250a_2^5b_2-434160b_2^2-543600a_2^2b_2^2-112875a_2^4b_2^2-312000a_2b_2^3$$
$$-171500a_2^3b_2^3-40200b_2^4-112875a_2^2b_2^4-26250a_2b_2^5+875b_2^6$$
$$+62208b_5+198720a_2^2b_5+50400a_2^4b_5+397440a_2b_2b_5+201600a_2^3b_2b_5$$
$$+198720b_2^2b_5+302400a_2^2b_2^2b_5+201600a_2b_2^3b_5+50400b_2^4b_5\big).$$

Moreover, we obtain

$$\begin{aligned} &\text{Resultant}\,[f_1, f_3, b_2] \\ =\ &114829757638306762275116811878400000000000000000000000000 \\ &\times(3+35a_2^2)^2(27+20a_2^2-18b_5)^4(2b_5-3)^6(2b_5+1)^{10}(6b_5-13)^4 \neq 0. \end{aligned}$$

This shows that for $0<\delta_1,\delta_2 \ll 1$, seven small-amplitude limit cycles can bifurcate from the origin $(0,0)$ of the perturbed system of (4.13). □

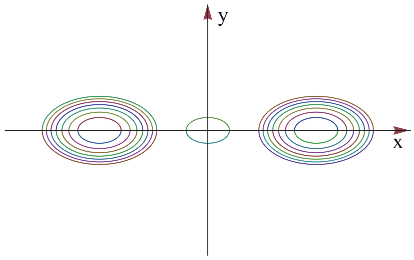

Fig. 8. Distribution of 15 limit cycles in system (4.12).

Theorem 4.6 yields that seven small-amplitude limit cycles can bifurcate from $(1, 0)$ of system of (4.12). So seven small-amplitude limit cycles can bifurcate from $(-1, 0)$ of system of (4.12) because the upper and lower half planes of this system are Z_2 equivalent. Now, for the conditions under which there exist 14 limit cycles around $(\pm 1, 0)$, we show that the origin of system (4.12) may be a center or focus. More precisely, we can prove that the first Lyapunov constant evaluated at the origin is given by

$$u_1 = -\frac{1}{4}(8a_2 + 24a_6 - b_2 - 3b_6)\pi . \tag{4.23}$$

Then, it is easy to verify that when $(\pm 1, 0)$ are 7th-order weak focus points,

$$u_1 = \frac{2b_5 - 3}{8(27 + 5(a_2 - b_2)^2 - 18b5)}(513a_2 + 80a_2^3 - 135b_2 + 150a_2^2b_2 + 60a_2b_2^2 - 10b_2^3 - 126a_2b_5 - 126b_2b_5)\pi \neq 0,$$

implying that when $(\pm 1, 0)$ of system (4.12) are 7th-order weak focus points, the origin is a first-order weak focus. Hence, system (4.12) can have at least 15 limit cycles, with 14 of them around the two symmetric singular points $(\pm 1, 0)$ and one around the origin $(0, 0)$.

Summarizing the above results gives the following theorem.

Theorem 4.7. *System (4.12) can have 15 limit cycles with the* $7 \bigcup 1 \bigcup 7$ *distribution around the singular points,* $(-1, 0), (0, 0)$ *and* $(1, 0)$, *respectively.*

The distribution of the 15 limit cycles is illustrated in Fig. 8.

5. Conclusion

In this paper, a modified, computationally efficient method is present for computing the focal values and period constants of switching systems associated with elementary singular points. We have proved using our method that a cubic switching system can have at least 15 limit cycles. This is a new best result obtained so far for cubic switching systems.

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