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Short communication

A note on the paper “Center and isochronous center conditions for switching systems associated with elementary singular points”

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ABSTRACT

In this note, we correct an error in our paper published in Communications in Nonlinear Science and Numerical Simulation in 2015 [5], in which the form of a Z_2-equivariant cubic planar switching system is not correct. In this note, we add one more switching line and use the same equations given in [5] to construct a correct Z_2-equivariant cubic system. Then, we prove that the new system can exhibit 15 limit cycles and therefore the conclusion given in [5] still holds.

1. Z_2-equivariant switching systems

In this note, we correct an error in our previously published paper in *Communications in Nonlinear Science and Numerical Simulation* [5]. In [5], an existing method is modified for computing the focal values and period constants of switching systems associated with elementary singular points. With this method, a cubic switching system is constructed to show the existence of 15 limit cycles under the assumption that the system is Z_2-equivariant. However, we recently found that the system presented in [5] is actually not Z_2-equivariant, and it can exhibit only 8 limit cycles. In this note, we add one more switching line on the y-axis, in addition to the one on the x-axis, and use the same equations given in [5] to construct a Z_2-equivariant cubic system. Then, we use the method described in [5] to prove that the new constructed system can exhibit 15 limit cycles. Therefore, the conclusion given in [5] still holds.

Consider the following planar system,

$$\frac{dx}{dt} = X(x,y), \quad \frac{dy}{dt} = Y(x,y), \tag{1.1}$$

which is called a Z_q-equivariant if it is invariant under a real planar counterclockwise rotation with angle $\frac{2\pi}{q}$. So a planar system is Z_2-equivariant if the following conditions hold:

$$X(-x,-y) = -X(x,y), \qquad Y(-x,-y) = -Y(x,y).$$

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In other words, a Z_2-equivariant planar system can always be written as

$$\frac{dx}{dt} = \sum_{k=0}^{\infty} X_{2k+1}(x, y), \qquad \frac{dy}{dt} = \sum_{k=0}^{\infty} Y_{2k+1}(x, y). \tag{1.2}$$

Thus, for a switching planar system, described by the following ordinary differential equations:

$$\begin{cases} \dfrac{dx}{dt} = F^{+}(x, y), \\ \dfrac{dy}{dt} = G^{+}(x, y), \end{cases} \quad (y > 0), \qquad \begin{cases} \dfrac{dx}{dt} = F^{-}(x, y), \\ \dfrac{dy}{dt} = G^{-}(x, y), \end{cases} \quad (y < 0), \tag{1.3}$$

which is a Z_2-equivariant if it is invariant under a real planar counterclockwise rotation with angle π, namely,

$$F^{+}(-x, -y) = -F^{-}(x, y), \qquad G^{+}(-x, -y) = -G^{-}(x, y). \tag{1.4}$$

In particular, for a Z_2-equivariant cubic switching system, given by

$$\begin{aligned} F^{+}(x, y) &= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ &\quad + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3; \\ G^{+}(x, y) &= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \\ &\quad + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3. \end{aligned} \tag{1.5}$$

$$\begin{aligned} F^{-}(x, y) &= A_{00} + A_{10}x + A_{01}y + A_{20}x^2 + A_{11}xy + A_{02}y^2 \\ &\quad + A_{30}x^3 + A_{21}x^2y + A_{12}xy^2 + A_{03}y^3; \\ G^{-}(x, y) &= B_{00} + B_{10}x + B_{01}y + B_{20}x^2 + B_{11}xy + B_{02}y^2 \\ &\quad + B_{30}x^3 + B_{21}x^2y + B_{12}xy^2 + B_{03}y^3, \end{aligned} \tag{1.6}$$

the conditions given in (1.4) yield

$$\begin{aligned} &a_{00} = -A_{00}, \quad a_{20} = -A_{20}, \quad a_{11} = -A_{11}, \quad a_{02} = -A_{02}, \\ &a_{10} = A_{10}, \quad a_{01} = A_{01}, \quad a_{30} = A_{30}, \quad a_{21} = A_{21}, \quad a_{12} = A_{12}, \\ &b_{00} = -B_{00}, \quad b_{20} = -B_{20}, \quad b_{11} = -B_{11}, \quad b_{02} = -B_{02}, \\ &b_{10} = B_{10}, \quad b_{01} = B_{01}, \quad b_{30} = B_{30}, \quad b_{21} = B_{21}, \quad b_{12} = B_{12}, \end{aligned} \tag{1.7}$$

which is different from that of planar cubic continuous systems. Therefore, the planar switching cubic system given in [5],

$$\begin{aligned} &\begin{cases} \dfrac{dx}{dt} = \delta(x - x^3) - y + 3x^2y - 2a_2xy^2 - 2a_3y^3, \\ \dfrac{dy}{dt} = -2\delta y + x - x^3 - 2a_5xy^2 - 2a_6y^3, \end{cases} \quad (y > 0), \\ &\begin{cases} \dfrac{dx}{dt} = \delta(x - x^3) - y + 3x^2y - 2b_2xy^2 - 2b_3y^3, \\ \dfrac{dy}{dt} = -2\delta y + x - x^3 - 2b_5xy^2 - 2b_6y^3, \end{cases} \quad (y < 0), \end{aligned} \tag{1.8}$$

is not a Z_2-equivariant system. Note that the equations given in [5] do not contain the small perturbation terms $\delta(x - x^3)$ and $2\delta y$ ($|\delta| \ll 1$), which are added here to help better understand how two more small-amplitude limit cycles are obtained around the singular points ($\pm$ 1, 0). In fact, the above system is Z_2-equivariant if and only if

$$a_2 = b_2, \quad a_3 = b_3, \quad a_5 = b_5, \quad a_6 = b_6,$$

under which the system becomes a continuous system. So for system (1.8) there can exist only 8 limit cycles rather than 15 limit cycles. More precisely, we have

Theorem 1.1. *For system (1.8), when the singular point (1,0) is a 7th-order weak focus, the singular point* $(-1, 0)$ *is a 1st-order focus. There can exist 8 limit cycles bifurcating from (* $\pm$ *1, 0) with distribution* $1 \cup 7$.

In the following, we will construct a new cubic Z_2-equivariant system with 4 switching lines and show that the method described in [5] can still be directly used. To achieve this, consider a discontinuous planar system with 4 switching lines on the x- and y-axes, described as follows:

$$\begin{cases} \dfrac{dx}{dt} = \delta(x - x^3) - y + 3x^2y - 2a_2xy^2 - 2a_3y^3, \\ \dfrac{dy}{dt} = -2\delta y + x - x^3 - 2a_5xy^2 - 2a_6y^3, \end{cases} \quad (x > 0,\ y > 0),$$

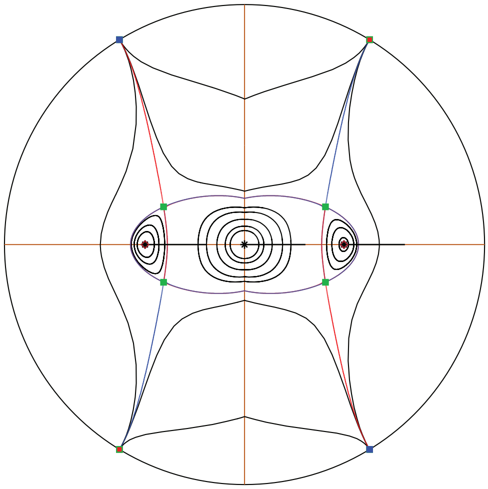

Fig. 1. Phase portrait of system (1.9), showing that ($\pm$ 1, 0) are centers of system (2.2) when $a_2 = -b_2 = a_3 = b_3 = a_5 = b_5 = a_6 = -b_6 = 1$ and the origin is also a center.

$$
\begin{cases} \dfrac{dx}{dt} = \delta(x - x^3) - y + 3x^2y - 2b_2xy^2 - 2b_3y^3, \\ \dfrac{dy}{dt} = -2\delta y + x - x^3 - 2b_5xy^2 - 2b_6y^3, \end{cases} \quad (x < 0,\ y > 0).
$$

$$
\begin{cases} \dfrac{dx}{dt} = \delta(x - x^3) - y + 3x^2y - 2a_2xy^2 - 2a_3y^3, \\ \dfrac{dy}{dt} = -2\delta y + x - x^3 - 2a_5xy^2 - 2a_6y^3, \end{cases} \quad (x < 0,\ y < 0),
$$

$$
\begin{cases} \dfrac{dx}{dt} = \delta(x - x^3) - y + 3x^2y - 2b_2xy^2 - 2b_3y^3, \\ \dfrac{dy}{dt} = -2\delta y + x - x^3 - 2b_5xy^2 - 2b_6y^3, \end{cases} \quad (x > 0,\ y < 0). \tag{1.9}
$$

It is easy to verify that the above system is Z_2-equivariant because it is invariant under a real planar counterclockwise rotation with angle π, and that ($\pm$ 1, 0) are two singular points. It is interesting to note that the equations in (1.9) are actually the exactly same as that in (1.8). The only difference is that we have added one more switching line on the y-axis.

Since Hopf bifurcation can always be considered in a small neighborhood of a singular point, we can restrict the Hopf bifurcation in the neighborhood of the two singular points ($\pm$ 1, 0). Thus, the analysis can be carried out as the Hopf bifurcation occurs in a switching system from a switching line. This implies that the focal values at (1,0) of system (1.9) are the same as that of system (1.8) at (1,0). However, since now system (1.9) is Z_2-equivariant, we have 14 limit cycles bifurcating from ($\pm$ 1, 0) with $7 \cup 7$ distribution.

When one of the conditions in Proposition 4.2 in [5] is satisfied, the singular points ($\pm$ 1, 0) are centers of system (1.9). Simulations with the aid of P5 programme for different parameter values are given in Figs. 1 through 5 to illustrate different cases when ($\pm$ 1, 0) are centers or fine foci.

In order to obtain one more small-amplitude limit cycle in the modified Z_2-equivariant switching system (1.9), we need to consider limit cycle bifurcation from the origin of system (1.9).

2. Computation of the focal values at the origin of Z_2-equivalent systems

As shown in [5], one more small-amplitude limit cycle can be obtained from the origin due to Hopf bifurcation. Therefore, the total number of limit cycles we can find from the system (1.9) is 15. However, the method used in [5] for computing the focal values cannot be applied for the switching systems with two switching lines. Therefore, we need to develop a new method.

For Z_2-equivalent systems, it is obvious that any closed trajectory around the origin must be symmetric with the origin. There are many methods for computing the focal values of smooth systems, and some approaches have been developed to compute the focal values of switching systems with one switching line. However, there are very few methods developed so far for computing the focal values of switching systems with more than one switching line or curve. The Melnikov method is the main tool for studying such problems. Limit cycles bifurcating from a vertex in a non-smooth planar system was considered in [1,2], and bifurcation of sliding periodic orbits in periodically forced discontinuous systems was studied in [3]. In [4], generalized Hopf bifurcation emerging from a corner in general planar piecewise smooth systems was investigated.

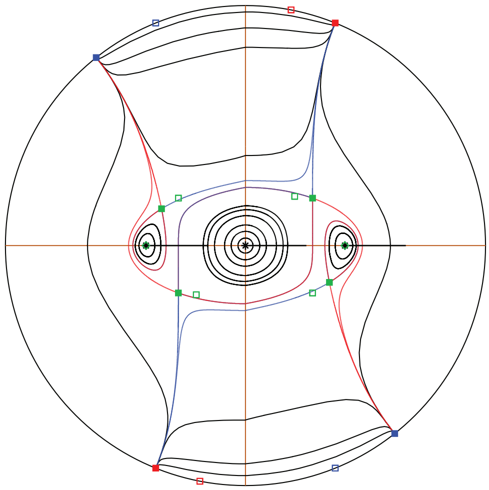

Fig. 2. Phase portrait of system (1.9), showing that (± 1, 0) are centers of system (2.2) when $a_2 = a_6 = 0$, $b_2 = a_5 = b_5 = \frac{3}{2}$, $b_6 = \frac{1}{2}$ and the origin is also a center.

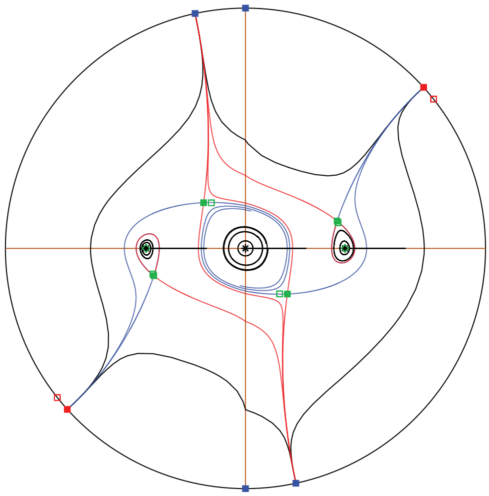

Fig. 3. Phase portrait of system (1.9), showing that (± 1, 0) are centers of system (2.2) when $a_2 = b_2 = 3$, $a_3 = 0$, $a_5 = b_5 = \frac{3}{2}$, $a_6 = b_6 = -1$ and the origin is also a center.

Recently, limit cycle bifurcations in a class of perturbed planar discontinuous systems with four switching lines were investigated by using the Melnikov method [6]. In this section, we present a method for computing the focal values at the origin of switching systems with two switching lines. Certainly, this method can be also used for computing the focal values at the origin of smooth systems or switching systems with one switching line (Figs. 2–5).

The classical method to solve center problem is based on the computation of focal values, with the procedure described as follows. With the polar coordinates transformation,

$$x = r\cos\theta, \qquad y = r\sin\theta, \tag{2.1}$$

the general differential system,

$$\begin{aligned} \frac{dx}{dt} &= \delta x - y + \sum_{k=2}^{n} X_k(x,y) \equiv X(x,y), \\ \frac{dy}{dt} &= x + \delta y + \sum_{k=2}^{n} Y_k(x,y) \equiv Y(x,y), \end{aligned} \tag{2.2}$$

can be transformed to

$$\frac{dr}{dt} = r\Big(\delta + \sum_{k=2}^{n} \varphi_{k+2}(\theta) r^k\Big),$$

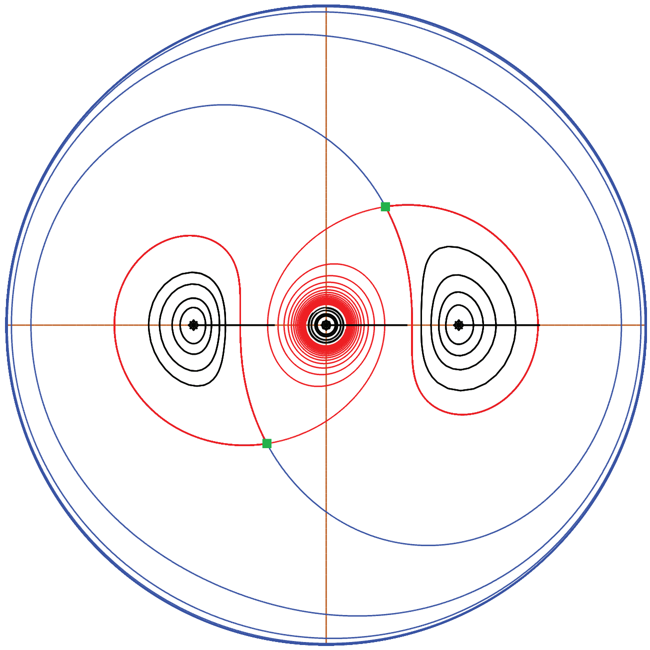

Fig. 4. Phase portrait of system (1.9), showing that $(\pm 1, 0)$ are centers of system (2.2) when $a_2 = b_2 = a_3 = b_3 = a_5 = b_5 = a_6 = b_6 = -\frac{1}{2}$ and the origin is a first-order focus.

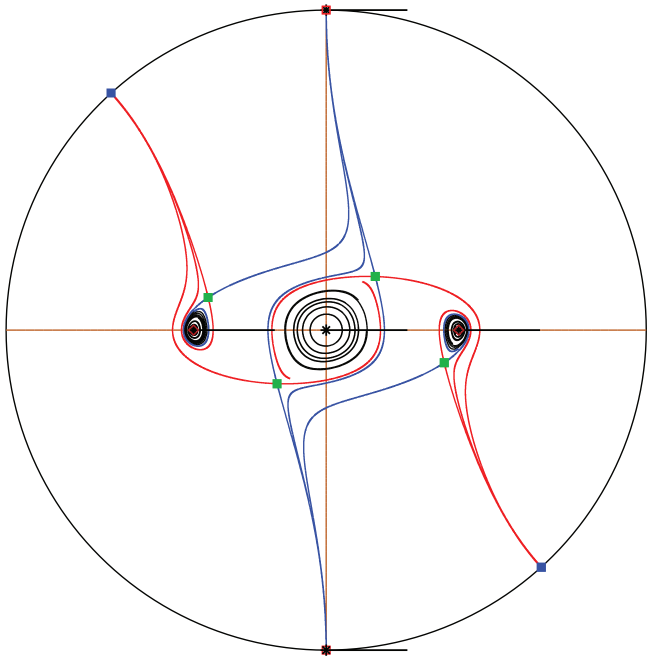

Fig. 5. Phase portrait of system (1.9), showing that $(\pm 1, 0)$ are fine foci of system (2.2) when $a_2 = b_2 = -3$, $a_3 = b_3 = 0$, $a_5 = b_5 = \frac{5}{2}$, $a_6 = b_6 = -1$ and the origin is a center.

$$\frac{d\theta}{dt} = 1 + \sum_{k=2}^{n} \psi_{k+2}(\theta) r^k, \tag{2.3}$$

where $\varphi_k(\theta)$, $\psi_k(\theta)$ are polynomials of $\cos\theta$ and $\sin\theta$, given by

$$\varphi_k(\theta) = \cos\theta X_{k-1}(\cos\theta, \sin\theta) + \sin\theta Y_{k-1}(\cos\theta, \sin\theta),$$

$$\psi_k(\theta) = \cos\theta Y_{k-1}(\cos\theta, \sin\theta) - \sin\theta X_{k-1}(\cos\theta, \sin\theta).$$

From Eq. (2.3) we have

$$\frac{dr}{d\theta} = \frac{r(\delta + \sum_{k=2}^{n} \varphi_{k+2}(\theta) r^k)}{1 + \sum_{k=2}^{n} \psi_{k+2}(\theta) r^k}, \tag{2.4}$$

whose expansion around $r = 0$ can be expressed in the form of

$$\frac{dr}{d\theta} = r \sum_{k=1}^{\infty} R_k(\theta) r^k. \tag{2.5}$$

By the method of small parameters of Poincaré, the general solution of (2.5) can be obtained as

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} v_k(\theta) h^k,$$

where $v_1(0) = 1,\ v_k(0) = 0,\ \forall k \geq 2$. Now, substituting the above solution $r = \tilde{r}(\theta, h)$ into (2.5) yields

$$\begin{aligned} v_1'(\theta) &= R_0(\theta) v_1(\theta), \\ v_2'(\theta) &= R_0(\theta) v_2(\theta) + R_1(\theta) v_1^2(\theta), \\ &\ \vdots \\ v_m'(\theta) &= R_0(\theta)\Omega_{1,m}(\theta) + R_1(\theta)\Omega_{2,m}(\theta) + \cdots + R_{m-1}(\theta)\Omega_{m,m}(\theta). \end{aligned} \tag{2.6}$$

Thus, we may solve $v_k(\theta)$ one by one, yielding

$$\begin{aligned} v_1(\theta) &= e^{\oint_0^{\vartheta} R_0(\varphi) d\varphi}, \\ v_2(\theta) &= 2 v_1(\theta) \oint_0^{\vartheta} R_1(\varphi) v_1(\varphi) d\varphi, \\ &\ \vdots \\ v_m(\theta) &= v_1(\theta) \oint_0^{\vartheta} \frac{R_1(\varphi)\Omega_{2,m}(\varphi) + \cdots + R_{m-1}(\varphi)\Omega_{m,m}(\varphi)}{v_1(\varphi)} d\varphi. \end{aligned} \tag{2.7}$$

Furthermore, we define the successive function as

$$\Delta(h) = \tilde{r}(2\pi, h) - h, \tag{2.8}$$

which in turn gives the condition to define a center, as

$$\tilde{r}(2\pi, h) = h. \tag{2.9}$$

Now, we consider Z_2-equivalent switching systems with the x-axis and the y-axis as the two switching lines, which can be written as

$$\begin{aligned} &\begin{cases} \dfrac{dx}{dt} = F_1(x, y), \\ \dfrac{dy}{dt} = G_1(x, y), \end{cases} \quad (x > 0,\ y > 0), \\ &\begin{cases} \dfrac{dx}{dt} = F_2(x, y), \\ \dfrac{dy}{dt} = G_2(x, y), \end{cases} \quad (x < 0,\ y > 0). \\ &\begin{cases} \dfrac{dx}{dt} = F_1(x, y), \\ \dfrac{dy}{dt} = G_1(x, y), \end{cases} \quad (x < 0,\ y < 0), \\ &\begin{cases} \dfrac{dx}{dt} = F_2(x, y), \\ \dfrac{dy}{dt} = G_2(x, y), \end{cases} \quad (x > 0,\ y < 0). \end{aligned} \tag{2.10}$$

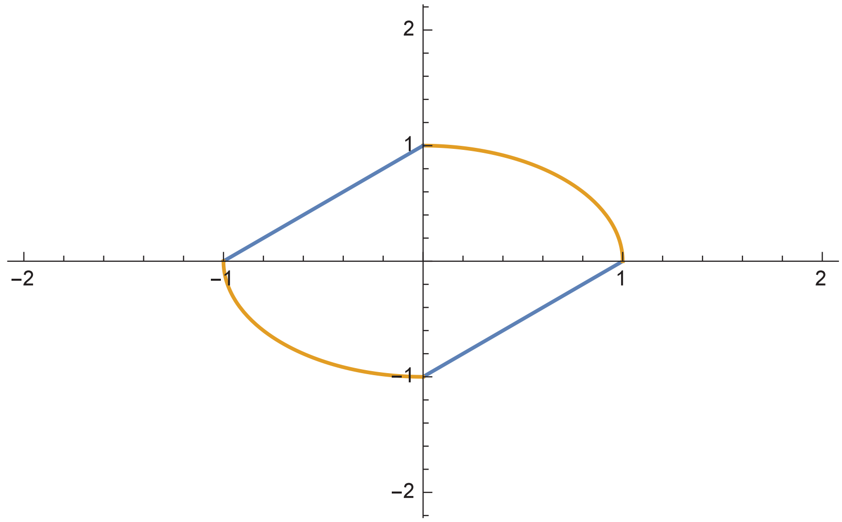

Fig. 6. Symmetry of a trajectory for Z_2-equivalent systems.

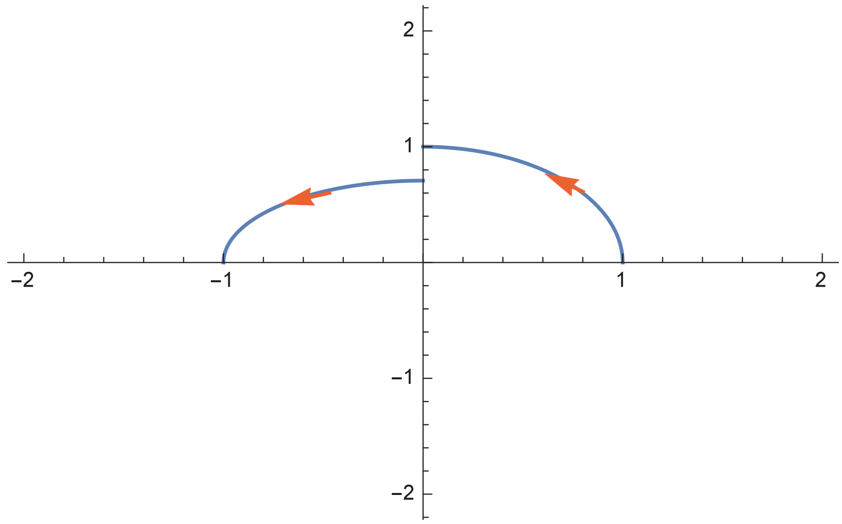

Fig. 7. A trajectory in the second quadrant reaches the point $(-h, 0)$.

Because of the symmetry in Z_2-equivalent systems, we only need to compute the half return map for the upper half plane. If the trajectories from $(h, 0)$ can reach $(-h, 0)$, the trajectories is a closed orbit because of the symmetry, see Fig. 6.

We note that it is difficult to compute the trajectories in the second quadrant with the initial condition $(0, y_0)$, associated with the origin. But we can compute the trajectory in the second quadrant which reaches $(-h, 0)$ by transforming this trajectory to the first quadrant, as shown in Fig. 7.

The main steps in the computation can be summarized as follows:

1. Introduce the transformation: $x \to -x,\ t \to -t$ into the equation in the second quadrant, see Figs. 8 and 9.
2. For the two systems in the first quadrant, we define the successive function as

$$\Delta(h) = \tilde{r}_1\left(\frac{\pi}{2}, h\right) - \tilde{r}_2\left(\frac{\pi}{2}, h\right). \tag{2.11}$$

Definition 2.1. Define

$$\Delta(h) = \sum_{k=1}^{n}\left(u_k\left(\frac{\pi}{2}\right) - v_k\left(\frac{\pi}{2}\right)\right)h^k = \sum_{k=1}^{n} V_k h^k, \tag{2.12}$$

where V_k is called the kth-order focal value at the origin of the switching system (2.10).

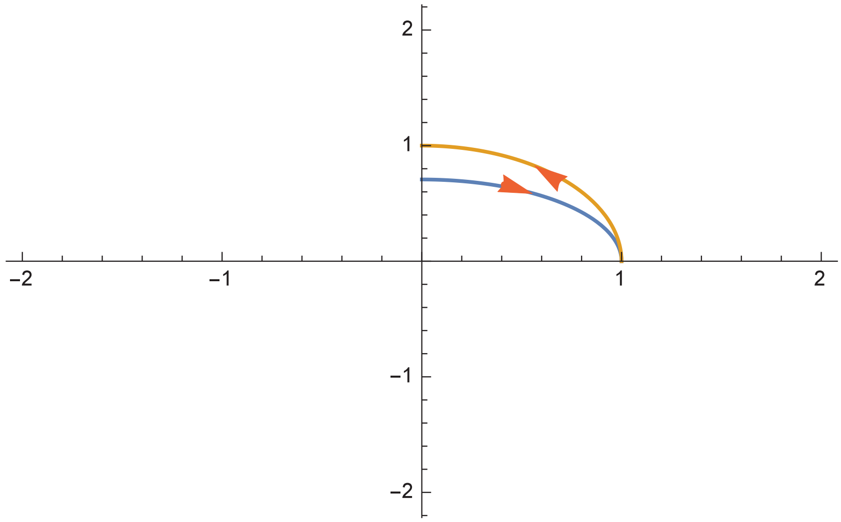

Fig. 8. The trajectory in the second quadrant (see Fig. 7) is transformed to the first quadrant with $x \to -x$.

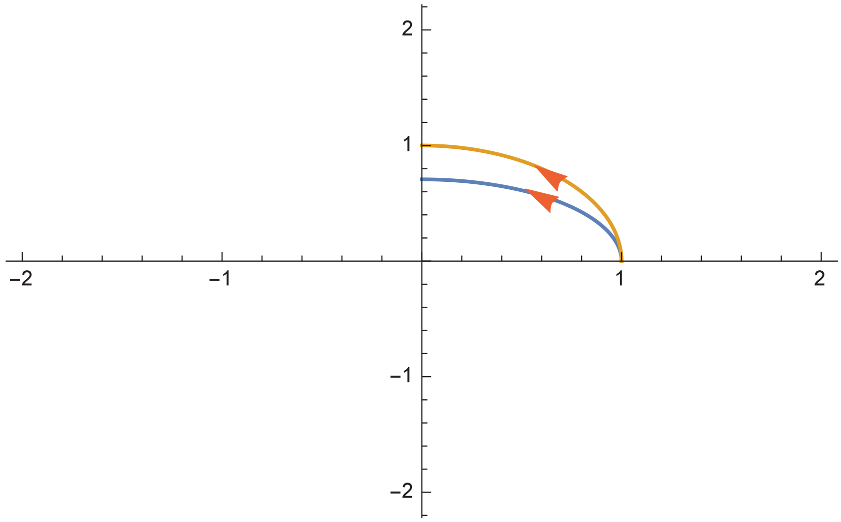

Fig. 9. The transformed trajectory (see Fig. 8) changes its moving direction with $t \to -t$, leaving the point $(h, 0)$.

By using this method for system $(1.9)|_{\delta=0}$, the first focal values at the origin is obtained as

$$V_1 = \tfrac{1}{8}(-4a_3 - 4a_5 + 4b_3 + 4b_5 - (a_2 + 3a_6 + b_2 + 3b_6)\pi).$$

When the two singular points $(\pm 1, 0)$ are 7th-order weak focus of system (1.9), the origin of the system is a first-order weak focus. Therefore, the total number of limit cycles we can find from the system (1.9) is still 15.

Remark 2.1. This method described above can be used only for computing the focal values of the origin of Z_2-equivalent smooth systems or switching systems with one or two switching lines. It cannot be used to compute the focus values around, for example, $(\pm 1, 0)$.

As an example, consider the following switching system with one switching line,

$$\begin{cases} \dfrac{dx}{dt} = -y + y^2 + \dfrac{3}{2}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y, \\ \dfrac{dy}{dt} = x + b_{02}y^2 - x^3 + b_{12}xy^2 + b_{03}y^3 + b_{40}x^4 + b_{31}x^3y, \end{cases} \quad (y > 0),$$
$$\begin{cases} \dfrac{dx}{dt} = -y - y^2 + \dfrac{3}{2}x^2y + a_{12}xy^2 + a_{03}y^3 - a_{40}x^4 - a_{31}x^3y, \\ \dfrac{dy}{dt} = x - b_{02}y^2 - x^3 + b_{12}xy^2 + b_{03}y^3 - b_{40}x^4 - b_{31}x^3y, \end{cases} \quad (y < 0). \tag{2.13}$$

By using the half-return map method (see [5]), the first eight focal values are obtained as

$$
\begin{aligned}
\lambda_1 &= \frac{8}{3} b_{02},\\
\lambda_2 &= \frac{1}{4}(a_{12}+3b_{03})\pi,\\
\lambda_3 &= \frac{4}{5}(a_{31}+b_{40}),\\
\lambda_4 &= -\frac{1}{8}(3b_{03}+2b_{03}b_{12}+b_{40})\pi,\\
\lambda_5 &= \frac{2}{105} b_{03}(-3+36a_{03}-48a_{40}+34b_{12}+24a_{03}b_{12}+24b_{12}^2-12b_{31}),\\
\lambda_6 &= -\frac{1}{384} b_{03}(3+2b_{12})(-39+60a_{03}+90a_{40}+79b_{12}+60a_{03}b_{12}+60b_{12}^2)\pi,\\
\lambda_7 &= \frac{1}{2835} b_{03}b_{03}(3+2b_{12})(-1563-1752a_{03}+2592b_{03}^2+296b_{12}\\
&\quad +480a_{03}b_{12}+480b_{12}^2),\\
\lambda_8 &= -\frac{1}{92160} b_{03}(3+2b_{12})(21195+81,252a_{03}-1440a_{03}^2-6480b_{03}^2\\
&\quad +28,152b_{12}+9528a_{03}b_{12}-10,080a_{03}^2b_{12}+25,920b_{03}^2b_{12}\\
&\quad +20,408b_{12}^2-6720a_{03}b_{12}^2+3360b_{12}^3)\pi,
\end{aligned}
$$

If we use the new method developed in this notes, we obtain the first eight focal values as $\frac{1}{2}\lambda_k$, $k = 1, 2, \ldots, 8$. This is clear that because the new method only computes the half part (one quadrant) of the above focus values which are for the up-half plane.

Summarizing the results in this section and previous section, we obtain the following theorem.

Theorem 2.1. *For the Z_2-equivariant system (1.9) with two switching lines at the x-axis and the y-axis, there exist 15 small-amplitude limit cycles around (± 1, 0) and (0,0) with the distribution* $7 \cup 1 \cup 7$.

Therefore, the conclusion given in [5] for the existence of 15 limit cycles for Z_2-equivariant cubic switching systems is still true.

Declaration of Competing Interest

The authors of this manuscript, Feng Li and Pei Yu, certify that they have NO affiliations with or involvement in any organization or entity with any financial interest, or non-financial interest in the subject matter or materials discussed in this manuscript.

CRediT authorship contribution statement

Feng Li: Methodology, Writing - original draft. **Pei Yu:** Methodology, Investigation, Writing - review & editing.

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