International Journal of Bifurcation and Chaos, Vol. 26, No. 8 (2016) 1650141 (18 pages)

DOI: 10.1142/S0218127416501418

Coupled Oscillatory Systems with $\mathbb{D}_4$ Symmetry and Application to van der Pol Oscillators

Adrian C. Murza
Department of Mathematics and Computer Science, Transilvania University, Braşov, Strada Iuliu Maniu 50, 500091, Braşov, Romania
adrian_murza@hotmail.com

Pei Yu*
Department of Applied Mathematics, Western University, London, Ontario N6A 5B7, Canada
pyu@uwo.ca

Received March 14, 2016

In this paper, we study the dynamics of autonomous ODE systems with $\mathbb{D}_4$ symmetry. First, we consider eight weakly-coupled oscillators and establish the condition for the existence of stable heteroclinic cycles in most generic $\mathbb{D}_4$-equivariant systems. Then, we analyze the action of $\mathbb{D}_4$ on $\mathbb{C}^2$ and study the pattern of periodic solutions arising from Hopf bifurcation. We identify the type of periodic solutions associated with the pairs (H, K) of spatiotemporal or spatial symmetries, and prove their existence by using the H mod K Theorem due to Hopf bifurcation and the $\mathbb{D}_4$ symmetry. In particular, we give a rigorous proof for the existence of a fourth branch of periodic solutions in $\mathbb{D}_4$-equivariant systems. Further, we apply our theory to study a concrete case: two coupled van der Pol oscillators with $\mathbb{D}_4$ symmetry. We use normal form theory to analyze the periodic solutions arising from Hopf bifurcation. Among the families of the periodic solutions, we pay particular attention to the phase-locked oscillations, each of them being embedded in one of the invariant manifolds, and identify the in-phase, completely synchronized motions. We derive their explicit expressions and analyze their stability in terms of the parameters.

Keywords: Dihedral group; Hopf bifurcation; periodic solutions; heteroclinic cycles; spatiotemporal symmetry; van der Pol oscillator.

1. Introduction

This paper deals with two important solutions of equivariant ODE systems with $\mathbb{D}_4$ symmetry: heteroclinic cycles and spatiotemporal/spatial symmetric periodic motions. Such dynamical behaviors have been observed in practice, for example, in biological systems [Takamatsu *et al.*, 2003]. Two techniques have been developed to study heteroclinic cycles in $\mathbb{D}_n$-equivariant systems. The first one is used to consider perturbation of $\mathbb{D}_n$-equivariant ODEs with some Γ-equivariant terms, where Γ is an isotropy subgroup of $\mathbb{D}_n$ [Hou & Golubitsky, 1997]. The other one can be applied to identify invariant subspaces in flow, i.e. the fixed points of the isotropy subgroups of $\mathbb{D}_n$ [Buono *et al.*, 2000]. In this paper, we first use the methodology developed in [Ashwin & Swift, 1992] to study a generic $\mathbb{D}_4$-equivariant ODE system with weak coupling. It has been shown in [Ashwin & Swift, 1992] that even strongly coupled oscillator systems may have a

*Author for correspondence

weakly-coupled limit. Therefore, we assume that our system is formed by dissipative oscillators, namely every periodic orbit of the system is attracting and unique in the neighborhood of the orbit. In the case of no coupling, Ashwin and Swift [1992] have shown that there exists an attracting N-torus with one angle for each oscillator. The torus is generally hyperbolic, and so under small coupling it persists with a slow evolution of phase difference. When the coupling parameter takes small values (called weak coupling), averaging the equations introduces an approximate decoupling between the fast and slow variations of phases. This is the key idea used in proving the existence of stable heteroclinic cycles in weakly coupled $\mathbb{D}_4$-equivariant systems. As far as we know, heteroclinic cycles in $\mathbb{D}_4$-equivariant systems have been only found in Euler equations [Swift, 1988]. So our general result obtained in this paper is new.

In this contribution, we are also interested in establishing a criterion which can be applied to identify the periodic solutions predicted by the H mod K Theorem [Golubitsky & Stewart, 2003] in a four-dimensional ODE, equivariant under the action of $\mathbb{D}_4 \times \mathbb{S}^1$ group, which are obtained from the Equivariant Hopf Theorem [Golubitsky *et al.*, 1988; Golubitsky & Stewart, 2003]. The H mod K Theorem provides a complete set of possible periodic solutions based exclusively on the structure of the group Γ acting on the differential equation; while the Equivariant Hopf Theorem guarantees the existence of families of small-amplitude periodic solutions bifurcating from the origin for all **C**-axial subgroups of $\Gamma \times \mathbb{S}^1$. However, not always all solutions predicted by the H mod K Theorem can be obtained by the generic Hopf bifurcation theory [Golubitsky & Stewart, 2003]. In this article, we identify the periodic solutions predicted by the H mod K Theorem, which occur due to Hopf bifurcations from the trivial equilibrium when the $\mathbb{D}_4$ symmetry group acts on $\mathbb{C}^2$. We noticed that Swift [1988] mentioned the existence of a fourth branch of periodic solutions in $\mathbb{D}_4$-equivariant systems without proof. We give a rigorous proof for this solution.

Further, we apply our established theory with normal form theory to analyze the periodic solutions of two coupled van der Pol oscillators. Systems of two coupled van der Pol oscillators with or without delay have been intensively studied over the last two decades [Barrón & Sen, 2009; Camacho *et al.*, 2014; Kawahara, 1980; Low *et al.*, 2003; Paccosi *et al.*, 2014; Rand & Holmes, 1980; Uwate *et al.*, 2010]. Our main interest in this paper is focused on the specific $\mathbb{D}_4$-equivariant coupled van der Pol systems with particular attention on periodic solutions, which were only discussed by Swift [1988]. It is noted that Swift used the normal form truncated to cubic order and mentioned the existence of different types of periodic solutions, according to the results given in [Golubitsky *et al.*, 1988]. It should be pointed out that Swift [1988] only mentioned the existence of phase-locked solutions and invariant 2D tori. In this paper, we will present a more detailed analysis on the periodic solutions by using the method of normal forms. Normal form theory has been widely applied to study dynamical behavior of nonlinear systems, for example, see the books [Chow & Hale, 1982; Guckenheimer & Holmes, 1993; Chow *et al.*, 1994; Kuznetsov, 1998; Han & Yu, 2012], and computation methods discussed in [Yu, 1998; Bi & Yu, 1999; Yu, 2003]. In particular, the normal form for double Hopf bifurcation is given in [Bi & Yu, 1999; Yu, 2003]. Recently, the simplest normal form theory has been developed to further simply the conventional normal forms (e.g. see [Yu, 1999; Yu & Leung, 2003; Gazor & Yu, 2012]). The phase-locked periodic solutions have also been considered in a double pendulum system [Yu & Bi, 1998]. We analyze the periodic solutions that bifurcate from the unique equilibrium — the origin. We obtain a number of families of periodic solutions due to Hopf bifurcation. We characterize the periodic solutions which are embedded in three invariant manifolds of the coupled system. With special attention, we obtain three sets of phase-locked synchronized quasi-periodic solutions, which are located on a 2D torus, embedded in one of the invariant manifolds. In addition, we derive the explicit forms of the phase-locked solutions in terms of the parameters, showing which solutions are completely synchronized in both amplitude and phase. Stability of these solutions is also explicitly determined on parameters, verified by simulations.

This article is organized as follows. In Sec. 2, we construct a generic $\mathbb{D}_4$-equivariant system under the assumption of weak coupling limit to generate stable heteroclinic cycles. In Sec. 3, we review the action of $\mathbb{D}_4$ and $\mathbb{D}_4 \times \mathbb{S}^1$ on $\mathbb{C}^2$, [Swift, 1988; Golubitsky *et al.*, 1988], and find a condition for a fourth branch of periodic solutions in systems with $\mathbb{D}_4$ symmetry. Moreover, we identify the pairs (H, K) of spatiotemporal/spatial symmetries of the periodic solutions predicted by the H mod K Theorem, which can be obtained at a primary Hopf bifurcation. In Sec. 4, we investigate in detail the periodic solutions in a system of two coupled van der Pol oscillators with $\mathbb{D}_4$ symmetry. Simulations are given in Sec. 5 to verify the analytical predictions. Finally, the conclusion is given in Sec. 6.

2. Weakly-Coupled Oscillators

In this section, we construct an oscillatory system with the $\mathbb{D}_4$ symmetry group, which is described by a Cayley diagram. A Cayley diagram consists of a set of nodes with arrows between them. The vertices or nodes of the graph represent the group elements and the arrows show how the generators act on the elements of the group. Let $J \subset \mathbb{D}_4$ be the generating set of $\mathbb{D}_4$. This implies that

$$\begin{aligned} &\text{(a)}\ \ J \text{ generates } \mathbb{D}_4,\\ &\text{(b)}\ \ J \text{ is finite}\quad \text{and} \\ &\text{(c)}\ \ J = J^{-1}. \end{aligned} \tag{1}$$

Since $\mathbb{D}_4$ is finite, the assumptions (b) and (c) are automatically satisfied. The Cayley graph of $\mathbb{D}_4$, as shown in Fig. 1, is a directed colored graph whose arrows indicate the action on the group elements and the vertices denote the group elements. As shown in [Ashwin & Stork, 1994] the generating set J is a set of "colors" of the directed edges, and the group elements κ_i and ζ_j are connected through an edge from κ_i to ζ_j of color $c \in J$ if and only if $\kappa_i = \zeta_j$.

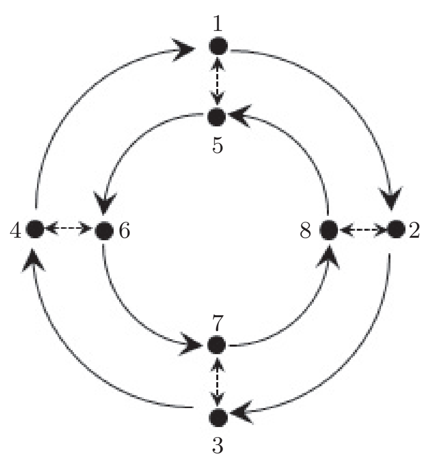


Fig. 1. A Cayley graph of the $\mathbb{D}_4$ group showing two generators: The solid arrows representing the left-multiplication with κ and the dotted arrows the left-multiplication with ζ.

2.1. *Eight oscillators with weak coupling*

Following the work of Ashwin and Stork [1994], we may identify the vertices as cells with a certain dynamics and the edges as couplings between the cells. Consequently, we can construct an ODE system which has the $\mathbb{D}_4$ symmetry, with two generators ζ and κ having act as:

$$\begin{aligned} \zeta &:= (1\ 2\ 3\ 4)(5\ 6\ 7\ 8),\\ \kappa &:= (1\ 5)(2\ 8)(3\ 7)(4\ 6). \end{aligned} \tag{2}$$

If we assign coupling between the cells related by the permutations in (2), we can build the following pairwise system in $\mathbb{R}^8$ with the $\mathbb{D}_4$ symmetry,

$$\begin{aligned} \dot{x}_1 &= f(x_1) + g(x_4, x_1) + h(x_5, x_1),\\ \dot{x}_2 &= f(x_2) + g(x_1, x_2) + h(x_8, x_2),\\ \dot{x}_3 &= f(x_3) + g(x_2, x_3) + h(x_7, x_3),\\ \dot{x}_4 &= f(x_4) + g(x_3, x_4) + h(x_6, x_4),\\ \dot{x}_5 &= f(x_5) + g(x_8, x_5) + h(x_6, x_1),\\ \dot{x}_6 &= f(x_6) + g(x_7, x_8) + h(x_2, x_5),\\ \dot{x}_7 &= f(x_7) + g(x_6, x_7) + h(x_3, x_8),\\ \dot{x}_8 &= f(x_8) + g(x_5, x_6) + h(x_4, x_7) \end{aligned} \tag{3}$$

where the dot denotes differentiation with respect to time t, $f : \mathbb{R} \to \mathbb{R}$ and $g, h : \mathbb{R}^2 \to \mathbb{R}$. Using the idea of Ashwin and Stork [1994] we may think of f, g and h as being generic functions assuring that the isotropy of the vector field under the action of $\mathbb{O}_8$ is generically $\mathbb{D}_4$. In the case of weak coupling, system (3) can be rewritten as an ODE system in the form of

$$\dot{x}_i = f(x_i) + \epsilon g_i(x_1, \ldots, x_8), \quad i = 1, 2, \ldots, 8, \tag{4}$$

where $x_i \in \mathbb{R}$ and g commutes with the permutation action of $\mathbb{D}_4$ on $\mathbb{R}^8$, both f and g_i are of the class $\mathbf{C}^\infty$. The constant ϵ represents the weak coupling. Similar to [Ashwin & Swift, 1992] or [Ashwin & Stork, 1994] we assume that each state of system (4) has a hyperbolic stable limit cycle.

It has been shown in [Ashwin & Swift, 1992] that in the case of weak coupling, one should not just consider irreducible representations of $\mathbb{D}_4$. In our case, there are eight stable hyperbolic limit cycles at $\epsilon = 0$, implying that the asymptotic dynamics of system (4) is decomposed into the asymptotic dynamics of eight limit cycles. This way, we can embed the flow on an eight-dimensional torus $\mathbb{T}^8$. Again, we assume the hyperbolicity on each of the eight limit cycles for small enough values of ϵ, which justifies the expression of the dynamics of the system as an ODE system in terms of eight phases, i.e. an ODE system on $\mathbb{T}^8$ which is $\mathbb{D}_4$-equivariant. There are certain similarities between the $\mathbb{D}_4$ and $\mathbb{Q}_4$ groups acting on $\mathbb{C}^2$ as discussed in [Ashwin *et al.*, 1994], but they are not isomorphic. When the coupling parameter ϵ is small, it is possible to average the equations and introduce an approximate decoupling between the fast and slow variations of phases. This can be seen as introducing phase shift symmetry which acts on $\mathbb{T}^8$ by a translation along the diagonal; $R_\psi(\phi_1, \ldots, \phi_8) := (\phi_1 + \psi, \ldots, \phi_8 + \psi)$, for $\psi \in \mathbb{S}^1$. We are particularly interested in the three-dimensional fixed-point spaces as shown in Table 1, which are invariant under $\mathbb{Z}_2$ or $\widetilde{\mathbb{Z}_2}$ action. Proving the existence of heteroclinic cycles in these two fixed-point spaces is the same, so in the following we will only consider $\mathbb{Z}_2$.

2.2. *Analysis of a family of vector fields in* Fix($\mathbb{Z}_2$)

For convenience, we define the coordinates in Fix($\mathbb{Z}_2$) by using the following basis,

$$
\begin{aligned}
e_1 &= \frac{1}{2}(1, 1, 1, 1, -1, -1, -1, -1),\\
e_2 &= \frac{1}{2}(1, -1, 1, -1, -1, 1, -1, 1),\\
e_3 &= \frac{1}{2}(1, -1, 1, -1, 1, -1, 1, 1)
\end{aligned}
\tag{5}
$$

and consider the space spanned by $\{e_1, e_2, e_3\}$, parameterized by $\{\psi_1, \psi_2, \psi_3\} : \sum_{n=1}^{3} \psi_n e_n$. With these coordinates, we construct the following family of three-dimensional differential systems which satisfy the symmetry of Fix($\mathbb{Z}_2$).

$$
\begin{aligned}
\dot{\psi}_1 &= \sin(\psi_1)\cos(\psi_2) + \epsilon \sin(4\psi_1)\cos(4\psi_2),\\
\dot{\psi}_2 &= \sin(\psi_2)\cos(\psi_3) + \epsilon \sin(4\psi_2)\cos(4\psi_3),
\end{aligned}
$$

Table 1. Isotropy subgroups and fixed points for the $\mathbb{D}_4 \times \mathbb{S}^1$ action on $\mathbb{T}^8$.

Σ	Fix(Σ)	Generators	dim Fix(Σ)
$\mathbb{D}_4$	$(0,0,0,0,0,0,0,0,0)$	$(\zeta,0),(\kappa,0)$	0
$\widetilde{\mathbb{D}_4^a}$	$(0,0,0,0,\pi,\pi,\pi,\pi)$	$(\zeta,\pi),(\kappa,\pi)$	0
$\widetilde{\mathbb{D}_4^b}$	$(0,\pi,0,\pi,0,\pi,0,\pi)$	$(\kappa\zeta,\pi),(\kappa,\pi)$	0
$\widetilde{\mathbb{D}_4^c}$	$(0,\pi,0,\pi,\pi,0,\pi,0)$	$(\kappa\zeta,\pi),(\zeta^2,\pi)$	0
$\mathbb{Z}_4^0$	$(0,0,0,0,\phi,\phi,\phi,\phi)$	$(\zeta,0)$	1
$\widetilde{\mathbb{Z}_4^\pi}$	$(0,\pi,0,\pi,\phi,\phi+\pi,\phi,\phi+\pi)$	(ζ,π)	1
$\widetilde{\mathbb{Z}_4^{\pi/2}}$	$\left(0,\frac{3\pi}{2},\pi,\frac{\pi}{2},\phi,\phi+\frac{\pi}{2},\phi+\pi,\phi+\frac{3\pi}{2}\right)$	$\left(\zeta,\frac{\pi}{2}\right)$	1
$\widetilde{\mathbb{D}_2^a}$	$(0,\phi,0,\phi,\phi,0,\phi,0)$	$(\kappa,0),((0,0),(\pi,\pi))$	1
$\widetilde{\mathbb{D}_2^b}$	$(0,\phi,0,\phi,\phi+\pi,\pi,\phi+\pi,\pi)$	$(\kappa,\pi),((0,0),(\pi,\pi))$	1
$\widetilde{\mathbb{D}_2^c}$	$\left(0,\phi,\pi,\phi+\pi,\phi+\frac{3\pi}{2},\frac{3\pi}{2},\phi+\frac{\pi}{2},\frac{\pi}{2}\right)$	$\left(\kappa,\frac{\pi}{2}\right),((0,0),(\pi,\pi))$	1
$\widetilde{\mathbb{D}_2^d}$	$(0,\phi,0,\phi,0,\phi,0,\phi)$	$(\zeta^2,0),((0,0),(\pi,\pi))$	1
$\widetilde{\mathbb{D}_2^e}$	$(0,\phi,0,\phi,\pi,\phi+\pi,\pi,\phi+\pi)$	$(\zeta^2,\pi),((0,0),(\pi,\pi))$	1
$\widetilde{\mathbb{D}_2^f}$	$\left(0,\phi,\pi,\phi+\pi,\frac{\pi}{2},\phi+\frac{\pi}{2},\frac{3\pi}{2},\phi+\frac{3\pi}{2}\right)$	$\left(\zeta^2,\frac{\pi}{2}\right),((0,0),(\pi,\pi))$	1
$\mathbb{Z}_2$	$(0,\phi_1,0,\phi_1,\phi_2,\phi_3,\phi_2,\phi_3)$	κ	3
$\widetilde{\mathbb{Z}_2}$	$(0,\phi_1,\pi,\phi_1+\pi,\phi_2,\phi_3,\phi_2+\pi,\phi_3+\pi)$	ζ^2	3

$$\dot{\psi}_3 = \sin(\psi_3)\cos(\psi_1) + \epsilon\sin(4\psi_3)\cos(4\psi_1) + q(1+\cos(\psi_1-\psi_2))\sin(4\psi_3). \tag{6}$$

Since the planes $\psi_i = 0 \pmod{\pi}$, $i = 1, 2, 3$, are invariant under the flow of (6), it is clear that $(\pi, 0, 0)$, $(0, \pi, 0)$ and $(0, 0, \pi)$ are equilibria of (6). To check the existence of heteroclinic cycles in system (6), we first linearize the system at the equilibria (i.e. the zero-dimensional fixed points). Without loss of generality, we assume that $\mathrm{Fix}(\mathbb{Z}_2)$ is attracting and therefore the stability. We analyze the dynamics within the fixed point space $\mathrm{Fix}(\mathbb{Z}_2)$. In particular, we will prove that the eigenvalues of the linearization in each case are of opposite signs (see Table 2), indicating the existence of such a heteroclinic cycle between them. We use the criteria of Krupa and Melbourne [1995] to study the stability of the heteroclinic cycle.

We have the following theorem.

Theorem 1. *There exists a heteroclinic cycle in system (6) in the following way*:

$$\cdots \xrightarrow{\widetilde{\mathbb{Z}_4^{\pi}}} \widetilde{\mathbb{D}_4^a} \xrightarrow{\widetilde{\mathbb{Z}_4^{\pi/2}}} \widetilde{\mathbb{D}_4^b} \xrightarrow{\widetilde{\mathbb{Z}_4^0}} \widetilde{\mathbb{D}_4^c} \xrightarrow{\widetilde{\mathbb{Z}_4^{\pi}}} \cdots.$$

The heteroclinic cycle is

(A) *asymptotically stable if* $-\frac{1}{4} < \epsilon < 0$ *and* $q < \frac{1}{8} + \frac{\epsilon}{2}$;
(B) *or unstable but essentially asymptotically stable if* $-\frac{1}{4} < \epsilon < 0$ *and* $\frac{1}{8} + \frac{\epsilon}{2} < q < \frac{1}{2} - \frac{(1+4\epsilon)^3}{8(-1+4\epsilon)^2}$;
(C) *or unstable if* $\frac{1}{4} > \epsilon > 0$.

Proof. In $\mathrm{Fix}(\mathbb{Z}_2)$ and $\mathrm{Fix}(\widetilde{\mathbb{Z}_2})$ there are no extra fixed points if $|\epsilon| < \frac{1}{4}$ and $|\epsilon + 2q| < \frac{1}{4}$. The stability is given by $\rho = \prod_{i=1}^{3} \rho_i$, where $\rho_i = \min\{c_i/e_i, 1 - t_i/e_i\}$, e_i is the expanding eigenvector at the ith point of the cycle, $-c_i$ is the contracting eigenvector and t_i is the tangential eigenvector of the linearization. For the heteroclinic cycle we have

$$\rho_1 = \rho_2 = \frac{-1+4\epsilon}{1+4\epsilon},$$

$$\rho_3 = \begin{cases} \dfrac{1-4\epsilon}{1+4\epsilon} & \text{if } q < \dfrac{1}{8} + \dfrac{\epsilon}{2}, \\[2ex] \dfrac{2-8q}{1+4\epsilon} & \text{if } q > \dfrac{1}{8} + \dfrac{\epsilon}{2}. \end{cases}$$

Then the proof follows by applying Theorem 2.4 in [Krupa & Melbourne, 1995]. ■

Remark 2.1. For any $\frac{1}{4} < \epsilon < 0$, we have $\frac{1}{8} + \frac{\epsilon}{2} < q < \frac{1}{2} - \frac{(1+4\epsilon)^3}{8(-1+4\epsilon)^2}$, and therefore there exist values of q to yield essentially asymptotic stable heteroclinic connections. In consequence, there exists an attracting heteroclinic cycle even though the linear stability of $\mathrm{Fix}(\widetilde{\mathbb{D}_2^c})$ has an expanding transverse eigenvalue.

Table 2. Eigenvalues of the flow of Eq. (6) at the four nonconjugate zero-dimensional fixed points.

$\mathrm{Fix}(\Sigma)$	(ψ_1, ψ_2, ψ_3)	λ_1	λ_2	λ_3
$\mathbb{D}_4$	$(0, 0, 0)$	$1 + 4\epsilon$	$1 + 4\epsilon$	$1 + 4(\epsilon + 2q)$
$\widetilde{\mathbb{D}_4^a}$	$(\pi, 0, 0)$	$-1 + 4\epsilon$	$-1 + 4\epsilon$	$1 + 4\epsilon$
$\widetilde{\mathbb{D}_4^b}$	$(0, \pi, 0)$	$-1 + 4\epsilon$	$-1 + 4\epsilon$	$1 + 4\epsilon$
$\widetilde{\mathbb{D}_4^c}$	$(0, 0, \pi)$	$-1 + 4\epsilon$	$1 + 4\epsilon$	$-1 + 4(\epsilon + 2q)$

3. The H mod K Theorem versus the Equivariant Hopf Theorem

Consider the system

$$\dot{x} = f(x, \lambda), \tag{7}$$

where $x \in \mathbb{C}^2$, $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $f : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2$ is smooth and commutes with $\mathbb{D}_4$:

$$f(\sigma \cdot x, \lambda) = \sigma \cdot f(x, \lambda), \quad \sigma \in \mathbb{D}_4.$$

We assume that at the Hopf bifurcation point $(df)_{(0,0)}$ has eigenvalues $\pm i$. Since $\mathrm{Fix}(\mathbb{D}_4) = \{0\}$, it follows that $f(0, \lambda) \equiv 0$. As in [Dias & Paiva, 2010], our goal is to study the generic existence of periodic solutions of (7) near the bifurcation point $(x, \lambda) = (0, 0)$. In addition, we assume that f is in Birkhoff normal form, i.e. it commutes with $\mathbb{S}^1$, thus we may assume that $\mathbb{S}^1$ acts on $\mathbb{C}^2$ by

$$\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2), \quad \theta \in \mathbb{S}^1. \tag{8}$$

We call $(\gamma, \theta) \in \Gamma \times \mathbb{S}^1$ a spatiotemporal symmetry of the solution $x(t)$. A spatiotemporal symmetry of $x(t)$ for which $\theta = 0$ is called a spatial symmetry, since it fixes the point $x(t)$ for any $t > 0$. The group of all spatiotemporal symmetries of $x(t)$ is denoted by

$$\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1.$$

As shown in [Golubitsky & Stewart, 2003], the symmetry group $\Sigma_{x(t)}$ can be identified with a pair of subgroups H and K of Γ and a homomorphism $\Theta : H \to \mathbb{S}^1$ with kernel K. Define

$$\begin{aligned} K &= \{\gamma \in \Gamma : \gamma x(t) = x(t)\ \forall\, t\} \quad \text{and} \\ H &= \{\gamma \in \Gamma : \gamma x(t) = \{x(t)\}\ \forall\, t\}. \end{aligned} \tag{9}$$

The subgroup $K \subseteq \Sigma_{x(t)}$ is the group of spatial symmetries of $x(t)$ and the subgroup H consists of those symmetries that preserve the trajectory of $x(t)$, i.e. the spatial parts of the spatiotemporal symmetries of $x(t)$. The groups $H \subseteq \Gamma$ and $\Sigma_{x(t)} \subseteq \Gamma \times \mathbb{S}^1$ are isomorphic; the isomorphism is in fact just the restriction to $\Sigma_{x(t)}$ of the projection of $\Gamma \times \mathbb{S}^1$ onto Γ. Therefore, the group $\Sigma_{x(t)}$ can be written as

$$\Sigma^{\Theta} = \{(h, \Theta(h)) : h \in H, \Theta(h) \in \mathbb{S}^1\}.$$

Moreover, we call Σ^{Θ} a twisted subgroup of $\Gamma \times \mathbb{S}^1$. In our case, Γ is the group $\mathbb{D}_4$ and the H mod K Theorem states necessary and sufficient conditions for the existence of a periodic solution to a Γ-equivariant system of ODEs with specified spatiotemporal symmetries $K \subset H \subset \Gamma$. Recall that the isotropy subgroup Σ_x of a point $x \in \mathbb{R}^n$ consists of group elements that fix x, that is, they satisfy

$$\Sigma_x = \sigma \in \Gamma : \sigma x = x.$$

As mentioned in the introduction, our main tools for characterizing the spatial and spatiotemporal oscillation patterns arising in a $\mathbb{D}_4$-equivariant system, are the H mod K Theorem and the Equivariant Hopf Theorem. For convenience, we state them below.

Let $N(H)$ be the normalizer of H in Γ, satisfying $N(H) = \{\gamma \in \Gamma : \gamma H = H\gamma\}$, and $\mathrm{Fix}(K) = \{x \in \mathbb{R}^n : kx = x\ \forall\, k \in K\}$.

Definition 3.1. Let $K \subset \Gamma$ be an isotropy subgroup. The variety L_K is defined by

$$L_K = \bigcup_{\gamma \notin K} \mathrm{Fix}(\gamma) \cap \mathrm{Fix}(K).$$

Theorem 2. H mod K Theorem [Golubitsky & Stewart, 2003]. *Let Γ be a finite group acting on $\mathbb{R}^n$. There is a periodic solution to some Γ-equivariant system of ODEs on $\mathbb{R}^n$ with spatial symmetries K and spatiotemporal symmetries H if and only if the following conditions hold*:

(a) *H/K is cyclic*;
(b) *K is an isotropy subgroup*;
(c) $\dim \mathrm{Fix}(K) \geq 2$. *If* $\dim \mathrm{Fix}(K) = 2$, *then either $H = K$ or $H = N(K)$*;
(d) *H fixes a connected component of* $\mathrm{Fix}(K) \backslash L_K$, *where L_K appears as in Definition 3.1 above.*

Moreover, if the conditions (a)–(d) *hold, the system can be chosen so that the periodic solution is stable.*

Definition 3.2. The pair of subgroups (H, K) is called admissible if the pair satisfies conditions (a)–(d) in the H mod K Theorem, that is, if there exist periodic solutions to some Γ-equivariant system with (H, K) symmetry.

Theorem 3 Equivariant Hopf Theorem [Golubitsky & Stewart, 2003]. *Let a compact Lie group Γ act Γ-simply, orthogonally, and nontrivially on $\mathbb{R}^{2m}$. Assume that*

(i) *$f : \mathbb{R}^{2m} \times \mathbb{R} \to \mathbb{R}^{2m}$ is Γ-equivariant, implying that $f(0, \lambda) = 0$ and $(df)_{(0,\lambda)}$ have eigenvalues $\sigma(\lambda) \pm i\rho(\lambda)$, each with multiplicity m*;
(ii) *$\sigma(0) = 0$ and $\rho(0) = 1$*;
(iii) *$\sigma'(0) \neq 0$, called transversality*; *and*
(iv) *$\Sigma \subseteq \Gamma \times \mathbb{S}^1$ is a $\mathbf{C}$-axial subgroup.*

Then, there exists a unique branch of periodic solutions with period $\approx 2\pi$ emanating from the origin, with spatiotemporal symmetries Σ.

3.1. $\mathbb{D}_4$ *and* $\mathbb{D}_4 \times \mathbb{S}^1$ *actions on* $\mathbb{C}^2$

In this section, we review the standard action of $\mathbb{D}_4$ and $\mathbb{D}_4 \times \mathbb{S}^1$ on $\mathbb{C}^2$, the corresponding isotropy lattice and the isotropy subgroups. We follow Golubitsky *et al.* [1988] and list in Table 3 the possible pairs (H, K) for the action of $\mathbb{D}_4$ on $\mathbb{C}^2$, while the isotropy lattices for the action of $\mathbb{D}_4$ and $\mathbb{D}_4 \times \mathbb{S}^1$ on $\mathbb{C}^2$ are shown in Fig. 2. Assume that $\Gamma = \mathbb{D}_4$ acts on $\mathbb{C}^2$ in the standard way as the symmetries of the square, as in [Golubitsky *et al.*, 2004, p. 17]. The action of $\mathbb{D}_4 \times \mathbb{S}^1$, on $(z_1, z_2) \in \mathbb{C}^2$ is as follows:

$$\begin{aligned} &\gamma \cdot (z_1, z_2) = (e^{i\gamma} z_1, e^{-i\gamma} z_2), \quad \gamma \in \mathbb{Z}_4, \\ &\kappa \cdot (z_1, z_2) = (z_2, z_1) \quad \text{and} \\ &\mathbb{S}^1 \text{ acts defined as in (8).} \end{aligned} \tag{10}$$

3.2. *Spatial and spatiotemporal symmetries*

The equivariant normal form for a $\mathbb{D}_4 \times \mathbb{S}^1$ smooth function $f : \mathbb{C}^2 \to \mathbb{C}^2$, truncated to the cubic order,

Table 3. Possible pairs (H, K) for the H mod K Theorem in the action of $\mathbb{D}_4$ on $\mathbb{C}^2$.

K	Generators of K	H	Generators of H	Fix(K)	dim
$\mathbb{D}_4$	$\{\zeta, \kappa\}$	$\mathbb{D}_4$	$\{\zeta, \kappa\}$	$\{(x, x)\}$	1
$\mathbb{D}_2$	$\{\kappa, \rho\}$	$\mathbb{D}_4$	$\{\zeta, \kappa\}$	$\{(x, x)\}$	1
$\mathbb{Z}_4$	$\{\zeta\}$	$\mathbb{D}_4$	$\{\zeta, \kappa\}$	$\{(x, x)\}$	1
$\mathbb{Z}_2$	$\{\kappa\}$	$\mathbb{D}_4$	$\{\zeta, \kappa\}$	$\{(z, z)\}$	2
$\mathbb{Z}_2$	$\{\rho\}$	$\mathbb{D}_4$	$\{\zeta, \kappa\}$	$\{(z, z)\}$	2
$\mathbb{1}$	$\{\mathbb{1}\}$	$\mathbb{D}_4$	$\{\zeta, \kappa\}$	$\{(z_1, z_2)\}$	4
$\mathbb{D}_2$	$\{\kappa, \rho\}$	$\mathbb{D}_2$	$\{\kappa, \rho\}$	$\{(x, x)\}$	1
$\mathbb{Z}_2$	$\{\kappa\}$	$\mathbb{D}_2$	$\{\kappa, \rho\}$	$\{(z, z)\}$	2
$\mathbb{Z}_2$	$\{\rho\}$	$\mathbb{D}_2$	$\{\kappa, \rho\}$	$\{(x_1, x_2)\}$	2
$\mathbb{1}$	$\{\mathbb{1}\}$	$\mathbb{D}_2$	$\{\kappa, \rho\}$	$\{(z_1, z_2)\}$	4
$\mathbb{Z}_4$	$\{\zeta\}$	$\mathbb{Z}_4$	$\{\zeta\}$	$\{(x, x)\}$	1
$\mathbb{Z}_2$	$\{\rho\}$	$\mathbb{Z}_4$	$\{\zeta\}$	$\{(x_1, x_2)\}$	2
$\mathbb{1}$	$\{\mathbb{1}\}$	$\mathbb{Z}_4$	$\{\zeta\}$	$\{(z_1, z_2)\}$	4
$\mathbb{Z}_2$	$\{\kappa\}$	$\mathbb{Z}_2$	$\{\kappa\}$	$\{(z, z)\}$	2
$\mathbb{1}$	$\{\mathbb{1}\}$	$\mathbb{Z}_2$	$\{\kappa\}$	$\{(z_1, z_2)\}$	4
$\mathbb{Z}_2$	$\{\rho\}$	$\mathbb{Z}_2$	$\{\rho\}$	$\{(x_1, x_2)\}$	2
$\mathbb{1}$	$\{\mathbb{1}\}$	$\mathbb{Z}_2$	$\{\rho\}$	$\{(z_1, z_2)\}$	4
$\mathbb{1}$	$\{\mathbb{1}\}$	$\mathbb{1}$	$\{\mathbb{1}\}$	$\{(z_1, z_2)\}$	4

is given by [Swift, 1988]

$$\begin{aligned}\dot{z}_1 &= (\lambda + i\omega)z_1 + (A|z_1|^2 + |z_2|^2 \\ &\quad + B|z_1|^2)z_1 + C\bar{z}_1 z_2^2 \\ \dot{z}_2 &= (\lambda + i\omega)z_2 + (A|z_1|^2 + |z_2|^2 \\ &\quad + B|z_2|^2)z_2 + C\bar{z}_2 z_2^1.\end{aligned} \tag{11}$$

It follows from the generic bifurcation theory that the form (11) determines the behavior of each branching solution if the nondegeneracy conditions on the complex parameters are satisfied [Swift, 1988]. Here, λ is the bifurcation parameter and ω denotes the perturbation of the period from the critical frequency at Hopf bifurcation. The isotropy subgroups of $\mathbb{D}_4 \times \mathbb{S}^1$ are defined in Table 4 [Golubitsky *et al.*, 1988]. Up to conjugacy, there are three **C**-axial subgroups. It follows from the Equivariant Hopf Theorem that there are at least three branches of periodic solutions which may occur in such a system with $\mathbb{D}_4$ symmetry.

In the following, we prove that when certain condition of the complex parameters A, B and C is satisfied, there exists a fourth branch of periodic solutions corresponding to the isotropy subgroup $\mathbb{Z}_2^c$. More precisely we have the following result.

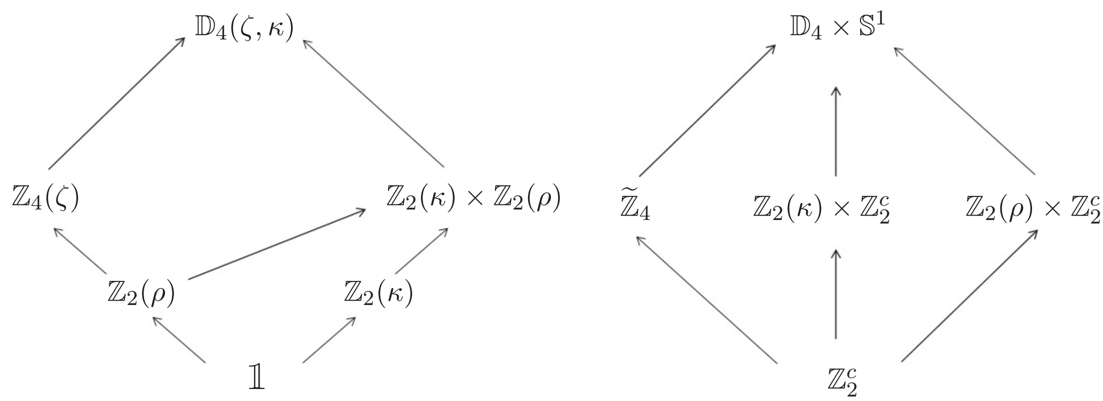


Fig. 2. Isotropy lattices for the $\mathbb{D}_4$ (left) and $\mathbb{D}_4 \times \mathbb{S}^1$ (right) groups, acting on $\mathbb{C}^2$. Note that $\rho = \zeta^2$.

Table 4. Spatial and spatiotemporal symmetries of solutions arising from the trivial equilibrium due to primary Hopf bifurcation for the action of $\mathbb{D}_4 \times \mathbb{S}^1$ on $\mathbb{C}^2$.

Isotropy Subgroup Generators, $n \equiv 0 \pmod 4$	Subgroup of $\Sigma \subset \mathbb{D}_n \times \mathbb{S}^1$ Generators	Spatiotemporal Symmetries H Generators	Spatial Symmetries K Generators	Fixed Point Space	Dimension
$\mathbb{D}_4 \times \mathbb{S}^1$	$\{\gamma, \kappa, \theta\}$	$\{\gamma, \kappa\}$	$\{(0,0)(\pi,\pi)\}$	$\{(0,0)\}$	0
$\widetilde{\mathbb{Z}}_4 = \{(\gamma, -\gamma) : \gamma \in \mathbb{Z}_4\}$	$\{-\gamma, \gamma\}$	$\{\gamma\}$	$\{(0,0)(\pi,\pi)\}$	$\{(z_1, 0)\}$	2
$\mathbb{Z}_2(\kappa) \times \mathbb{Z}_2^c$	$\{\kappa, [(0,0)(\pi,\pi)]\}$	$\{\kappa, [(0,0)(\pi,\pi)]\}$	$\{\kappa, [(0,0)(\pi,\pi)]\}$	$\{(z_1, z_1)\}$	2
$\mathbb{Z}_2(\rho) \times \mathbb{Z}_2^c$	$\{\rho, [(0,0)(\pi,\pi)]\}$	$\{\rho, [(0,0)(\pi,\pi)]\}$	$\{\rho, [(0,0)(\pi,\pi)]\}$	$\{(z_1, e^{2\pi i/4} z_1)\}$	2
$\mathbb{Z}_2^c$	$\{(0,0)(\pi,\pi)\}$	$\{(0,0)(\pi,\pi)\}$	$\{(0,0)(\pi,\pi)\}$	$\mathbb{C}^2$	4

Theorem 4. *Consider the normal form (11). If $|B|^2 \geq |C|^2$, then the system admits a branch of periodic solutions that bifurcate from $(0,0)$ with the symmetry of $\mathbb{Z}_2^c$.*

Proof. We begin by introducing a change of coordinates: $z_1 = r_1 e^{i\phi_1}$, $z_2 = r_1 e^{i\phi_2}$ into (11) to obtain the following equations at the equilibrium:

$$\begin{aligned}
&(\lambda + i\omega) r_1 e^{i\phi_1} + (A r_1^2 + r_2^2 + B r_1^2) r_1 e^{i\phi_1} \\
&\quad + C r_1 r_2^2 e^{i(2\phi_2 - \phi_1)} = 0, \\
&(\lambda + i\omega) r_1 e^{i\phi_1} + (A r_1^2 + r_2^2 + B r_2^2) r_2 e^{i\phi_2} \\
&\quad + C r_1^2 r_2 e^{i(2\phi_1 - \phi_2)} = 0,
\end{aligned}$$

which in turn yield

$$\begin{aligned}
&(\lambda + i\omega) + (A r_1^2 + r_2^2 + B r_1^2) + C r_2^2 e^{2i(\phi_2 - \phi_1)} = 0, \\
&(\lambda + i\omega) + (A r_1^2 + r_2^2 + B r_1^2) + C r_1^2 e^{2i(\phi_1 - \phi_2)} = 0.
\end{aligned} \tag{12}$$

Let $\phi = \phi_2 - \phi_1$. Then, combining the two equations in (12) we obtain

$$r_1^2 (B - C e^{2\phi i}) = r_2^2 (B - C e^{-2\phi i}). \tag{13}$$

It follows from (13) that

$$\begin{aligned}
r^2 = \frac{r_2^2}{r_1^2} &= \frac{B - C e^{2\phi i}}{B - C e^{-2\phi i}} \\
&= \frac{(B - C e^{2\phi i})\overline{(B - C e^{-2\phi i})}}{|B - C e^{-2\phi i}|^2} \\
&= \frac{(B - C e^{2\phi i})(\overline{B} - \overline{C} e^{2\phi i})}{|B - C e^{-2\phi i}|^2} \geq 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathcal{I}m((B - C e^{2\phi i})(\overline{B} - \overline{C} e^{2\phi i})) = 0 \quad \text{and} \\
&\mathcal{R}e((B - C e^{2\phi i})(\overline{B} - \overline{C} e^{2\phi i})) \geq 0.
\end{aligned} \tag{14}$$

Let $B = B_r + iB_i$ and $C = C_r + iC_i$. Then, Eq. (14) yields

$$\begin{aligned}
&2(C_r^2 + C_i^2)\cos(2\phi) = 2(B_i C_i + B_r C_r), \\
&B_r^2 + B_i^2 - 2(B_i C_i + B_r C_r)\cos(2\phi) \\
&\qquad + (C_r^2 + C_i^2)(\cos(2\phi)^2 - \sin(2\phi)^2) \geq 0.
\end{aligned} \tag{15}$$

Substituting the first equation in (15) into the second, we obtain

$$|B|^2 \geq |C|^2. \tag{16}$$

Therefore, there always exist solutions to (11) in the form of $z_1 = r_1 e^{i\phi_1}$, $z_2 = r_1 e^{i\phi_2}$ as long as the condition (16) is satisfied, and they do not depend on ϕ_1, ϕ_2. In particular, we analyze the action of the group $\mathbb{Z}_2^c$ on $z_1 = r_1 e^{i\phi_1}$, $z_2 = r_2 e^{i\phi_2}$ under the condition (16). Recall that $\mathbb{Z}_2^c = \{(0,0), (\pi,\pi)\}$. Thus,

$$\begin{aligned}
(\pi,\pi) \cdot (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}) &= (r_1 e^{i(\phi_1 + 2\pi)}, r_2 e^{i\phi_2}) \\
&= (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}), \\
(0,0) \cdot (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}) &= (r_1 e^{i\phi_1}, r_2 e^{i\phi_2}).
\end{aligned}$$

In consequence, the solution is fixed by $\mathbb{Z}_2^c$ for any ϕ_1, ϕ_2. ■

Moreover, we have the following theorem.

Theorem 5. *For the representation of the group $\mathbb{D}_4$ on $\mathbb{C}^2$, the pairs of subgroups (H, K), satisfying the conditions of the H mod K Theorem, occur as spatiotemporal and spatial symmetries, respectively, of periodic solutions arising through a primary Hopf bifurcation, as shown in Table 5.*

Proof. First, we discuss why the mentioned pairs (H, K) are admissible. For the case (i), the subgroup $\mathbb{Z}_2(\kappa)$ of $\mathbb{D}_4(\zeta, \kappa)$ projects into the subgroup $\mathbb{Z}_2(\kappa) \times \mathbb{Z}_2^c$ of $\mathbb{D}_4 \times \mathbb{S}^1$. The conditions (a)–(d) of the H mod K Theorem (in Theorem 2) are satisfied.

Table 5. Spatiotemporal symmetries of the solutions arising from the trivial equilibrium due to primary Hopf bifurcation for the action of $\mathbb{D}_4 \times \mathbb{S}^1$ on $\mathbb{C}^2$ with the corresponding solution types.

Case	H	K	Solution Types
(i)	$\mathbb{Z}_2(\kappa)$	$\mathbb{Z}_2(\kappa)$	$x_1(t), x_1(t), x_2(t), x_2(t)$
(ii)	$\mathbb{Z}_2(\rho)$	$\mathbb{Z}_2(\rho)$	$x_1(t), x_2(t), x_3(t), x_2(t)$
(iii)	$\mathbb{Z}_4(\zeta)$	$\mathbb{1}$	$x_1(t), x_1\left(t+\frac{1}{4}\right), x_1\left(t+\frac{2}{4}\right), x_1\left(t+\frac{3}{4}\right)$
(iv)	$\mathbb{1}$	$\mathbb{1}$	$x_1(t), x_2(t), x_3(t), x_4(t)$

Since $\mathbb{Z}_2(\kappa) \times \mathbb{Z}_2^c$ has two-dimensional fixed-point subspace (see Table 4), there are periodic solutions arising from a primary Hopf bifurcation with the corresponding symmetry. A similar analysis can be applied to the case (ii) for the subgroup $\mathbb{Z}_2(\rho)$. The case (iii) corresponds to the projection of the subgroup $\mathbb{Z}_4(\zeta)$ of $\mathbb{D}_4(\zeta, \kappa)$ acting on $\mathbb{C}^2$ into the corresponding cyclic subgroup $\widetilde{\mathbb{Z}}_4$ of $\mathbb{D}_4 \times \mathbb{S}^2$. Since it is **C**-axial, there is a periodic solution arising from a Hopf bifurcation, with symmetry $\mathbb{Z}_4$. Finally, the Case (iv) corresponds to the projection of the identity subgroup of $\mathbb{D}_4$, into the isotropy subgroup $\mathbb{Z}_2^c$ of $\mathbb{D}_4 \times \mathbb{S}^1$. In this case, it follows from Theorem 4 that there are periodic solutions arising from a primary Hopf bifurcation, with symmetry $\mathbb{Z}_2^c$.

Next, we analyze why the remaining (H, K) pairs in Table 3 cannot correspond to periodic solutions from primary Hopf bifurcations. The first six items in Table 3 correspond to $H = \mathbb{D}_4(\kappa, \rho)$, which is not an isotropy subgroup of $\mathbb{D}_4 \times \mathbb{S}^1$. Similarly, the next four items in Table 3 correspond to $H = \mathbb{D}_2(\kappa, \rho)$, which is also not an isotropy subgroup of $\mathbb{D}_4 \times \mathbb{S}^1$. In the 13th item in Table 3, we have $H = K = \mathbb{Z}_4(\zeta)$. In this case, K cannot be $K = \mathbb{Z}_4(\zeta)$ since this isotropy subgroup has a one-dimensional fixed-point subspace, which does not satisfy hypothesis (c) of the H mod K Theorem. The 14th item in Table 3 corresponds to $(H, K) = (\mathbb{Z}_4(\zeta), \mathbb{Z}_2(\rho))$. Since in this case K has a two-dimensional fixed-point subspace, it has to be either $H = K$ or $H = N(K)$, where $N(K)$ is the normalizer of K in $\mathbb{D}_4$. For the former, $H = K$ is not possible because $H = \mathbb{Z}_4(\zeta) \neq \mathbb{Z}_2(\rho)$; and the latter is also not possible because $N(K) = \mathbb{D}_2(\kappa, \rho) \neq \mathbb{Z}_4(\zeta)$.

The cases $(H, K) = (\mathbb{Z}_2(\kappa), \mathbb{1})$ and $(H, K) = (\mathbb{Z}_2(\rho), \mathbb{1})$ in Table 5 are more interesting. Their analysis is similar and so we only study one of them, say $(H, K) = (\mathbb{Z}_2(\kappa), \mathbb{1})$. If $X(t) = (z_1(t), z_2(t))$ is a 2π-periodic solution of a $\mathbb{D}_4$-invariant differential equation, with $H = \mathbb{Z}_2(\kappa)$, $K = \mathbb{1}$, then for some $\theta \neq 0 \pmod{2\pi}$, and for all $t > 0$ we have $z_1(t + \theta) = z_1(t)$ and $z_2(t) = z_2(t + 2\theta)$. If one of $z_1(t)$ and $z_2(t)$ is identically zero, then it follows from the isotropy subgroups of $\mathbb{D}_4 \times \mathbb{S}^1$ in Table 5 that the fixed-point subspaces of all isotropy subgroups are zero-dimensional. The only other possibility is that all coordinates of $X(t)$ are nonzero, i.e. $X(t) = (z_1(t), z_2(t + \theta))$ where $\theta = \frac{\rho}{2}$, with θ-periodic function z_1 and 2θ-periodic function $z_2(t)$. In this case, if $X(t)$ bifurcates from an equilibrium in the normal form (11), then at the bifurcation point the eigenvalues would be $\pm 2\pi i/\theta$ and $\pm \pi i/\theta$, leading to 2:1 resonance. This last situation is not compatible with the normal form (11), since the linearization of (11) at the origin is a (complex) multiple of the identity.

For the isotropy subgroups $K = \mathbb{1}$, with four-dimensional fixed-point subspaces, we need to verify condition (d) of the H mod K Theorem. For every $\gamma \in \Gamma$, it is clear that $\dim \mathrm{Fix}(\gamma)$ is even, and thus L_K is always the union of a finite number of subspaces with even dimension. Hence, L_K has even codimension in $\mathrm{Fix}(K)$ and consequently, $\mathrm{Fix}(K) \backslash L_K$ is connected and condition (d) in the H mod K Theorem is applicable to all isotropy subgroups with four-dimensional fixed-point subspaces. ■

4. Case Study: Two Coupled van der Pol Oscillators

In this section, we apply the theory established in the previous sections and use normal form theory to consider the following two coupled planar van der Pol oscillators,

$$\begin{aligned} \ddot{x} &= -x + (\lambda - x^2 - \alpha w^2)\dot{x}, \\ \ddot{w} &= -w + (\lambda - w^2 - \alpha x^2)\dot{w}, \end{aligned} \tag{17}$$

where (x, w) are the coordinates in the plane $\mathbb{R}^2$. System (17) is equivariant under the $\mathbb{D}_4$ action, generated by $(x, w) \mapsto (-x, w)$ and $(x, w) \mapsto (w, x)$.

The two coupled van der Pol oscillators have been studied for several decades (e.g. see [Rand & Holmes, 1980; Paccosi *et al.*, 2014]), and found in real applications [Kawahara, 1980; Camacho *et al.*, 2014]. Even three coupled van der Pol oscillators [Uwate *et al.*, 2010] and four coupled van der Pol oscillators [Barrón & Sen, 2009] have also been explored

To initiate the linear analysis of this system, it is useful to introduce new variables, $x = x_1$, $\dot{x} = x_2$, $w = x_3$, $\dot{w} = x_4$, into system (17) to obtain

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= -x_1 + (\lambda - x_1^2 - \alpha x_3^2)x_2,\\
\dot{x}_3 &= x_4,\\
\dot{x}_4 &= -x_3 + (\lambda - x_3^2 - \alpha x_1^2)x_4.
\end{aligned} \tag{18}$$

We observe that the unique equilibrium point of system (18) is given by

$$\mathrm{E}_0 : (x_1, x_2, x_3, x_4) = (0, 0, 0, 0). \tag{19}$$

The Jacobian matrix of system (18) evaluated at E_0 has two pairs of complex conjugate eigenvalues:

$$\frac{1}{2}(\lambda - \sqrt{-4+\lambda^2})\ \text{(twice)} \quad \text{and}$$
$$\frac{1}{2}(\lambda + \sqrt{-4+\lambda^2})\ \text{(twice)}.$$

Therefore, at $\lambda = 0$, the four eigenvalues will cross the imaginary axis simultaneously and the linearized system behaves as two coupled oscillators. However, it can be shown that at the critical point $\lambda = 0$, the origin of system (18) is actually globally asymptotically stable if $\alpha > 0$. To achieve this, we construct the Lyapunov function: $V = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$, and hence along the trajectory of system (18) we have $\frac{dV}{dt}|_{(18)} = -[(x_1^2 + \alpha x_3^2)x_2^2 + (x_3^2 + \alpha x_1^2)x_4^2] \le 0$. The equal sign holds only if $x_1 = x_3 = 0$ or $x_2 = x_4 = 0$; but none of them is the solution of (18) except the trivial equilibrium solution $x_1 = x_2 = x_3 = x_4 = 0$. Thus, by applying the LaSalles principle [LaSalle, 1976], we conclude that the origin of system (18) is globally asymptotically stable.

The corresponding normal form of system (18), given in polar coordinates, can be obtained as [Bi & Yu, 1999; Yu, 2003]:

$$\begin{aligned}
\dot{r}_1 &= r_1\left[\frac{1}{2}\lambda - \frac{1}{8}r_1^2 - \frac{1}{8}\alpha(2-\cos 2\phi)r_2^2 + \cdots\right],\\
\dot{r}_2 &= r_2\left[\frac{1}{2}\lambda - \frac{1}{8}r_2^2 - \frac{1}{8}\alpha(2-\cos 2\phi)r_1^2 + \cdots\right],\\
\dot{\theta}_1 &= 1 - \frac{1}{8}\alpha r_2^2 \sin 2\phi - \frac{11}{256}r_1^4\\
&\quad - \frac{\alpha}{128}[3(4-\alpha) - 2(1-2\alpha)\cos 2\phi]r_1^2 r_2^2\\
&\quad - \frac{\alpha}{256}(9\alpha + 2\cos 2\phi)r_2^4 + \cdots,\\
\dot{\theta}_2 &= 1 + \frac{1}{8}\alpha r_1^2 \sin 2\phi - \frac{11}{256}r_2^4\\
&\quad - \frac{\alpha}{128}[3(4-\alpha) - 2(1-2\alpha)\cos 2\phi]r_1^2 r_2^2\\
&\quad - \frac{\alpha}{256}(9\alpha + 2\cos 2\phi)r_1^4 + \cdots,
\end{aligned} \tag{20}$$

where $\phi = \theta_1 - \theta_2$, and $\cdots$ represents higher-order terms.

4.1. *Periodic and phase-locked periodic solutions*

Consider the normalized system (18). First, it is easy to see that there are three invariant manifolds, described by

$$\begin{aligned}
&\mathrm{M}_1 : \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_3 = x_4 = 0\},\\
&\mathrm{M}_2 : \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 = x_2 = 0\},\\
&\mathrm{M}_3 : \{(x_1, x_2, x_3, x_4) \in R^4 \mid x_1 = x_3, x_2 = x_4\}.
\end{aligned} \tag{21}$$

Next, it is clear that the trivial equilibrium of (18), E_0, is asymptotically stable (unstable) for $\lambda < 0$ ($\lambda > 0$).

To find possible periodic solutions bifurcating from E_0, we use the normal form (20). Letting $r_2 = 0$ and ignoring the phase equation of θ_2, we obtain a family of periodic solutions due to a Hopf bifurcation, given by

$$\mathrm{P}_1 : (x_1, x_2, x_3, x_4) = (r\cos\theta, r\sin\theta, 0, 0), \tag{22}$$

where r and θ are described by the following normal form equations:

$$\dot{r} = \frac{1}{8}r(4\lambda - r^2 + \cdots), \quad \dot{\theta} = 1 - \frac{11}{256}r^4 + \cdots. \tag{23}$$

The periodic solution P_1 is embedded in the invariant manifold M_1. It can be seen from (23) that the Hopf bifurcation is supercritical and thus the bifurcating limit cycles, which exist for $\lambda > 0$, should be stable, restricted to the invariant manifold M_1. That is, for $\lambda > 0$, as long as the initial conditions are chosen in the form of $(x_1, x_2, x_3, x_4) = (x_1, x_2, 0, 0)$, where x_1 and x_2 are not simultaneously taken zero, the stable periodic solution remains on M_1. However, for the whole four-dimensional space, $(x_3, x_4) = (0, 0)$ is unstable due to $\lambda > 0$. Therefore, if at least one of x_3 and x_4 is chosen nonzero in the initial condition, the periodic solution P_1 becomes unstable.

Similarly, another family of periodic solutions due to another Hopf bifurcation is given by

$$\mathrm{P}_2 : (x_1, x_2, x_3, x_4) = (0, 0, r\cos\theta, r\sin\theta), \quad (24)$$

where r and θ are determined by the following normal form (23). For the same reason, when $\lambda > 0$, the periodic solution P_2 is stable, embedded in the invariant manifold M_2, if the initial condition is chosen in the form $(x_1, x_2, x_3, x_4) = (0, 0, x_3, x_4)$, with x_3 and x_4 not being simultaneously zero, but becomes unstable in the whole four-dimensional space when at least one of x_3 and x_4 is nonzero in the initial condition.

Further, a family of quasi-periodic solutions is given by

$$\begin{aligned}\mathrm{Q}_3 : (x_1, x_2, x_3, x_4) = (&r_1\cos\theta_1, r_2\sin\theta_1,\\ &r_2\cos\theta_2, r_2\sin\theta_2),\end{aligned} \quad (25)$$

where r_1, r_2, θ_1 and θ_2 are described by (20) with $r_1 r_2 \neq 0$.

One important set of the quasi-periodic solutions is the so-called phase-locked solutions, i.e. $\theta_1 - \theta_2 = \text{const}$ and that is why it is called "phase-locked". Therefore, combining the last two equations in (20) yields the following system:

$$\begin{aligned}
\dot r_1 &= \frac{1}{128} r_1[64\lambda - 16r_1^2 - 16\alpha(2-\cos 2\phi)r_2^2 + \cdots],\\
\dot r_2 &= \frac{1}{128} r_2[64\lambda - 16r_2^2 - 16\alpha(2-\cos 2\phi)r_1^2 + \cdots],\\
\dot\phi &= -\frac{1}{256}(r_1^2 + r_2^2)\{32\alpha\sin 2\phi + (r_1^2 - r_2^2)\\
&\quad \times [11 - \alpha(9\alpha + 2\cos 2\phi)]\} + \cdots.
\end{aligned} \quad (26)$$

Letting $\dot r_1 = \dot r_2 = \dot\phi = 0$ yields the truncated algebraic equations (for $r_1^2 + r_2^2 \neq 0$):

$$\begin{aligned}
&64\lambda - 16r_1^2 - 16\alpha(2-\cos 2\phi)r_2^2 = 0,\\
&64\lambda - 16r_2^2 - 16\alpha(2-\cos 2\phi)r_1^2 = 0,\\
&32\alpha\sin 2\phi + (r_1^2 - r_2^2)\\
&\quad \times [11 - \alpha(9\alpha + 2\cos 2\phi)] = 0.
\end{aligned} \quad (27)$$

Combining the first two equations of (27) results in

$$16(r_1^2 - r_2^2)[1 + \alpha(\cos 2\phi - 2)] = 0,$$

which in turn gives two solutions:

$$r_1 = r_2 \quad \text{or} \quad \cos 2\phi = 2 - \frac{1}{\alpha}. \quad (28)$$

First consider $r_1 = r_2$. For this case, the third equation of (27) yields $\sin 2\phi = 0$, which gives possible solutions $2\phi = 0, \pi$ and 2π. When $2\phi = 0, 2\pi$, $\cos 2\phi = 1$, and when $2\phi = \pi$, $\cos 2\phi = -1$. Thus, using the first equation of (27) we obtain

$$r_1 = r_2 = 2\sqrt{\frac{\lambda}{1+\alpha}}, \quad (\lambda > 0, \alpha > -1), \quad \text{for } \phi = 0, \pi \quad (29)$$

and

$$r_1 = r_2 = 2\sqrt{\frac{\lambda}{1+3\alpha}}, \quad \left(\lambda > 0, \alpha > -\frac{1}{3}\right), \quad \text{for } \phi = \frac{\pi}{2}. \quad (30)$$

The corresponding families of phase-locked solutions are approximated by

$$(x_1, x_2, x_3, x_4) = \begin{cases} 2\sqrt{\dfrac{\lambda}{1+\alpha}}(\cos\omega t, \sin\omega t, \cos\omega t, \sin\omega t), & \text{or}\\ 2\sqrt{\dfrac{\lambda}{1+\alpha}}(\cos(\omega t + \pi), \sin(\omega t + \pi), \\ \quad \cos\omega t, \sin\omega t), \end{cases} \quad (31)$$

where $\omega = 1 - \frac{11}{16}\lambda^2$ for $\alpha > -1$; and

$$(x_1, x_2, x_3, x_4) = 2\sqrt{\frac{\lambda}{1+3\alpha}}\left(\cos\left(\omega t + \frac{\pi}{2}\right), \sin\left(\omega t + \frac{\pi}{2}\right), \cos\omega t, \sin\omega t\right), \quad (32)$$

where $\omega = 1 - \frac{(11+26\alpha-5\alpha^2)\lambda^2}{16(1+3\alpha)^2}$ for $\alpha > -\frac{1}{3}$.

Note in (31) that the first solution with $\phi = 0$ is called an "in-phase" periodic solution, while the second solution with $\phi = \pi$ is called "out-phase" periodic solution. All the three phase-locked periodic solutions given in (31) and (32) are located on a 2D torus which is embedded in the invariant manifold M_3. All of these solutions imply that the two oscillators are synchronized. But only for the in-phase solution, the two oscillators are completely synchronized with both amplitude and phase. The second solution in (31) with $\phi = \pi$ and the solution in (32) with $\phi = \frac{\pi}{2}$ are only synchronized in amplitude. However, these two synchronized motions are not distinctive in the phase space as shown in Figs. 5 and 7.

It should also be pointed out that when $\alpha = 0$, the first two equations of (18) are decoupled from the last two equations of (18), which may yield the periodic solution P_1 or P_2, or the phase-locked solutions given in (31) and (32).

Next, consider $\cos 2\phi = 2 - \frac{1}{\alpha}$, for which the first equation of (27) yields

$$r_1^2 + r_2^2 = 4\lambda \Rightarrow r_2^2 = 4\lambda - r_1^2. \tag{33}$$

Substituting the above solution for r_2^2 and (28) for $\cos 2\phi$ into the third equation of (27) and solving for $\sin 2\phi$ we obtain

$$\sin 2\phi = \frac{-1}{16\alpha}(r_1^2 - 2\lambda)(1-\alpha)(9\alpha + 13). \tag{34}$$

Then, the identity $\sin^2 2\phi + \cos^2 2\phi = 1$ yields the following equation:

$$\begin{aligned} &\frac{1}{256\alpha^2}(1-\alpha)[(1-\alpha)(13+9\alpha)^2 r_1^4 \\ &\quad - 4\lambda(1-\alpha)(13+9\alpha)^2 r_1^2 \\ &\quad + 4\lambda^2(1-\alpha)(13+9\alpha)^2 + 256(1-3\alpha)] \\ &= 0. \end{aligned} \tag{35}$$

First note that $\alpha = 1$ satisfies (35), for which $\cos 2\phi = 1$ and $\sin 2\phi = 0$, which indicates that this phase-locked solution is either in-phase ($\phi = 0$) or out-phase ($\phi = \pi$). Thus, the phase-locked solutions are given by (for $\alpha = 1$)

$$\begin{aligned} &(x_1, x_2, x_3, x_4) \\ &= \begin{cases} (r_1 \cos\omega t, r_1 \sin\omega t, r_2 \cos\omega t, r_2 \sin\omega t), & \text{or} \\ (r_1 \cos(\omega t + \pi), r_1 \sin(\omega t + \pi), \\ \qquad r_2 \cos\omega t, r_2 \sin\omega t), \end{cases} \end{aligned} \tag{36}$$

where $\omega = 1 - \frac{11}{16}\lambda^2$, $r_1^2 + r_2^2 = 4\lambda$. Note that the particular solution of (36) for $r_1 = r_2$ is included in (31).

Now, consider $\alpha \neq 1$. The phase-locked solutions can be solved from the second factor in the square bracket of (35), which is a quadratic polynomial in r_1^2. The discriminant of the polynomial is $\Delta = -1024(3\alpha - 1)(\alpha - 1)(13 + 9\alpha)^2$, which is required to be non-negative in order to get real solutions from the polynomial, yielding $\frac{1}{3} \le \alpha < 1$, which guarantees that the quadratic polynomial has positive solutions:

$$\begin{aligned} &(r_1^{\pm}, r_2^{\mp}) = (\sqrt{2(\lambda \pm \lambda^*)}, \sqrt{2(\lambda \mp \lambda^*)}), \quad \text{where} \\ &\lambda^* = \frac{8}{13 + 9\alpha}\sqrt{\frac{3\alpha - 1}{1 - \alpha}}. \end{aligned} \tag{37}$$

For $\alpha = \frac{1}{3}$, $r_1 = r_2 = \sqrt{2\lambda}$ for which $\sin 2\phi = 0$, $\cos 2\phi = -1$ yielding $\phi = \frac{\pi}{2}$, and so the phase-locked solution is given by

$$\begin{aligned} (x_1, x_2, x_3, x_4) = \sqrt{2\lambda}\Big(&\cos\Big(\omega t + \frac{\pi}{2}\Big), \\ &\sin\Big(\omega t + \frac{\pi}{2}\Big), \cos\omega t, \sin\omega t\Big), \end{aligned} \tag{38}$$

where $\omega = 1 - \frac{43}{144}\lambda^2$. However, it is easy to see that the above solution is actually included in the solution given in (32).

Finally, for $\frac{1}{3} < \alpha < 1$, the positivity of r_1 and r_2 requires $\lambda > \lambda^*$. For this case, we have two sets of phase-locked solutions:

$$\begin{aligned} (x_1, x_2, x_3, x_4) = (&r_1^{\pm}\cos(\omega t \pm \phi), r_1^{\pm}\sin(\omega t \pm \phi), \\ &r_2^{\mp}\cos\omega t, r_2^{\mp}\sin\omega t), \end{aligned} \tag{39}$$

where

$$\begin{aligned} \omega = &-\frac{1}{64(13+9\alpha)^2}[(19\alpha^2 + 12\alpha + 13) \\ &\times (13+9\alpha)^2\lambda^2 - 512(\alpha+5)(3\alpha+5)], \end{aligned} \tag{40}$$

$$\phi = \frac{1}{2}\tan^{-1}\left[\frac{\sqrt{(1-\alpha)(3\alpha-1)}}{1-2\alpha}\right].$$

Summarizing the above results we have the following theorem.

Theorem 6. *For system (18), the periodic solutions are given by (22)–(24); and the phase-locked periodic solutions are given by (29)–(32) for equal amplitudes in the two oscillators, and the second equation in (28), (33), (34), (36)–(40) for different amplitudes in the two oscillators.*

4.2. *Stability analysis*

It has been shown that the trivial equilibrium E_0 is stable (unstable) for $\lambda < 0$ ($\lambda > 0$), and that the periodic solutions P_1 and P_2 are stable for $\lambda > 0$, if restricted to the invariant manifolds M_1 and M_2, respectively.

In order to find stability of the phase-locked solutions, we use the Jacobian matrix of (26), given by

$$J(r_1, r_2, \phi) = \begin{bmatrix} \frac{\lambda}{2} - \frac{3r_1^2}{8} - \frac{1}{8}\alpha r_2^2(2 - \cos 2\phi) & -\frac{1}{4}\alpha r_1 r_2(2 - \cos 2\phi) & -\frac{1}{4}\alpha r_1 r_2^2 \sin 2\phi \\ -\frac{1}{4}\alpha r_1 r_2(2 - \cos 2\phi) & \frac{\lambda}{2} - \frac{3r_2^2}{8} - \frac{1}{8}\alpha r_1^2(2 - \cos 2\phi) & -\frac{1}{4}\alpha r_1^2 r_2 \sin 2\phi \\ -C_1 r_1 - C_2 & -C_1 r_2 + C_2 & C_3 \end{bmatrix}, \quad (41)$$

where

$$C_1 = \frac{1}{128}[32\alpha \sin 2\phi + (r_1^2 - r_2^2) \times (11 - 9\alpha^2 - 2\alpha \cos 2\phi)],$$

$$C_2 = \frac{1}{128}(r_1^2 + r_2^2)(11 - 9\alpha^2 - 2\alpha \cos 2\phi),$$

$$C_3 = -\frac{\alpha}{64}(r_1^2 + r_2^2)[16 \cos 2\phi + (r_1^2 - r_2^2) \sin 2\phi].$$

First, consider solution (31). Evaluating the Jacobian J on solution (29) we obtain the characteristic polynomial,

$$P_1(\xi) = (\xi + \lambda)\left(\xi + \frac{2\lambda\alpha}{1+\alpha}\right)\left(\xi + \frac{\lambda(1-\alpha)}{1+\alpha}\right),$$

which shows that the phase-locked solution (31) is stable for $-1 < \alpha < 1$, and whether the solution is in-phase or out-phase, depending upon the initial conditions, as illustrated in the simulations.

Next, consider solution (32). Similarly, evaluating the Jacobian on solution (30) yields the characteristic polynomial,

$$P_2(\xi) = (\xi + \lambda)\left(\xi + \frac{\lambda(1-3\alpha)}{1+3\alpha}\right)\left(\xi - \frac{2\lambda\alpha}{1+3\alpha}\right).$$

Thus, for $-\frac{1}{3} < \alpha < 0$, the phase-locked solution (32) with phase difference $\frac{\pi}{2}$ is stable.

For $\alpha = 1$, it is easy to get the characteristic polynomial $P_3(\xi) = \xi\,(\xi + \lambda)^2$, which is a critical case. For $\alpha = \frac{1}{3}$, we obtain the characteristic polynomial $P_4(\xi) = \xi(\xi + \lambda)(\xi - \frac{\lambda}{3})$, showing that the phase-locked solution is unstable for $\alpha = \frac{1}{3}$.

Finally, for $\frac{1}{3} < \alpha < 1$, evaluating the Jacobian on the phase-locked solution (39), we obtain the characteristic polynomial,

$$P_5(\xi) = \xi^3 + \frac{2\lambda(9\alpha+1)(\alpha+1)}{9\alpha+13}\xi^2 + \frac{\lambda^2(18\alpha^2 + 11\alpha - 11)}{9\alpha+13}\xi - \frac{1}{2}\lambda(3\alpha - 1)(1 - \alpha)(\lambda^2 - \lambda^{*2}).$$

Since the existence of this solution requires $\lambda > \lambda^*$, this implies that $P_5(\xi)$ is an unstable polynomial. Hence, the phase-locked solution (39) is unstable when it exists for $\frac{1}{3} < \alpha < 1$.

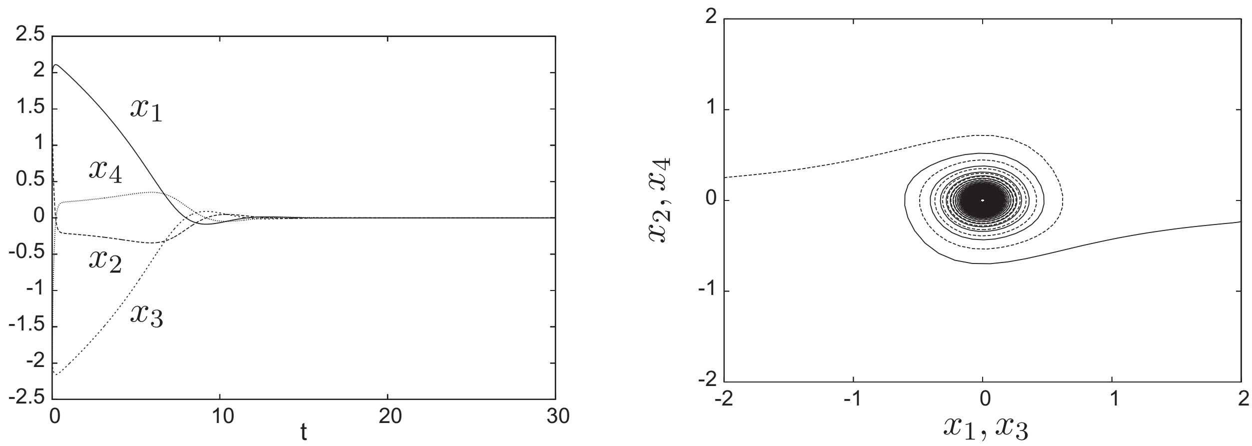
We summarize the above results to obtain the following result.

Theorem 7. *For system (18), the periodic solutions* P_1 *and* P_2 *given in (22)–(24) are stable for* $\lambda > 0$, *if restricted to the invariant manifolds* M_1 *and* M_2, *respectively. The phase-locked periodic solutions given by (29) and (31) with phase difference either* 0 *or* π *are stable for* $\alpha \in (-1, 1)$, *while that given by (30) and (32) with phase difference* $\frac{\pi}{2}$ *is stable for* $\alpha \in (-\frac{1}{3}, 0)$.

5. Simulations

In this section, we present simulations to verify the results we obtained in the previous section. First, we depict the cases for $\lambda \le 0$ with $\alpha = 1$, and two values for $\lambda : \lambda = -1$ and $\lambda = 0$. The simulations are given in Fig. 3, showing the convergence to the origin for both cases.

Next, the stable periodic solutions P_1 and P_2 for the parameter values $\alpha = 1$, $\lambda = 1$ are shown in Figs. 4(a) and 4(b) with the initial conditions



(a) (b)

Fig. 3. Simulated trajectories of system (18) for $\alpha = 1$ with the initial condition $(x_1, x_2, x_3, x_4) = (2, 1.5, -2, -2)$, converging to the origin when (a) $\lambda = -1$ and (b) $\lambda = 0$.

$(x_1, x_2, x_3, x_4) = (2, -1.5, 0, 0)$ and $(x_1, x_2, x_3, x_4) = (0, 0, 2, -1.5)$, respectively.

For solution (31) which is stable for $\alpha \in (-1, 1)$, we take the following two sets of parameter values for simulation:

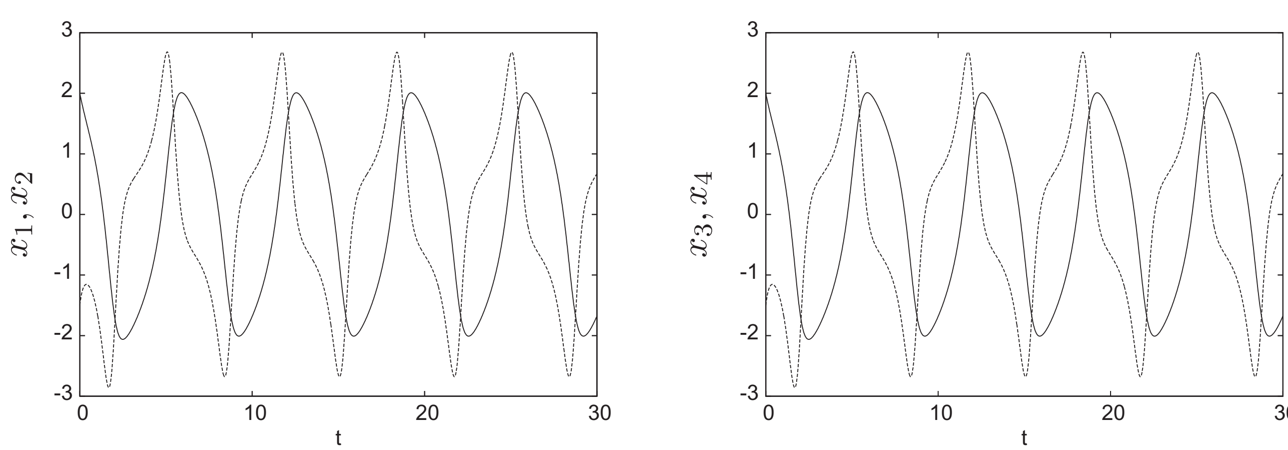
$$(\alpha, \lambda) = (0.5, 0.5) \quad \text{and} \quad (-0.5, 0.5),$$

with the initial conditions:

$$(x_1, x_2, x_3, x_4) = \begin{cases} (2, 1, 2.5, 1.5), & \text{for in-phase motion,} \\ (2, 1, -2.5, 1.5), & \text{for out-phase motion.} \end{cases}$$

The simulations for $(\alpha, \lambda) = (0.5, 0.5)$ are shown in Fig. 5, where the in-phase phase-locked solution is depicted in Figs. 5(a) and 5(b); while the out-phase phase-locked solution is given in Figs. 5(c) and 5(d). The corresponding phase portraits for the in-phase and out-phase locked solutions are shown in Figs. 6(a) and 6(b), respectively.

For the parameter values of $(\alpha, \lambda) = (-0.5, 0.5)$, we have found that if we use the same initial condition $(x_1, x_2, x_3, x_4) = (2, 1, 2.5, 1.5)$ or $(x_1, x_2, x_3, x_4) = (2, 1, -2.5, 1.5)$, the phase-locked solution (31) is unstable. However, as long as the initial condition is chosen such that $x_1 = x_3$, $x_2 = x_4$, the trajectory converges to the phase-locked solution. For example, with the initial condition $(x_1, x_2, x_3, x_4) = (2, 1, 2, 1)$, we obtain the in-phase solution as shown in Figs. 7(a) and 7(b); and



(a) (b)

Fig. 4. Simulated trajectories of system (18) for $\alpha = 1$, $\lambda = 1$: (a) converging to the periodic solution P_1 on the invariant manifold M_1, with the initial condition $(x_1, x_2, x_3, x_4) = (2, -1.5, 0, 0)$ and (b) converging to the periodic solution P_2 on the invariant manifold M_2, with the initial condition $(x_1, x_2, x_3, x_4) = (0, 0, 2, -1.5)$.

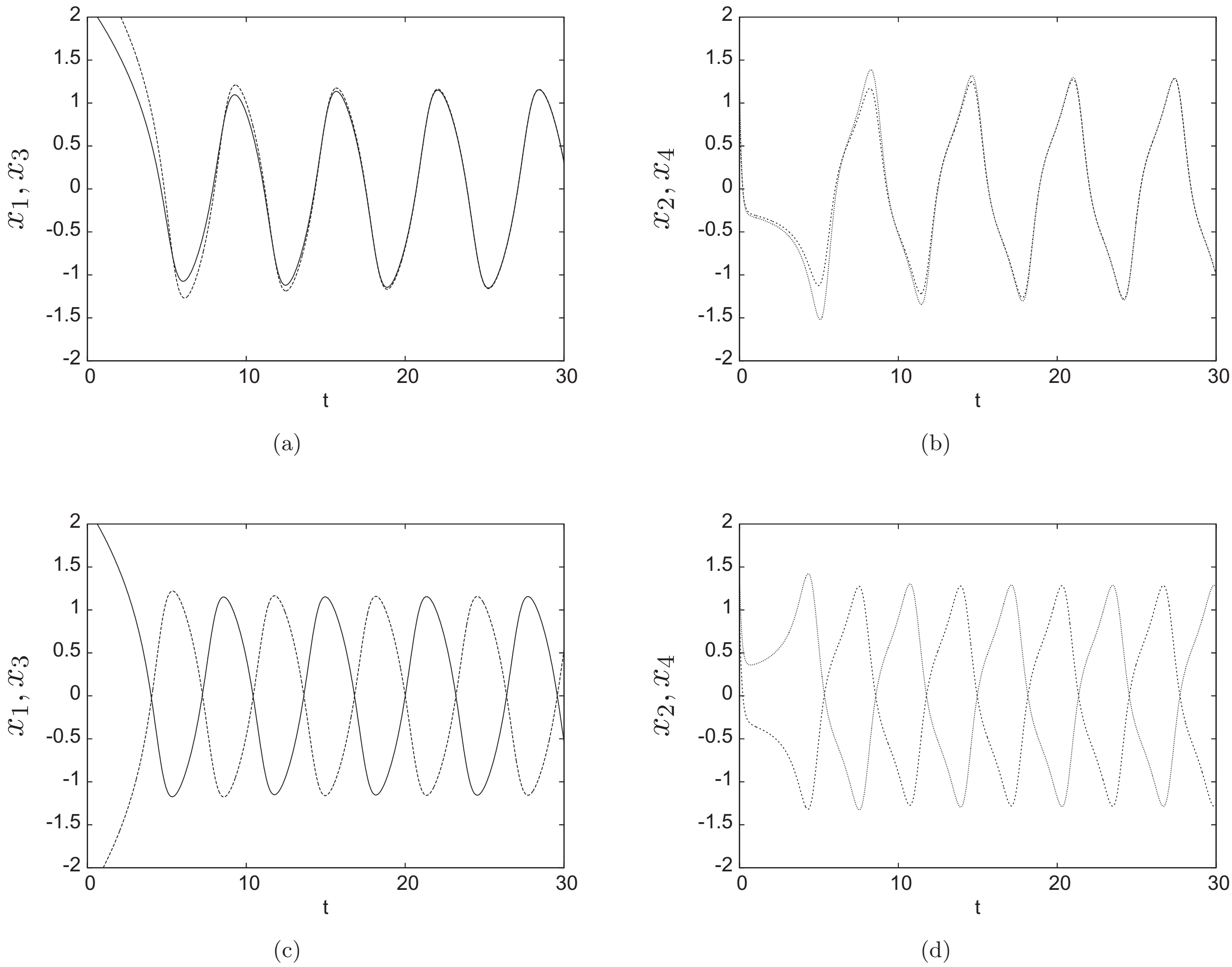


Fig. 5. Simulated phase-locked solutions of system (18) for $\alpha = 0.5$, $\lambda = 0.5$: (a) and (b) in-phase solution with the initial condition $(x_1, x_2, x_3, x_4) = (2, 1, 2.5, 1.5)$, (c) and (d) out-phase solution with the initial condition $(x_1, x_2, x_3, x_4) = (2, 1, -2.5, 1.5)$.

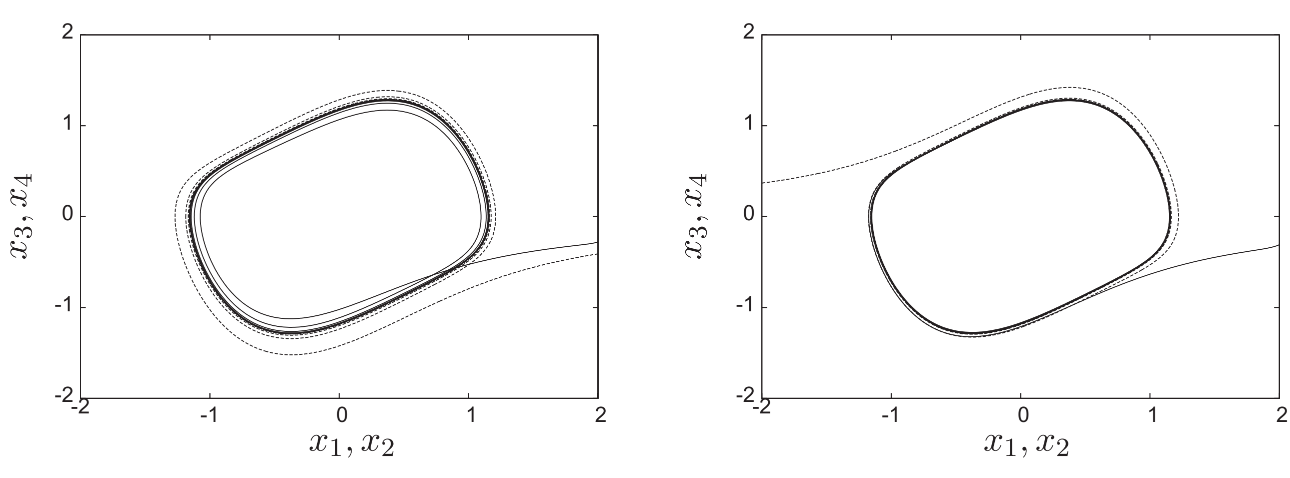
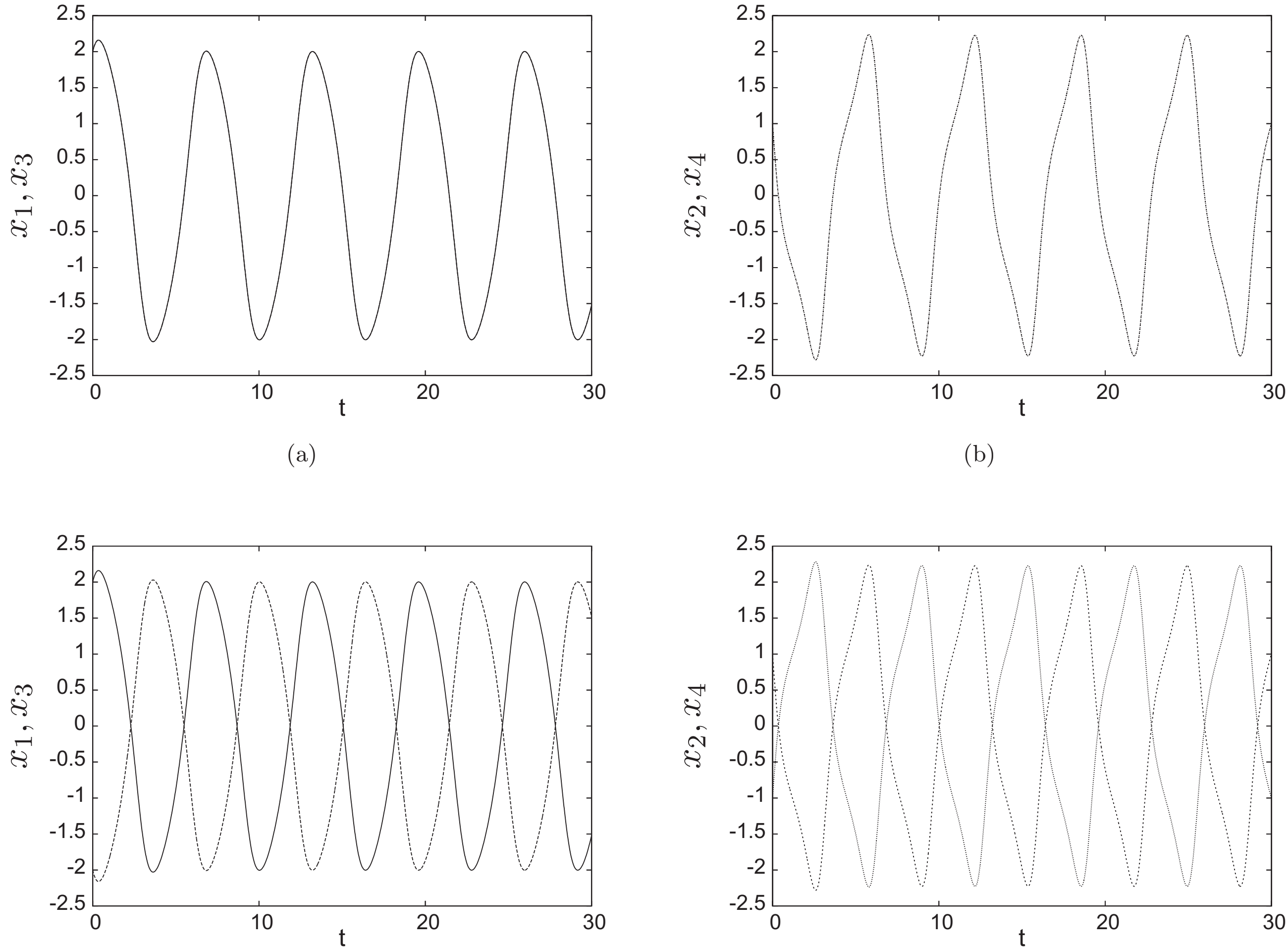


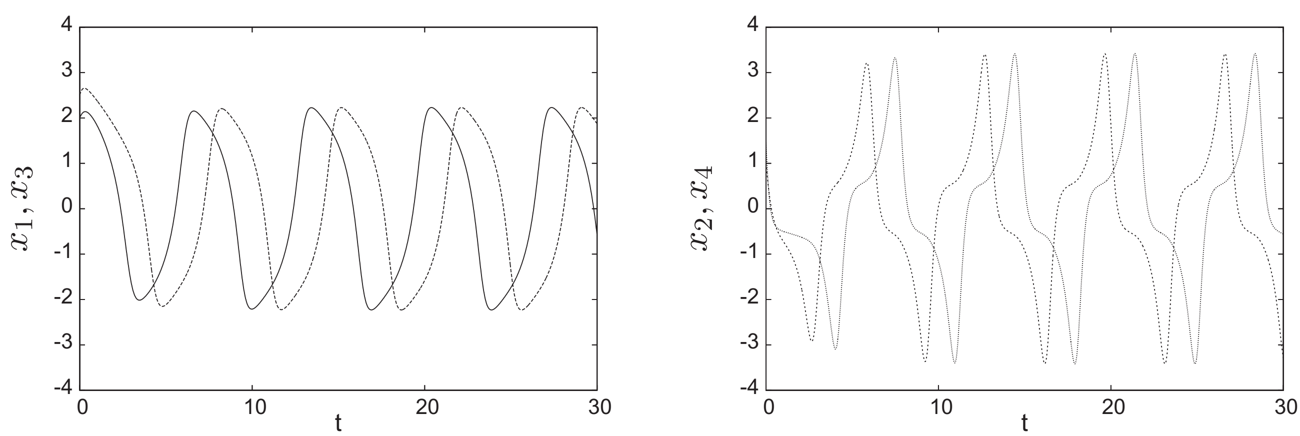
Fig. 6. Simulated phase portraits of system (18) for $\alpha = 0.5$, $\lambda = 0.5$: (a) the in-phase solution in Figs. 5(a) and 5(b) and (b) the out-phase solution in Figs. 5(c) and 5(d).



(a) (b)

(c) (d)

Fig. 7. Simulated phase-locked solutions of system (18) for $\alpha = -0.5$, $\lambda = 0.5$: (a) and (b) the in-phase solution with the initial condition $(x_1, x_2, x_3, x_4) = (2, 1, 2, 1)$, (c) and (d) the out-phase solution with the initial condition $(x_1, x_2, x_3, x_4) = (2, 1, -2, -1)$.



(a) (b)

Fig. 8. Simulated phase-locked solutions of system (18) with phase difference $\frac{\pi}{2}$ for $\alpha = -0.2$, $\lambda = 0.5$ using the initial condition $(x_1, x_2, x_3, x_4) = (2, 1, 2.5, 1.5)$.

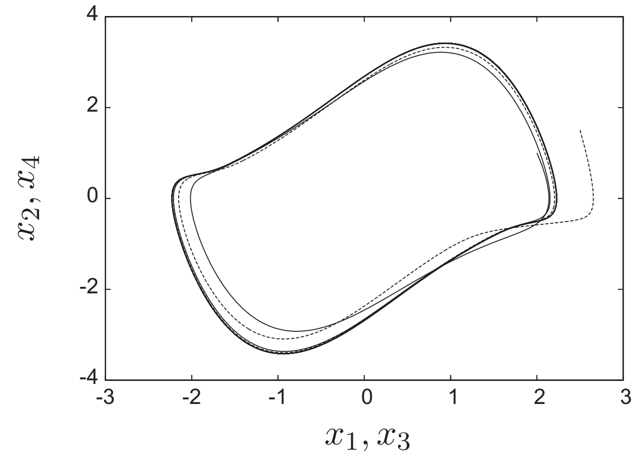


Fig. 9. Simulated phase portrait of system (18) with phase difference $\frac{\pi}{2}$ for $\alpha = -0.2$, $\lambda = 0.5$, corresponding to the time history given in Fig. 8.

the out-phase solution as shown in Figs. 7(c) and 7(d) by using the initial condition $(x_1, x_2, x_3, x_4) = (2, 1, -2, -1)$.

For the phase-locked solution (32) which is stable for $-\frac{1}{3} < \alpha < 0$, we choose $\alpha = -0.2$ and $\lambda = 0.5$. Note that this set of values also satisfy the phase-locked solution (31). Thus, for this set of values, by using appropriated initial conditions, we may get three possible phase-locked solutions with the phase differences: $\frac{\pi}{2}$ as well as 0 and π. It has been noted that the trajectory converges to the phase-locked solution (32) with any initial conditions, see the simulated time history and phase portrait given in Figs. 8 and 9, respectively; while the trajectory converges to the phase-locked solution (31) with $\phi = 0$ if the initial condition satisfies $x_1 = x_3$, $x_2 = x_4$, but with $\phi = \pi$ if $x_1 = -x_3$, $x_2 = -x_4$, as shown in Fig. 7.

Summarizing the above results we obtain that for $\alpha \in (0, 1)$, the phase-locked solution (31) with phase difference $\phi = 0$ or π is stable for any initial conditions (see the example shown in Fig. 5); while for $\alpha \in (-\frac{1}{3}, 0)$, the phase-locked solution (32) with phase difference $\phi = \frac{\pi}{2}$ is stable for any initial conditions (see the example shown in Fig. 8).

6. Conclusion

In this paper, we have studied the oscillatory behavior of $\mathbb{D}_4$-equivariant systems. Starting from the Cayley graph of the $\mathbb{D}_4$ group, in Theorem 1 we have shown the existence of heteroclinic cycles in the most generic $\mathbb{D}_4$-equivariant systems with the weak-coupling assumption. Then, we applied the H mod K Theorem and the Equivariant Hopf Theorem to analyze the oscillation patterns of the periodic solutions in a four-dimensional $\mathbb{D}_4$-equivariant dynamical system. We are the first to provide (in Theorem 5) the corresponding pair (H, K) of spatiotemporal/spatial symmetries to each of these patterns. We identified three invariant manifolds and are first to obtain three types of phase-locked synchronized quasi-periodic motions, each of them is located on a 2D torus, embedded in one of the invariant manifolds. We have derived explicit expressions for each of the phase-locked solutions and their stability conditions in terms of the system parameters. In addition, we have shown that although all the phase-locked solutions imply that the two oscillators are synchronized, only the in-phase solutions are completely synchronized with both amplitude and phase. Simulations are presented to verify the analytical predictions.

Stimulated by this work, in our future research the methodology developed in this paper can be generalized to study the oscillatory behavior of systems with other symmetries such as the $\mathbb{D}_n$ group (with n odd), and the $\mathbb{Z}_m \times \mathbb{Z}_n$ group, (with m and n coprimes).

Acknowledgment

P. Yu would like to acknowledge the funding support for this research, received from the Natural Science and Engineering Research Council of Canada (NSERC No. R2686A02).

References

Ashwin, P. & Swift, J. W. [1992] "The dynamics of n identical oscillators with symmetric coupling," *J. Nonlin. Sci.* **2**, 69–108.

Ashwin, P. & Stork, P. [1994] "Permissible symmetries of coupled cell networks," *Math. Proc. Camb. Phil. Soc.* **116**, 27–36.

Ashwin, P., Dangelmayr, G., Stewart, I. & Wegelin, M. [1994] "Oscillator networks with the symmetry of the unit quaternion group," *Dynamics, Bifurcation and Symmetry, Series C: Mathematical and Physical Sciences*, Nato Science Series *C* (Springer, Netherlands).

Barrón, M. & Sen, M. [2009] "Synchronization of four coupled van der Pol oscillators," *Nonlin. Dyn.* **56**, 357–367.

Bi, Q. & Yu, P. [1999] "Symbolic computation of normal forms for semi-simple cases," *J. Comput. Appl. Math.* **102**, 195–220.

Buono, P. L., Golubitsky, M. & Palacios, A. [2000] "Heteroclinic cycles in rings of coupled cells," *Physica D* **143**, 74–108.

Camacho, E., Rand, R. H. & Howland, H. [2014] "Dynamics of two van der Pol oscillators coupled via a bath," *Int. J. Solids Struct.* **41**, 2133–2143.

Chow, S. N. & Hale, J. K. [1982] *Methods of Bifurcation Theory* (Springer, NY).

Chow, S. N., Li, C. C. & Wang, D. [1994] *Normal Forms and Bifurcation of Planar Vector Fields* (Cambridge University Press, Cambridge).

Dias, A. P. S. & Paiva, R. C. [2010] "A note on Hopf bifurcation with dihedral group symmetry," *Glasgow Math. J.* **48**, 41–51.

Gazor, M. & Yu, P. [2012] "Spectral sequences and parametric normal forms," *J. Diff. Eqs.* **252**, 1003–1031.

Golubitsky, M., Stewart, I. & Schaeffer, D. [1988] *Singularities and Groups in Bifurcation Theory II*, Applied Mathematical Sciences, Vol. 69 (Springer-Verlag, NY).

Golubitsky, M. & Stewart, I. [2003] *The Symmetry Perspective: From Equilibrium to Chaos in Phase Space and Physical Space* (Birkhäuser-Verlag, Berlin).

Golubitsky, M., Shiau, L. J. & Török, A. [2004] "Symmetry and pattern formation on the visual cortex," *Dynamics and Bifurcation of Patterns in Dissipative Systems*, World Scientific Series on Nonlinear Science Series, Vol. 12 (World Scientific, Singapore).

Guckenheimer, J. & Holmes, P. [1993] *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, 4th edition (Springer, NY).

Han, M. & Yu, P. [2012] *Normal Forms, Melnikov Functions, and Bifurcations of Limit Cycles* (Springer, NY).

Hou, C. & Golubitsky, M. [1997] "An example of symmetry breaking to heteroclinic cycles," *J. Diff. Eqs.* **133**, 30–48.

Kawahara, T. [1980] "Coupled van der Pol oscillators — A model of excitatory and inhibitory neural interactions," *Biol. Cybern.* **39**, 37–43.

Krupa, M. & Melbourne, I. [1995] "Asymptotic stability of heteroclinic cycles in systems with symmetry," *Erg. Th. Dyn. Syst.* **15**, 121–147.

Kuznetsov, Yu. A. [1998] *Elements of Applied Bifurcation Theory*, 2nd edition (Springer, NY).

LaSalle, J. P. [1976] *The Stability of Dynamics Systems* (SIAM, Philadelphia).

Low, L. A., Reinhall, P. G. & Storti, D. W. [2003] "An investigation of coupled van der Pol oscillators," *J. Vibr. Acoust.* **125**, 162–169.

Paccosi, R. G., Figliola, A. & Galán-Vioque, J. [2014] "A bifurcation approach to the synchronization of coupled van der Pol oscillators," *SIAM J. Appl. Dyn. Syst.* **13**, 1152–1167.

Rand, R. H. & Holmes, P. J. [1980] "Bifurcation of periodic motions in two weakly coupled van der Pol oscillators," *Int. J. Non-Lin. Mech.* **15**, 387–399.

Swift, J. W. [1988] "Hopf bifurcation with the symmetry of the square," *Nonlinearity* **1**, 333–377.

Takamatsu, A., Tanaka, R., Yamamoto, T. & Fujii, T. [2003] "Control of oscillation patterns in a symmetric coupled biological oscillator system," *Proc. AIP Conf.*, pp. 230–235.

Uwate, Y., Nishio, Y. & Stoop, R. [2010] "Synchronization in three coupled van der Pol oscillators with different coupling strength," *2010 Int. Workshop on Nonlinear Circuits, Communication and Signal Processing*, NCSP'10, Waikiki, Hawaii, March 3–5, 2010.

Yu, P. [1998] "Computation of normal forms via a perturbation technique," *J. Sound Vibr.* **211**, 19–38.

Yu, P. & Bi, Q. [1998] "Analysis of non-linear dynamics and bifurcations of a double pendulum," *J. Sound Vibr.* **217**, 691–736.

Yu, P. [1999] "Simplest normal forms of Hopf and generalized Hopf bifurcations," *Int. J. Bifurcation and Chaos* **9**, 1917–1939.

Yu, P. [2003] "A simple and efficient method for computing center manifold and normal forms associated with semi-simple cases," *Dyn. Contin. Discr. Impuls. Syst. Ser. B: Appl. Algorith.* **10**, 273–286.

Yu, P. & Leung, A. Y. T. [2003] "The simplest normal form of Hopf bifurcation," *Nonlinearity* **16**, 277–300.