DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **26**, Number **10**, October **2021**

doi:10.3934/dcdsb.2020375

pp. **5661–5679**

PARAMETER IDENTIFICATION ON ABELIAN INTEGRALS TO ACHIEVE CHEBYSHEV PROPERTY

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(Communicated by Dongmei Xiao)

Abstract. Chebyshev criterion is a powerful tool in the study of limit cycle bifurcations in dynamical systems based on Abelian integrals, but it is difficult when the Abelian integrals involve parameters. In this paper, we consider the Abelian integrals on the periodic annuli of a Hamiltonian with one parameter, arising from the generalized Liénard system, and identify the parameter values such that the Abelian integrals have Chebyshev property. In particular, the bounds on the number of zeros of the Abelian integrals are derived for different parameter intervals. The main mathematical tools are transformations and polynomial boundary theory, which overcome the difficulties in symbolic computations and analysis, arising from large parametric-semi-algebraic systems.

1. **Introduction and main result.**

The classical Liénard system,

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{1}$$

was proposed to model oscillating circuits at the primary development of radio and vacuum tube technology [17]. It has extensive applications in science and engineering, for example, the Van der Pol oscillator, $\ddot{x} + (1 - x^2)\dot{x} + x = 0$ in which $f(x) = 1 - x^2$, was proposed by Van der Pol [29] to model the oscillation of electrical circuits, by Fitzhugh [7] and Nagumo [21] to model the action of potentials of neurons, and later by Cartwright *et al.* [3] to model dynamical behaviours of other physical and biological systems, such as the two plates in a geological fault.

2020 *Mathematics Subject Classification.* Primary: 34C07, 34C10; Secondary: 34C23.

Key words and phrases. Hilbert's 16th problem, Liénard system, limit cycle, heteroclinc loops, bifurcation.

Pei Yu was supported by the Natural Sciences and Engineering Research Council of Canada (No. R2686A02). Xianbo Sun was partially supported by the Ontario Graduate Scholarship, NSFC(11861009), NSFGX(2018GXNSFAA138198) and Program for Innovative Team of GUFE(2018-2021).

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On the other hand, many researchers have realized that system (1) has limitation on modeling most recently developed nonlinear problems due to the linear restoring force. Therefore, the system (1) was generalized to a more general version with nonlinear restoring force,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{2}$$

where $f(x)$ and $g(x)$ are univariate polynomials. System (2) is usually called generalized Liénard system and used to model various kinds of oscillations such as limit cycles in natural sciences [6]. In reality, the planar form,

$$\dot{x} = y, \quad \dot{x} = -g(x) - f(x)y, \tag{3}$$

is often obtained by simplifying higher dimensional dynamical systems to study local bifurcations, such as Bogdanov-Takens bifurcation investigated in a recent unified SIR and HIV disease Model [35]. Even though system (3) has a simple expression, it is still very difficult to determine the number of limit cycles and their location bifurcating in this system. As a matter of fact, Smale proposed [22] 18 open mathematical problems for the 21th century, formally in reply to a request from Arnold (Arnold's inspiration came from Hilbert's problems [13]), one of them asks for the number of limit cycles that system (1) can have. However, even for the simplest version of the Hilbert's 16th problem, it is still not easy to solve. Arnold proposed [1] a weaker version of Hilbert's 16th problem, which, instead of considering general polynomial differential systems, studies the number of limit cycles emerging from the periodic annuli by perturbing Hamiltonian systems. The corresponding system is

$$\dot{x} = H_y(x, y) + \epsilon p(x, y), \quad \dot{x} = -H_x(x, y) + \varepsilon q(x, y), \tag{4}$$

where $\varepsilon > 0$ is sufficiently small, $H(x, y)$, $p(x, y)$ and $q(x, y)$ are polynomials. System $(4)_{\epsilon=0}$ is a Hamiltonian system with the first integral or Hamiltonian function $H(x, y)$, which defines a family of periodic orbits $\{\Gamma_h\} = \{(x, y)|H(x, y) = h\}$. The number of limit cycles of system (4) is estimated by a return map constructed on the periodic annulus [11], and further can be estimated by the zeros of its first order approximation, called Abelian integral or Melnikov function, defined as

$$M(h) = \oint_{\Gamma_h} q(x, y)dx - p(x, y)dy, \quad h \in J, \tag{5}$$

where J is an open interval ensuring that $H(x, y) = h \in J$ defines a family of closed orbits. So far, only quadratic systems of (4) have been completely solved, see [4]. Based on Smale's version (3) and Arnold's version (4), a further simpler version has been studied on the Poincaré bifurcation of limit cycles for the following generalized Liénard system with a weak damping effect,

$$\dot{x} = y, \quad \dot{x} = -g(x) - \varepsilon f(x)y, \tag{6}$$

which belongs to the perturbed Hamiltonian system (4) with $p = 0$ and $q(x, y) = f(x)y$, and so $H(x, y) = \frac{y^2}{2} + \int g(x)dx$. The corresponding Abelian integral is reduced to the simple form,

$$A(h) = \oint_{\Gamma_h} f(x)dy, \quad h \in J. \tag{7}$$

There have been many works reported on system (6) which has degrees 2, 3, 4, and degrees 5 and 7 with symmetry. The results are summarized in a recent publication [26]. When system (6) has symmetry and degree 7, it has the following normal form,

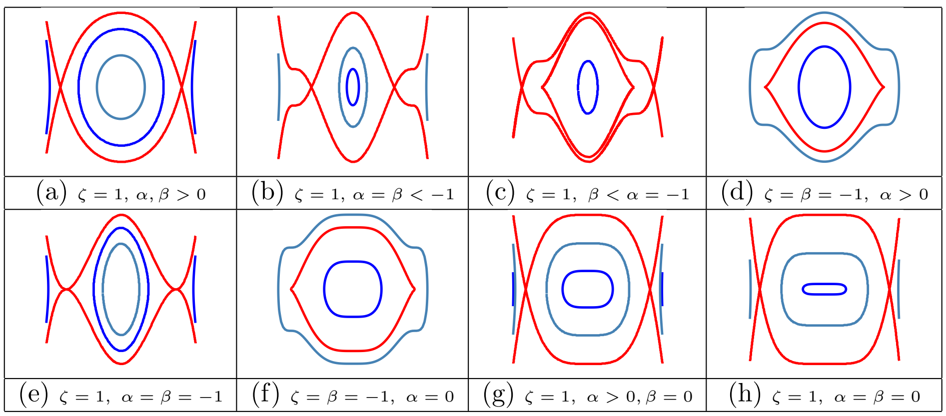
$$\dot{x} = y, \quad \dot{y} = \zeta x(x^2-1)(x^2+\alpha)(x^2+\beta) + \varepsilon(a_0 + a_1x^2 + a_2x^4 + a_3x^6)y, \tag{8}$$

where $\zeta = \pm 1$, α and β are real numbers. The Hamiltonian of system $(8)_{\varepsilon=0}$ is

$$\mathbb{H}(x,y) = \frac{y^2}{2} + \frac{\zeta\alpha\beta}{2}x^2 + \frac{\zeta(\alpha+\beta-\alpha\beta)}{2}x^4 + \frac{\zeta(1-\alpha-\beta)}{4}x^6 - \frac{\zeta}{2}x^8. \tag{9}$$

We assume system $(8)_{\epsilon=0}$ has at least one unique heteroclinic loop, which has eight different topological phase portraits, as shown in Table 1.

Table 1. Eight different topological phase eortraits of system $(8)_{\epsilon=0}$ with heteroclinic loops (red curves).



The case (a) of system (8) was considered in [23] to show that the Abelian integral of system (8) has at most 4 zeros when $\alpha = \beta = 2$, and 4 zeros can be detected by considering the asymptotic expansion of the Abelian integral near the boundary of the periodic annulus, see [9, 10, 12, 25]. Recently, Zhu *et al.* pointed out that the corresponding Abelian integral has at most 4 zeros, but only 3 zeros were obtained by choosing $\alpha = 1$ and $\beta = 2$, see [36]. For case (b), Yang *et al.* [33] fixed $\alpha = \beta = -2$ and proved that the corresponding Abelian integral has at most 4 zeros which can be detected by applying the same method used in [23], and in the same article, the exact same result was obtained for case (c) by fixing $\alpha = -2$ and $\beta = -1$. For cases (e) and (f), it has been proved that the corresponding Abelian integrals have at most 4 zeros but only 3 zeros have been reached, see [14, 24, 27]. It should be noted that all of these results were obtained by applying the Chebyshev criterion [8, 19]. A combination of techniques has been used in [23, 27] to show the advantage in solving the problem. The bounds given in [23, 33, 36] were obtained only for system (8) with the parameters α and β being fixed. It is unknown whether the bounds on the number of zeros of Abelian integrals depend on the parameters α and β, and if it does, then what is the parameter values such that the Abelian integrals have Chebyshev property?

In this paper, we study system (8) for cases (a), (b) and (d), restricted to one parameter α. This parameter α involved in the Hamiltonian causes a great deal of difficulty in bounding the number of zeros of the Abelian integral. For convenience, in the following, we use the notations for all cases (a), (b) and (d) of system (8):

(1) $\mathbb{H}(x,y)$ for the Hamiltonian of system (8)$_{\varepsilon=0}$;
(2) $\Upsilon^\dagger$ for the heteroclinic loop, satisfying $\mathbb{H}(\Upsilon^\dagger)=\mathbb{H}(-1,0)$;
(3) Υ_h for the closed orbits, satisfying $\Upsilon_h=\{(x,y)|x\in(-1,1),\ \mathbb{H}(x,y)=h,\ 0<h<\mathbb{H}(\Upsilon^\dagger)\}$;
(4) Υ_* for the center at $(0,0)$, satisfying $\mathbb{H}(0,0)=0$.
(5) $\mathcal{P}$ for the periodic annulus formed by $\{\Upsilon_h\}$.

The corresponding Abelian integral constructed on $\mathcal{P}$ as the bifurcation function of the perturbed system (8) is given by

$$\begin{aligned}\mathbb{I}(h,\delta)=&\oint_{\Upsilon_h}(a_0+a_1x^2+a_2x^4+a_3x^6)y\,dx\\ :=&\ a_0\,\mathbb{I}_0(h)+a_1\,\mathbb{I}_1(h)+a_2\,\mathbb{I}_2(h)+a_3\,\mathbb{I}_3(h),\end{aligned}\tag{10}$$

for $h\in(0,\mathbb{H}(\Upsilon^\dagger))$, where $\delta=(a_0,a_1,a_2,a_3)$, $\mathbb{I}_i(h)=\oint_{\Upsilon_h}x^{2i}y\,dx,\ i=0,1,2,3.$

Our main results are stated in the following three theorems.

Theorem 1.1. *For $\zeta=\beta=-1$ and $h\in(0,\frac{1+4\alpha}{24})$, with multiplicity counted, $\mathbb{I}(h,\delta)$ has at most three zeros if $\alpha\in[1,3]$ and four zeros if $\alpha\in(3,+\infty)$.*

Theorem 1.2. *For $\zeta=1$, $\beta=\alpha$ and $h\in(0,\frac{1+4\alpha+6\alpha^2}{24})$, with multiplicity counted, $\mathbb{I}(h,\delta)$ has at most three zeros if $\alpha\in(\alpha_4,\alpha_{10})$ and four zeros if $\alpha\in(-\infty,\alpha_4)\cup(\alpha_{10},-1)\cup(\alpha^*,+\infty)$, where α_4, α_{10} and α^* are given in the proof.*

Theorem 1.3. *For $\zeta=\beta=1$ and $h\in(0,\frac{3+8\alpha}{24})$, $\mathbb{I}(h,\delta)$ has at most four zeros counting multiplicity if $\alpha\in\left[\frac{3}{8},+\infty\right)$.*

With system (8), Theorem 1.1 is for case (d) with $\alpha\geq1$; Theorem 1.2 covers case (b), and case (a) with $\beta=\alpha>\alpha^*>0$; and Theorem 1.3 gives the bounds for part of case (a) with $\beta=1$ and $\alpha\geq\frac{3}{8}$. Combining Poincaré-Pontryagin Theorem and Theorems 1.1, 1.2 and 1.3, we can also obtain the number of limit cycles emerging from $\mathcal{P}$.

We have noticed that the zero problem on the Abelian integral $\mathbb{I}(h,\delta)$ is more difficult due to the parameter α. The main mathematical tool used in proving our main results is Grau's criterion [8, 19] on a set of integrals. This idea reduces the problem to detecting the roots of some semi-algebraic sets. The parameter α in $\mathbb{H}(x,y)$ also appears in the semi-algebraic sets. However, it is not easy to determine the roots of a parametric semi-algebraic set even if some methodologies have been developed, such as parametric regular chains [5], zero classification and zero counting [34]. These methods are applicable to parametric-semi-algebraic set systems with fewer variables and fewer terms. We have experienced a larger parametric semi-algebraic set with more than 1000 terms, causing computers to crash on symbolic computation. In this work, in order to overcome the difficulty, we will introduce a series of variable transformations, to show the positivity of some relative polynomials. We will also utilize the polynomial boundary theory to detect the branch points of the parametric semi-algebraic system. It will be shown that the transformations (even if they are simple) and the polynomial boundary theory are very powerful, with which the parameter range ensuring the integrals to have Chebyshev property is obtained. We have thus simplified the complicated symbolic computations and made concise proofs.

The rest of the paper is organized as follows. Definitions, Chebyshev criterion and polynomial boundary theory are presented in the next section. In section 3, we prove Theorem 1.1 for which we mainly determine the range of the parameter α in

$\mathbb{H}(x,y)$ such that $\{\mathbb{I}_0(h),\mathbb{I}_1(h),\mathbb{I}_2(h),\mathbb{I}_3(h)\}$ has the Chebyshev property. In section 4, we prove Theorem 1.2, and in section 5, we give an outline of proof for Theorem 1.3. Finally, conclusion is drawn in section 6.

2. Preliminary.

The main tool used in this paper is an algebraic method to bound the number of zeros of linear combination of a set of integrals on a family of closed orbits, see [8, 19]. It is a general version of the method of determining the monotonicity of two Abelian integrals [16, 18]. A brief introduction is given below.

Definition 2.1. A set of analytic functions $\{l_0(x),l_1(x),l_2(x),...,l_{n-1}(x)\}$ for x belonging to some interval J is called a Chebyshev system (T-system for short) if any nontrivial linear combination,

$$k_0l_0(x)+k_1l_1(x)+\cdots+k_{n-1}l_{n-1}(x),$$

has at most $n-1$ isolated zeros on J; called a T-system with accuracy k if any nontrivial linear combination,

$$k_0l_0(x)+k_1l_1(x)+\cdots+k_{n-1}l_{n-1}(x),$$

has at most $n-1+k$ isolated zeros accounting multiplicity on J; and called extended complete Chebyshev system (ECT-system for short) if any nontrivial linear combination,

$$k_0l_0(x)+k_1l_1(x)+\cdots+k_{i-1}l_{i-1}(x),$$

has at most $i-1$ isolated zeros accounting multiplicity on J for all $i=1,\cdots,n$.

One sufficient condition ensuring that $\{l_0(x),l_1(x),l_2(x),...,l_{n-1}(x)\}$ is an ECT-system is non-vanishing of the Wronskians $W[l_0(x),\cdots,l_i(x)]$, $i=0,1,\cdots,n-1$, that is,

$$W[l_0(x),\cdots,l_i(x)]\neq 0,\quad \text{for}\quad i=0,1,\cdots,n-1.$$

Let the analytic Hamiltonian have the form $\mathcal{H}(x,y)=\Psi(x)+\frac{1}{2}y^2$ and define a family of closed curves $\{L_h\}=\{(x,y)|\mathcal{H}(x,y)=h\}$. The closed curves $\{Ł_h\}$ surround an element center at the origin and form an annulus. $\mathcal{H}(x,y)$ has a local minimum assumed to be 0 at $(0,0)$. The projection of $\{Ł_h\}$ on the x-axis is an interval (x_l,x_r) with $x_l<0<x_r$. There exists an analytic involution $z(x)$ such that $\Phi(x)=\Phi(z(x))$ for all $x\in(x_l,x_r)$, with $(x^*,0)$ and $(z(x^*),0)$ being the intersection points of the closed curve L_{h^*} with the x-axis, satisfying $\mathcal{H}(x^*,0)=\mathcal{H}(z(x^*),0)=h^*$. In particular, we have $z(x)=-x$ if $\Psi(x)$ is an even function and the related system is symmetric with respect to the y-axis.

Let $\xi_i(x)$, $i=0,1,\ldots,n-1$, be analytic on (x_l,x_r) and the ordered integral set $\{A_0(h),A_1(h),\cdots,A_{n-1}(h)\}$ are well defined on $\{L_h\}$ by

$$A_i(h)=\int_{L_h}\xi_i(x)y^{2s-1}dx,\quad i=0,1,2,\cdots,n-1,$$

where $s\in\mathbb{N}$, $h\in(\mathcal{H}(0,0),\mathcal{H}(x_r,0)):=(0,h_0)$. Further define

$$l_i(x):=\frac{\xi_i(x)}{\Phi'(x)}-\frac{\xi_i(-x)}{\Phi'(-x)}. \tag{11}$$

Then, we have

Lemma 2.2. [8] $\{A_0(h),A_1(h),\cdots,A_{n-1}(h)\}$ *is an* ECT-*system on* $(0,h_0)$ *if* $s>n-2$, *and* $\{l_0(x),l_1(x),\cdots,l_{n-1}(x)\}$ *is an* ECT-*system on* $(x_l,0)$ *or* $(0,x_r)$.

Lemma 2.3. [19] *$\{A_0(h), A_1(h), \cdots, A_{n-1}(h)\}$ is a* T*-system with accuracy k on $(0, h_0)$ if $s > n-2+k$, $\{l_0(x), l_1(x), \cdots, l_{n-2}(x)\}$ is an* ECT*-system, and the Wronskian $W[l_0(x), l_1(x), \cdots, l_{n-1}(x)]$ has k zeros accounting multiplicity on $(x_l, 0)$ or $(0, x_r)$.*

We give a brief introduction of polynomial boundary theory. Let k be a field, $x_1 \prec x_2 \prec \cdots \prec x_n$ be n ordered variables and $R = k[x_1, \cdots, x_n]$ be the polynomial ring on k. The greatest variable x_i in $f(x_1, \cdots, x_i)$ is called its main variable, denoted by $mvar(f)$. The coefficient of the main variable of f is called the leading coefficient, denoted by $lc(f)$.

Definition 2.4. (Semi-Algebraic Systems). A semi-algebraic system (SAS for short) is a conjunctive polynomial formula of the following form,

$$\text{SAS}, \quad \mathcal{S}: \begin{cases} p_1(x_1, \cdots, x_n) = 0, \cdots, p_s(x_1, \cdots, x_n) = 0, \\ g_1(x_1, \cdots, x_n) \geq 0, \cdots, g_r(x_1, \cdots, x_n) \geq 0, \\ g_{r+1}(x_1, \cdots, x_n) > 0, \cdots, g_t(x_1, \cdots, x_n) > 0, \\ h_1(x_1, \cdots, x_n) \neq 0, \cdots, h_m(x_1, \cdots, x_n) \neq 0, \end{cases} \tag{12}$$

where $n, s \geq 1$, $t \geq r \geq 0$, $m \geq 0$, and all $p_i, g_i, h_i \in R(u, x)$ are polynomials with integer coefficients.

An SAS is called a parametric SAS if $s < n$ (s indeterminates are viewed as independent variables and the other $n-s$ indeterminates are treated as parameters, denoted by $u = (x_{s+1}, \cdots, x_n)$). An SAS is usually denoted by $[F, N, P, H]$, where $F = [p_1, \cdots, p_s]$, $N = [g_1, \cdots, g_r]$, $P = [g_{r+1}, \cdots, g_t]$ and $H = [h_1, \cdots, h_m]$.

There exist several well-known methods, such as the Ritt-Wu method, Gröbner basis method and subresultant method [2, 30, 31], which enable us to transform an SAS (12) equivalently to one or more triangular sets: $\mathrm{T_1}, \cdots, \mathrm{T_j}$ in the form of

$$\mathrm{T_j}: \begin{cases} f_1^j(u, x_1) = 0, \\ f_2^j(u, x_1, x_2) = 0, \\ f_3^j(u, x_1, x_2, x_3) = 0, \\ \qquad \vdots \\ f_s^j(x_1, \cdots, x_s) = 0, \\ g_1(x_1, \cdots, x_n) \geq 0, \cdots, g_r(x_1, \cdots, x_n) \geq 0, \\ g_{r+1}(x_1, \cdots, x_n) > 0, \cdots, g_t(x_1, \cdots, x_n) > 0, \\ h_1(x_1, \cdots, x_n) \neq 0, \cdots, h_m(x_1, \cdots, x_n) \neq 0, \end{cases} \tag{13}$$

where $\{f_1^j(u, x_1), f_2^j(u, x_1, x_2), f_3^j(u, x_1, x_2, x_3), \cdots, f_s^j(x_1, x_2, x_3, \cdots, x_s)\}$ is a triangular set (TS for short), or a normal ascending chain.

Let $\mathrm{dis}(f_i)$ denote the discriminant of a polynomial f_i with respect to x_i, $\mathrm{res}(\cdot, \diamond, x_j)$ denote the Sylvester resultant of $\cdot$ and $\diamond$ with respect to x_j, and $\gcd(f_1, f_2, \cdots, f_i)$ denote the greatest common factor of $f_1, f_2, \cdots, f_i$.

Definition 2.5. (Border Polynomial of TS) Consider the TS $\mathrm{T_j}$. For convenience, f_i^j is denoted by f_i(only for this definition). The following polynomial is called border polynomial of (13):

$$\begin{aligned}\mathrm{B_{T_j}} &= lc(f_1)\mathrm{dis}(f_1)\prod_{2\le i\le s}\mathrm{res}(lc(f_i)\mathrm{dis}(f_i); f_{i-1},\cdots,f_1)\prod_{1\le j\le t}\mathrm{res}(g_j; f_s,\cdots,f_1)\\ &\quad\times\prod_{1\le k\le m}\mathrm{res}(h_k; f_s,\cdots,f_1),\end{aligned}$$

where

$$\mathrm{res}(*; f_i,\cdots,f_1)=\mathrm{res}(\cdots(\mathrm{res}(\underline{\mathrm{res}(*,f_i,x_i)},f_{i-1},x_{i-1}),\cdots),f_1,x_1).$$

For two TSs: $\mathrm{T_j}$ and $\mathrm{T_{\tilde{j}}}$, let

$$r_i^{j\tilde{j}}=\gcd\big(\mathrm{res}(f_i^j; f_i^{\tilde{j}}, f_{i-1}^{\tilde{j}},\cdots,f_1^{\tilde{j}}),\mathrm{res}(f_i^{\tilde{j}}; f_i^j, f_{i-1}^j,\cdots,f_1^j)\big),\ 1\le i\le s,$$

and

$$C_{j\tilde{j}}=\gcd(r_1^{j\tilde{j}},\cdots,r_s^{j\tilde{j}}).$$

Definition 2.6. (Border Polynomial of an SAS) If a parametric SAS $\mathcal{S}$ is transformed equivalently to regular TSs $\{\mathrm{T_1},\cdots,\mathrm{T_l}\}$, then

$$\mathrm{B_S}=\prod_{1\le j\le\tilde{j}\le l}C_{j\tilde{j}}\prod_{j=1}^{l}\mathrm{B_{T_j}}$$

is called the border polynomial of $\mathcal{S}$.

Lemma 2.7. [32, 34] *The number of distinct real solutions of the semi-algebraic system* S *is invariant in each connected component of the complement of* $\mathrm{B_S}=0$ *in* R^{n-s}.

Remark 2.8. When the parameter values satisfy the boundary $\mathrm{B_S}=0$, it is usually called degenerate case, for which it should be analyzed by other methods, see [32, 34]. Based on the above described idea, Yang and Xia [32, 34] developed a practical method for computing the border polynomial of S, which has been included into the Maple software package.

3. **Proof of Theorem 1.1.** In this section, we first study the Abelian integral of system (8) for case (d) and determine the range of α, such that the sets $\{\mathbb{I}_0(h),\mathbb{I}_1(h),\mathbb{I}_2(h),\mathbb{I}_3(h)\}$ is a T-system by Lemma 2.2. We rewrite

$$\mathbb{E}(x)=\mathbb{H}(x,0)=\frac{\alpha}{2}x^2+\frac{1-2\alpha}{4}x^4+\frac{\alpha-2}{6}x^6+\frac{x^8}{8}.$$

The projection of the period annulus $\mathcal{P}$ on the x-axis is $(-1,1)$. Note that $x\mathbb{E}'(x)>0$ for all $x\in(-1,1)\setminus\{0\}$. There exists an analytic involution $z(x)=-x$ defined by $\mathbb{E}(x)=\mathbb{E}(z(x))$. Our goal is to prove that the vector space $\{\mathbb{I}_0(h),\mathbb{I}_1(h),\mathbb{I}_2(h),\mathbb{I}_3(h)\}$ has Chebyshev property by applying Lemma 2.2. However, the condition $s>n-2$ in Lemma 2.2 is not satisfied and thus the power in the integrand of $\mathbb{I}_i(h)$ $(i=0,1,2,3)$ needs to be increased. To achieve this, we have the following result.

Lemma 3.1. *For* $i=0,1,2,3$*, we have*

$$8h^3\mathbb{I}_i(h)=\int_{\Upsilon_h}f_i(x)y^7dx:=\bar{\mathbb{I}}_i(h),$$

where $f_i(x)=\frac{x^{2i}\sum_{j=0}^{15}p_j(\alpha)x^{2j}}{181440(x^2-1)^9(x^2+\alpha)^6}$ *and* $p_j(\alpha)$*,* $i=0,1,\cdots,15$*, are polynomials in* α.

Proof. First, note that $2\mathbb{E}(x)+y^2=2h$ holds on each closed curve Υ_h. Thus we obtain

$$\begin{aligned}8h^3\mathbb{I}_i(h) &= \int\limits_{\Upsilon_h}(2\mathbb{E}(x)+y^2)^3x^{2i}ydx\\ &= \int\limits_{\Upsilon_h}8x^{2i}\mathbb{E}^3(x)ydx+\int\limits_{\Upsilon_h}12x^{2i}\mathbb{E}^2(x)y^3dx+\int\limits_{\Upsilon_h}6x^{2i}\mathbb{E}(x)y^5dx+\int\limits_{\Upsilon_h}x^{2i}y^7dx,\end{aligned} \tag{14}$$

for $i=0,1,2,3$. Noticing that the functions $\frac{8x^{2i}\mathbb{E}^3(x)}{\mathbb{E}'(x)}$ are analytic near $x=0$, and applying Lemma 4.1 in [8] we have

$$\begin{aligned}\int\limits_{\Upsilon_h}8x^{2i}\mathbb{E}^3(x)ydx &= \int\limits_{\Upsilon_h}\mathbb{G}_i(x)y^7dx,\\ \int\limits_{\Upsilon_h}12x^{2i}\mathbb{E}^2(x)y^3dx &= \int\limits_{\Upsilon_h}\overline{\mathbb{G}}_i(x)y^7dx,\\ \int\limits_{\Upsilon_h}6x^{2i}\mathbb{E}(x)y^5dx &= \int\limits_{\Upsilon_h}\overline{\overline{\mathbb{G}}}_i(x)y^7dx,\end{aligned} \tag{15}$$

where

$$\begin{aligned}\mathbb{G}_i(x) &= \frac{x^{2i}g_i(x)}{181440(x^2-1)^9(x^2+\alpha)^6},\\ \overline{\mathbb{G}}_i(x) &= \frac{x^{2i}\overline{g}_i(x)}{1680(x^2-1)^6(x^2+\alpha)^4},\\ \overline{\overline{\mathbb{G}}}_i(x) &= \frac{x^{2i}\overline{\overline{g}}_i(x)}{28(x^2-1)^3(x^2+\alpha)^2},\end{aligned}$$

in which the polynomials $g_i(x)$, $\overline{g}_i(x)$ and $\overline{\overline{g}}_i(x)$ have degrees 30, 20 and 10, respectively, which are omitted here for brevity. Combining (14) and (15) finishes the proof of Lemma 3.1. □

By Lemma 3.1, we need only determine the range of α such that $\{\overline{\mathbb{I}}_0,\overline{\mathbb{I}}_1,\overline{\mathbb{I}}_2,\overline{\mathbb{I}}_3\}$ forms a Chebyshev system on $(0,\frac{4\alpha+1}{24})$. In order to apply Lemma 2.2, we set

$$\mathbb{L}_i(x)=\Big(\frac{f_i}{\mathbb{E}'}\Big)(x)-\Big(\frac{f_i}{\mathbb{E}'}\Big)(z(x)),\quad i=0,1,2,3, \tag{16}$$

where $0<x<1$ and $z(x)=-x$. Then, for $i=0,1,2,3$,

$$\mathbb{L}_i(x)=-\frac{2x^{2\,i}l_i(x)}{181440\,(x^2-1)^{11}\,(x^2+\alpha)^7\,x},$$

with $l_i(x)$ being a univariate polynomial of degree 30 in x with parameters α and i.

Now we only need to analyze the related Wronskians to verify if the ordered set of the criterion functions $\{\mathbb{L}_1(x),\mathbb{L}_2(x),\mathbb{L}_3(x),\mathbb{L}_0(x)\}$ form an ECT-system. A direct computation leads to the following lemma.

Lemma 3.2.
$$W[\mathbb{L}_1(x)] = \frac{x w_1(x,\alpha)}{90720\,(x^2-1)^{11}(x^2+\alpha)^7},$$
$$W[\mathbb{L}_1(x),\mathbb{L}_2(x)] = \frac{x^3 w_2(x,\alpha)}{4115059200\,(x^2-1)^{22}(x^2+\alpha)^{13}},$$
$$W[\mathbb{L}_1(x),\mathbb{L}_2(x),\mathbb{L}_3(x)] = \frac{x^6 w_3(x,\alpha)}{3110984755200\,(x^2-1)^{32}(x^2+\alpha)^{18}},$$
$$W[\mathbb{L}_1(x),\mathbb{L}_2(x),\mathbb{L}_3(x),\mathbb{L}_0(x)] = \frac{x^2 w_4(x,\alpha)}{1175952237465600\,(x^2-1)^{42}(x^2+\alpha)^{22}},$$
where the univariate polynomials $w_1(x,\alpha)$, $w_2(x,\alpha)$, $w_3(x,\alpha)$ and $w_4(x,\alpha)$ have degrees 30, 58, 82 *and* 104, *respectively, and α is a positive parameter.*

We have the following result.

Lemma 3.3. *The Wronskians on $\mathbb{L}_0(x)$, $\mathbb{L}_1(x)$, $\mathbb{L}_2(x)$ and $\mathbb{L}_3(x)$ have the following properties:*

(i) *both the Wronskians $W[\mathbb{L}_1(x)]$ and $W[\mathbb{L}_1(x),\mathbb{L}_2(x)]$ do not vanish for $x \in (0,1)$ when $\alpha \in (0,+\infty)$;*
(ii) *$W[\mathbb{L}_1(x),\mathbb{L}_2(x),\mathbb{L}_3(x)]$ does not vanish for $x \in (0,1)$ when $\alpha \in [1,+\infty)$;*
(iii) *$W[\mathbb{L}_1(x),\mathbb{L}_2(x),\mathbb{L}_3(x),\mathbb{L}_0(x)]$ has no zeros for $x \in (0,1)$ when $\alpha \in [1,3]$, and has exact one simple zero when $\alpha \in (3,+\infty)$.*

Proof. Due to Lemma 3.2, it is suffice to prove that the results hold for $w_i(x,\alpha)$, $i = 1,2,3,4$. We first prove assertion (i). Introducing $x = \frac{1}{1+X} \in (0,1)$ with $X > 0$, we have
$$w_1(x,\alpha) = w_1\Big(\frac{1}{1+X},\alpha\Big) = \frac{1}{(1+X)^{30}} \sum_{i=0,j=0}^{30,6} c_{i,j} X^i \alpha^j,$$
with all the coefficients c_{ij} being nonnegative and not zero identically. Therefore, $w_1(\frac{1}{1+X},\alpha)$ has no zeros on $\{(X,\alpha)|X>0,\alpha>0\}$. This implies that $w_1(x,\alpha)$ has no zeros for $x \in (0,1)$ when $\alpha \in (0,+\infty)$. Similarly,
$$w_2(x,\alpha) = w_2\Big(\frac{1}{1+X},\alpha\Big) = \frac{1}{(1+X)^{58}} \sum_{i=0,j=0}^{58,11} \bar{c}_{i,j} X^i \alpha^j,$$
with all the coefficients $\bar{c}_{ij}$ being nonnegative and not zero identically. Then, $w_2(\frac{1}{1+X},\alpha)$ has no zeros on $\{(X,\alpha)|X>0,\alpha>0\}$. This implies that $w_2(x,\alpha)$ has no zeros for $x \in (0,1)$ when $\alpha \in (0,+\infty)$.

Next, consider $w_3(x,\alpha)$ for assertion (ii). A direct substitution yields
$$w_3(0,\alpha) = 266294200054579200\,\alpha^{15}, \quad w_3(1,\alpha) = 1146880\,(\alpha-1)(\alpha+1)^5(4\alpha+1)^9,$$
and so
$$w_3(0,\alpha)w_3(1,\alpha) < 0 \ \text{ if } \ \alpha \in (0,1).$$
Hence, $w_3(x,\alpha)$ has at least one zero for $x \in (0,1)$ when $\alpha \in (0,1)$.

When $\alpha \in [1,+\infty)$, we introduce $\alpha = b+1$ with $b \ge 0$ and $x = \frac{1}{1+X}$ into $w_3(x,\alpha)$ to obtain
$$w_3(x,\alpha) = w_3\Big(\frac{1}{1+X},1+b\Big) = \frac{-1}{(1+X)^{82}} \sum_{i=0,j=0}^{82,15} \tilde{c}_{i,j} X^i b^j,$$

with all the coefficients $\widetilde{c}_{ij} \geq 0$ and $\widetilde{c}_{i0} \neq 0$. Then, $w_3\big(\frac{1}{1+X}, 1+b\big)$ has no positive zeros on $\{(X,b)|X>0, b\geq 0\}$. Therefore, $w_3(x,\alpha)$ has no zeros for $x\in(0,1)$ when α belongs to $[1,+\infty)$.

Finally, we prove assertion (iii). A direct computation shows that

$$w_4(0,\alpha) = 1767000490346161373184 0\alpha^{18},$$
$$w_4(1,\alpha) = -7340032\,(\alpha-3)\,(\alpha+1)^5\,(4\,\alpha+1)^{12},$$

and so,

$$w_4(0,\alpha)>0, \quad w_4(1,\alpha)<0 \quad \text{for} \quad \alpha\in(3,+\infty).$$

Hence, for $x\in(0,1)$, $w_4(x,\alpha)$ has at least one zero when $\alpha\in(3,+\infty)$, and maybe no zeros when $\alpha\in[1,3]$.

When $\alpha\in[1,3]$, we introduce $x=\frac{1}{1+X}$ and $\alpha=b+1$ with $X>0$ and $b\in[0,2]$ into $w_4(x,\alpha)$ to obtain

$$w_4(x,\alpha) = w_4\Big(\frac{1}{1+X}, 1+b\Big) = \frac{1}{(1+X)^{104}} \sum_{i=0,j=0}^{104,18} \widehat{c}_{i,j} X^i b^j,$$

where the coefficients $\widehat{c}_{ij}$'s have different signs, which can be rewritten as

$$\sum_{i=0,j=0}^{104,18} \widehat{c}_{i,j} X^i b^j = \sum_{i=0}^{104} g_i(b)\, X^i,$$

where $g_i(b)$, $i=0,1,\cdots,104$, are polynomials and have the degrees less than or equal to 18. The coefficients in $g_i(b)$ for $i=14,15,\cdots,104$ are all positive, implying that

$$g_i(b)\geq 0 \ \text{ on } \ [0,2] \quad \text{for} \ \ i=14,15,\ldots,104.$$

The coefficients in each $g_i(b)$, $i=0,1,\cdots,13$, have different signs. $g_0(b)$, $g_1(b)$ and $g_2(b)$ are given below, others are omitted for briefness.

$$g_0(b) = -7340032\,(b-2)(b+2)^5\,(4\,b+5)^{12},$$
$$g_1(b) = -7340032\,(100\,b+171)\,(b-2)\,(b+2)^4\,(4\,b+5)^{12},$$
$$g_2(b) = -917504\,\big(39375\,b^3+55178\,b^2-156567\,b-229058\big)\,(b+2)^3\,(4\,b+5)^{12}.$$

Applying Sturm's Theorem with a sample point $b^*\in(0,2)$ to $g_i(b)$ $(i=0,1,2,3,4,\cdots,13)$, we have,

$$g_i(b)\geq 0 \ \text{ on } \ [0,2] \quad \text{for} \ \ i=0,1,2,\ldots,13.$$

So, all $g_i(b)$'s are nonnegative. Then $\sum_{i=0}^{104} g_i(b)X^i$ has no positive roots for $X\in(0,+\infty)$ if $b\in[0,2]$. This implies that $w_4(x,\alpha)$ has no zeros for $x\in(0,1)$ when $\alpha\in[1,3]$.

When $\alpha\in(3,+\infty)$, we have

$$\frac{\partial w_4}{\partial x}(x,\alpha) = \frac{\partial w_4}{\partial x}\Big(\frac{1}{1+X}, 1+b\Big) = -\frac{1}{(1+X)^{103}} \sum_{i=0,j=0}^{102,18} c^{\dagger}_{i,j} X^i b^j,$$

where the coefficients $c^{\dagger}_{ij}$'s are nonnegative. Therefore, $\frac{\partial w_4}{\partial x}(x,\alpha)\leq 0$ and so $w_4(x,\alpha)$ is monotone on $(0,1)$ when $\alpha\in(3,+\infty)$. Hence, $w_4(x,\alpha)$ has a unique simple zero in $(0,1)$ when $\alpha\in(3,+\infty)$.

The proof of Lemma 3.3 is complete. $\square$

Proof. (**Theorem 1.1**) Lemma 3.3 shows that $\{\mathbb{L}_1(x), \mathbb{L}_2(x), \mathbb{L}_3(x), \mathbb{L}_0(x)\}$ is an ECT-system for $x \in (0,1)$ when $\alpha \in [1,3]$, and a Chebyshev system with accuracy 1 when $\alpha \in (3,+\infty)$. Then, Theorem 1.1 is proved by applying Lemmas 2.2 and 2.3. $\square$

Remark 3.4. **(i)** In the above analysis, we mainly apply Lemma 3.1 with the formula $8h^3\mathbb{I}_i(h) = \int_{\Upsilon_h} f_i(x)y^7dx$ to prove not only for that $\{\mathbb{L}_1(x), \mathbb{L}_2(x), \mathbb{L}_3(x), \mathbb{L}_0(x)\}$ is an ECT-system for $x \in (0,1)$ when $\alpha \in [1,3]$, but also for that it is a T-system with accuracy 1 when $\alpha \in (3,+\infty)$. It should be pointed out that one may use the simpler formula $4h^2\mathbb{I}_i(h) = \int_{\Upsilon_h} f_i^*(x)y^5dx$, where $f_i^*(x)$ is some relative function, to verify the former, but not applicable for the latter. Therefore, we have used the same formula $8h^3\mathbb{I}_i(h) = \int_{\Upsilon_h} f_i(x)y^7dx$ in the analysis for brievity.

(ii) The Chebyshev system theory is invalid for case (d) when $\alpha \in (0,1)$, and so the upper bound is unknown for this case. However, we may study the asymptotic expansion of $\mathbb{I}(h)$ near the center and apply the zero bifurcation in [11] to prove that $\mathbb{I}(h)$ has three zeros near $h = 0$.

4. **Proof of Theorem 1.2.** In this section, we prove Theorem 1.2. For briefness, we omit the details on the portraits of system $(8)_{\alpha=\beta}$ and the construction of the Wronskians, since they are similar to that for proving Theorem 1.1, and the methods and arguments are completely same. Similar to (16) and Lemma 3.3, we denote the determining functions for system $(8)_{\alpha=\beta}$ as $\mathcal{L}_i(x)$, $i = 0,1,2,3$, and the four Wronskians are obtained by direct computation, as given below.

Lemma 4.1.
$$
\begin{aligned}
W[\mathcal{L}_1(x)] &= \frac{x m_1(x,\alpha)}{90720\,(x-1)^7\,(x+1)^7\,(x^2+\alpha)^{11}},\\
W[\mathcal{L}_1(x), \mathcal{L}_3(x)] &= \frac{x^5 m_2(x,\alpha)}{2057529600\,(x-1)^{13}\,(x+1)^{13}\,(x^2+\alpha)^{22}},\\
W[\mathcal{L}_1(x), \mathcal{L}_3(x), \mathcal{L}_0(x))] &= \frac{x^2\mathcal{S}_1(x,\alpha)}{3110984755200\,(x-1)^{18}(x+1)^{18}(x^2+\alpha)^{32}},\\
W[\mathcal{L}_1(x), \mathcal{L}_3(x), \big(\lambda\mathcal{L}_0(x)+\mathcal{L}_2(x)\big)] &= \frac{x^2(\lambda\mathcal{S}_1(x,\alpha)+\mathcal{S}_2(x,\alpha))}{3110984755200\,(x-1)^{18}(x+1)^{18}(x^2+\alpha)^{32}},
\end{aligned}
$$
where $m_1(x,\alpha)$, $m_2(x,\alpha)$, $\mathcal{S}_1(x,\alpha)$ and $\mathcal{S}_2(x,\alpha)$ are polynomials having the degrees 39*,* 58*,* 82 *and* 86*, respectively.*

Furthermore, we have the following result.

Lemma 4.2. *The Wronskians on $\mathcal{L}_0(x)$, $\mathcal{L}_1(x)$, $\mathcal{L}_2(x)$ and $\mathcal{L}_3(x)$ have the following properties:*

- (i) *$W[\mathcal{L}_1(x)]$ does not vanish for $x \in (0,1)$ when $\alpha \in (-\infty,-1) \cup (0,+\infty)$;*
- (ii) *$W[\mathcal{L}_1(x), \mathcal{L}_3(x)]$ does not vanish for $x \in (0,1)$ when $\alpha \in (-\infty,-1) \cup [1,+\infty)$;*
- (iii) *$W[\mathcal{L}_1(x), \mathcal{L}_3(x), \mathcal{L}_0(x))]$ does not vanish when $\alpha \in (-\infty,-1) \cup (\alpha^*,+\infty)$, where α^* is isolated in $\big[\frac{2835}{2048}, \frac{5671}{4096}\big]$.*
- (iv) *When $\alpha \in (-\infty,\alpha_1) \cup (\alpha^*,+\infty)$, $W[\mathcal{L}_1(x), \mathcal{L}_3(x), \big(\lambda\mathcal{L}_0(x)+\mathcal{L}_2(x)\big)]$ has exactly two simple zeros for $x \in (0,1)$ if $\lambda \in (0,\lambda^*] \cup [\lambda^{**},\lambda^r)$, one simple zero if $\lambda \in (\lambda^r,0]$, and no zeros if $\lambda \in (\lambda^*,+\infty) \cup (-\infty,\lambda^{**})$, where α_1 is given in (17), λ^*, λ^{**} and λ^r are given in the proof.*

(v) *$W[\mathcal{L}_1(x), \mathcal{L}_3(x), \big(\lambda\mathcal{L}_0(x) + \mathcal{L}_2(x)\big)]$ has at most two zeros for $x \in (0,1)$ when $\alpha \in (\alpha_1, \alpha_4) \cup (\alpha_{10}, -1)$, where α_1, α_4 and α_{10} are given in (17).*

(vi) *When $\alpha \in (\alpha_4, \alpha_{10})$, $W[\mathcal{L}_1(x), \mathcal{L}_3(x), \big(\lambda\mathcal{L}_0(x) + \mathcal{L}_2(x)\big)]$ has one zero for $x \in (0,1)$ if $\lambda \in (0, \lambda^r)$, and no zeros if $\lambda \in (-\infty, 0] \cup [\lambda^r, +\infty)$.*

There are some α_i's appearing in the following discussion, which are isolated in some smaller intervals with the estimated values, as given in (17). These α_i's divide the parameter space $\mathbb{R}$ into several connected components.

$$
\begin{aligned}
\alpha_1 &\approx -15.73222285 \in \big[-\tfrac{-4124109}{262144}, -\tfrac{2062053}{131072}\big],\\
\alpha_2 &\approx -6.186140508 \in \big[-\tfrac{415147741}{67108864}, -\tfrac{207571399}{33554432}\big],\\
\alpha_3 &\approx -6.185783063 \in \big[-\tfrac{25337}{4096}, -\tfrac{101347}{16384}\big],\\
\alpha_4 &\approx -4.037807739 \in \big[-\tfrac{16539}{4096}, -\tfrac{264621}{65536}\big],\\
\alpha_5 &\approx -4.013960571 \in \big[-\tfrac{65765}{16384}, -\tfrac{16441}{4096}\big],\\
\alpha_6 &\approx -3.679527580 \in \big[-\tfrac{120571}{32768}, -\tfrac{60285}{16384}\big],\\
\alpha_7 &\approx -3.637824056 \in \big[-\tfrac{3814535}{1048576}, -\tfrac{61032557}{16777216}\big],\\
\alpha_8 &\approx -3.623110488 \in \big[-\tfrac{15934611303989}{4398046511104}, -\tfrac{3983650523795}{1099511627776}\big],\\
\alpha_9 &\approx -3.195096115 \in \big[-\tfrac{1756523309131}{549755813888}, -\tfrac{14052177264239}{4398046511104}\big],\\
\alpha_{10} &\approx -3.138081101 \in \big[-\tfrac{102829}{32768}, -\tfrac{205655}{65536}\big],\\
\alpha^{\dagger} &\approx \quad 0.327886656 \in \big[\tfrac{2750511}{8388608}, \tfrac{1375257}{4194304}\big],\\
\alpha^{\dagger\dagger} &\approx \quad 1.000000079 \in \big[1, \tfrac{2097153}{2097152}\big],\\
\alpha^{\ddagger} &\approx \quad 1.000670638 \in \big[\tfrac{33576653}{33554432}, \tfrac{134308967}{134217728}\big].
\end{aligned}
\tag{17}
$$

For being concise for the proof of Lemma 4.2, we give four claims, based on which we first prove Lemma 4.2, and then the four claims.

Claim 1: $\mathcal{S}_1(x,\alpha)$ has no zeros for $x \in (0,1)$ if $\alpha \in (\alpha^*, +\infty)$, where α^* is isolated in $\big[\frac{2835}{2048}, \frac{5671}{4096}\big]$ and given in the proof.

Claim 2: $\mathcal{S}_3(x,\alpha)$, which is given in (18), has two zeros counting multiplicity $x^*(\alpha), x^{**}(\alpha) \in (0,1)$ if $\alpha \in (-\infty, \alpha_1) \cup (\alpha^*, +\infty)$.

Claim 3: $\mathcal{S}_3(x,\alpha)$ has a unique zero $x^*(\alpha) \in (0,1)$ if $\alpha \in (\alpha_1, \alpha_4) \cup (\alpha_{10}, -1)$.

Claim 4: $\mathcal{S}_3(x,\alpha)$ has no zeros for $x \in (0,1)$ if $\alpha \in (\alpha_4, \alpha_{10})$.

Proof. (**Lemma 4.2**) It is suffice to show that the conclusions are true for the main numerators of four Wronskians, $m_1(x,\alpha)$, $m_2(x,\alpha)$, $S_1(x,\alpha)$ and $S_2(x,\alpha)$. We prove the six cases one by one.

(i) Introducing $x = \frac{1}{1+X} \in (0,1)$ with $X > 0$, we have

$$m_1(x,\alpha) = m_1\Big(\frac{1}{1+X}, \alpha\Big) = \frac{1}{(1+X)^{20}} \sum_{i=0,j=0}^{20,6} d_{i,j} X^i \alpha^j,$$

with all coefficients $d_{ij} \geq 0$. Therefore, $m_1(\frac{1}{1+X}, \alpha)$ has no zeros on $\{(X,\alpha)|X > 0, \alpha > 0\}$, implying that $m_1(x,\alpha)$ has no zeros on $\{(x,\alpha)|0 < x < 1, \alpha > 0\}$.

Similarly, the transformation, $x = \frac{1}{1+X}$, $\alpha = -1 - b$ with $X, b > 0$, is introduced to prove that $m_1(x,\alpha)$ has no zeros on $\{(x,\alpha)|0 < x < 1, \alpha < -1\}$. Hence, assertion (i) holds.

(ii) Substituting $x = 0$ and $x = 1$ into $m_2(x, \alpha)$ shows that

$$m_2(0, \alpha) = -2407897497600\alpha^{18}$$
$$m_2(1, \alpha) = -5760\ (\alpha - 1)\,(\alpha + 1)^5\left(6\,\alpha^2 + 4\,\alpha + 1\right)^6.$$

Therefore, $m_2(x, \alpha)$ has at least one zero for $x \in (0, 1)$ if $\alpha \in (0, 1)$. When $\alpha \in [1, +\infty)$, we introduce $\alpha = b + 1$ and $x = \frac{1}{1+X}$ with $b \geq 0$ and $X > 0$ to obtain

$$m_2(x, \alpha) = m_2\Big(\frac{1}{1+X}, 1 + b\Big) = \frac{-1}{(1 + X)^{58}} \sum_{i=0,j=0}^{58,18} \widetilde{d}_{i,j} X^i b^j,$$

with all coefficients $\widetilde{d}_{i,j} \geq 0$ and some $\widetilde{d}_{i,0} > 0$. Therefore, $\sum_{i=0,j=0}^{58,18} \widetilde{d}_{i,j} X^i b^j$ has no positive zeros on $\{(X, b)|X > 0, b \geq 0\}$, and thus $m_2(x, \alpha)$ has no zeros on $\{(x, \alpha)|x \in (0, 1), \alpha \geq 1\}$.

Similarly, introducing the transformation, $x = \frac{1}{1+X}$, $\alpha = -1 - b$ with $X, b > 0$, shows that $m_2(x, \alpha)$ has no zeros on $\{(x, \alpha)|0 < x < 1, \alpha < -1\}$. Hence, the conclusion (ii) is true.

(iii) For $\alpha < -1$, we can introduce the transformation, $x = \frac{1}{1+X}$, $\alpha = -1 - b$ with $X, b > 0$, to prove that $\mathcal{S}_1(x, \alpha)$ has no zeros on $\{(x, \alpha)|0 < x < 1, \alpha < -1\}$ as follows:

$$\mathcal{S}_1(x, \alpha) = \mathcal{S}_1\Big(\frac{1}{1+X}, -1 - b\Big) = \frac{1}{(1 + X)^{82}} \sum_{i=0,j=0}^{82,26} \hat{d}_{i,j} X^i b^j,$$

where the coefficients $\hat{d}_{ij}$'s have different signs, 25 of them are negative, and other 2201 ones are positive. We rewrite the polynomial as

$$\sum_{i=0,j=0}^{82,26} \hat{d}_{i,j} X^i b^j = \sum_{i=0}^{82} \hat{g}_i(b) X^i,$$

where $\hat{g}_i(b)$, $i = 0, 1, \cdots, 104$, are lengthy polynomials with degrees less than or equal to 26. The coefficients in $\hat{g}_i(b)$ for $i = 8, 9, \cdots, 104$ are all positive, implying that

$$\hat{g}_i(b) > 0 \ \text{ in } \ (0, +\infty) \quad \text{for} \ \ i = 8, 9, \cdots, 104.$$

The coefficients in each $g_i(b)$ for $i = 0, 1, \cdots, 7$, have different signs. However, applying Sturm's Theorem with a sample point $b^* \in (0, \infty)$ to $\hat{g}_i(b)$ $(i = 0, 1, \cdots, 7)$, we have,

$$\hat{g}_i(b) > 0 \ \text{ in } \ (0, \infty) \quad \text{for} \ \ i = 0, 1, \cdots, 7.$$

So, all $g_i(b)$'s are positive. Thus, $\sum_{i=0}^{82} \hat{g}_i(b) X^i$ has no positive roots for $X \in (0, +\infty)$ if $b \in (0, \infty)$, namely $\mathcal{S}_1(x, \alpha)$ has no zeros for $x \in (0, 1)$ when $\alpha \in (-\infty, -1)$.

For $\alpha > 0$, we only need to consider $\alpha > 1$ because we know that the Wronskians $W[\mathcal{L}_1(x), \mathcal{L}_3(x)]$ do not vanish by applying the Chebyshev criterion. We cannot obtain any exact partition for $\alpha > 1$ by the existing methods. We will apply polynomial boundary theory in computer algebra to $\mathcal{S}_1(x, \alpha)$, resulting in Claim 1 to be proved, and so the proof for (iii) is complete.

(iv) We have shown that $\mathcal{S}_1(x, \alpha) > 0$ for $x \in (0, 1)$ if $\alpha \in (-\infty, -1) \cup (\alpha^*, +\infty)$. Then $\lambda\mathcal{S}_1(x, \alpha) + \mathcal{S}_2(x, \alpha) = 0$ well defines a smooth function $\lambda(x, \alpha)$ for

$x \in (0,1)$ with a parameter $\alpha \in (-\infty,-1)\cup(\alpha^*,+\infty)$, as $\lambda(x,\alpha) = -\frac{S_2(x,\alpha)}{S_1(x,\alpha)}$, which has the derivative,

$$\lambda'(x,\alpha) = \frac{4x^3(x^2-1)\mathcal{S}_3(x,\alpha)}{(S_1(x,\alpha))^2}, \tag{18}$$

where $\mathcal{S}_3(x,\alpha)$ is a polynomial with 1590 terms and has the degree 162 in x and 52 in the parameter α. The following results are obtained from a direct computation,

$$\lambda(0^+,\alpha) = 0^+, \quad \lambda(1^-,\alpha) = -\frac{3\,\alpha^3+33\,\alpha^2-23\,\alpha+27}{15\,\alpha^3+85\,\alpha^2+45\,\alpha-265},$$

and $\lambda'(0^+,\alpha) = 0^+$.

Assume $\alpha \in (-\infty,\alpha_1)\cup(\alpha^*,+\infty)$ and thus Claim 2 holds. Then we can show that $\lambda(x,\alpha)$ is increasing from $(0,0)$ to a maximum point (x^*,λ^*), then decreasing to a minimum point (x^{**},λ^{**}), and then increasing again to the right endpoint $(1,\lambda^r)$, where $\lambda^* = \lambda(x^*,\alpha) > 0$, $\lambda^{**} = \lambda(x^{**},\alpha) < 0$ and $\lambda^r = \lambda(1^-,\alpha) = -\frac{3\,\alpha^3+33\,\alpha^2-23\,\alpha+27}{15\,\alpha^3+85\,\alpha^2+45\,\alpha-265} < 0$. For any fixed λ, the number of roots of $\lambda\mathcal{S}_1(x,\alpha)+\mathcal{S}_2(x,\alpha)$ is exactly the number of intersection points of the curve $\{(x,\lambda) : \lambda = \lambda(x,\alpha),\ x \in (0,1)\}$ with the segment $\{(x,\lambda) : x \in (0,1)\}$. Thus, the conclusion (iv) in Lemma 4.2 holds.

(v) Assume $\alpha \in (\alpha_1,\alpha_4)\cup(\alpha_{10},-1)$ and Claim 3 holds. We can prove that $\lambda(x,\alpha)$ is increasing from $(0,0)$ to a maximum point (x^*,λ^*), then decreasing to the right endpoint $(1,\lambda^r)$, where $\lambda^* = \lambda(x^*,\alpha) > 0$. For any fixed λ, the number of roots of $\lambda\mathcal{S}_1(x,\alpha)+\mathcal{S}_2(x,\alpha)$ is exactly the number of intersection points of the curve $\{(x,\lambda) : \lambda = \lambda(x,\alpha),\ x \in (0,1)\}$ with the segment $\{(x,\lambda) : x \in (0,1)\}$. Thus, the assertion (v) in Lemma 4.2 is true.

(vi) Furthermore, assume $\alpha \in (\alpha_4,\alpha_{10})$ and Claim 4 holds. Then, we obtain that $\lambda(x,\alpha)$ is monotone in $(0,1)$, increasing from $(0,0)$ to the right endpoint $(1,\lambda^r)$. Hence, the item (vi) in Lemma 4.2 is proved.

The remaining part in proving Lemma 4.2 is to prove Claims 1-4. We will prove these claims by applying the polynomial boundary theory in computer algebra to parametric semi-algebraic systems $\{\mathcal{S}_1(x,\alpha), x > 0, 1-x > 0\}$ and $\{\mathcal{S}_3(x,\alpha), x > 0, 1-x > 0\}$.

Proof. (**Claim 1**) Computing the boundary of the parametric semi-algebraic system, $\{\mathcal{S}_1(x,\alpha), x > 0, 1-x > 0\}$, we obtain seven polynomials,

$$\begin{aligned}
bp_1(\alpha) &= \alpha,\\
bp_2(\alpha) &= \alpha+1,\\
bp_3(\alpha) &= \alpha+4,\\
bp_4(\alpha) &= \alpha^2+\frac{2}{3}\alpha+\frac{1}{6},\\
bp_5(\alpha) &= 3\,\alpha^3+17\,\alpha^2+9\,\alpha-53,
\end{aligned}$$

and $bp_6(\alpha)$ and $bp_7(\alpha)$ are lengthy polynomials with degrees 72 and 250, respectively. The boundary set $\{bp_i(\alpha)\}$ has a total of two roots in $(0,+\infty)$: $\alpha^\star$ and α^*, dividing the parameter space $\mathbb{R}$ into 3 intervals, where $\alpha^\star$ (≈ 1.384385705240) is the root of $bp_5(\alpha)$, isolated in

$$\left[\frac{438729342548815344426106865133}{316912650057057350374175801344}, \frac{175491737019526137770442746061 3}{126765060022822940149670320537 6}\right],$$

and α^* (≈ 1.384385814301) is the root of $bp_7(\alpha)$, isolated in

$$\left[\frac{10460123470104333670091 5}{7555786372591432341913 6}, \frac{5230061735052166835047 1}{3777893186295716170956 8}\right].$$

Taking any three sample points from the three intervals, $(0, \alpha^\star)$, $(\alpha^\star, \alpha^*)$ and $(\alpha^*, +\infty)$, and applying Sturm's theorem show that $\mathcal{S}_1(x, \alpha_{\text{sample}})$ has respectively one, two and zero simple roots in $(0, 1)$. Hence, $\mathcal{S}_1(x, \alpha)$ has no zeros in $(0, 1)$ when $\alpha \in (\alpha^*, +\infty)$ by applying polynomial boundary theory (Lemma 2.7). □

Proof. (**Claims 2-4**) Consider the parametric semi-algebraic system $\{\mathcal{S}_3(x, \alpha), x > 0, 1 - x > 0\}$. We obtain its boundary set consisting of 12 polynomials in α. The bifurcation roots of the boundary set in $\mathbb{R}$ are

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ -4,\ \alpha_6,\ \alpha_7,\ \alpha_8,\ \alpha_9,\ \alpha_{10}\ \ -1,\ 0,\ \alpha^\dagger,\ \frac{1}{3},\ 1,\ \alpha^{\dagger\dagger},\ \alpha^\ddagger,$$

where the estimated values and bounded intervals are given in (17).

To prove the three claims, we first study the number of roots of $\mathcal{S}_3(x, \alpha)$ when $\alpha > 0$, and we only need to consider $\alpha \in (\alpha^*, +\infty) \subset (\alpha^\ddagger, +\infty)$ since it has been shown in the proofs for (iv)–(vi) of Lemma 4.2 that $\mathcal{S}_1 \neq 0$. Choosing a sample point $\alpha^\diamond \in (\alpha^*, +\infty)$ shows that $\mathcal{S}_3(x, \alpha^\diamond)$ has two simple zeros in $(0, 1)$. Hence, $\mathcal{S}_3(x, \alpha)$ has two simple zeros in $(0, 1)$ when $\alpha \in (\alpha^*, +\infty)$ by Lemma 2.7.

Next, we determine the number of roots of $\mathcal{S}_3(x, \alpha)$ when $\alpha \in (-\infty, -1)$. Since α_i, $i = 1, 2, \cdots, 10$, and -4 divide the interval $(-\infty, -1)$ into 12 open subintervals: $(-\infty, \alpha_1)$, (α_i, α_{i+1}), $i = 1, 2, \ldots, 9$, $(\alpha_{10}, -1)$, we choose one sample point from each of the open intervals and applying Lemma 2.7 to obtain the following results. $\mathcal{S}_3(x, \alpha)$ has two simple zeros in $(0, 1)$ when $\alpha \in (-\infty, \alpha_1)$, a unique simple zero in $(0, 1)$ when $\alpha \in (\alpha_1, \alpha_2) \cup (\alpha_2, \alpha_3) \cup (\alpha_3, \alpha_4) \cup (\alpha_{10}, -1)$, and no zeros in $(0, 1)$ when $\alpha \in (\alpha_4, \alpha_5) \cup (\alpha_5, -4) \cup (-4, \alpha_6) \cup (\alpha_6, \alpha_7) \cup (\alpha_7, \alpha_8) \cup (\alpha_8, \alpha_9) \cup (\alpha_9, \alpha_{10})$.

As discussed above, the number of roots of a parametric semi-algebraic system when the parameter is on the boundary needs further analysis. However, the boundary points α_i cannot be determined exactly, and also it is not reliable to count the zeros of $\mathcal{S}_3(x, \alpha_i)$ by using the estimated values of α_i, $i = 0, 1, \cdots, 10$.

To complete the proof for Claim 3, we show that $\mathcal{S}_3(x, \alpha_i)$ for $i = 2, 3$ has a unique zero in $(0, 1)$. Applying the polynomial boundary theory, we can show that $\frac{d\mathcal{S}_3}{dx}(x, \alpha)$ has a unique zero in $(0, 1)$ when α is located in an open integral (α_1^*, α_4^*) which contains (α_1, α_4), implying that $\mathcal{S}_3(x, \alpha)$ has a unique critical point. A direct computation shows that $\mathcal{S}_3(0^+, \alpha) < 0$, $\mathcal{S}_3(1^-, \alpha) > 0$ and $\mathcal{S}_3'(0^+, \alpha) > 0$. Hence, $\mathcal{S}_3(x, \alpha)$ has a unique zero when $\alpha \in (\alpha_1, \alpha_4)$. This completes the proof for Claim 3.

Finally, to finish the proof for Claim 4, we introduce the transformations, $\alpha = \alpha_{10}^l + \frac{(\alpha_4^r - \alpha_{10}^l)}{1+b}$ and $x = \frac{1}{1+X}$, where α_4^r is the right-end point of the interval isolating α_4, and α_{10}^l is the left-end point of the interval isolating α_{10}, $b > 0$ and $X > 0$. With these transformations, we obtain,

$$\mathcal{S}_3(x, \alpha) = \frac{\mathcal{Q}(X, b)}{(1+y)^{162}(1+b)^{52}}.$$

The coefficients of the polynomial $\mathcal{Q}(X, b)$ are nonnegative and not identically zero. Thus, $\mathcal{Q}(X, b)$ has no roots in $\{(X, b) | X > 0, b > 0\}$, and so $\mathcal{S}_3(x, \alpha)$ has no zeros in $(0, 1)$ when $\alpha \in (\alpha_4, \alpha_{10})$. □

Therefore, the proof for Lemma 4.2 is finished. □

Proof. (**Theorem 1.2**) Consider the Abelian integral given in (10). When $a_2 = 0$, $\mathbb{I}(h, \delta) = a_1\mathbb{I}_1(h) + a_3\mathbb{I}_3(h) + a_0\mathbb{I}_0(h)$ has at most two zeros by combining (i)-(iii) in Lemma 4.2 with Lemma 2.2; when $a_2 \neq 0$, without loss of generality, we assume $a_2 = 1$. Further, we introduce a combination and replace a_0 by λ, then $\mathbb{I}(h, \delta) = a_1\mathbb{I}_1(h) + a_3\mathbb{I}_3(h) + (\lambda\mathbb{I}_0(h) + \mathbb{I}_2(h))$. Combining (i), (ii) and (iv)–(vi) in Lemma 4.2 with Lemma 2.3 finishes the proof of Theorem 1.2. $\square$

Remark 4.3. **(i)** To prove Theorem 1.2, a combination of two Abelian integrals has been introduced in $\mathbb{I}(h, \delta)$ to obtain an upper bound as 4 on the number of zeros of $\mathbb{I}(h)$. It is noted that the combination has advantage over directly applying Lemma 2.3 and the symbolic computation analysis to the set $\{\mathbb{I}_0(h), \mathbb{I}_1(h), \mathbb{I}_2(h), \mathbb{I}_3(h)\}$ for analyzing the zeros of the wronskian of four determining functions similar as $\mathcal{L}_1$, $\mathcal{L}_3$, $\mathcal{L}_0$ and $\mathcal{L}_2$. A direct application of Lemma 2.3 and the symbolic computation analysis can only reach the upper bound to be 5, when $\alpha \in (-\infty, \alpha_1) \cup (\alpha^*, +\infty)$. The combination introduced leads to a comprehensive analysis on the ratio of the two Wronskians and its derivative, which includes the major factor $\mathcal{S}_3(x, \alpha)$ given in (18). It is surprising to find that $\mathcal{S}_3(x, \alpha)$ can be factorized as $\mathcal{S}_3(x, \alpha) = 5W[\mathcal{L}_1, \mathcal{L}_3]W[\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_0, \mathcal{L}_2]$. For briefness, we do not present the very lengthy forms of $W[\mathcal{L}_1, \mathcal{L}_3]$ and $W[\mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_0, \mathcal{L}_2]$. Trying other combinations such as $a_2\mathbb{I}_2 + \mathbb{I}_0$ has also been carried out, however no better results than that in the proof of Theorem 1.2 are obtained on the range of α and on the number of zeros of $\mathbb{I}(h)$.

(ii) When $\alpha \in [\alpha_4, \alpha_{10}] \cup (-1, 0) \cup (0, \alpha^*)$, the upper bound in Theorem 1.2 is unknown because the second Wronskian has a zero and so the Chebyshev criterion is not applicable. However, it is not difficult to find three zeros of $\mathbb{I}(h)$ near $h = 0$ by applying the asymptotic expansion of $\mathbb{I}(h)$ near $h = 0$, and the zero bifurcation method [11].

5. An outline of proof for Theorem 1.3.

In this section, we outline the proof for Theorem 1.3. The related determination functions for system $(8)_{\beta=1}$ are denoted by $\mathscr{L}_i(x)$, $i = 0, 1, 2, 3$. Similarly, we have two preliminary lemmas.

Lemma 5.1.

$$W[\mathscr{L}_2(x)] = \frac{x^3 q_1(x,\alpha)}{90720\,(x-1)^7\,(x+1)^7\,(x^2+\alpha)^7},$$

$$W[\mathscr{L}_2(x), \mathscr{L}_3(x)] = \frac{x^7 q_2(x,\alpha)}{4115059200\,(x-1)^{13}\,(x+1)^{13}\,(x^2+\alpha)^{13}},$$

$$W[\mathscr{L}_2(x), \mathscr{L}_3(x), \mathscr{L}_0(x))] = \frac{x^4 S_1(x,\alpha)}{3110984755200\,(x-1)^{18}(x+1)^{18}(x^2+\alpha)^{18}},$$

$$W[\mathscr{L}_2(x), \mathscr{L}_3(x), \big(a_0\mathscr{L}_0(x)+\mathscr{L}_1(x)\big)] = \frac{x^4(\eta S_1(x,\alpha) + S_2(x,\alpha))}{3110984755200\,(x-1)^{18}(x+1)^{18}(x^2+\alpha)^{32}},$$

where $q_1(x,\alpha)$, $q_2(x,\alpha)$, $S_1(x,\alpha)$ and $S_2(x,\alpha)$ are polynomials with degrees 36, 66, 90 and 92, respectively.

Lemma 5.2. (i) $W[\mathscr{L}_2(x)]$ *does not vanish for* $x \in (0,1)$ *when* $\alpha \in (0, +\infty)$*;*

(ii) $W[\mathscr{L}_2(x), \mathscr{L}_3(x)]$ *does not vanish for* $x \in (0,1)$ *when* $\alpha \in (\alpha^{\maltese}, +\infty)$*, where* $\alpha^{\maltese} \in [0, \frac{1}{256}]$*;*

(iii) $W[\mathscr{L}_2(x), \mathscr{L}_3(x), \mathscr{L}_0(x))]$ *does not vanish when* $\alpha \in (\frac{3}{8}, +\infty)$*;*

(iv) *if* $\alpha \in (\alpha^{\maltese}, +\infty)$*, there exist some values* a_0^**,* a_0^{**} *and* a_0^r *depending on* α *such that* $W[\mathscr{L}_1(x), \mathscr{L}_3(x), \big(a_0\mathscr{L}_0(x) + \mathscr{L}_2(x)\big)]$ *has two zeros when* $a_0 \in [a_0^*, 0) \cup (a_0^r, a_0^{**}]$*, one zero when* $a_0 \in [0, \lambda^r]$*, and no zeros when* $a_0 \in (-\infty, a_0^*) \cup (a_0^{**}, +\infty)$*.*

Proof. (**Theorem 1.3**) When $a_1 = 0$, it can be shown by Lemmas 5.2 that the set $\{\mathscr{L}_2(x), \mathscr{L}_3(x), \mathscr{L}_0(x)\}$ is a Chebyshev system for $x \in (0, 1)$ provided $\alpha \in (\frac{3}{8}, +\infty)$. When $a_1 \neq 0$, without loss of generality, we assume $a_1 = 1$. Lemma 5.2 shows that the set $\{\mathscr{L}_2(x), \mathscr{L}_3(x), (a_0\mathscr{L}_0(x) + \mathscr{L}_1(x))\}$ is a Chebyshev system with accuracy 1 provided $\alpha \in (\frac{3}{8}, +\infty)$. Then Theorem 1.3 is proved by applying Lemma 2.3. □

Remark 5.3. An upper bound on the maximal number of zeros of the Abelian integral cannot be obtained by using the Chebyshev criterion. However, it is not difficult to show that $\mathbb{I}(h)$ has at least three zeros near $h = 0$ by applying the asymptotic expansion and the zero bifurcation method [11].

6. **Conclusion.** In this paper, the parameter range ensuring the Abelian integrals on a symmetric Hamiltonian to have Chebyshev property is bounded. The bounds on the number of zeros of the Abelian integrals of parametric Hamiltonians have not been well studied by directly using Chebyshev criterion, due to the difficulty in computation and analysis. We have developed a new method to overcome the difficulty arising from the large parametric-semi-algebraic systems, for which the traditional techniques in symbolic computation fail. We provide a rigorous proof that the exact bound on the number of zeros of the Abelian integral of system (8) is three for case (d), when the parameter α in the Hamiltonian is located in a bounded interval. In addition, the results in [14, 23, 36] by fixing parameters in (8) are extended to one-parametric system (8). The analytical tools used in this paper are a series of transformations of variables and polynomial boundary theory in computer algebra. However, for system (8) there exist more open problems on the number of zeros of $\mathbb{I}(h, \delta)$, which needs further research: (1) what is the bound and the parameter range for case (a) when system (8) has two free parameters α and β? (2) what is the number of zeros of $\mathbb{I}(h, \delta)$ on the periodic annulus outside the cusp-cycle for cases (c) and (d)? (3) what is the exact bound for cases (e) and (f), 3 or 4? (4) how to bound the number of the zeros of $\mathbb{I}(h, \delta)$ for cases (g) and (h), for which the algebraic method fails? To solve these problems, new methodologies and efficient symbolic algorithms need to be developed.

Conflict of interest. The authors declare that they have no conflict of interest.

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Received May 2020; revised August 2020.

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