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The monotonicity of ratios of some Abelian integrals ☆

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In this paper, we study the monotonicity of the ratio of the Abelian integrals, $\oint_{\gamma_i(h)} ydx$ and $\oint_{\gamma_i(h)} xydx$, in an interval, where $i = 1, 2$, and $\gamma_i(h)$ is a compact component of some hyperelliptic curves with genus 2 as $h \in \gamma_i(h)$. We give positive answers to the two conjectures proposed by Wang et al. (2014) [18].

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1. Introduction and main results

The well-known weak Hilbert's 16th problem [1] asks for the maximal number of isolated zeros of the following Abelian integral,

$$I(h) = \oint_{\Gamma_h} f(x,y)dx - g(x,y)dy, \quad h \in \Sigma,$$

where $f(x,y)$ and $g(x,y)$ are polynomials of degree m, Γ_h is a compact component of the level set $\{H(x,y) = h, h \in \Sigma\}$, where Σ represents an interval and the Hamiltonian $H(x,y)$ is an $(n+1)$th-degree polynomial. This open problem is extremely difficult, and researchers choose some simpler forms of $H(x,y)$, $f(x,y)$ and $g(x,y)$ to study, see [2] for a relatively new survey work.

Suppose that the Hamiltonian function $H(x,y)$ has the form $H(x,y) = y^2 + P_{n+1}(x)$, where $P_{n+1}(x)$ is a polynomial of degree $n+1$. In this case, the Abelian integral is usually called elliptic integral if $[\frac{n}{2}] < 2$, and hyperelliptic integral if $[\frac{n}{2}] \geq 2$. When $n = 2$, Petrov [3] proved that $I(h)$ has at most $m-1$ zeros and this upper bound is sharp for arbitrary m. When $n = 3$, Dumortier and Li [4–7] obtained the exact upper bound of the number of zeros of $I(h)$ for $m \leq 3$. Later, it was proved that the number of zeros of $I(h)$ is linearly dependent on m, see [8,9] and references therein. When $n > 3$, $I(h)$ is a hyperelliptic integral, and it is rather difficult to find the upper bound of the number of zeros of such an $I(h)$. However, it is still very interesting and important to study the hyperelliptic integrals with some small m such as $m = 2$. In this respect, if the ratio of two Abelian integrals,

$$\frac{\oint_{\Gamma(h)} ydx}{\oint_{\Gamma(h)} xydx},$$

is monotonic (in other words, the integrals $\oint_{\Gamma(h)} ydx$ and $\oint_{\Gamma(h)} xydx$ have Chebyshev property), then the Abelian integral $I(h)$ has at most one isolated zero when $f(x,y) = (\alpha_0 + \alpha_1 x)y$ and $g(x,y) = 0$.

In [10], Li and Zhang first provided a criterion to determine the monotonicity of the ratio of two Abelian integrals. It is very convenient to be used for determining the exact upper bound of the associated Abelian integral when $m = 2$. Grau et al. [11] generalized Li and Zhang's criterion to deal with the Chebyshev property of m Abelian integrals with $m > 2$, see [12–15]. Gavrilov and Iliev [16] studied the hyperelliptic curve with $H(x,y) = y^2 + P_5(x)$, and proved that the ratio of the two complete Abelian integrals of the first kind,

$$\frac{\oint_{\Gamma(h)} \frac{1}{y}dx}{\oint_{\Gamma(h)} \frac{x}{y}dx},$$

is monotonic. Later, Liu and Xiao [17] established a more useful criterion for proving the monotonicity of the ratio of two Abelian integrals, $\oint_{\Gamma(h)} ydx$ and $\oint_{\Gamma(h)} xydx$. Here $\Gamma(h)$ is the compact component of $y^2 + \Psi(x) = h$, when $\Psi(x)$ is analytic with a local minimum at the center of the corresponding Hamiltonian system. As an application, they obtained the sufficient and necessary conditions for monotonicity of the ratio of $\oint_{\Gamma(h)} ydx$ and $\oint_{\Gamma(h)} xydx$ on the hyperelliptic closed curves defined by $\{(x, y)|y^2 + P_5(x) = h\}$.

In [18], Wang et al. studied the monotonicity of the ratio of $\oint_{\Gamma(h)} ydx$ and $\oint_{\Gamma(h)} xydx$ on the hyperelliptic curves, given by

$$H(x, y) = y^2 + \int x(x - \alpha)(x - \beta)(x - \gamma)(x - 1)dx \tag{1.1}$$

and

$$H^*(x, y) = y^2 - \int x(x - \alpha)(x - \beta)(x - \gamma)(x - 1)dx. \tag{1.2}$$

The corresponding Hamiltonian systems are respectively described by

$$\dot{x} = -2y, \quad \dot{y} = x(x - \alpha)(x - \beta)(x - \gamma)(x - 1) \tag{1.3}$$

and

$$\dot{x} = 2y, \quad \dot{y} = x(x - \alpha)(x - \beta)(x - \gamma)(x - 1), \tag{1.4}$$

where $0 \le \alpha \le \beta \le \lambda \le 1$.

Moreover, in [18], the authors gave a complete classification of hyperelliptic curves and investigated the monotonicity of the ratios of the two Abelian integrals on these hyperbolic curves. A number of good results were obtained in [18]. In particular, when $\alpha = \beta = 0$ and $\lambda < 1$, system (1.3) is reduced to

$$\dot{x} = -2y, \quad \dot{y} = x^3(x - \lambda)(x - 1) \tag{1.5}$$

with the Hamiltonian function,

$$H(x, y) = y^2 + \tfrac{\lambda}{4}\, x^4 - \tfrac{1+\lambda}{5}\, x^5 + \tfrac{1}{6}\, x^6. \tag{1.6}$$

When $\beta = \lambda = 0$ and $0 < \alpha < 1$, system (1.3) becomes

$$\dot{x} = -2y, \quad \dot{y} = x(x - \alpha)(x - 1)^3, \tag{1.7}$$

with the Hamiltonian function

$$\mathcal{H}(x, y) = y^2 + \tfrac{\alpha}{2}\, x^2 - \tfrac{1+3\alpha}{3}\, x^3 + \tfrac{3+3\alpha}{4}\, x^4 - \tfrac{3+\alpha}{5}\, x^5 + \tfrac{1}{6}\, x^6. \tag{1.8}$$

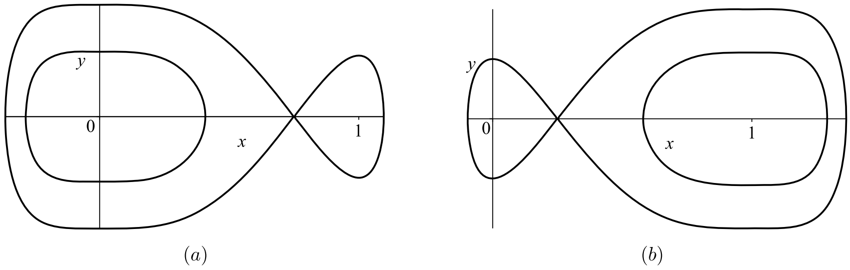

Fig. 1. The level set of $H(x,y) = h$: (a) for system (1.5) and (b) for system (1.7).

The two compact components of the level sets of $H(x,y) = h$ and $\mathcal{H}(x,y) = h$ surrounding the nilpotent point $(0,0)$ of systems (1.5) and the nilpotent point $(1,0)$ of system (1.7), are denoted by $\gamma_1(h)$ and $\gamma_2(h)$, as shown in Figs. 1(a) and 1(b), respectively. It should be pointed out that it is more difficult to analyze the bifurcation and related problems for these degenerate cases.

Let

$$I_0(h) = \oint_{\gamma_1(h)} ydx, \quad I_1(h) = \oint_{\gamma_1(h)} xydx, \quad \mathcal{I}_0(h) = \oint_{\gamma_2(h)} ydx, \quad \text{and } \mathcal{I}_1(h) = \oint_{\gamma_2(h)} xydx.$$

It has been proved in [18] that

Theorem WXY. (i) $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0, H(\lambda,0))$ *for* $\lambda \in (0, \frac{2}{3}]$, *and* (ii) $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ *is monotonic in the interval* $(\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ *for* $\alpha \in [\frac{1}{3}, 1)$.

Note that no answers are given in [18] for part (i) of Theorem WXY when $\lambda \in (\frac{2}{3}, 1)$ and for part (ii) when $\alpha \in (0, \frac{1}{3})$. Instead, the authors proposed the following conjecture.

Conjecture. (i) $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0, H(\lambda,0))$ *for* $\lambda \in (\frac{2}{3}, 1)$, *and* (ii) $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ *is monotonic in the interval* $(\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ *for* $\alpha \in (0, \frac{1}{3})$.

The aim of this paper is to give a positive answer to the above conjecture. Our main results are given in the following two theorems.

Theorem A. $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0, H(\lambda,0))$ *for* $\lambda \in (\frac{2}{3}, 1)$.

Theorem B. $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ *is monotonic in the interval* $(\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ *for* $\alpha \in (0, \frac{1}{3})$.

Combining Theorems WXY, A and B shows that $\frac{I_1(h)}{I_0(h)}$ is monotonic in the interval $(0, H(\lambda,0))$ for $\lambda \in (0,\ 1)$, and $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ is monotonic in the interval $(\mathcal{H}(1,0), \mathcal{H}(\alpha,0))$ for $\alpha \in (0\,1)$. Thus, any non-trivial linear combination, $a_0 I_0 + a_1 I_1$ (or $a_0\mathcal{I}_0 + a_1\mathcal{I}_1$), has at most one zero. By the Poincaré theorem (see [19]), system (1.5) (or system (1.7)) perturbed by $(a_0 + a_1 x)y$ (or $(\alpha_0 + \alpha_1 x)y$) can have at most one limit cycle, which can be reached.

Theorems A and B will be proved in Sections 2 and 3, respectively. The proofs are based on the criteria given in [17] and theory on the boundary of polynomial algebraic systems [20,21]. The techniques developed in this paper greatly simplify the analysis and can be applied to other types of differential equations.

2. Proof of Theorem A

Let

$$\Phi(x) = H(x,y) - y^2 = \tfrac{\lambda}{4}\,x^4 - \tfrac{1+\lambda}{5}\,x^5 + \tfrac{1}{6}\,x^6.$$

It is not difficult to prove that $\Phi'(x)x > 0$ and there exist two analytic functions $\mu(h)$ and $\nu(h)$ satisfying

$$\Phi(\mu(h)) \equiv \Phi(\nu(h)) \equiv h, \quad a_\lambda < \mu(h) < 0 < \nu(h) < \lambda,$$

where $\lambda \in (-0.43708017\cdots, 0)$ with $\Phi(a_\lambda) = \Phi(\lambda)$. Further, define the function

$$U(h) = \mu(h) + \nu(h).$$

Then, in the following, we will show that $U'(h) \neq 0$ in $(0, H(\lambda, 0))$ for $\lambda \in (\frac{2}{3}, 1)$, and thus the conclusion is true by using the criterion in [17].

$\Phi(\nu) = h$ and $\Phi'(\nu) > 0$ imply that $\nu'(h) > 0$ in $(0, H(\lambda, 0))$. Therefore, $\nu(h)$ has an inverse function $h = h^{-1}(\nu)$, which is substituted into $\mu(h)$ to yield $\mu(h) = \mu(\nu)$, where $\mu(\nu)$ is defined by $\Phi(\mu) - \Phi(\nu) = 0$, satisfying $a_\lambda < \mu < 0 < \nu < \lambda$. Factorizing $\Phi(\mu) - \Phi(\nu)$ gives $-\frac{\nu-\mu}{60} q(\nu, \mu, \lambda)$, where

$$\begin{aligned} q(\nu,\mu,\lambda) = {} & 12\,(\lambda+1)(\mu^4 + \mu^3\nu + \mu^2\nu^2 + \mu\,\nu^3 + \nu^4) - 15\lambda(\mu+\nu)(\mu^2+\nu^2) \\ & - 10(\mu+\nu)(\mu^2+\nu^2+\mu\nu)(\mu^2+\nu^2-\mu\nu). \end{aligned}$$

In fact, $\mu(\nu)$ is determined by $q(\mu, \nu, \lambda)$. Hence,

$$U'(h) = \left[\tfrac{d\mu}{d\nu} + 1\right]\nu'(h) = \left[-\tfrac{q_\mu(\nu,\mu,\lambda)}{q_\nu(\nu,\mu,\lambda)} + 1\right]\nu'(h) = 2(\mu-\nu)\,\tfrac{U_1(\nu,\mu,\lambda)}{U_2(\nu,\mu,\lambda)}\nu'(h),$$

where

$$\begin{aligned} U_1(\nu,\mu,\lambda) = {} & 6(\lambda+1)(3\,\mu^2 + 4\,\mu\,\nu + 3\,\nu^2) - 15\,\lambda\,(\mu+\nu) \\ & - 10\,(2\mu^3 + 3\,\mu^2\nu + 3\,\mu\,\nu^2 + 2\,\nu^3), \\ U_2(\nu,\mu,\lambda) = {} & 12\,\lambda\,(4\,\mu^3 + 3\,\mu^2\nu + 2\,\mu\,\nu^2 + \nu^3) - 15\,\lambda\,(3\,\mu^2 + 2\,\mu\,\nu + \nu^2) \\ & - 10\,(5\mu^4 + 4\,\mu^3\nu + 3\,\mu^2\nu^2 + 2\,\mu\,\nu^3 + \nu^4) + 12\,(4\,\mu^3 + 3\,\mu^2\nu + 2\,\mu\,\nu^2 + \nu^3). \end{aligned}$$

It is suffice to prove $U_i(\nu,\mu,\lambda)\neq 0$ for $i=1,2$ on

$$D:\ \left\{(\nu,\mu,\lambda)\middle|\kappa_\lambda<\mu<0<\nu<\lambda,\ \tfrac{2}{3}<\lambda<1\right\}.$$

Computing the resultant between U_2 and q with respect to ν gives $r_0 = -1296000000000\mu^{12}(\mu-1)^4(\lambda-\mu)^4$, which has no zeros on D. Therefore, U_2 and q have no common roots on D, which implies that $U_2(\nu,\mu,\lambda)\neq 0$ on D.

Similarly, computing the resultant between U_1 and q with respect to μ and ν respectively, we obtain

$$r_1(\nu,\lambda)=g(\nu,\lambda)\quad\text{and}\quad r_2(\mu,\lambda)=g(\mu,\lambda), \tag{2.1}$$

where $g(\omega,\lambda)$ is a polynomial, given by

$$\begin{aligned}
&g(\omega,\lambda)\\
&=-54000\omega^4(\lambda+1)\big[64\lambda^2\omega^4(81\lambda^4-162\lambda^3\omega+261\lambda^2\omega^2-300\lambda\omega^3+125\omega^4)\\
&\quad-16\lambda\omega^3(648\lambda^5-567\lambda^4\omega+990\lambda^3\omega^2-951\lambda^2\omega^3-900\lambda\omega^4+875\omega^5)\\
&\quad+8\omega^2(648\lambda^6+1620\lambda^5\omega-1521\lambda^4\omega^2+2790\lambda^3\omega^3-5997\lambda^2\omega^4+1800\lambda\omega^5+1000\omega^6)\\
&\quad-12\omega\,(270\lambda^6+747\lambda^5\omega-60\lambda^4\omega^2+801\lambda^3\omega^3-1860\lambda^2\omega^4-1268\lambda\omega^5+1600\omega^6)\\
&\quad+18\,(225\lambda^6+210\lambda^5\omega+303\lambda^4\omega^2+40\lambda^3\omega^3-676\lambda^2\omega^4-880\lambda\omega^5+928\omega^6)\\
&\quad-\ 27\,(325\,\lambda^5-140\,\lambda^4\omega+332\,\lambda^3\omega^2-480\,\lambda^2\omega^3-336\,\lambda\,\omega^4+384\,\omega^5)\\
&\quad+2\,(2025\,\lambda^4-1620\,\lambda^3\omega+2592\,\lambda^2\omega^2-5184\,\lambda\,\omega^3+2592\,\omega^4)\big].
\end{aligned}$$

Taking $\nu=\frac{\lambda}{1+t}$ and $\lambda=\frac{3}{4}+\frac{1/4}{1+s}$ yields

$$g(\nu,\lambda)=\tfrac{3375(4+3s)^8(8+7s)}{10456576(1+s)^{15}(1+t)^{12}}\,g^*(t,s),$$

where all coefficients of $g^*(t,s)$ are positive with $g^*(0,0)=1600$. Therefore, $g^*(t,s)>0$ on $\{(t,s)|t\in(0,+\infty),s\in[0,+\infty]\}$, which implies that U_1 and q have no common roots on

$$D_1:\ \left\{(\nu,\mu,\lambda)|\kappa_\lambda<\mu<0<\nu<\lambda,\ \tfrac{3}{4}\leq\lambda<1\right\}.$$

Hence, $U_1(\nu,\mu,\lambda)\neq 0$ on D_1, and so we have

Proposition 2.1. $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0,H(\lambda,0))$ *for* $\lambda\in[\frac{3}{4},1)$.

The remaining task is to investigate the problem on the region:

$$D\backslash D_1=\left\{(\nu,\mu,\lambda)|\kappa_\lambda<\mu<0<\nu<\lambda,\ \tfrac{2}{3}<\lambda<\tfrac{3}{4}\right\}.$$

We will apply the following techniques in polynomial theory, see [20,21] for more details.

Let k be a field, $x_1 \prec x_2 \prec \cdots \prec x_n$ be n ordered variables and $R = k[x_1, \cdots, x_n]$ be the polynomial ring on k. The greatest variable x_i in $f(x_1, \cdots, x_i)$ is called its main variable, denoted by $mvar(f)$. The coefficient of the main variable of f is called leading coefficient, denoted by $lc(f)$.

Definition 2.1 *(Semi-algebraic systems).* A semi-algebraic system (SAS for short) is a conjunctive polynomial formula of the following form:

$$\text{SAS},\quad \mathrm{S}: \begin{cases} p_1(x_1, \cdots, x_n) = 0, \cdots, p_s(x_1, \cdots, x_n) = 0, \\ g_1(x_1, \cdots, x_n) \geq 0, \cdots, g_r(x_1, \cdots, x_n) \geq 0, \\ g_{r+1}(x_1, \cdots, x_n) > 0, \cdots, g_t(x_1, \cdots, x_n) > 0, \\ h_1(x_1, \cdots, x_n) \neq 0, \cdots, h_m(x_1, \cdots, x_n) \neq 0, \end{cases} \tag{2.2}$$

where $n, s \geq 1$, $t \geq r \geq 0$, $m \geq 0$, all $p_i, g_i, h_i \in R(u, x)$ are polynomials with integer coefficients.

An SAS is called a parametric SAS if $s < n$ (s indeterminates are viewed as independent variables and the other $n - s$ indeterminates are treated as parameters, denoted by $u = (x_{s+1}, \cdots, x_n)$). An SAS is usually denoted by $[F, N, P, H]$, where $F = [p_1, \cdots, p_s]$, $N = [g_1, \cdots, g_r]$, $P = [g_{r+1}, \cdots, g_t]$ and $H = [h_1, \cdots, h_m]$.

There exist several well-known methods, such as the Ritt-Wu method, Gröbner basis method and subresultant method [22–24], which enable us to transform an SAS (2.2) equivalently to one or more TSAs: $\mathrm{T_1}, \cdots, \mathrm{T_l}$ in the form of

$$\text{TSA},\quad \mathrm{T_j}: \begin{cases} f_1^j(u, x_1) = 0, f_2^j(u, x_1, x_2) = 0, f_3^j(u, x_1, x_2, x_3) = 0, \cdots, \\ \quad f_s^j(x_1, \cdots, x_s) = 0, \\ g_1(x_1, \cdots, x_n) \geq 0, \cdots, g_r(x_1, \cdots, x_n) \geq 0, \\ g_{r+1}(x_1, \cdots, x_n) > 0, \cdots, g_t(x_1, \cdots, x_n) > 0, \\ h_1(x_1, \cdots, x_n) \neq 0, \cdots, h_m(x_1, \cdots, x_n) \neq 0, \end{cases} \tag{2.3}$$

where $\{f_1^j(u, x_1), f_2^j(u, x_1, x_2), f_3^j(u, x_1, x_2, x_3), \cdots, f_s^j(x_1, x_2, x_3, \cdots, x_s)\}$ is a triangular set, or a normal ascending chain.

Let $\mathrm{dis}(f_i)$ denote the discriminant of a polynomial f_i with respect to x_i, $\mathrm{res}(\cdot, \diamond, x_j)$ denote the Sylvester resultant of $\cdot$ and $\diamond$ with respect to x_j, and $\gcd(f_1, f_2, \cdots, f_i)$ denote the greatest common factor of $f_1, f_2, \cdots, f_i$.

Definition 2.2 *(Border polynomial of TSA).* Consider the parametric semi-algebraic system (2.3) TSA: $\mathrm{T_j}$. For convenience, f_i^j is denoted by f_i (only for this definition). The following polynomial is called border polynomial of (2.3):

$$\mathrm{B}_{\mathrm{T_j}} = lc(f_1)\mathrm{dis}(f_1)\prod_{2\le i\le s}\mathrm{res}(lc(f_i)\mathrm{dis}(f_i);f_{i-1},\cdots,f_1)\prod_{1\le j\le t}\mathrm{res}(g_j;f_s,\cdots,f_1)$$
$$\times\prod_{1\le k\le m}\mathrm{res}(h_k;f_s,\cdots,f_1),$$

where

$$\mathrm{res}(*;f_i,\cdots,f_1)=\mathrm{res}(\cdots(\mathrm{res}(\underline{\mathrm{res}(*,f_i,x_i)},f_{i-1},x_{i-1}),\cdots),f_1,x_1).$$

For two TSA: $\mathrm{T_j}$ and $\mathrm{T_{\tilde{j}}}$, let

$$r_i^{j\tilde{j}}=\gcd\big(\mathrm{res}(f_i^j;f_i^{\tilde{j}},f_{i-1}^{\tilde{j}},\cdots,f_1^{\tilde{j}}),\mathrm{res}(f_i^{\tilde{j}};f_i^j,f_{i-1}^j,\cdots,f_1^j)\big),\ 1\le i\le s,\quad\text{and}$$
$$C_{j\tilde{j}}=\gcd(r_1^{j\tilde{j}},\cdots,r_s^{j\tilde{j}}).$$

Definition 2.3 *(Border polynomial of SAS).* If a parametric SAS S is transformed equivalently to regular TSAs $\{\mathrm{T}_1,\cdots,\mathrm{T_l}\}$, then

$$\mathrm{B_S}=\prod_{1\le j\le\tilde{j}\le l}C_{j\tilde{j}}\prod_{j=1}^{l}\mathrm{B}_{\mathrm{T_j}}$$

is called the border polynomial of S.

Lemma 2.1. *[20,21] The number of distinct real solutions of the semi-algebraic system* S *is invariant in each connected component of the complement of* $\mathrm{B_S}=0$ *in* R^{n-s}*.*

Remark 2.1. When the parameter values satisfy the boundary $\mathrm{B_S}=0$, it is usually called degenerate case, for which it should be analyzed by other methods, see [20,21].

Based on the above described idea, Yang and Xia [20,21] developed a practical method for computing the border polynomial of S, which has been included into the computer algebra system − Maple.

To complete the proof of Theorem A, we construct the following semi-algebraic system to assure that the following semi-algebraic system,

$$\text{SAS},\quad \mathrm{S_A}:\begin{cases}q(\nu,\mu,\lambda)=0,\quad q(\lambda,\kappa,\quad\lambda)=0,\quad U_1(\nu,\mu,\lambda)=0,\\ \nu>0,\quad -\mu>0,\quad \lambda-\nu>0,\quad \mu-\kappa>0,\quad -\kappa>0,\end{cases}\tag{2.4}$$

has no roots. Computing its border polynomial we obtain

$$\mathrm{B}_{\mathrm{S_A}}=\Big(\lambda^9-\tfrac{5}{2}\lambda^8+\tfrac{41641\,\lambda^7}{24592}+\tfrac{15855\,\lambda^6}{6148}-\tfrac{693\,\lambda^5}{212}+\tfrac{575721\,\lambda^4}{196736}-\tfrac{4279635\,\lambda^3}{1573888}$$
$$-\tfrac{395847\,\lambda^2}{196736}+\tfrac{1563705\,\lambda}{393472}-\tfrac{133407}{98368}\Big)$$

$$\times \left(\lambda^8 - \frac{121\,\lambda^7}{38} + \frac{337543\,\lambda^6}{57456} - \frac{443339\,\lambda^5}{57456} + \frac{792185\,\lambda^4}{76608} - \frac{443339\,\lambda^3}{57456} \right.$$
$$\left. + \frac{337543\,\lambda^2}{57456} - \frac{121\,\lambda}{38} + 1\right)$$
$$\times \left(\lambda^6 - \tfrac{4}{3}\lambda^5 - \frac{13\,\lambda^4}{48} + \frac{2939\,\lambda^3}{864} - \frac{13\,\lambda^2}{48} - \tfrac{4}{3}\lambda + 1\right)\left(\lambda^2 - \tfrac{7}{4}\lambda + 1\right)\left(\lambda^2 + \tfrac{\lambda}{2} + 1\right)$$
$$\times (\lambda + 1)\left(\lambda - \tfrac{4}{3}\right)\left(\lambda - \tfrac{3}{2}\right)\left(\lambda - \tfrac{2}{3}\right)\left(-\tfrac{3}{4} + \lambda\right)\lambda.$$

It can be shown that $B_{S_A} = 0$ has a unique root, $\lambda^* = 0.70513143\cdots$, on $(\frac{2}{3}, \frac{3}{4})$, which is actually the root of the first factor of B_{S_A}. Therefore, the complement of $B_{S_A} = 0$ restrict to $\lambda \in (\frac{2}{3}, \frac{3}{4})$ is $(\frac{2}{3}, \lambda^*) \cup (\lambda^*, \frac{3}{4})$.

By Lemma 2.1, the number of zeros of (2.4) is invariant for $\lambda \in (\frac{2}{3}, \lambda^*)$ and for $\lambda \in (\lambda^*, \frac{3}{4})$. Therefore, we may choose a $\lambda \in (\lambda^*, \frac{3}{4})$ to investigate if $U_1(\nu, \mu, \lambda)$ vanishes on

$$D_2 = \left\{(\nu, \mu, \lambda) | \kappa_\lambda < \mu < 0 < \nu < \lambda, \lambda^* < \lambda < \tfrac{3}{4}\right\},$$

and take a $\lambda \in (\frac{2}{3}, \lambda^*)$ to investigate if $U_1(\nu, \mu, \lambda)$ vanishes on

$$D_3 = \left\{(\nu, \mu, \lambda) | \kappa_\lambda < \mu < 0 < \nu < \lambda, \tfrac{2}{3} < \lambda < \lambda^*\right\}.$$

First, taking $\lambda = \frac{72}{100} \in (\lambda^*, \frac{3}{4})$ and substituting it into (2.1) yields

$$r_2(\mu, \tfrac{72}{100}) = \frac{10336\,x^8}{5} - \frac{5333376\,x^7}{625} + \frac{5061371904\,x^6}{390625} - \frac{79435137024\,x^5}{9765625} + \frac{250937411616\,x^4}{244140625}$$
$$+ \frac{190180493568\,x^3}{244140625} - \frac{50040379584\,x^2}{244140625} + \frac{4225074048\,x}{48828125} - \frac{442158912}{9765625},$$

which has no roots on $(-\frac{9}{25}, 0) = (-0.36, 0)$, while

$$\kappa_\lambda = \kappa_{\frac{72}{100}} \in \left[-\frac{92651}{262144}, -\frac{46325}{131072}\right] \approx [-0.3534355164, -0.3534317017]$$

when $\lambda = \frac{72}{100}$. Hence, $r_2(\mu, \frac{72}{100}) \neq 0$ on $(\kappa_{\frac{72}{100}}, 0)$ which implies that U_1 and q have no common roots on D_2, and so $U_1(\nu, \mu, \lambda) \neq 0$ on D_2. Therefore, the following result holds.

Proposition 2.2. $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0, H(\lambda, 0))$ *for* $\lambda \in (\lambda^*, \frac{3}{4})$.

Next, we choose a value of $\lambda \in (\frac{2}{3}, \lambda^*)$ to investigate if $U_1(\nu, \mu, \lambda)$ vanishes on D_3. Taking $\lambda = \frac{7}{10} \in (\frac{2}{3}, \lambda^*)$ and substituting it into (2.1) gives

$$r_1(\nu, \tfrac{7}{10}) = r(\nu) \qquad \text{and} \qquad r_2(\mu, \tfrac{7}{10}) = r(\mu), \tag{2.5}$$

where

$$r(\omega) = 2120\,\omega^8 - \frac{43248\,\omega^7}{5} + \frac{8172924\,\omega^6}{625} - \frac{26278668\,\omega^5}{3125} + \frac{46507833\,\omega^4}{31250} + \frac{7359912\,\omega^3}{15625}$$
$$- \frac{1901151\,\omega^2}{15625} + \frac{157437\,\omega}{3125} - \frac{64827}{2500}.$$

By applying Sturm's Theorem, we obtain that $r_1(\nu, \frac{7}{10})$ has a unique root $\nu_1 \in [0, \frac{7}{10}]$, and $r_2(\mu, \frac{7}{10})$ has a unique root $\mu_1 \in [\kappa_{\frac{7}{10}}, 0]$. By real root isolating, $\nu_1 \in [\frac{57315}{131072}, \frac{114631}{262144}] \approx [0.4372787476, 0.4372825623]$, $\mu_1 \in [-\frac{86919}{262144}, -\frac{43459}{131072}] \approx [-0.3315696716, -0.3315658569]$, where $\kappa_{\frac{7}{10}} = -0.34603108\cdots$. Therefore, if $U_1(\nu, \mu, \frac{7}{10})$ and $q_1(\nu, \mu, \frac{7}{10})$ have a common root on D_3 with $\lambda = \frac{7}{10}$, the root must be in the regions defined by

$$\widetilde{D}: \ \left[\tfrac{57315}{131072}, \tfrac{114631}{262144}\right] \times \left[-\tfrac{86919}{262144}, -\tfrac{43459}{131072}\right].$$

In the following, we will prove that $U_1(\nu, \mu, \frac{7}{10})$ and $q(\nu, \mu, \frac{7}{10})$ have no common roots by showing that $q(\nu, \mu, \frac{7}{10}) \neq 0$ on $\widetilde{D}$.

The resultant $\mathrm{res}(\frac{\partial q}{\partial \nu}, \frac{\partial q}{\partial \mu}, \mu)$ has no roots on $[\frac{57315}{131072}, \frac{114631}{262144}]$ by Sturm's Theorem, implying that there is no maximal or minimum value inside $\widetilde{D}$. Thus, the maximal and minimum values of $q(\nu, \mu, \frac{7}{10})$ are reached on the boundaries of $\widetilde{D}$. However, a direct computation shows that $q(\nu, \mu, \frac{7}{10}) > 0$ on the four boundaries of $\widetilde{D}$ including the four intersections, indicating that both the maximal and minimum values of $q(\nu, \mu, \frac{7}{10})$ are positive. Hence, $q(\nu, \mu, \frac{7}{10}) \neq 0$ on $\widetilde{D}$, leading to $U_1(\nu, \mu, \frac{7}{10}) \neq 0$ on $\widetilde{D}$. The above discussion gives the following proposition.

Proposition 2.3. $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0, H(\lambda, 0))$ *for* $\lambda \in (\frac{2}{3}, \lambda^*)$.

The rest of this section is to prove $U_1(\nu, \mu, \lambda^*) \neq 0$ on

$$D_4 = \{(\nu, \mu, \lambda) | \kappa_\lambda < \mu < 0 < \nu < \lambda, \lambda = \lambda^*\}.$$

Recall that λ^* is the root of the first factor of $\mathrm{B_{S_A}}$, denoted by

$$\begin{aligned} w(\lambda) = {} & \lambda^9 - \tfrac{5}{2}\lambda^8 + \tfrac{41641\,\lambda^7}{24592} + \tfrac{15855\,\lambda^6}{6148} - \tfrac{693\,\lambda^5}{212} + \tfrac{575721\,\lambda^4}{196736} - \tfrac{4279635\,\lambda^3}{1573888} - \tfrac{395847\,\lambda^2}{196736} \\ & + \tfrac{1563705\,\lambda}{393472} - \tfrac{133407}{98368}. \end{aligned} \tag{2.6}$$

By computation and Sturm's Theorem, we can show that the resultant $\mathrm{res}(r_2, w, \lambda)$ has a unique zero $\mu_1^* = -0.34794635\cdots$ in $(-1, 0)$, and the resultant $\mathrm{res}(p(\kappa, \lambda), w, \lambda)$ has a unique zero $\kappa_{\lambda^*} = -0.34794635\cdots$ in $(-1, 0)$.

If $\mu_1^* = \kappa_{\lambda^*}$, then $r_2(\mu, \lambda^*)$ has no roots in the interval $(\kappa_{\lambda^*}, 0)$, implying that there are no common roots of $U_1(\nu, \mu, \lambda^*)$ and $q(\nu, \mu, \lambda^*)$ on D_4. Thus, $U_1(\nu, \mu, \lambda^*) \neq 0$ on D_4. In fact, it is true that $\mu_1^* = \kappa_{\lambda^*}$, because $\mathrm{res}(r_2, w, \lambda)$ and $\mathrm{res}(p(\kappa, \lambda), w, \lambda)$ have only one common factor, given by

$$\begin{aligned} cf = {} & 2477123436544\,\mu^{18} - 22294110928896\,\mu^{17} + 88660409909248\,\mu^{16} \\ & - 208881731469312\,\mu^{15} + 339714939006976\,\mu^{14} - 439074987073536\,\mu^{13} \\ & + 503505982218240\,\mu^{12} - 516729993408000\,\mu^{11} + 456405726382272\,\mu^{10} \\ & - 350788104117504\,\mu^{9} + 238040277061248\,\mu^{8} - 133699127790168\,\mu^{7} \end{aligned}$$

$$+ 58621983725097\,\mu^6 - 20391635790324\,\mu^5 + 3828815281827\,\mu^4$$
$$+ 1375428475098\,\mu^3 - 983422004562\,\mu^2 + 354781508544\,\mu - 142379421192,$$

which has a unique root ($= -0.34794635\cdots$) in $(-1, 0)$, while other factors of $\mathrm{res}(r_2, w, \lambda)$ and $\mathrm{res}(p, w, \lambda)$ have no common roots in $(-1, 0)$. Thus, we have

Proposition 2.4. $\frac{I_1(h)}{I_0(h)}$ *is monotonic in the interval* $(0, H(\lambda, 0))$ *for* $\lambda = \lambda^*$.

Combining Propositions 2.1–2.4, we have proved Theorem A.

3. Proof of Theorem B

In this section, we prove Theorem B corresponding to system (1.7). We simply transform system (1.7) to (1.5), and then the proof for Theorem A also works for Theorem B. To achieve this, taking the transformation, $x = -\widetilde{x} + 1$, $y = \widetilde{y}$ and $dt = -d\tau$, into system (1.7), and still use x and y for $\widetilde{x}$ and $\widetilde{y}$, we obtain a new system,

$$\dot{x} = -2y, \quad \dot{y} = x^3(x - (1 - \alpha))(x - 1) \tag{3.1}$$

with the Hamiltonian function, $\widetilde{\mathcal{H}}(x, y) = y^2 + \frac{1-\alpha}{4}x^4 - \frac{1+1-\alpha}{5}x^5 + \frac{1}{6}x^6$. The center level set $\gamma_2(h)$ of $\mathcal{H}(x, y) = h$ has been transformed to the origin, which is exactly the same as $\gamma_1(h)$. So it is obvious that system (3.1) is exactly the same as system (1.5) if denoting $1 - \alpha = \lambda$, and $\mathcal{I}_0(h) = I_0(h)$, $\mathcal{I}_1(h) = I_1(h)$. Therefore, $\frac{\mathcal{I}_1(h)}{\mathcal{I}_0(h)}$ on $(\mathcal{H}(1, 0), \mathcal{H}(\alpha, 0))$ has the same monotonicity as $\frac{I_1(h)}{I_0(h)}$ on $(0, H(\lambda, 0))$. The proof of Theorem B is complete.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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