International Journal of Bifurcation and Chaos, Vol. 27, No. 12 (2017) 1750192 (10 pages)

DOI: 10.1142/S0218127417501929

Global Existence and Uniqueness of Periodic Waves in a Population Model with Density-Dependent Migrations and Allee Effect

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Received September 26, 2017

We report a new result on the traveling wave solutions of a biological invasion model with density-dependent migrations and Allee effect. It has been shown in the literature that such a model can exhibit one periodic wave solution by using Hopf bifurcation theory. In this paper, global bifurcation theory is applied to prove that there exists maximal one periodic solution which can be reached in a large feasible parameter regime. The basic idea used in our technique is to examine the monotonicity of the ratio of related Abelian integrals. Especially, the existence condition for the solution near a homoclinic loop is obtained.

Keywords: Biological invasion model; isolated traveling wave; Abelian integral; limit cycle; homoclinic orbit; heteroclinic orbit.

1. Introduction

Once a new species appears in the new environment, its survival is the main issue: will the population density eventually increase, or will the population become extinct soon because of hostility of the new environment? If the species survives, then what is the population dynamics, and in particular, is there any oscillating motion? To understand this nonlinear phenomenon, researchers have studied the traveling front propagation in the population dynamical models for the single-species invasion [Hengeveld, 1989; Davis, 2009; Petrovskii & Li, 2006; Petrovskii & Venturino, 2008]. To be more realistic, such models should allow for migrations, due to environmental effects (density-independent) and biological mechanisms (also density-dependent) as well as the Allee effect on population growth.

Allee effect guarantees the growth of the biological invasion when the per capita growth rate is larger than some density value. When the Allee effect is sufficiently strong, the population experiences extinction when the rate is below some critical threshold [Schreiber, 2003]. In contrast, the population with a weak Allee effect does not have such

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a threshold and it surpasses to grow. The dynamics of the corresponding mathematical models reveal that an Allee effect can change or slow the pattern of range expansion, see [Davis *et al.*, 2004]. Some theoretical studies [Kot *et al.*, 1996; Lewis & Kareiva, 1993] and empirical results [Veit & Lewis, 1996] have shown that the Allee effect can adversely affect invasion rate of the spreading species. When considering the single-species invasion, the diffusion equation is the best tool. However, due to the complexity of the environment many other terms should be included in the simplified model. Petrovskii and Li [2003] built the following nonlinear advection-diffusion-reaction transport equation, describing the propagation of traveling population fronts of single-species invasion:

$$\frac{\partial U(X,T)}{\partial T} + (A_0 + 2A_1 U)\frac{\partial U}{\partial X} = D\frac{\partial^2 U}{\partial X^2} + \alpha U(U - U_0)(K - U), \quad (1)$$

whose physical behavior can vary strongly depending on the parameter values. Here, U denotes the population density, K is the species carrying capacity (therefore $K > U > 0$), and U_0 measures the Allee effect (U_0 satisfies $0 < U_0 < K$ if considering strong Allee effect and $U_0 = K$ is a limit case), α is a coefficient, and the parameters A_0 and A_1 represent respectively the speed of advection due to the impact of water or wind current and the speed of migration due to biological mechanisms.

Petrovskii and Li [2003] obtained the analytical solutions and show that the direction of the front propagation can be different depending on the parameter values and thus on the relative intensity of the density-dependent factors. In [Sherratt, 2012], an efficient numerical continuation method was developed to study small-amplitude periodic traveling waves due to Hopf bifurcation for a more generalized system of (1). In [Almeida *et al.*, 2006], a finite element method is applied to (1) to show, from the view point of numerical computation, that the density-dependent character plays the major role in obtaining more accurately approximate solutions for the biological meaningfulness.

For simplicity, assuming $D > 0$, $\alpha > 0$, $A_0 > 0$, $A_1 > 0$ and introducing the transformation,

$$u = \frac{U}{K}, \quad t = T\alpha K^2, \quad x = X\sqrt{\frac{\alpha K^2}{D}},$$

into (1) yields the dimensionless model,

$$u_t + (a_0 + a_1 u)u_x = u_{xx} - \beta u + (1+\beta)u^2 - u^3, \quad (2)$$

where the new parameters are defined as $\beta = U_0 K^{-1}$, $a_0 = A_0 K^{-1}(\alpha D)^{-1/2}$ and $a_1 = 2A_1 K^{-1} \times (\alpha D)^{-1/2}$. Obviously, all β, a_0 and a_1 take positive values. However, for being biologically meaningful, β must be chosen from the interval $[0, 1]$.

An interesting pattern of the propagation about the species invasion is the periodic oscillation on the density of invasion population, which is modeled by the periodic traveling waves, see [Sherratt & Smith, 2008]. Wang and Huang [2014] studied Hopf bifurcation in system (1) and proved that system (1) can have at most one small-amplitude periodic traveling wave solution near the steady state $(\phi, y) = (1, 0)$ for the parameter values satisfying [see system (7)]

$$e^{a_0+a_1-c} - 1 \ll \left|\frac{a_1(1-\rho^2)\pi}{2\rho^5}\right|, \quad \frac{a_1(1-\rho^2)\pi}{2\rho^5} < 0, \quad (3)$$

where $\rho^2 = \beta - 1$, by assuming that a_0 is sufficiently close to c, and $|a_1|$ is sufficiently small. Note that $\rho^2 \geq 0$ implies $\beta \geq 1$, and thus the result may only have mathematical interest.

Even though some good results have been obtained for system (1), it is still a major concern to assess the physical behavior regarding species invasion for interplay among diffusion, advection and reaction phenomena. In this paper, we pay attention to the periodic oscillation on the density of invasion population caused by the interaction between the advection due to the impact of wind or water current and the migration due to biological mechanisms. We consider the global bifurcation and prove that system (1) can have maximal one bounded periodic traveling wave for any $\beta \in (0, 1)$. This indicates that the solution we obtain is global, unlike that obtained due to Hopf bifurcation which is restricted to the vicinity of the Hopf critical point. Moreover, our analysis shows that the largest amplitude of periodic solutions can be reached near the solitary solution. In order to give a comparison with the result given in [Wang & Huang, 2014], we take the same assumption used in [Wang & Huang, 2014] that a_0 is sufficiently close to c, and $|a_1|$ is sufficiently small. Our results reveal that the ratio between $a_0 - c$ and a_1 should be in an interval depending on β, for which the invasion population density can have oscillation; the amplitude and distribution of the periodic wave depend on the ratio; the periodic wave tends to have Hopf bifurcation and homoclinic bifurcation when the ratio is at the boundary of the interval, see Sec. 3; and the variation of invasion population density must approximate the periodic oscillation with time and distance.

In the next section, we deduce system (7) from (1), and present the global bifurcation theory. In Sec. 3, we prove our main results. Simulation is given in Sec. 4, and finally, conclusion is drawn in Sec. 5.

2. System Reduction and Poincaré Bifurcation

To study the traveling waves that system (1) [or (2)] may exhibit, assume a continuous traveling solution of model (1) is given in the form of

$$u(x,t) = \phi(\xi), \quad \xi = x - ct, \tag{4}$$

for $\xi \in (-\infty, +\infty)$, satisfying

$$\lim_{\xi\to\infty} \phi(\xi) = m, \quad \lim_{\xi\to-\infty} \phi(\xi) = n, \tag{5}$$

where c is the propagation speed of a wave. Substituting (4) into (2) we obtain

$$\phi''(\xi) = (a_0 - c + a_1 u(\xi))\phi'(\xi) + \beta\phi(\xi) - (1+\beta)\phi^2(\xi) + \phi^3(\xi). \tag{6}$$

Further, let $\phi = u(\xi)$, $y = \phi'(\xi)$. Then, (6) becomes a dynamical system described by the following ordinary differential equations:

$$\begin{aligned} \frac{d\phi}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \phi(\phi-1)(\phi-\beta) + (a_0 - c + a_1\phi)y. \end{aligned} \tag{7}$$

$u(x,t)$ is a solitary wave solution if $m = n$ and a kink or anti-kink solution if $m \neq n$. The solitary wave solution of (1) corresponds to a homoclinic orbit of (7), while the kink (or anti-kink) wave solution of (1) corresponds to a heteroclinic orbit (or so-called connecting orbit) of (7). A periodic orbit of (7) corresponds to a periodic traveling wave solution of (1). A limit cycle (isolated periodic orbit) of (7) corresponds to an isolated periodic traveling wave solution of (1).

It is easy to see that system (7) has three singular points $(0,0)$, $(1,0)$ and $(\beta,0)$ denoted by $S_i(\phi_i, 0)$, $i = 1,2,3$, respectively. Since the population density U is positive and it is easy to verify that $(0,0)$ is a saddle for $\beta > 0$, we only need to investigate the dynamics of system (7) on the right-half plane of the ϕ–y plane, and in particular to focus on the dynamical behavior near the three singular points. Because in this paper, we are restricted to the parameter values when a_0 is sufficiently close to c, and $|a_1|$ is sufficiently small, we may take the following rescaling,

$$a_0 - c = \varepsilon\alpha_0 \quad \text{and} \quad a_1 = \varepsilon\alpha_1, \tag{8}$$

where α_0 and α_1 are bounded parameters, and $0 < \varepsilon \ll 1$ denotes small perturbations. Then system (7) can be rewritten as

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \phi(\phi-1)(\phi-\beta) + \varepsilon(\alpha_0 + \alpha_1\phi)y. \tag{9}$$

The unperturbed system $(9)_{\epsilon=0}$ has the Hamiltonian function,

$$H(\phi, y) = \frac{y^2}{2} - \frac{\beta}{2}\phi^2 + \frac{\beta+1}{3}\phi^3 - \frac{1}{4}\phi^4. \tag{10}$$

By the theory of planar dynamical systems (e.g. see [Guckenheimer & Holmes, 1983; Han & Yu, 2012; Chow & Hale, 1981]), the phase portraits of system $(9)_{\epsilon=0}$ can be classified into five cases with the closed orbits defined by the following functions:

$$\Gamma_h:\ H(\phi, y) = h \begin{cases} \text{(a)}\ \ h \in \left(\dfrac{\beta^3(\beta-2)}{12}, 0\right) & \text{for } \beta \in \left(0, \dfrac{1}{2}\right), \\ \text{(b)}\ \ h \in \left(-\dfrac{1}{64}, 0\right) & \text{for } \beta = \dfrac{1}{2}, \\ \text{(c)}\ \ h \in \left(\dfrac{\beta^3(\beta-2)}{12}, \dfrac{1-2\beta}{12}\right) & \text{for } \beta \in \left(\dfrac{1}{2}, 1\right), \\ \text{(d)}\ \ \text{No closed orbits} & \text{for } \beta = 0, \\ \text{(e)}\ \ \text{No closed orbits} & \text{for } \beta = 1. \end{cases} \tag{11}$$

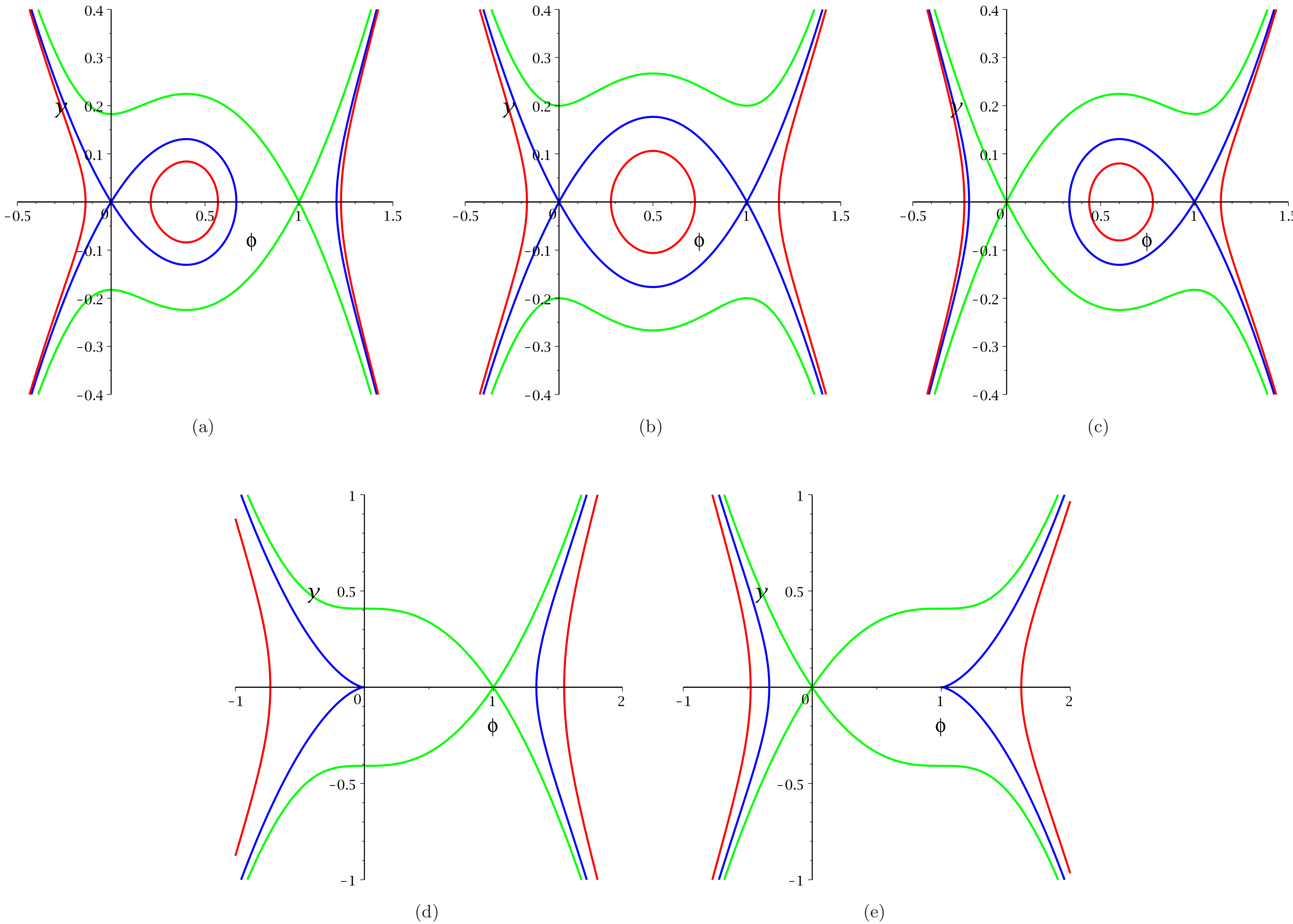


Fig. 1. The phase portraits of system $(9)_{\epsilon=0}$ for (a) $\beta = \frac{2}{5} \in (0, \frac{1}{2})$, (b) $\beta = \frac{1}{2}$, (c) $\beta = \frac{3}{5} \in (\frac{1}{2}, 1)$, (d) $\beta = 0$ and (e) $\beta = 1$.

Five typical graphs are shown in Fig. 1, each of them corresponds to one value of β taken from one of the five intervals.

In this paper, we concentrate on general cases, and leave the degenerate cases $\beta = 1$ and $\beta = 0$ (each has a nilpotent critical point) for further study. These critical cases may exhibit some interesting dynamical behaviors. Therefore, in the following we only study the Cases (a)–(c). For Case (a), $H(\phi, y) = \frac{\beta^3(\beta-2)}{12}$ defines an elementary center $(\beta, 0)$ and $H(\phi, y) = 0$ defines a homoclinic loop, denoted by Γ_{h^*} passing through the hyperbolic saddle $(0, 0)$. For Case (c), $H(\phi, y) = \frac{\beta^3(\beta-2)}{12}$ defines an elementary center $(\beta, 0)$ and $H(\phi, y) = \frac{1-2\beta}{12}$ defines a homoclinic loop, denoted by Γ_{h^*} passing through the hyperbolic saddle $(1, 0)$.

Now, suppose one closed orbit of system (9) is transversal to the positive ϕ-axis at $A(h) = (a(h), 0)$, where the positive ϕ-axis can be parameterized by h. Let $B(h) = (b(h), 0)$ be the first intersection point of the closed orbit starting from $A(h)$ with the positive ϕ-axis. Then, the displacement function of system (9) can be obtained as [Han & Yu, 2012]

$$d(h, \epsilon) = \int_{\widehat{AB}} dH = \varepsilon(I(h, \delta) + O(\epsilon)), \qquad (12)$$

where $\delta = (\alpha_0, \alpha_1)$, and

$$\begin{aligned} I(h, \delta) &= \oint_{\Gamma_h} (\alpha_0 + \alpha_1 \phi) y d\phi \\ &= \alpha_0 I_0(h) + \alpha_1 I_1(h), \end{aligned} \qquad (13)$$

with $I_0 = \oint y d\phi$ and $I_1(h) = \oint \phi y d\phi$. By Poincaré bifurcation theory [Han & Yu, 2012], the number of zeros of $I(h, \delta)$ corresponds to the number of limit cycles of system (9), which is the number of isolated periodic traveling waves. If the ratio $I_1(h)/I_0(h)$ is monotonic, then $I(h, \delta)$ has at most one zero, which can be reached. In fact, we have the following result.

Theorem 1. *For system (9), the ratio $I_1(h)/I_0(h)$ is monotonic for $\beta \in (0, 1)$.*

To prove Theorem 1, we will apply a result obtained by Li and Zhang [1996] in the study of the weak Hilbert's 16th problem. In [Li & Zhang, 1996], a criterion is given for determining the monotonicity of the ratio, $\frac{\oint_{\Gamma_h} f_2(x) y dx}{\oint_{\Gamma_h} f_1(x) y dx}$, where Γ_h is a family of ovals described by

$$H(x, y) = \Phi(x) + \Psi(y),$$

which surrounds the origin, and $\Phi(x)$ and $\Psi(x)$ are polynomials. Then, Γ_h represents a closed orbit of the Hamiltonian system,

$$\dot{x} = \Psi'(y), \quad \dot{y} = -\Phi'(x).$$

As an application, the following result is obtained in [Li & Zhang, 1996].

Lemma 1 [Li & Zhang, 1996]. *Let Γ_h be the ovals surrounding the origin of $H(x, y) = \frac{y^2}{2} + ax^2 + bx^3 + cx^4$, where $a > 0$ and b, $c < 0$. Then, the ratio $\frac{\oint_{\Gamma_h} xydx}{\oint_{\Gamma_h} ydx}$ is monotonic on $(0, h_1)$, where $h_1 = H(x_1, 0)$ with $x_1 > 0$ satisfying $H_x(x_1, y) = 0$.*

3. Proof of Theorem 1

Proof of Theorem 1. We first prove Cases (a) and (c), and then Case (b). For Case (a): $0 < \beta < \frac{1}{2}$, introducing the transformation $w = -\phi + \beta$ and $t = -\tau$ into system (9) yields

$$\begin{aligned} \frac{dw}{d\tau} &= y, \\ \frac{dy}{d\tau} &= w(w-\beta)(w-\beta+1) \\ &\quad + \varepsilon(-\alpha_0 - \beta\alpha_1 + \alpha_1 w)y, \end{aligned} \qquad (14)$$

with the following Hamiltonian function for the unperturbed system $(14)_{\varepsilon=0}$,

$$\begin{aligned} \tilde{H}(w, y) &= H(\beta - w, y) \\ &= \frac{\beta^3(\beta-2)}{12} + \frac{y^2}{2} + \frac{\beta(1-\beta)}{2} w^2 \\ &\quad + \frac{2\beta - 1}{3} w^3 - \frac{1}{4} w^4. \end{aligned} \qquad (15)$$

The Abelian integral of (14) is

$$\begin{aligned} \tilde{I}(h) &= \alpha_0 \oint_{\tilde{\Gamma}_h} -ydw + \alpha_1 \oint_{\tilde{\Gamma}_h} (-\beta + x) y dw \\ &\equiv \alpha_0 \tilde{I}_0(h) + \alpha_1 \tilde{I}_1(h), \end{aligned} \qquad (16)$$

where $\tilde{\Gamma}_h = \{\tilde{H} = h \,|\, h \in (\frac{\beta^3(\beta-2)}{12}, 0)\}$.

We will prove

$$\tilde{I}_0(h) = I_0(h), \quad \tilde{I}_1(h) = I_1(h).$$

From the transformation, we have

$$\tilde{I}_0(h) = \oint_{\tilde{\Gamma}_h} -ydw = \oint_{\tilde{\Gamma}_h} -y^2 d\tau = \oint_{\tilde{\Gamma}_h} y^2 dt = \oint_{\Gamma_h} y^2 dt = \oint_{\Gamma_h} yd\phi = I_0(h)$$

and

$$\tilde{I}_1(h) = \oint_{\tilde{\Gamma}_h} (-\beta + w) ydw = \oint_{\tilde{\Gamma}_h} (-\beta + w) y^2 d\tau = \oint_{\tilde{\Gamma}_h} (\beta - w) y^2 dt = \oint_{\Gamma_h} \phi y^2 dt = \oint_{\Gamma_h} \phi y d\phi = I_1(h).$$

Therefore,

$$\frac{I_1(h)}{I_0(h)} = \frac{\tilde{I}_1(h)}{\tilde{I}_0(h)} = \frac{\oint_{\tilde{\Gamma}_h} (-\beta + w) ydw}{\oint_{\tilde{\Gamma}_h} -ydw} = \beta - \frac{\oint_{\tilde{\Gamma}_h} wydw}{\oint_{\tilde{\Gamma}_h} ydw}.$$

Then, by Lemma 3.1, we know that $\frac{\oint_{\tilde{\Gamma}_h} wydw}{\oint_{\tilde{\Gamma}_h} ydw}$ is monotonic, and thus $I_1(h)/I_0(h)$ is monotonic.

For Case (c): $\frac{1}{2} < \beta < 1$, under the transformation $w = \phi - \beta$, system (9) becomes

$$\frac{dw}{dt} = y, \quad \frac{dy}{dt} = w(w+\beta)(w+\beta-1) + \varepsilon(\alpha_0 + \beta\alpha_1 + \alpha_1 w) y. \tag{17}$$

Then, by applying the same procedure used in proving Case (a), to system (17), we can show that $I_1(h)/I_0(h)$ is monotonic.

Finally, we consider Case (b): $\beta = \frac{1}{2}$. Similarly, we introduce the transformation $w = \phi - \frac{1}{2}$ into system (9) to obtain

$$\frac{dw}{dt} = y, \quad \frac{dy}{dt} = w\left(w + \frac{1}{2}\right)\left(w - \frac{1}{2}\right) + \varepsilon\left(\alpha_0 + \frac{\alpha_1}{2} + \alpha_1 w\right) y, \tag{18}$$

with the Hamiltonian function,

$$\overline{H}(w, y) = H\left(w + \frac{1}{2}, y\right) = -\frac{1}{64} + \frac{y^2}{2} + \frac{1}{8}w^2 - \frac{w^4}{4}, \tag{19}$$

for the unperturbed system $(18)_{\varepsilon=0}$. The Abelian integral of system (18) is given by

$$\overline{I}(h) = \alpha_0 \oint_{\overline{\Gamma}_h} ydw + \alpha_1 \oint_{\overline{\Gamma}_h} \left(\frac{1}{2} + w\right) ydw \equiv \alpha_0 \overline{I}_0(h) + \alpha_1 \overline{I}_1(h), \tag{20}$$

where $\overline{\Gamma}_h = \{\overline{H} = h \mid h \in (-\frac{1}{64}, 0)\}$.

We will prove

$$\overline{I}_0(h) = I_0(h), \quad \overline{I}_1(h) = I_1(h).$$

With the transformation, a direct computation shows that

$$\overline{I}_0(h) = \oint_{\overline{\Gamma}_h} ydw = \oint_{\overline{\Gamma}_h} y^2 d\tau = \oint_{\overline{\Gamma}_h} y^2 dt = \oint_{\Gamma_h} y^2 dt = \oint_{\Gamma_h} yd\phi = I_0(h)$$

and

$$\overline{I}_1(h) = \oint_{\overline{\Gamma}_h} \left(\frac{1}{2} + w\right) ydw = \oint_{\overline{\Gamma}_h} \left(\frac{1}{2} + w\right) y^2 d\tau = \oint_{\overline{\Gamma}_h} \left(\frac{1}{2} + w\right) y^2 dt = \oint_{\Gamma_h} \phi y^2 dt = \oint_{\Gamma_h} \phi y d\phi = I_1(h).$$

Hence,

$$\frac{I_1(h)}{I_0(h)} = \frac{\overline{I}_1(h)}{\overline{I}_0(h)} = \frac{\oint_{\overline{\Gamma}_h} \left(\frac{1}{2} + w\right) ydw}{\oint_{\overline{\Gamma}_h} ydw} = \frac{1}{2} + \frac{\oint_{\overline{\Gamma}_h} wydw}{\oint_{\overline{\Gamma}_h} ydw}.$$

By symmetry of the unperturbed system $(18)_{\varepsilon=0}$, $\oint_{\overline{\Gamma}_h} wydw = 0$, and thus,

$$\frac{I_1(h)}{I_0(h)} \equiv \frac{1}{2}.$$

This completes the proof for Theorem 1. ∎

(i) Existence of periodic solution for Cases (a) and (c). For Cases (a) and (c),

$$\lim_{\Gamma_h\to\Gamma_c}\frac{I_1(h)}{I_0(h)}=\beta \quad\text{and}\quad \lim_{\Gamma_h\to\Gamma_h}\frac{I_1(h)}{I_0(h)}=f^*(\beta).$$

Therefore, $\frac{I_1(h)}{I_0(h)}\in(\beta,f^*(\beta))$. Because

$$\begin{aligned}I(h)&=\alpha_0I_0(h)+\alpha_1I_1(h)\\&=I_0(h)\left(\alpha_0+\alpha_1\frac{I_1(h)}{I_0(h)}\right)\\&=\alpha_1I_0(h)\left(\frac{\alpha_0}{\alpha_1}+\frac{I_1(h)}{I_0(h)}\right),\end{aligned}$$

$I(h)$ has a zero at h^* if choosing

$$\frac{\alpha_0}{\alpha_1}=-\frac{I_1(h^*)}{I_0(h^*)}. \tag{21}$$

The Implicit Function Theorem shows that $d(h,\epsilon)$ has a zero near h^*. Therefore, there exists a unique limit cycle for system (7), and so system (1) has a unique periodic wave.

It follows from [Han & Yu, 2012] that $\frac{\alpha_0}{\alpha_1}=-\beta$ is the critical value for Hopf bifurcation, and $\frac{\alpha_0}{\alpha_1}=-f^*(\beta)$ is the critical value for homoclinic bifurcation.

(ii) Nonexistence of periodic solution for Case (b). For Case (b),

$$\begin{aligned}I(h)&=\alpha_0I_0(h)+\alpha_1I_1(h)\\&\equiv\left(\alpha_0+\frac{\alpha_1}{2}\right)I_0(h)\\&=\frac{3\alpha_0I_0(h)}{2}.\end{aligned}$$

Because

$$I_0'(h)=\oint_{\Gamma_h}dt>0 \quad\text{and}\quad \lim_{h\to\frac{\beta^3(\beta-2)}{12}}I_0(h)=0,$$

we conclude that $I_0(h)\neq0$. Hence, $I(h)\neq0$ if $\alpha_0+\frac{\alpha_1}{2}\neq0$.

When $\alpha_0+\frac{\alpha_1}{2}=0$, system (18) is time reversible. Therefore, $(0,0)$ is a center of system (18) surrounded by a family of closed orbits, which implies that $(\beta,0)$ is a center of system (9) surrounded by a family of closed orbits in the annulus $\{\Gamma_h\}$.

4. Simulations

In this section, we present simulations to verify the theoretical results we obtained in the previous sections. It has been shown that there exists a periodic solution if taking ε sufficiently small, $\beta=\frac{1}{4}$ and α_i satisfying $\frac{\alpha_0}{\alpha_1}\in(-0.25,-0.2202322892)$. In the following, we fix $\beta=\frac{1}{4}$ and choose three different initial values $(x,0)$ to get their corresponding Hamiltonian values, and then obtain the ratio of the Abelian integrals, by which we will decide the ratio of α_0 and α_1 according to the relationship (21). For simplicity, we choose $\alpha_1=1$. Using

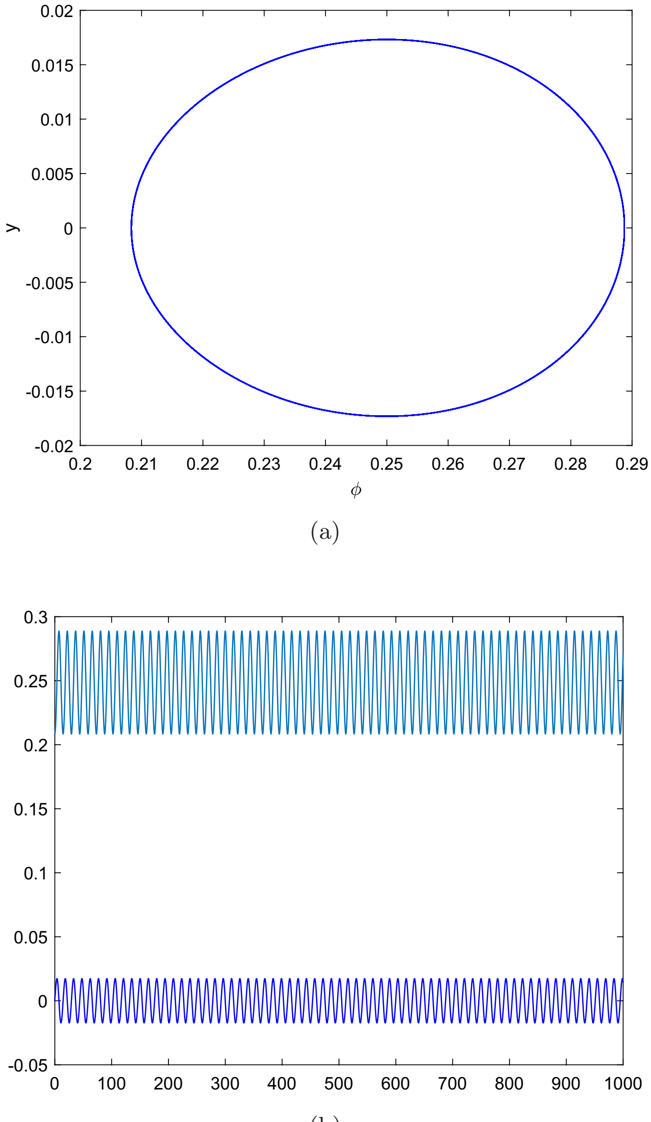
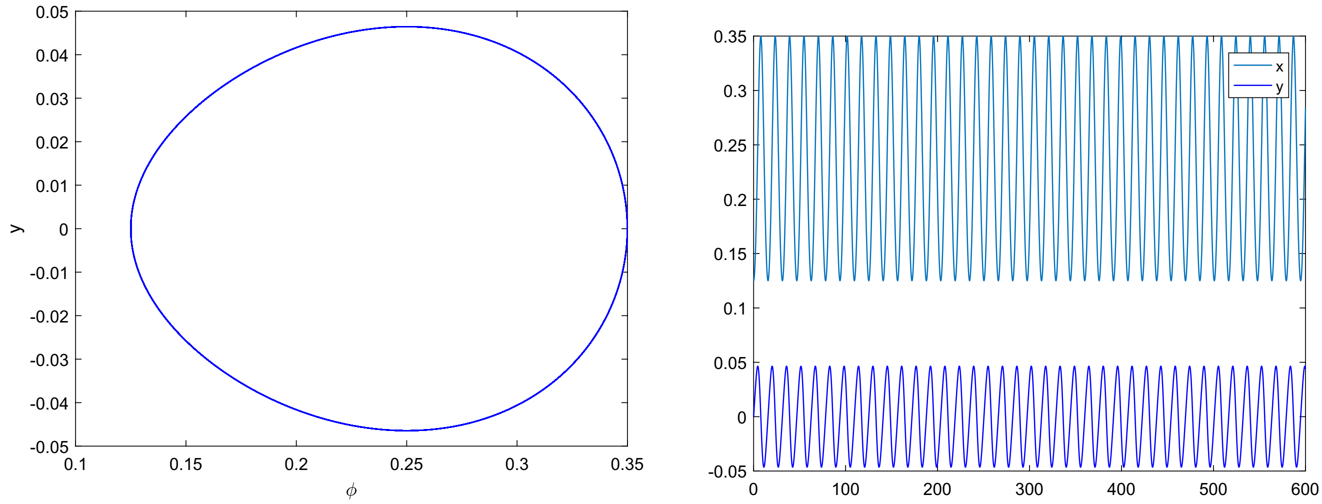


Fig. 2. Simulated periodic solution of system (9) with $\beta=\frac{1}{4}$, $\varepsilon=0.001$, $\alpha_0=-0.2489121908$, $\alpha_1=1$ and the initial value $(w,y)=(\frac{5}{24},0)$.



(a) (b)

Fig. 3. Simulated periodic solution of system (9) with $\beta = \frac{1}{4}$, $\varepsilon = 0.001$, $\alpha_0 = -0.2409185309$, $\alpha_1 = 1$ and the initial value $(w, y) = (\frac{1}{8}, 0)$.

the fixed values of α_0 and α_1, we solve (9) numerically. The following three simulations correspond to the three cases (a)–(c), as shown in (11).

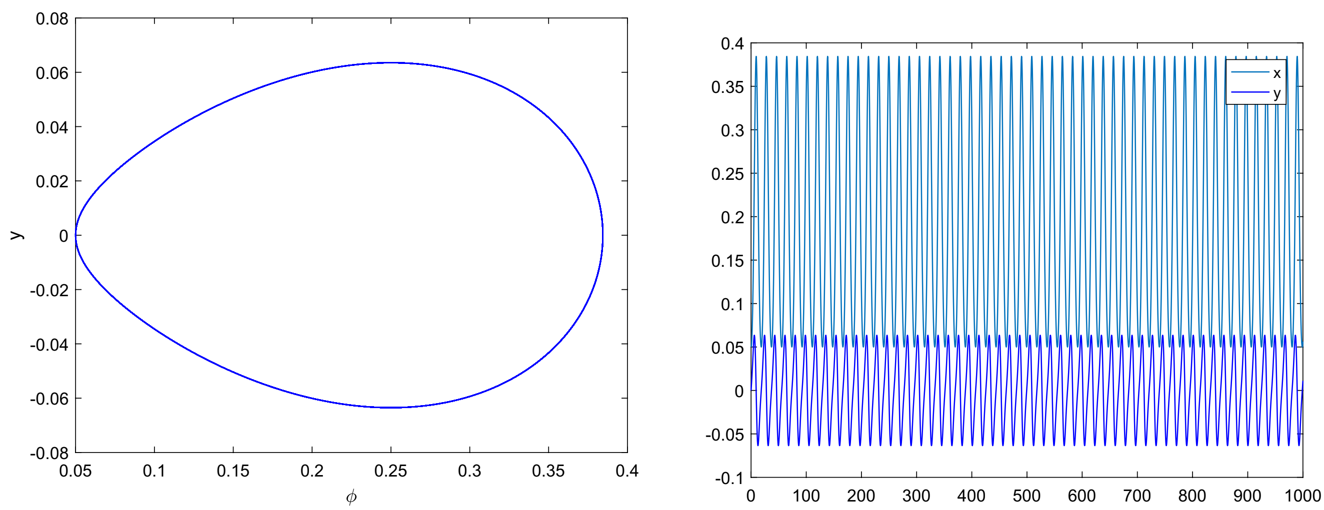
(a) Taking $w = \frac{5}{24}$, we get $\mathcal{H}(\frac{5}{24}, 0) = -\frac{2825}{1327104}$. A direct computation shows that

$$\frac{I_1\left(-\dfrac{2825}{1327104}\right)}{I_0\left(-\dfrac{2825}{1327104}\right)} \approx 0.2489121908.$$

Then we take $\varepsilon = 0.001$, $\alpha_0 = -0.2489121908$, $\alpha_1 = 1$ and choose the initial value $(\phi, y) = (\frac{5}{24}, 0)$ to obtain the simulation, as shown in Fig. 2.

(b) Taking $w = \frac{1}{8}$, we have $\mathcal{H}(\frac{1}{8}, 0) = -\frac{59}{49152}$, and

$$\frac{I_1\left(-\dfrac{59}{49152}\right)}{I_0\left(-\dfrac{59}{49152}\right)} \approx 0.2409185309.$$



(a) (b)

Fig. 4. The periodic solution of (9) with $\beta = \frac{1}{4}$, $\varepsilon = 0.001$, $\alpha_0 = -0.2280338310$, $\alpha_1 = 1$ and the initial value $(w, y) = (\frac{1}{20}, 0)$, $t \in (0, 1000)$.

Then we take $\varepsilon = 0.001$, $\alpha_0 = -0.2409185309$, $\alpha_1 = 1$ with the initial value $(\phi, y) = (\frac{1}{8}, 0)$ to obtain the simulation, as shown in Fig. 3.

(c) Taking $w = \frac{1}{20}$, we get $\mathcal{H}(\frac{1}{20}, 0) = -\frac{503}{1920000}$ and

$$\frac{I_1\left(-\dfrac{503}{1920000}\right)}{I_0\left(-\dfrac{503}{1920000}\right)} \approx 0.2280338310.$$

Further, taking $\varepsilon = 0.001$, $\alpha_0 = -0.2280338310$, $\alpha_1 = 1$ and the initial value $(\phi, y) = (\frac{1}{20}, 0)$, we obtain the simulation, as shown in Fig. 4.

The results shown in Figs. 2–4 indicate a good agreement between the simulation and analytical prediction.

5. Conclusion

In this paper, we have extended the existing local result on oscillation (periodic traveling wave) of invasion population density to a global result. It has been shown that in a population model with density-dependent migrations and Allee effect, maximal one oscillation (periodic traveling wave) can exist globally.

Moreover, when the invasion speed c of exotic species close enough to the values a_0 and a_1 is relatively small, we have shown that (i) the ratio between $a_0 - c$ and a_1 should be in an interval depending on β, for which the invasion population density can have oscillation; (ii) the amplitude and distribution of the periodic wave depend on the ratio; (iii) the periodic wave tends to the Hopf bifurcation and homoclinic bifurcation; and (iv) the variation of invasion population density approximates the periodic oscillation with distance and time. Since a_0 corresponds to the speed of advection A_0 due to the impacts of wind and water current, etc., a_1 to the speed of migration A_1 due to the biological mechanisms, and β represents the "measure" of the Allee effect, the special patterns of invasion spread found from the analysis in this paper can be discovered in reality.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 11461001), the Natural Science Foundation of Guangxi Province (No. 2015jjBA10052), the School-Based Research Project on the Key Discipline Development and Research in GUFE in 2016 (Nos. 2016YK16 and 2016ZDKT05), and the Natural Science and Engineering Research Council of Canada (No. R2686A02).

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