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ON CHAOS OF THE LOGISTIC MAPS

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Abstract. This paper is concerned with chaos of a family of logistic maps. It is first proved that a regular and nondegenerate snap-back repeller implies chaos in the sense of both Devaney and Li-Yorke for a map in a metric space. Based on this result, it is shown that the logistic system is chaotic in the sense of both Devaney and Li-Yorke, and has uniformly positive Lyapunov exponents in an invariant set for a certain parameter interval with a lower bound less than a specific value, at which the unique 3-periodic orbit appears. In addition, it shows the exact parameter range for the existence of an asymptotically stable 3-periodic point, and consequently the exact parameter range for the biggest periodic window, i.e., 3-periodic window, in the period-doubling bifurcation diagram.

Keywords. Logistic map; Chaos; 3-periodic window; Snap-back repeller; Lyapunov exponent.

1 Introduction

The well-known logistic system is given by

$$x_{n+1} = f(x_n), \quad n \geq 0, \tag{1}$$

where $f(x) := r\,x\,(1-x)$ is the logistic map and $r > 0$ is a parameter.

System (1) has been extensively studied for a very long time. The logistic model, first developed by the sociologist and mathematician, P. F. Verhulst, was to describe a population growth with limited resources from year n to year $n+1$. The first term, $r\,x_n$, represents the reproduction tendency that is proportional to the nth-year population and the second term, $1-x_n$, denotes the need of coexistence and the sharing of the limited resources.

System (1) has been used in many books as a prototype of dynamical systems because it is not only one of the simplest nonlinear systems, but also exhibits amazingly rich dynamical phenomena. The global behavior of the process in dependence on the parameter was first studied in 1976 by R. M. May [15]. Over the last three decades, the logistic map and unimodal maps have attracted a great deal of interest from many mathematicians and

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scientists, and many good results have been obtained, as summarized in recent papers and monographs (cf. [2, 3, 5, 8, 9, 13, 15, 19, 20, 25]). However, there are still some unsolved problems about system (1) due to its complexity.

In recent years, logistic maps often appeared in many partial difference equations (cf. e.g. [6, 7, 16, 17, 26]). Therefore, it is very important to study the dynamics of system (1) in more detail not only because of mathematical interest but also because such studies could provide more useful information for analyzing practical problems and for a better understanding of nonlinear dynamical systems in general.

For convenience, in the following we first briefly review some known and outstanding results (see [4, 5, 19]).

The dynamical behaviors of system (1) are very simple for $x < 0$ and $x > 1$, that is, $f^n(x) \to -\infty$ as $n \to \infty$ for all $x \in (-\infty, 0) \cup (1, \infty)$ and for $r > 1$. So, all possible interesting and complex dynamical behaviors occur in the unit interval $[0, 1]$. System (1) has exactly two fixed points,

$$z_1 := 0, \quad z_2 := 1 - r^{-1}.$$

Obviously, $z_2 \in (0, 1)$ for $r > 1$. The fixed point z_1 is unstable for $r > 1$, while the other fixed point z_2 is asymptotically stable for $r \in (1,\, 3]$, but unstable for $r > 3$. The 2-periodic bifurcation begins at $r_2 := 3$. The unique 2-periodic orbit is asymptotically stable for $r \in (3,\, r_4]$, where $r_4 := 1 + 6^{1/2}$, and unstable for $r > r_4$. The 2^2-periodic orbit starts at $r = r_4$ and is asymptotically stable for $r \in (r_4,\, r_8]$ with $r_8 \approx 3.54409$ but unstable for $r > r_8$. This process of period-doubling bifurcation continues to periods $2^3,\, 2^4, 2^5, \ldots$, and finally gives rise to chaos at $r_\infty \approx 3.569946$. This is the famous period-doubling scenario leading to chaos. However, this is only confirmed by numerical simulation results, as shown in Figure 1, and has not been proved rigorously. We shall point out that a system may not be chaotic in general if it has only periodic points of order $\{2^j\}_{j=1}^{\infty}$. Smital [23] has shown that there are continuous interval maps f and g such that f and g have only periodic points of order $\{2^j\}_{j=1}^{\infty}$, f is but g is not chaotic in the sense of Li-Yorke (see Definition 2).

Based on the result in [24], completely proved by Graczyk and Swiatek [9], the set of parameter values of r such that system (1) has an asymptotically stable periodic orbit is dense in $(0,\, 4]$ and consists of countably infinitely many nontrivial intervals. So, there are countably infinitely many periodic windows in the period-doubling bifurcation diagram. It is noted that the 3-periodic window is the biggest periodic window (see Figure 1). An asymptotically stable 3-periodic orbit bifurcates at $r_3 := 1 + 8^{1/2} \approx 3.828427$ (cf. [2, 5, 20]). Recently, Gordon [8] gave an upper bound for the values of r that support stable 3-periodic orbits, $r = \hat{r}_3$, where

$$\begin{aligned} \hat{r}_3 : \quad &= 1 + \left[\tfrac{11}{3} + \left(\tfrac{1915}{54} + \tfrac{5\sqrt{201}}{2} \right)^{1/3} + \left(\tfrac{1915}{54} - \tfrac{5\sqrt{201}}{2} \right)^{1/3} \right]^{1/2} \\ &\approx 3.841499. \end{aligned}$$

Is the value $\hat{r}_3$ the infimum of those upper bounds? Does an asymptotically stable 3-periodic orbit always exists for every $r \in [r_3, \hat{r}_3)$? These two questions are related to the exact range of parameter r for the 3-periodic window. It turns out that the answers are positive and detailed proofs are given in Section 3.

When $r \geq r_3$, 3-periodic orbits always exist (see Theorem 3). So, by the well-known Li-Yorke theorem [12], periodic orbits of all orders always exist and system (1) is chaotic in the sense of Li-Yorke for every $r \geq r_3$. Is it also chaotic in the sense of Devaney (see Definition 3) in this case? By Proposition 52 in [3, Chapter VI], a continuous map g in an interval I has a periodic point whose period is not a power of 2 if and only if there exist a positive integer n and an infinite closed subset $X \subset I$ such that X is invariant under g^n and the restriction of g^n to X is topological mixing, and consequently g^n is topologically transitive in X. Based on the results in [1], [3, Chapter VI, Lemma 41], and [27], a continuous map with topological transitivity in an interval has a dense set of periodic points and sensitive dependence on initial conditions in this interval. This means that a continuous map with topological transitivity in an interval is chaotic in the sense of Devaney. It is noted that the connectedness of the interval is very important in their proofs. However, the above closed subset X is not connected in general. So, it is not certain whether the map g is chaotic in the sense of Devaney if g has a periodic point whose period is not a power of 2. Therefore, it can not be assured that system (1) is chaotic in the sense of Devaney for every $r_3 \leq r \leq 4$. It is known that system (1) is chaotic on a Cantor set in the sense of Devaney and has a dense orbit in the Cantor set for $r > 4$ (see [4] for $r > 2+5^{1/2}$, and [11, 18] for $r > 4$). In addition, it is seen from Figure 2 that the Lyapunov exponents are negative for $3 < r < r_\infty$, and oscillate between positive and negative values for $r_\infty < r < 4$. For the special value $r = 4$, the Lyapunov exponent is equal to $\ln 2$ [18]. In Section 4, by employing snap-back repellers, we shall prove that there exists a compact and perfect invariant set D containing a Cantor set such that system (1) is chaotic in the sense of both Devaney and Li-Yorke, with a dense orbit in the invariant set D for every $r \in [r_0, 4]$, where $r_0 = 3 + (\frac{2}{3})^{1/2} \approx 3.816496 < r_3$. Further, it will be shown that system (1) has uniformly positive Lyapunov exponents in the invariant set D for every $r \in [r_0, 4]$.

2 Preliminaries

In this section, we introduce some concepts and lemmas, which will be used in the following sections.

Definition 1. [5, Chapter 1, Definitions 1.4 and 1.5]. Let $F : I \to I$ be a map and I be an interval in $\mathbf{R}$.

(i) A point $x_0 \in I$ is said to be a k-periodic point of F if $F^k(x_0) = x_0$

and $F^j(x_0) \neq x_0$ for $1 \leq j \leq k-1$. Further, the set $\{x_j\}_{j=0}^{k-1}$ is said to be a k-periodic orbit of F, where $x_{j+1} = F(x_j)$ for $0 \leq j \leq k-2$.

(ii) Let x_0 be a k-periodic point of F. Then x_0 is said to be stable if it is a stable fixed point of F^k; x_0 is said to be asymptotically stable if it is an asymptotically stable fixed point of F^k; and x_0 is said to be unstable if it is an unstable fixed point of F^k.

An asymptotically stable periodic point is also called an attracting periodic point or a periodic sink. A periodic point (orbit) of map F is also called a periodic point (orbit) of the corresponding system:

$$x_{n+1} = F(x_n), \quad n \geq 0.$$

Lemma 1. [5, Chapter 1, Theorem 1.6]. *Let $F : I \to I$ be a map and I be an interval in* $\mathbf{R}$. *And let $x_0 \in I$ be a k-periodic point of F and F be continuously differentiable at each point of the periodic orbit. Then the following statements hold:*

(i) *the periodic point x_0 is asymptotically stable if $|(F^k)'(x_0)| < 1$;*

(ii) *the periodic point x_0 is unstable if $|(F^k)'(x_0)| > 1$.*

Now we present the definitions of chaos in the sense of Li-Yorke and Devaney, which have often been used in recent years.

Definition 2. Let $S \subset X$ be a set with at least two points and a map $F : S \to X$, where (X, d) is a metric space. Then S is called a scrambled set of F if for any two different points $x,\, y \in S$,

(i) $\liminf\limits_{n\to\infty} d(F^n(x),\, F^n(y)) = 0$;

(ii) $\limsup\limits_{n\to\infty} d(F^n(x),\, F^n(y)) > 0$.

F is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set S of F.

Note that there are three conditions in the original characterization of chaos in Li-Yorke's theorem [12]. Besides the above conditions (i) and (ii), the third one is that for all $x \in S$ and for all periodic points p of F,

$$\limsup_{n\to\infty} d\left(F^n(x),\, F^n(p)\right) > 0.$$

But conditions (i) and (ii) together imply that the scrambled set S contains at most one point x that does not satisfy the above condition. So, the third condition is not essential and can be removed.

Definition 3 [4]. Let V be a subset of a metric space (X, d). A map $F : V \subset X \to V$ is said to be chaotic on V in the sense of Devaney if

(i) F is topologically transitive;

(ii) the periodic points of F are dense in V;

(iii) F has sensitive dependence on initial conditions.

Properties (i) and (ii) together imply property (iii) if F is continuous in V [1]. So, property (iii) is redundant in the above definition. Under some conditions, chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke.

For convenience, some definitions of relevant concepts given in [21] are listed below. Denote by $\bar{B}_s(z) = \{x \in X : d(x,z) \leq s\}$, $B_s(z) = \{x \in X : d(x,z) < s\}$ the closed ball and the open ball of radius $s > 0$ centered at z, respectively.

Definition 4 [21, Definitions 2.1-2.6]**.** Let $F : X \to X$ be a map, where (X, d) is a metric space.

(i) A point $z \in X$ is called an expanding fixed point of F in $\bar{B}_s(z)$ for some constant $s > 0$, if $F(z) = z$ and there exists a constant $\lambda > 1$ such that

$$d(F(x),\, F(y)) \geq \lambda\, d(x,y), \quad \forall\, x,\, y \in V,$$

where the constant λ is called an expanding coefficient of F on V. Furthermore, z is called a regular expanding fixed point of F in $\bar{B}_s(z)$ if z is an interior point of $F(B_s(z))$.

(ii) Assume that z is an expanding fixed point of F in $\bar{B}_s(z)$ for some $s > 0$. Then z is said to be a snap-back repeller of F if there exists a point $x_0 \in B_s(z)$ with $x_0 \neq z$ and $F^m(x_0) = z$ for some positive integer m. Furthermore, z is said to be a nondegenerate snap-back repeller of F if there exist positive constants μ and s_0 such that $B_{s_0}(x_0) \subset B_s(z)$ and

$$d(F^m(x),\, F^m(y)) \geq \mu\, d(x,y), \quad \forall x, y \in \bar{B}_{s_0}(x_0);$$

z is called a regular snap-back repeller of F if $F(B_s(z))$ is open and there exists a positive constant δ_0 such that $B_{\delta_0}(x_0) \subset B_s(z)$ and z is an interior point of $F^m(B_\delta(x_0))$ for any positive constant $\delta \leq \delta_0$.

Lemma 2 [21, Theorem 4.2]**.** *Let (X, d) be a metric space in which each bounded and closed subset is compact. Assume that $F : X \to X$ has a regular and nondegenerate snap-back repeller z, associated with x_0, m, and s as specified in Definition 4, F is continuous in $\bar{B}_s(z)$, and F^m is continuous in a neighborhood of x_0. Then, for each neighborhood U of z, there exist a positive integer n and a Cantor set $\wedge \subset U$ such that $F^n : \wedge \to \wedge$ is topologically conjugate to the one-sided symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$. Consequently, F^n is chaotic on $\wedge$ in the sense of Devaney.*

Theorem 1. *Let all the assumptions in Lemma 2 hold. Further, assume that* F *is continuous in some neighborhoods of* $x_1, \ldots, x_{m-1}$*, where* $x_j = F^j(x_0)$ *for* $1 \le j \le m-1$*. Then there exists a compact and perfect invariant set* $D \subset X$ *containing a Cantor set such that* F *is chaotic in the sense of Devaney on* D *as well as in the sense of Li-Yorke, and it has a dense orbit in* D*.*

Proof. By Lemma 2 and the proof of [21, Theorem 4.1], there exist a Cantor set $\wedge$ and an integer $n > m$ such that $F^n : \wedge \to \wedge$ is topologically conjugate to the one-sided symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$, and F^j is continuous in $\wedge$ for $1 \le j \le n$. Since $(\sigma, \sum_2^+)$ has a dense set of periodic points and a dense orbit [4, Part 1, Proposition 6.6], F^n has also a dense set of periodic points and a dense orbit in $\wedge$. Set

$$D = \{F^j(x) : x \in \wedge,\ 0 \le j \le n-1\}.$$

Then $D \supset \wedge$ and F is continuous in D. It is easy to verify that D is a compact and perfect invariant set of F by the continuity of F, the compactness and perfectness of the Cantor set $\wedge$, and the invariance of $\wedge$ under F^n.

Next, consider the denseness of the periodic point set of F in D. For any point $x \in D$ and any neighborhood V of x, there exist $y \in \wedge$ and an integer j, $0 \le j \le n-1$, such that $F^j(y) = x$. By the continuity of F, $(F^j)^{-1}(V)$ is a neighborhood of y. So, there exists a k-periodic point $p \in \wedge$ of F^n such that $p \in (F^j)^{-1}(V)$. This implies that $F^j(p) \in V$ and $F^{nk}(F^j(p)) = F^j(F^{nk}(p)) = F^j(p)$. So, $F^j(p) \in D$ is a periodic point of F and is in V. Therefore, the periodic point set of F is dense in D.

Finally, we show that F has a dense orbit in D. Suppose $\{(F^n)^i(y_0)\}_{i=0}^{\infty}$ is a dense orbit of F^n in $\wedge$. One can prove that $\{F^i(y_0)\}_{i=0}^{\infty}$ is dense in D by applying a similar argument to that used in the above discussion. Therefore, F has a dense orbit in D and, consequently, is topologically transitive in D. Hence, F is chaotic in the sense of Devaney on D. In addition, since $\wedge$ is infinite and compact, and F is continuous, surjective, topologically transitive, and has a periodic point in $\wedge$, F is chaotic in the sense of Li-Yorke by Theorem 4.1 in [10]. This completes the proof.

Now, we turn to study the Lyapunov exponents of the map F in D in Theorem 1 for the special case of $X = \mathbf{R}$.

Definition 5. Let $F : \mathbf{R} \to \mathbf{R}$ be a map and $O^+(x_0) = \{x_j\}_{j=0}^{\infty}$ be the (forward) orbit of a point $x_0 \in \mathbf{R}$, where $x_{j+1} = F(x_j)$ for $j \ge 0$. Assume that F is differentiable at every point on the orbit. Then

$$\lambda_F(x_0) = \limsup_{k\to\infty} \frac{1}{k} \ln\left|(F^k)'(x_0)\right| = \limsup_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k-1} \ln|F'(x_j)|$$

is called the Lyapunov exponent of F at x_0 or on the orbit $O^+(x_0)$.

Theorem 2. *Let* $F : \mathbf{R} \to \mathbf{R}$ *be a map with a fixed point* $z \in \mathbf{R}$. *Assume that*

(i) F *is continuously differentiable in* $(z - a, z + a)$ *and* $|F'(x)| > 1$ *for all* $x \in (z - a, z + a)$ *for some constant* $a > 0$*;*

(ii) z *is a snap-back repeller of* F *with* $F^m(x_0) = z$ *for some* $x_0 \in (z - a, z + a)$, $x_0 \neq z$, *and some positive integer* m. *Furthermore,* F *is continuously differentiable in some neighborhoods of* $x_0, x_1, \ldots, x_{m-1}$, *and* $F'(x_j) \neq 0$, *where* $x_j = F(x_{j-1})$ *for* $0 \leq j \leq m - 1$.

Then there exists a compact and perfect invariant set D *containing a Cantor set such that* F *is chaotic in the sense of Devaney on* D *as well as in the sense of Li-Yorke, and the Lyapunov exponent of* F *is uniformly positive in* D*, bounded from below by* $n^{-1} \ln \lambda > 0$ *for some integer* $n > m$ *and some constant* $\lambda > 1$.

Proof. By assumptions (i) and (ii), z is a regular and nondegenerate snap-back repeller. Further, by Theorem 4.4 in [22] and the proof of Theorem 4.1 in [21] (these results are related to maps in Banach and complete metric spaces), there exist a Cantor set $\wedge \subset (z - a,\, z + a)$ and a positive integer $n > m$ such that $F^n : \wedge \to \wedge$ is topologically conjugate to the symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$, F^j is continuously differentiable in $\wedge$ for $1 \leq j \leq n$, and F^n is expanding in distance in $\wedge$; that is,

$$|F^n(x) - F^n(y)| \geq \lambda\, |x - y| \quad \forall x, y \in \wedge \tag{2}$$

for some constant $\lambda > 1$. It follows from (2) and the perfectness of $\wedge$ that

$$|(F^n)'(x)| \geq \lambda \quad \forall x \in \wedge,$$

which implies that the Lyapunov exponent of F^n at every point in $\wedge$ is larger than or equal to $\ln \lambda > 0$.

By the proof of Theorem 1, $D = \{F^j(x) : x \in \wedge,\ 0 \leq j \leq n - 1\}$ is a compact and perfect invariant set of F, containing the Cantor set $\wedge$, and F is chaotic in the sense of Devaney on D as well as in the sense of Li-Yorke. Since F^j is continuously differentiable in $\wedge$ for $1 \leq j \leq n$, F is continuously differentiable in D. For every point $y_0 \in D$, there exist $x_0 \in \wedge$ and j, $0 \leq j \leq n - 1$, such that $y_0 = F^j(x_0)$. Denote by $O^+(x_0) = \{x_i\}_{i=0}^{\infty}$ the orbit of x_0 for the map F, where $x_{i+1} = F(x_i)$ for $i \geq 0$. It is evident that $\{x_{in}\}_{i=0}^{\infty}$ is the orbit of x_0 for the map F^n. From

$$(F^i)'(x) = F'(F^{i-1}(x))\, F'(F^{i-2}(x)) \cdots F'(x),$$

it follows that for $i \geq 0$,

$$\ln |(F^n)'(x_{in})| = \sum_{l=0}^{n-1} \ln |F'(x_{in+l})|\, .$$

Hence,

$$\lambda_{F^n}(x_0) = \limsup_{k\to\infty} \frac{1}{k}\sum_{i=0}^{k-1} \ln |(F^n)'(x_{in})| = n \limsup_{k\to\infty} \frac{1}{kn}\sum_{i=0}^{kn-1} \ln |F'(x_i)|,$$

which implies that

$$\lambda_F(y_0) = \lambda_F(x_0) \ge n^{-1}\,\lambda_{F^n}(x_0) \ge n^{-1}\ln\lambda > 0.$$

The proof is finished.

3 Exact Parameter Range for the Existence of an Asymptotically Stable 3-Periodic Point

In this section, the exact parameter range is studied for the existence of an asymptotically stable 3-periodic point of system (1). The parameter r is restricted to $r > 3$ throughout this section.

We first discuss the existence of 3-periodic orbits. Obviously, $x \in [0,1]$ is a 3-periodic point of system (1) if and only if $f^3(x) = x$, $f(x) \neq x$, and $f^2(x) \neq x$. Setting

$$g(x) := \frac{f^3(x) - x}{f(x) - x},$$

we have

$$\begin{aligned} g(x) = \;& r^6x^6 - (3\,r+1)\,r^5x^5 + r^4(3\,r^2+4\,r+1)\,x^4 \\ & -r^3(r^3+5\,r^2+3\,r+1)\,x^3 + r^2(2\,r^3+3\,r^2+3\,r+1)\,x^2 \\ & -r\,(r^3+2\,r^2+2\,r+1)\,x + r^2+r+1. \end{aligned}$$

Then $x \in [0,\,1]$ is a 3-periodic point of system (1) if and only if x is a real root of the following equation:

$$g(x) = 0. \tag{3}$$

By the algebraic fundamental theory, Eq. (3) has six roots, x_k $(1 \le k \le 6)$, which may be complex.

Next, consider some general properties of the roots of Eq. (3). Let x be a root of Eq. (3). Then x satisfies $f^3(x) = x$ and $f^j(x) \neq x$ for $j = 1, 2$. This implies that $f(x)$ and $f^2(x)$ are also roots of Eq. (3), and x, $f(x)$, and $f^2(x)$ are mutually different roots of Eq. (3). So, any root of Eq. (3) is at most of multiplicity 2. If Eq. (3) has a real root of multiplicity 2, then it has exactly three different real roots, which are all of multiplicity 2 and consist of the only 3-periodic orbit of system (1). Since all the coefficients of Eq. (3) are real, complex roots of Eq. (3) appear in complex conjugation when Eq. (3) has complex roots. Therefore, all the roots are real if Eq. (3) has a real root.

Now, we show that Eq. (3) has a real root if and only if $r \geq r_3 = 1+8^{1/2}$. Assume the contrary, suppose that Eq. (3) has no real roots. Let $x_3 = f(x_1)$, $x_5 = f(x_3)$, $x_2 = \bar{x}_1$, $x_4 = f(x_2)$, $x_6 = f(x_4)$, and $x_j = \alpha_j + i\beta_j$ for $j = 1, 3, 5$. Then $\beta_j \neq 0$ and $x_{j+1} = \bar{x}_j$ for $j = 1, 3, 5$. The discriminant of g is

$$d = \prod_{1 \leq i < j \leq 6} (x_i - x_j)^2. \tag{4}$$

Expanding the right-hand side of the above equation yields

$$\begin{aligned} d = \quad & -4^3 \beta_1^2\, \beta_2^2\, \beta_3^2\, |x_1 - x_3|^4\, |x_1 - \bar{x}_3|^4\, |x_1 - x_5|^4 \\ & \times |x_1 - \bar{x}_5|^4\, |x_3 - x_5|^4\, |x_3 - \bar{x}_5|^4, \end{aligned}$$

which indicates that d is non-positive. It is seen that $d < 0$ for all $r \geq r_3$ if and only if Eq. (3) has no repeated complex roots for all $r \geq r_3$. Since x_1, x_3, and x_5 are mutually different, it suffices to show that $x_1 \neq \bar{x}_3$, $x_1 \neq \bar{x}_5$, and $x_3 \neq \bar{x}_5$. We only prove that the inequality $x_1 \neq \bar{x}_3$ holds. The other two inequalities can be proved with similar arguments. Suppose that $x_1 = \bar{x}_3$. It then follows from $x_3 = f(x_1)$ that $x_1 = f(\bar{x}_1)$, that is,

$$\alpha_1 + i\,\beta_1 = r\,(\alpha_1 - i\,\beta_1)\,(1 - \alpha_1 + i\,\beta_1).$$

By comparing the real and imaginary parts in the above equation, we get

$$\alpha_1 = r\,(\alpha_1 - \alpha_1^2 + \beta_1^2), \quad \beta_1 = r\,\beta_1\,(2\,\alpha_1 - 1). \tag{5}$$

From the second relation in (5) and noticing $\beta_1 \neq 0$, we have

$$\alpha_1 = \frac{1 + r}{2\,r}.$$

Substituting the above result into the first relation in (5) yields

$$\beta_1^2 = \frac{(1 + r)\,(3 - r)}{4\,r^2},$$

which contradicts the assumption $r \geq r_3$. Hence, $d < 0$ for all $r \geq r_3$. On the other hand, the discriminant d can be found as (with the aid of Maple)

$$d = r^{30}(r^2 - 5\,r + 7)^2\,(r^2 - 2\,r - 7)^3\,(r^2 + r + 1)^2, \tag{6}$$

which suggests that $d \geq 0$ when $r \geq r_3$. This is a contradiction. Therefore, Eq. (3) has a real root when $r \geq r_3$. Conversely, if Eq. (3) has a real root, then all its roots are real. So, it follows from (4) and (6) that $d \geq 0$, implying $r \geq r_3$. Hence, Eq. (3) has a real root if and only if $r \geq r_3$.

Further, it is seen from (6) that Eq. (3) has six different real roots if and only if $r > r_3$. Consequently, system (1) has two different 3-periodic orbits if and only if $r > r_3$. It is evident that these 3-periodic orbits fall in the interval $(0, 1)$. In addition, again from (6), it follows that $d = 0$ if and only if $r = r_3$, that is, Eq. (3) has repeated roots if and only if $r = r_3$. This implies that system (1) has a unique 3-periodic orbit in this case, which was first proved by Myrberg in 1958. Later, Saha and Strogatz [20], Bechhoeffer [2], and Gordon [8] simplified the proof.

In summary, the following result has been obtained.

Theorem 3. *System (1) has a unique* 3*-periodic orbit if and only if* $r = 1 + 8^{1/2}$ *and it has two different* 3*-periodic orbits if and only if* $r > 1 + 8^{1/2}$.

We turn to the stability of the 3-periodic orbits. It has been shown that the value $\hat{r}_3 \approx 3.841499$ (defined as in Section 1) is an upper bound for the r values that support stable 3-periodic orbits [8]. We further prove that system (1) has an asymptotically stable 3-periodic orbit for any $r \in (r_3, \hat{r}_3)$.

For convenience, first recall some results of [8]. Let $\{x_1, x_2, x_3\}$ be a 3-periodic orbit of system (1), where $x_2 = f(x_1)$, $x_3 = f(x_2)$. Then they can be written as

$$x_n = \mu + \beta\,\omega^n + \bar{\beta}\,\bar{\omega}^n, \quad n = 1, 2, 3,$$

where ω is a complex cubic root of unity, and μ and β are constants, satisfying

$$\begin{aligned} 2\,\beta\,\bar{\beta} &= (1 - \frac{1}{r})\,\mu - \mu^2, \\ \bar{\beta}^2 &= (1 - 2\mu - \frac{\omega}{r})\,\beta, \\ \beta^2 &= (1 - 2\,\mu - \frac{\bar{\omega}}{r})\,\bar{\beta}, \end{aligned}$$

which yields

$$\begin{aligned} \mu &= \frac{3\,r + 1 \pm (r^2 - 2\,r - 7)^{1/2}}{6\,r}, \\ 2\,|\beta|^2 &= (1 - \frac{1}{r})\,\mu - \mu^2, \end{aligned} \tag{7}$$

where the signs "±" determine two different 3-periodic orbits when $r > r_3$. Denote

$$h(r) := (f^3)'(x_1).$$

Then

$$h(r) = f'(x_1)\,f'(x_2)\,f'(x_3) = r^3\,(1 - 2A + 4B - 8C),$$

where $A = x_1 + x_2 + x_3$, $B = x_1\,x_2 + x_1\,x_3 + x_2\,x_3$, and $C = x_1\,x_2\,x_3$. By the result of [8], $A = 3\,\mu$, $B = 3\,(\mu^2 - |\beta|^2)$, and $C = 1 - A + B - r^{-3}$, which together with the second relation in (7) implies that

$$h(r) = r^3\,\left(-7 + 24\,\mu - 18\,\mu^2 - \frac{6\,\mu}{r} + \frac{8}{r^3}\right).$$

Substituting the first relation in (7) into the above equation yields

$$h_1(r) := h(r) = -\,r\,(r-2)\,(r^2-2r-7)^{1/2} - (r^2-2\,r-8) \tag{8}$$

for choosing "$-$" in the first relation of (7) and

$$h_2(r) := h(r) = r\,(r-2)\,(r^2-2r-7)^{1/2} - (r^2-2\,r-8) \tag{9}$$

for choosing "+" in the first relation of (7). By the result of [8] and by a direct calculation, one can obtain

$$h_1(\hat{r}_3) = -1, \quad h_j(r_3) = 1, \quad j = 1,\, 2. \tag{10}$$

By (8), the derivative of $h_1(r)$ with respect to r can be obtained as

$$\begin{aligned} h_1'(r) = \;& -(r-1)\,(r^2-2r-7)^{-1/2} \\ & \times \left[\,3\,(r^2-2\,r-7) + 2\,(r^2-2\,r-7)^{1/2} + 7\,\right], \end{aligned}$$

which indicates that $h_1'(r) < 0$ when $r > r_3$. Thus, h_1 is decreasing in the interval $(r_3,\,\infty)$ and consequently, $|h_1(r)| < 1$ for $r \in (r_3,\,\hat{r}_3)$ and $h_1(r) < -1$ for $r > \hat{r}_3$ by using (10). Therefore, the corresponding 3-periodic point is asymptotically stable for $r \in (r_3,\,\hat{r}_3)$ by Lemma 1 and for $r = r_3$ by [5, Page 43], and thus it is true for any $r \in [r_3,\,\hat{r}_3)$.

Similarly, it follows from (9) that

$$\begin{aligned} h_2'(r) = \;& (r-1)\,(r^2-2r-7)^{-1/2} \\ & \times \left[\,3\,(r^2-2\,r-7) - 2\,(r^2-2\,r-7)^{1/2} + 7\,\right], \end{aligned}$$

which implies that $h_2'(r) > 0$ when $r > r_3$. Hence, h_2 is increasing in the interval $(r_3,\,\infty)$ and consequently, $h_2(r) > 1$ for $r \in (r_3,\,\infty)$ in view of (10). Thus, the corresponding 3-periodic point is unstable for any $r \in (r_3,\,\infty)$ by Lemma 1. Summarizing the above discussions gives the following result.

Theorem 4. *System (1) has an asymptotically stable 3-periodic point for any $r \in [r_3, \hat{r}_3)$ and has no stable 3-periodic points for any $r > \hat{r}_3$.*

Remark 1. It is known that all the periodic windows in the bifurcation diagrams (see Figure 1) represent the attracting periodic points of system (1) that occur at those parameter values. So, Theorem 4 implies that the range of the parameter r for the 3-periodic window is exactly between $r_3 \approx 3.828427$ and $\hat{r}_3 \approx 3.841499$ (see Figure 3). This result can clarify some non-precise statements in the literature on the estimation of the parameter range for the 3-periodic window based on computer simulations (cf. e.g. [5 (page 43), 19 (page 340)]).

4 Chaos

In this section, we study chaos of system (1) and discuss its Lyapunov exponent.

The following result is a direct consequence of Theorem 3 by the Li-Yorke theorem [12].

Theorem 5. *For any $r \in [r_3, \infty)$, the following results hold:*

(i) *for every positive integer k, there is a k-periodic point in $[0, 1]$;*

(ii) *there is an uncountable scrambled set $S \subset [0,1]$, containing no periodic points, which satisfies the following conditions:*

(ii_1) *for every $p, q \in S$ with $p \neq q$,*

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0$$

and

$$\liminf_{n\to\infty} |f^n(p) - f^n(q)| = 0;$$

(ii_2) *for every $p \in S$ and periodic points $q \in [0, 1]$,*

$$\limsup_{n\to\infty} |f^n(p) - f^n(q)| > 0.$$

Consequently, system (1) is chaotic in the sense of Li-Yorke for any $r \geq r_3$.

It is known that system (1) has an invariant Cantor set on which it is chaotic in the sense of Devaney and has a dense orbit for any $r > 4$ (introduced in Section 1). By Theorem 5, system (1) is chaotic in the sense of Li-Yorke for any $r \geq r_3$. However, it is not known if there is an invariant set, which is a Cantor set or contains a Cantor set, on which system (1) is chaotic in the sense of Devaney and it has a dense orbit for some values of $r \leq 4$. We now study this problem by finding a snap-back repeller of the system.

Throughout the rest of this section, the parameter r is restricted to $3 < r \leq 4$.

It is obvious that f is continuously differentiable in $\mathbf{R}$ with $f'(x) = r\,(1 - 2\,x)$. For $r > 3$,

$$f'(z_1) = r > 1, \quad f'(z_2) = 2 - r < -1.$$

So, they are both expanding fixed points of f. The maximal open expanding subintervals of $[0, 1]$ near z_1 and z_2 are $I_1 := \big(0, 2^{-1}(1 - r^{-1})\big)$ and $I_2 := \big(2^{-1}(1 + r^{-1}), 1\big)$, respectively. Especially, the interval $J :=$

$\big(2^{-1}(1+r^{-1}),\, 2^{-1}(3-5\,r^{-1})\big) \subset [0,\,1]$ is an open ball of radius $\delta = 2^{-1}(1-3\,r^{-1})$ centered at z_2.

Lemma 3. *For every $r \in [r_0,\,4]$ with $r_0 := 3+(2/3)^{1/2}$, there exists a point $x_0 \in J$ such that $f^4(x_0) = z_2$ and $f'(x_i) \neq 0$, where $x_i = f(x_{i-1})$ for $i = 1,\,2,\,3$. Consequently, z_2 is a nondegenerate and regular snap-back repeller of f for every $r \in [r_0,\,4]$.*

Proof. The proof is divided into four steps.

(i) Firstly, consider the following equation:

$$f(x) = r\,x\,(1-x) = z_2, \tag{11}$$

which has two solutions; one is z_2 and the other is

$$y_1 := \frac{1}{r}.$$

It is easy to verify that $y_1 \in I_1$ and

$$f'(y_1) = r - 2 \neq 0. \tag{12}$$

(ii) Secondly, consider the following equation:

$$r\,x\,(1-x) = y_1, \tag{13}$$

which can be rewritten as

$$\left(x-\frac{1}{2}\right)^2 = \frac{1}{4} - \frac{1}{r^2}.$$

So, Eq. (13) has two solutions,

$$x = \frac{1}{2} \pm \left(\frac{1}{4}-\frac{1}{r^2}\right)^{1/2}.$$

Set

$$y_2 := \frac{1}{2} + \left(\frac{1}{4}-\frac{1}{r^2}\right)^{1/2}. \tag{14}$$

Then

$$\frac{1}{2}\left(3-\frac{5}{r}\right) < y_2 < 1, \tag{15}$$

which implies that $y_2 \notin J$. In fact, the second inequality in (15) is easily derived from (14) and the first inequality is equivalent to

$$3\,r^2 - 20\,r + 29 = 3\left(r - \frac{10+13^{1/2}}{3}\right)\left(r - \frac{10-13^{1/2}}{3}\right) < 0,$$

due to the facts that $(10-13^{1/2})/3 < 3$ and $4 < (10+13^{1/2})/3$. In addition,

$$f'(y_2) \neq 0. \tag{16}$$

(iii) Thirdly, consider the following equation:

$$r\,x\,(1-x) = y_2, \tag{17}$$

which can be rewritten as

$$\left(x - \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{y_2}{r}.$$

This equation has real solutions not equal to $\frac{1}{2}$ if and only if

$$y_2 < \frac{r}{4}, \tag{18}$$

which, due to Eq. (14), is equivalent to

$$r^4 - 4\,r^3 + 16 = (r-2)\,g_1(r) > 0, \tag{19}$$

where $g_1(r) := r^3 - 2\,r^2 - 4\,r - 8$. Since $g_1'(r) = 3\,(r-2)\,(r+\frac{2}{3}) > 0$, $g_1(r)$ is increasing in the interval $(3,\,4]$. A direct computation shows that $g_1(r) = 0$ has only one real root,

$$\begin{aligned} r_1 : \quad &= \frac{2}{3} + \frac{2}{3}\left(19 + 3\,\times 33^{1/2}\right)^{1/3} + \frac{8}{3}\left(19 + 3 \times 33^{1/2}\right)^{-1/3} \\ &\approx 3.678573. \end{aligned}$$

It then follows that $g_1(r) > 0$ for all $r \in (r_1,\,4]$. Therefore, inequality (19), i.e. inequality (18), holds for all $r \in (r_1, 4]$. Consequently, for every $r \in (r_1,\,4]$, Eq. (17) has two solutions,

$$x = \frac{1}{2} \pm \left(\frac{1}{4} - \frac{y_2}{r}\right)^{1/2}.$$

Set

$$y_3 := \frac{1}{2} + \left(\frac{1}{4} - \frac{y_2}{r}\right)^{1/2}. \tag{20}$$

It then follows from (20) and (15) that for every $r \in (r_1,\,4]$,

$$\frac{1}{2} < y_3 < z_2 \tag{21}$$

and

$$f'(y_3) \neq 0. \tag{22}$$

(iv) Finally, consider the following equation:

$$r\,x\,(1-x) = y_3, \tag{23}$$

which can be rewritten as

$$\left(x - \frac{1}{2}\right)^2 = \frac{1}{4} - \frac{y_3}{r}.$$

Similarly to the above discussion, this equation has real solutions not equal to $\frac{1}{2}$ if and only if

$$y_3 < \frac{r}{4}. \tag{24}$$

It then follows from (20) that inequality (24) is equivalent to

$$y_2 > g_2(r), \tag{25}$$

where $g_2(r) := 16^{-1}\,r^2\,(4-r)$. Since $g_2'(r) = 16^{-1}\,r\,(8-3\,r) < 0$ for $r \in [3,\,4]$, $g_2(r)$ is decreasing in $[3,\,4]$ and, consequently, $g_2(r) < g_2(3) = \frac{9}{16}$ for all $r \in (3,\,4]$. On the other hand, it is seen from (14) that for any $r \in (3,\,4]$,

$$y_2 > \frac{1}{2} + \left(\frac{1}{4} - \frac{1}{3^2}\right)^{1/2} = \frac{1}{2} + \frac{5^{1/2}}{6} > g_2(3).$$

Hence, for all $r \in (3,\,4]$, inequality (25), i.e. inequality (24), holds. This suggests that for all $r \in (r_1,\,4]$, Eq. (23) has the following two real solutions:

$$x = \frac{1}{2} \pm \left(\frac{1}{4} - \frac{y_3}{r}\right)^{1/2}.$$

Set

$$x_0 := \frac{1}{2} + \left(\frac{1}{4} - \frac{y_3}{r}\right)^{1/2}. \tag{26}$$

Now, we show that for all $r \in [r_0,\,4]$,

$$z_2 < x_0 < \frac{1}{2}\left(3 - \frac{5}{r}\right). \tag{27}$$

Obviously, $r_0 > r_1$. On the one hand, the first inequality in (27) is equivalent to $y_3 < z_2$ from (26) and by using $z_2 = 1 - r^{-1}$. This, together with (21), implies that the first inequality holds for all $r \in [r_0,\,4]$. On the other hand, it follows again from (26) that the second inequality in (27) is equivalent to

$$y_3 > 5 - \frac{3\,r}{4} - \frac{25}{4\,r}. \tag{28}$$

Further, inequality (28) is equivalent to

$$\left(\frac{1}{4} - \frac{y_2}{r}\right)^{1/2} > g_3(r) \tag{29}$$

by using (20), where

$$g_3(r) := \frac{9}{2} - \frac{3\,r}{4} - \frac{25}{4\,r} = -\frac{3}{4\,r}\left(r - 3 - \left(\frac{2}{3}\right)^{1/2}\right)\left(r - 3 + \left(\frac{2}{3}\right)^{1/2}\right).$$

It can be seen that $g_3(r) \le 0$ for all $r \in [r_0, 4]$. Hence, inequality (29), i.e. inequality (28), holds for all $r \in [r_0, 4]$. Consequently, (27) is proved.

Setting $x_1 = y_3$, $x_2 = y_2$, and $x_3 = y_1$, one has that $f(x_{i-1}) = x_i$ and $f'(x_i) \neq 0$ for $i = 1, 2, 3$ from (12), (16), and (22); and $f^4(x_0) = z_2$. Hence, x_0 is a nondegenerate and regular snap-back repeller. This completes the proof.

The following result is a consequence of Lemmas 2 and 3.

Theorem 6. *For every $r \in [r_0, 4]$, $r_0 = 3 + (2/3)^{1/2}$ and for each neighborhood $U \subset [0, 1]$ of $z_2 = 1 - r^{-1}$, there exist a positive integer $n > 4$ and a Cantor set $\Lambda \subset U$ such that $f^n : \Lambda \to \Lambda$ is topologically conjugate to the symbolic dynamical system $\sigma : \sum_2^+ \to \sum_2^+$. Consequently, f^n is chaotic on Λ in the sense of Devaney.*

The following result is a consequence of Lemma 3 and Theorem 2.

Theorem 7. *For every $r \in [r_0, 4]$, where $r_0 = 3 + (2/3)^{1/2}$, there exists a compact and perfect invariant set $D \subset [0, 1]$, containing a Cantor set, such that system (1) is chaotic in the sense of Devaney on D as well as in the sense of Li-Yorke, with a dense orbit in D, and has uniformly positive Lyapunov exponents in D.*

5 Conclusion and Discussion

In this paper, it has been proved that two 3-periodic orbits always exist when the parameter r of the logistic map $f(x) = r\,x\,(1-x)$ is larger than $r_3 = 1 + 8^{1/2}$, at which a unique 3-periodic orbit appears. The exact parameter range of the existence of an asymptotically stable 3-periodic point is obtained and, consequently, the exact parameter range for the biggest periodic window, i.e., 3-periodic window, in the period bifurcation diagram is given. This result can clarify some non-precise statements in the literature about the parameter range for the 3-periodic window based on computer simulations.

It has been shown that the corresponding logistic map is chaotic in the sense of both Devaney and Li-Yorke, and has a dense orbit in a compact and perfect invariant set D, which contains a Cantor set, for the parameter interval $[r_0, 4]$ with $r_0 = 3 + (2/3)^{1/2} < r_3$. Further, it has been proved that the system has uniformly positive Lyapunov exponents in the invariant set D for the whole parameter interval $[r_0, 4]$. In addition, it has been proved that a regular and nondegenerate snap-back repeller implies chaos in the sense of both Devaney and Li-Yorke for a map in a metric space.

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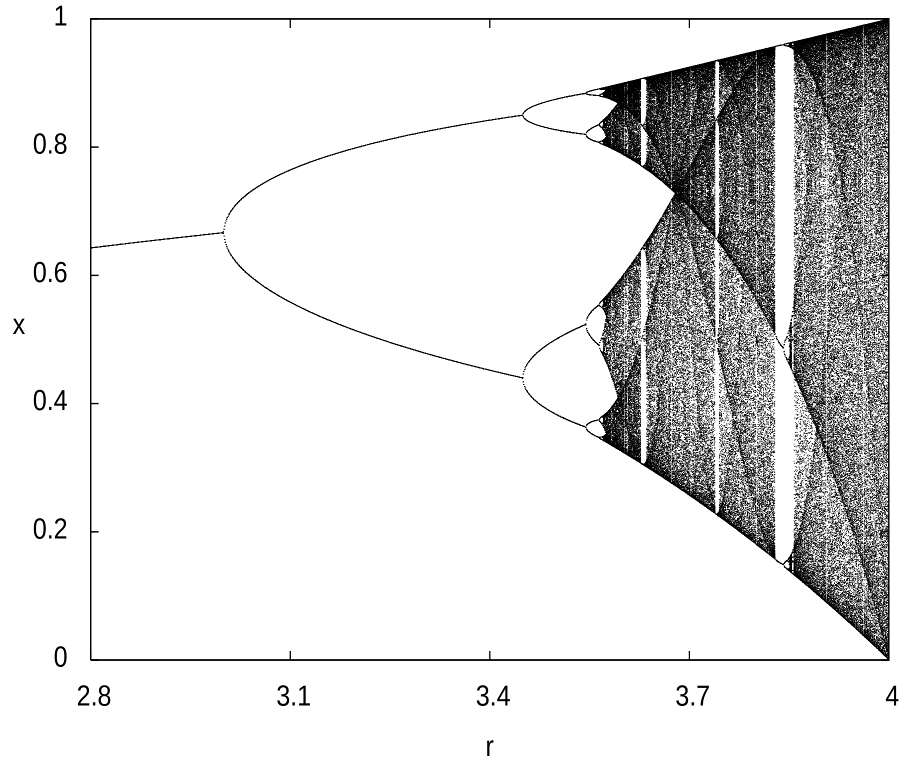

Figure 1: Bifurcation diagram of the logistic map for $2.8 \leq r \leq 4$.

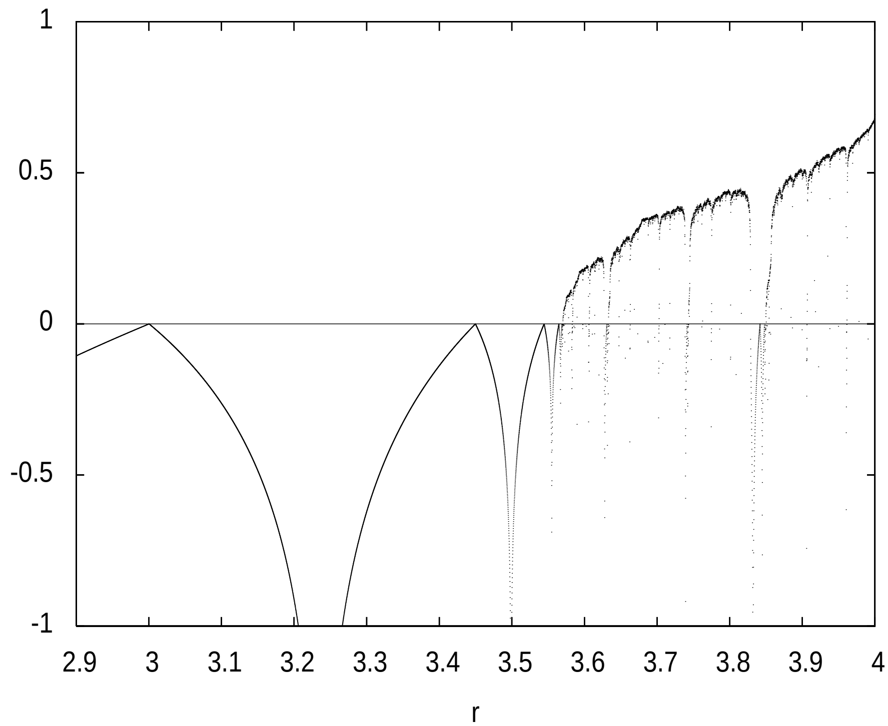

Figure 2: Lyapunov exponents for the logistic map when $2.9 \leq r \leq 4$.

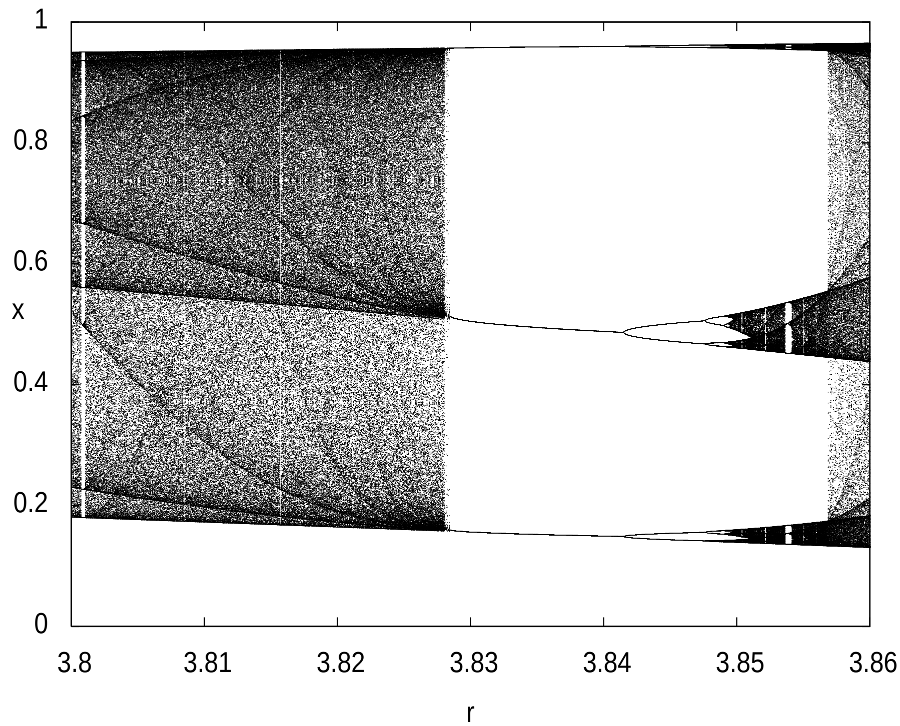

Figure 3: Bifurcation diagram of the logistic map for $3.8 \leq r \leq 3.86$.