DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **24**, Number **2**, February **2019**

doi:10.3934/dcdsb.2018341

pp. **965–987**

PERIODIC TRAVELING WAVES IN A GENERALIZED BBM EQUATION WITH WEAK BACKWARD DIFFUSION AND DISSIPATION TERMS

Xianbo Sun and Pei Yu*

Department of Applied Mathematics, Western University
London, Ontario, N6A 5B7, Canada

(Communicated by Xiaoqiang Zhao)

Abstract. In this paper, we consider a generalized BBM equation with weak backward diffusion, dissipation and Marangoni effects, and study the existence of periodic and solitary waves. Main attention is focused on periodic and solitary waves on a manifold via studying the number of zeros of some linear combination of Abelian integrals. The uniqueness of the periodic waves is established when the equation contains one coefficient in backward diffusion and dissipation terms, by showing that the Abelian integrals form a Chebyshev set. The monotonicity of the wave speed is proved, and moreover the upper and lower bounds of the limiting wave speeds are obtained. Especially, when the equation involves Marangoni effect due to imposed weak thermal gradients, it is shown that at most two periodic waves can exist. The exact conditions are obtained for the existence of one and two periodic waves as well as for the co-existence of one solitary and one periodic waves. The analysis is mainly based on Chebyshev criteria and asymptotic expansions of Abelian integrals near the solitary and singularity.

1. **Introduction.** Traveling waves in nonlinear wave equations can model many nonlinear complex phenomena in physics, chemistry, biology, mechanics, optics, etc. The wave profiles of long waves in shallow water with different conditions can be modeled by the famous Korteweg-de Vries (KdV) [26], Benjamin-Bona-Mahony [4], the Green-Naghdi [15] and Camassa-Holm [6] equations. In solving real world problems, certain relatively weak influences due to the existence of uncertainty or perturbation are unavoidable, for example in describing the shallow water waves in nonlinear dissipative media [8] and dispersive media [23]. In other words, one should add certain type of small terms in modelling the problems. Topper and Kawahara [41] studied the wave motions on a liquid layer over an inclined plane and established the following Partial Differential Equation (PDE),

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0, \tag{1}$$

for which the wave motion is assumed depending only on the gradient direction. When the inclined plane is relatively long and the surface tension is relatively weak,

2010 *Mathematics Subject Classification.* Primary: 34C25, 34C60; Secondary: 37C27.

Key words and phrases. Generalized Benjamin-Bona-Mahony equation, isolated periodic traveling wave, solitary wave, Abelian integral.

*Corresponding author: Pei Yu.

the u_{xx} and u_{xxxx} terms are relatively small, and the following equation is more appropriate for describing the real situation,

$$u_t + uu_x + u_{xxx} + \varepsilon(u_{xx} + u_{xxxx}) = 0, \tag{2}$$

where $0 < \varepsilon \ll 1$ represents small perturbations to the system. When $\varepsilon = 0$, the backward diffusion (u_{xx}) and dissipation (u_{xxxx}) vanish and (2) becomes the classical KdV equation [26], and so (2) is usually called a perturbed KdV equation. The KdV equation has played an important role in describing various physical problems, and many researchers have studied this equation and particularly paid attention to solitary and periodic waves. In 1993, Derks and Gils [9] discussed the uniqueness of traveling waves of equation (2). A year later, Ogawa [36] studied the existence of solitary and periodic waves of equation (2).

When the Marangoni effect is considered on the surface of a thin layer, additional nonlinearity in the form of $(uu_x)_x$ appears, see [12, 20]. For this model, Velarde *et.al.* [43] showed the consistent way of incorporating the Marangoni effect (heating the liquid layer from the air side) into the one-way long-wave assumption and derived the following equation:

$$u_t + 2\alpha_1 uu_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0, \tag{3}$$

which contains the nonlinear term $(uu_x)_x$ due to the Marangoni effect, describing the opposite to the Bénard convection [35, 44]. For the sake of completeness here, we notice that different cases by setting some parameters $\alpha_i = 0$ in Eq. (3) have been considered in many other works, for example [10, 24, 27, 28]. In particular, Mansour [31] studied the existence of solitary in Eq. (3) with all small non-vanishing parameters α_2, α_4 and α_5, and in addition established the existence of solitary for the following equation [32].

$$u_t + \alpha_1 u^2 u_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0.$$

It has been noted that solitary and traveling waves with periodic spatial profiles are very sensitive to weak external influence. For example, stationary periodic patterns in thermal convection may not be observed in a weakly windy circumstance [5]. However, the weak Marangoni effect may destabilize the waves [21], and different perturbations may generate different dynamics of systems, leading to, for example, broking the periodic traveling waves, changing its stability and yielding quasi-periodic motions on invariant tori, etc. One efficient way to deal with such problems is to apply bifurcation techniques from the view point of dynamical systems by taking the weak external effects as perturbations, and many good results have been obtained for certain nonlinear wave problems, see [48, 17, 22, 13].

In general, perturbations to a dynamical system may be classified into three types: periodic or quasi-periodic forcing, singular perturbation and regular perturbation. When a PDE is perturbed by quasi-periodic forcing terms, one method developed to investigate dynamics (quasi-periodic motion on some invariant tori) of the system is based on an infinite dimensional KAM theory. This theory is an extension of the well-known classical KAM theory, which was established by Kolmogorov [25], Anorld [2] and Moser [34]. It asserts that the majority of tori is persistent under perturbations if the Kolmogorov non-degenerate condition is satisfied.

When a perturbed system can be reduced to a singularly perturbed system, the first question is about the existence of traveling wave solutions of the system. There are lots of publications on this topic, such as singularly perturbed KdV equations [36, 3, 16, 30, 40], the perturbed dispersive-dissipative equations and reaction-diffusion systems [1, 49, 42]. One classical method to deal with singular perturbations is to apply Fenichel's theory (e.g. see [11]), which assures the existence of an invariant manifold and then the problem is reduced to a regular perturbed system on this manifold, see [36, 3, 16, 30, 40]. In these cases, the perturbation always has only one or two terms with lower degrees on the invariant manifold, see above mentioned references and also the works of Derks and Gils [9] and Ogawa [36, 38, 37].

However, very few problems can be directly reduced to regularly perturbed systems. Thus, perturbations are usually not restricted on manifolds. Moreover, there exist fewer mathematical tools which can be used to study the dynamics of perturbed systems, and yet, the analysis and computation based on these approaches are difficult to be used for proving the existence of periodic traveling waves. Thus, when Zhou *et al* studied the Burgers-Huxley equation [52] and Burgers-Fisher equation [53], they assumed that one coefficient in the equation and the wave speed are small so that these two small terms can be treated as two perturbations, which greatly simplifies the analysis and the proof on the existence of periodic waves [52, 53]. In general, if three or more perturbation terms are involved, the analysis becomes much more difficult.

After the works of Derks and Gils [9] and Ogawa [36], in 2014 Yan *et al.* [47] investigated the perturbed generalized KdV equation,

$$u_t + (u^n)_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0.$$

When $\epsilon = 0$ and $n = 2$, the above equation is reduced to the classical KdV equation,

$$u_t + (u^2)_x + u_{xxx} = 0.$$

Yan *et al.* [47] proved that there exists one periodic wave by choosing some wave speed c for sufficiently small $\epsilon > 0$. However, the uniqueness of the periodic wave is still open.

Another well-known model describing the propagation of surface water waves in a uniform channel is the Benjamin-Bona-Mahony (BBM) equation,

$$u_t + uu_x - u_{xxt} = 0.$$

This model describes surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma, acoustic waves in harmonic crystals, see [33]. Due to its wide applications and rich dynamics, researchers have developed many different forms of BBM equations which are usually called generalized BBM equations, see [46, 39, 51] and the reference therein. Wazwaz studied the following generalized BBM equation [46],

$$(u^m)_t + (u^n)_x + (u^l)_{xxx} = 0, \tag{4}$$

and found its compaction of dispersive structures.

More recently, Chen *et al.* [7] investigated a perturbed generalized BBM equation,

$$(u^2)_t + (u^3)_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0, \tag{5}$$

and established the existence of solitary waves and uniqueness of periodic waves. Both of the works [47] and [7] studied the perturbation problems restricted on manifolds, by using geometric singular perturbation theory. In [7], the authors applied Picard-Focus equations to determine the existence of periodic waves, and developed a good approach to prove that the dominating factor of the Melnikov function is monotonic, see Lemma 4.10 in [7]. Using the same approach, they also proved that the perturbed generalized defocusing mKdV equation,

$$u_t - u^2 u_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0,$$

has a unique periodic wave. However, this approach failed to deal with the unperturbed equation having a nilpotent saddle or more degenerate cusp, corresponding to $m > 2$ in (4). This is because, taking $m = 3$ for example, one needs to consider more terms in Lemma 4.10 in [7] in order to find some combination of the terms in order to prove the monotonicity of the dominating part of the Melnikov function. However, in general this is very difficult in higher degenerate cases, which is similar to dealing with the cases when more than two perturbation terms are involved in the equations.

In this paper, we study the BBM equation (4) for $m = 3$, $n = 4$ and $l = 1$ with two different kinds of weak dissipative effects P_1 and P_2, described by

$$(u^3)_t + (u^4)_x + u_{xxx} + \epsilon P_i = 0, \quad i = 1,\ 2, \tag{6}$$

where

$$P_1 = u_{xx} + u_{xxxx}, \quad P_2 = ((\alpha_0 + \alpha_1 u + \alpha_2 u^2)u_x)_x, \tag{7}$$

in which α_0, α_1 and α_2 are bounded parameters. P_1 describes the weak second and fourth derivative diffusions without Marangoni effect. P_2 describes a generalized Marangoni effect. The unperturbed problem has a more degenerate singularity which is a nilpotent saddle. Especially, for the case with weak Marangoni effect P_2, the problem is not restricted on a manifold but is reduced to a regular problem with more parameters. It can be seen from our reduction of the problem with P_1 that the weak Marangoni effect P_2 is equivalent to the compound dissipative-Marangoni effect $\gamma_0 u_{xx} + \gamma_1 u_{xxxx} + \gamma_2(uu_x)_x$. The main mathematical tools we will use in this paper are based on the relatively new theory of weak Hilbert's 16th problem and bifurcation theory.

The rest of this paper is organized as follows. In section 2, we give a reduction analysis and state our main results. In section 3, we present some perturbation theories and derive a special form of Abelian integral, also called Melnikov function, for periodic and solitary waves. It will be shown that our method without using Picard-Focus Equation is more effective compared to that used in the existing works. In section 4, we study the problem with perturbation P_1 by applying the Chebyshev criteria [14]. In section 5, we investigate the problem with perturbation P_2 and obtain the conditions on the existence of periodic waves. In particular, we derive the exact conditions on the existences of one and two periodic waves. Further, we establish a criterion on the co-existence of one solitary wave and one unique periodic wave. Conclusion is drawn in section 6.

2. Main results.

In this section, we present our main results for system (6). First, we consider system (6) with perturbation P_1, that is,

$$(u^3)_t + (u^4)_x + u_{xxx} + \epsilon(u_{xx} + u_{xxxx}) = 0. \tag{8}$$

Taking $\xi = x - ct$ into (8) yields

$$-3cu^2(\xi)u'(\xi) + 4u^3(\xi)u'(\xi) + u'''(\xi) + \epsilon(u''(\xi) + u''''(\xi)) = 0.$$

Then, integrating this equation and omitting the integral constant, we obtain

$$-cu^3 + u^4 + u''(\xi) + \epsilon(u'(\xi) + u'''(\xi)) = 0. \tag{9}$$

Further introducing the transformations $\xi = \frac{\tau}{c^{\frac{3}{2}}}$ and $u = c\mu$ into (9) yields

$$-\mu^3(\tau) + \mu^4(\tau) + \mu''(\tau) + \epsilon(c^{-\frac{3}{2}}u'(\tau) + c^{\frac{3}{2}}u'''(\tau)) = 0. \tag{10}$$

Similarly, system (6) with perturbation P_2 can be transformed to

$$-\mu^3(\tau) + \mu^4(\tau) + \mu''(\tau) + \epsilon(a_0 + a_1\mu(\tau) + a_2\mu^2(\tau))\mu'(\tau) = 0, \tag{11}$$

where $a_0 = c^{-\frac{3}{2}}\alpha_0$, $a_1 = c^{-\frac{1}{2}}\alpha_1$ and $a_2 = c^{\frac{1}{2}}\alpha_2$. Because $\alpha_i' s$ are independent, we will use $a_i' s$ in our analysis for convenience.

Correspondingly, the unperturbed system of (6) (with $\epsilon = 0$) is given by $-\mu^3(\tau) + \mu^4(\tau) + \mu''(\tau) = 0$, which is equivalent to the system,

$$\begin{aligned} \frac{d\mu}{d\tau} &= \nu, \\ \frac{d\nu}{d\tau} &= \mu^3 - \mu^4, \end{aligned} \tag{12}$$

which has a hyper-elliptic Hamiltonian function, given by

$$\mathcal{H}(\mu, \nu) = \frac{\nu^2}{2} - \frac{\mu^4}{4} + \frac{\mu^5}{5}, \tag{13}$$

satisfying $\mathcal{H}(1, 0) = -\frac{1}{20}$ and $\mathcal{H}(0, 0) = \mathcal{H}(\frac{5}{4}, 0) = 0$. The function $\mathcal{H} = h$ for $h \in (-\frac{1}{20}, 0)$ and $\mu \in (0, \frac{5}{4})$, depicted in Figure 1, shows a family of closed orbits surrounded by a homoclinic loop Γ_0, with a nilpotent saddle of order 1 at the origin.

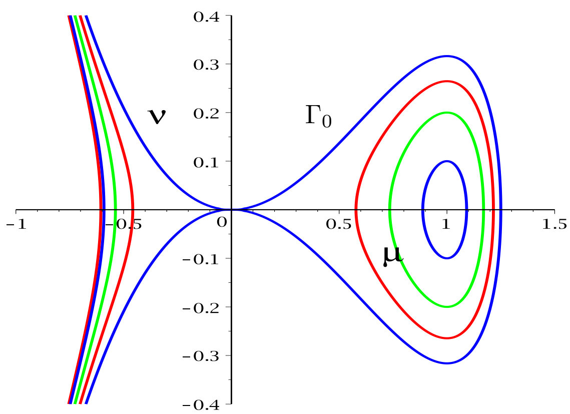

FIGURE 1. The portrait of system (12).

In order to state our main results clearly and systematically, we use the following notations: Γ_h denoting the closed curve defined by $\mathcal{H}(\mu, \nu) = h$; $\mu(\tau, h)$ representing the closed orbit of system (12) corresponding to Γ_h; $\mu(\tau, \epsilon, h, c(\epsilon, h))$ being the periodic wave of system (10) near Γ_h under the condition $c = c(\epsilon, h)$; $\mu(\tau, \epsilon, h)$ denoting the traveling wave of system (11) near Γ_h. Our main results are given in the following Theorem 2.1 and Theorem 2.2.

Theorem 2.1. *For the perturbed BBM equation (6) with perturbation P_1, the following holds.*

(i) For any sufficiently small $\epsilon > 0$ and any $h \in (-\frac{1}{20}, 0)$, there exists a smooth function $c(\epsilon, h)$ in ϵ and h such that system (6) has one unique isolated periodic

wave in a sufficiently small neighborhood of Γ_h*, given by* $u = c\mu(\tau, \epsilon, h, c(\epsilon, h))$*, satisfying*

$$\lim_{\epsilon\to 0} \mu(\tau, \epsilon, h, c) = \mu(\tau, h),$$
$$\frac{\partial}{\partial\tau}\mu(0, \epsilon, h, c) = \frac{\partial^2}{\partial\tau^2}\mu(0, \epsilon, h, c) = 0, \quad \frac{\partial^3}{\partial\tau^3}\mu(0, \epsilon, h, c) < 0,$$

and

$$\lim_{\epsilon\to 0} c(\epsilon, h) = c(h),$$

where $c(h)$ *is a monotonically increasing function in* h *satisfying* $1 < c(h) < \left(\frac{39}{25}\right)^{\frac{1}{3}}$.

(ii) *For any sufficiently small* $\epsilon > 0$*, there exists a critical wave speed* $c = (\frac{39}{25})^{\frac{1}{3}} + O(\epsilon)$ *such that system* (6) *has one solitary wave in a sufficiently small neighborhood of* Γ_0.

Theorem 2.2. *For any sufficiently small* $\epsilon > 0$*, the perturbed BBM equation* (6) *with perturbation* P_2 *has at most two isolated periodic waves. More precisely, we have the following results.*

(i) *The Abelian integral* $\mathcal{M}(h)$ *(given in (3.11)) has one unique simple zero for any* $h^* \in (-\frac{1}{20}, 0)$ *if and only if*

$$a_1 = \lambda \in (-\infty, -\tfrac{5}{3}] \bigcup [-\tfrac{10}{11}, +\infty) \quad \text{and} \quad a_0 = -\kappa(h^*, \lambda),$$

where $\kappa(h, \lambda)$ *is defined in* (45)*. Therefore, for any sufficiently small* $\epsilon > 0$*, system* (6) *has one unique isolated periodic wave* $u = c\mu(\tau, \epsilon, h^*)$*, in sufficiently small neighborhood of any closed curve* Γ_{h^*} *by taking* $a_1 = \lambda + O(\epsilon)$ *and* $a_0 = -\kappa(h^*, \lambda) + O(\epsilon)$*, satisfying*

$$\lim_{\epsilon\to 0} \mu(\tau, \epsilon, h^*) = \mu(\tau, h^*),$$
$$\frac{\partial}{\partial\tau}\mu(0, \epsilon, h^*) = \frac{\partial^2}{\partial\tau^2}\mu(0, \epsilon, h^*) = 0, \quad \frac{\partial^3}{\partial\tau^3}\mu(0, \epsilon, h^*) < 0.$$

(ii) *The Abelian integral* $\mathcal{M}(h)$ *has exactly two simple zeros* h_1 *and* h_2 *if and only if*

$$a_1 = \lambda \in \left(-\tfrac{5}{3}, -\tfrac{10}{11}\right) \quad \text{and} \quad a_0 = -\kappa(h_1, \lambda) = -\kappa(h_2, \lambda),$$

where

$$\kappa(h_1, \lambda) = \kappa(h_2, \lambda) \in \Big(\min_{h \in (-\frac{1}{20}, 0)} \{\kappa(h, \lambda)\},\ \min\{\kappa(-\tfrac{1}{20}, \lambda), \kappa(0, \lambda)\}\Big),$$

under which, for any sufficiently small $\epsilon > 0$*, system* (6) *has exactly two isolated periodic waves* $u_1 = c\mu(\tau, \epsilon, h_1)$ *and* $u_2 = c\mu(\tau, \epsilon, h_2)$*, in sufficiently small neighborhoods of the closed curves* Γ_{h_1} *and* Γ_{h_2} *by choosing* $a_1 = \lambda^* + O(\epsilon)$ *and* $a_0 = -\kappa(h_1, \lambda) + O(\epsilon)$*, satisfying*

$$\lim_{\epsilon\to 0} \mu(\tau, \epsilon, h_i) = \mu(\tau, h_i),$$
$$\frac{\partial}{\partial\tau}\mu(0, \epsilon, h_i) = \frac{\partial^2}{\partial\tau^2}\mu(0, \epsilon, h_i) = 0, \quad \frac{\partial^3}{\partial\tau^3}\mu(0, \epsilon, h_i) < 0,\ i = 1, 2.$$

(iii) *The Abelian integral* $\mathcal{M}(h)$ *has a unique zero at* $h = 0$ *if and only if*

$$a_0 = -\tfrac{5}{6}a_1 - \tfrac{25}{33}, \tag{14}$$

and further, under (14)*,* $\mathcal{M}(h)$ *has a unique simple zero in* $(-\frac{1}{20}, 0)$ *if and only if* $a_1 = \lambda^{**} \in (-\frac{16}{11}, -\frac{10}{11})$*. Therefore, for any sufficiently small* $\epsilon > 0$*, system* (6) *can have a solitary wave by taking* $a_0 = -\frac{5}{6}a_1 - \frac{25}{33} + O(\epsilon)$*, and co-existence of a solitary wave and a unique periodic wave by choosing* $a_0 = -\frac{5}{6}\lambda^{**} - \frac{25}{33} + O(\epsilon)$ *and* $a_1 = \lambda^{**} + O(\epsilon)$.

Before further analysis, in the next section we will present some definitions and lemmas in perturbation theory, which are needed for proving Theorems 2.1 and 2.2.

3. **Perturbation theory and analysis.**

Lemma 3.1 (Fenichel Criteria)**.** *Consider the system*

$$\begin{aligned} \dot{x} &= f_1(x, y, \epsilon), \\ \dot{y} &= \epsilon f_2(x, y, \epsilon), \end{aligned} \tag{15}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ and $0 < \epsilon \ll 1$ is a real parameter, f_1 and f_2 are C^∞ on the set $V \times I$, $V \subseteq \mathbb{R}^{n+1}$, I is an open interval containing zero. Assume that for $\epsilon = 0$, system (15) has a compact normally hyperbolic manifold M_0 which is contained in the set $f_1(x, y, 0) = 0$. The manifold M_0 is said to be normally hyperbolic if the linearization of (15) at each point in M_0 has exactly $\dim(M_0)$ eigenvalues on the imaginary axis. Then, for any $0 < r < +\infty$, there exists a manifold M_ϵ for ϵ sufficiently small such that the following conclusions hold.

(i) M_ϵ is locally invariant under the flow of (15).

(ii) M_ϵ is C^r in x, y and ϵ.

(iii) $M_\epsilon = \{(x, y) | x = h^\epsilon(y)\}$ for some C^r function h^ϵ, and y in some compact set K.

(iv) There exist locally invariant stable and unstable manifolds $W_s(M_\epsilon)$, $W_u(M^\epsilon)$, that lie within $\mathcal{O}(\epsilon)$ of, and are diffeomorphic to $W_s(M_0)$ and $W_u(M_0)$.

Definition 3.2. Suppose $f_0(x), f_1(x), \ldots, f_{n-1}(x)$ are analytic functions on an real open interval J.

(i) The family of sets $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ is called a Chebyshev system (T-system for short) provided that any nontrivial linear combination,

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{n-1} f_{n-1}(x),$$

has at most $n - 1$ isolated zeros on J.

(ii) An ordered set of n functions $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ is called complete Chebyshev system (CT-system for short) provided any nontrivial linear combination,

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{i-1} f_{i-1}(x),$$

has at most $i - 1$ zeros for all $i = 1, 2, \cdots, n$. Moreover it is called extended complete Chebyshev system (ECT-system for short) if the multiplicities of zeros are taken into account.

(iii) The continuous Wronskian of $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ at $x \in R$ is

$$W[f_0(x), f_1(x), \ldots, f_{k-1}(x)] = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1} \\ f_0'(x) & f_1'(x) & \cdots & f_{k-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix},$$

where $f'(x)$ is the first order derivative of $f(x)$ and $f^{(i)}(x)$ is the ith order derivative of $f(x)$, $i \geq 2$. The definitions imply that the function tuple $\{f_0(x), f_1(x), \cdots, f_{k-1}(x)\}$ is an ECT-system on J, so it is a CT-system on J, and thus a T-system on J. However the inverse is not true.

Let $\mathrm{res}(f_1, f_2)$ denote the resultant of $f_1(x)$ and $f_2(x)$, where $f_1(x)$ and $f_2(x)$ are two univariate polynomials of x on rational number field Q. As it is known, $\mathrm{res}(f_1(x), f_2(x)) = 0$ if and only if $f_1(x)$ and $f_2(x)$ have at least one common root.

Let $\mathrm{res}(f,g,x)$ and $\mathrm{res}(f,g,z)$ denote respectively the resultants between $f(x,z)$ and $g(x,z)$ with respect to x and z, where $f(x,z)$ and $g(x,z)$ are two polynomials in $\{x,z\}$ on rational number field. $\mathrm{res}(f,g,x)$ is a polynomial in z and $\mathrm{res}(f,g,z)$ is a polynomial in x. About the relation between the common roots of two polynomials and their resultants, the following result can be found in many works on polynomial algebra, such as [45]. For completeness we give a short proof.

Lemma 3.3 ([45]). *(i) Let (x_0,z_0) be a common root of $f(x,z)$ and $g(x,z)$. Then, $\mathrm{res}(f,g,x_0)=0$ and $\mathrm{res}(f,g,z_0)=0$. However, the inverse is not true.*

(ii) Let $\mathrm{res}(f,g,z)$ have a unique real root on some open interval (α,β), and $\mathrm{res}(f,g,x)$ have a unique real root on some open interval (γ,θ). Then, there exists at most one common real root of $f(x,z)$ and $g(x,z)$ on $(\alpha,\beta)\times(\gamma,\theta)$.

Proof. (ii) is obvious if (i) is true. So we only prove (i). A two-variable polynomial can be treated as one univariate polynomial of one variable with the other treated as a parameter. Taking $f(x,z)=f_z(x)$ and $g(x,z)=g_z(x)$ which are polynomials of x with parameter z. Let x_0 be the common root of $f_{z_0}(x)$ and $g_{z_0}(x)$. Then, $\mathrm{res}(f_{z_0}(x),g_{z_0}(x))=\mathrm{res}(f,g,z_0)=0$, where z_0 is the common root of $f_{x_0}(z)$ and $g_{x_0}(z)$, and therefore, $\mathrm{res}(f_{x_0}(z),g_{x_0}(z))=\mathrm{res}(f,g,x_0)=0$. $\square$

Let $H(x,y)=A(x)+\frac{y^2}{2}$ be an analytic function. Assume there exists a punctured neighborhood $\mathcal{P}$ of the origin foliated by ovals $\Gamma_h\subseteq\{(x,y)|H(x,y)=h,\ h\in(0,h_0),\ h_0=H(\partial\mathcal{P})\}$. The projection of $\mathcal{P}$ on the x-axis is an interval (x_l,x_r) with $x_l<0<x_r$. Under these assumptions it is easy to verify that $xA'(x)>0$ for all $x\in(x_l,x_r)\backslash\{0\}$, and $A(x)$ has a zero of even multiplicity at $x=0$ and there exists an analytic involution $z(x)$, defined by $A(x)=A(z(x))$, for all $x\in(x_l,x_r)$. Let

$$\mathcal{I}_i(h)=\oint_{\Gamma_h} f_i(x)y^{2s-1}dx,\ \text{for } h\in(0,h_0), \tag{16}$$

where $f_i(x)$, $i=0,1,\ldots,n-1$, are analytic functions on (x_l,x_r) and $s\in N$. Further, define

$$l_i(x):=\frac{f_i(x)}{A'(x)}-\frac{f_i(z(x))}{A'(z(x))}.$$

Then, we have

Lemma 3.4 ([14]). *Under the above assumption, $\{\mathcal{I}_0,\mathcal{I}_1,\cdots,\mathcal{I}_{n-1}\}$ is an ECT system on $(0,h_0)$ if $\{l_0,l_1,\cdots,l_{n-1}\}$ is an ECT system on $(x_l,0)$ or $(0,x_r)$ and $s>n-2$.*

Lemma 3.5 ([29]). *Under the above assumption, if the following conditions are satisfied:*
(i) $W[l_0,l_1,\ldots,l_i]$ does not vanish on $(0,x_r)$ for $i=0,1,\cdots,n-2$,
(ii) $W[l_0,l_1,\ldots,l_{n-1}]$ has k zeros on $(0,x_r)$ with multiplicities counted, and
(iii) $s>n+k-2$,
then, any nontrivial linear combination of $\{\mathcal{I}_0,\mathcal{I}_1,\cdots,\mathcal{I}_{n-1}\}$ has at most $n+k-1$ zeros on $(0,h_0)$ with multiplicities counted. In this case, $\{\mathcal{I}_0,\mathcal{I}_1,\cdots,\mathcal{I}_{n-1}\}$ is called a Chebyshev system with accuracy k on $(0,h_0)$, where $W[l_0,l_1,\ldots,l_i]$ denotes the Wronskian of $\{l_0,l_1,\ldots,l_i\}$.

Now we consider system (10), which can be rewritten in the form of

$$\frac{d\mu}{d\tau}=\nu,$$

$$\begin{aligned} \frac{d\nu}{d\tau} &= \omega, \\ \epsilon c^{\frac{3}{2}} \frac{d\omega}{d\tau} &= \mu^3 - \mu^4 - \omega - \frac{\epsilon}{c^{\frac{3}{2}}}\nu. \end{aligned} \tag{17}$$

Introducing the time scaling $\sigma = \frac{\tau}{\epsilon}$ into (17) yields

$$\begin{aligned} \frac{d\mu}{d\sigma} &= \epsilon\nu, \\ \frac{d\nu}{d\sigma} &= \epsilon\omega, \\ c^{\frac{3}{2}} \frac{d\omega}{d\sigma} &= \mu^3 - \mu^4 - \omega - \frac{\epsilon}{c^{\frac{3}{2}}}\nu. \end{aligned} \tag{18}$$

When $\epsilon > 0$, system (17) is equivalent to (18). System (17) is called the slow system, while system (18) is called the fast system.

The slow system (17) determines its critical manifold, which is a two-dimensional submanifold in $\mathbb{R}^3$:

$$M_0 = \{ (\mu, \nu, \omega) \in \mathbb{R}^3 \big| \, \omega = \mu^3 - \mu^4 \}.$$

The Jacobian matrix of the fast system (18) restricted on M_0 is given by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c^{-\frac{3}{2}}(3\mu^2 - 4\mu^3) & 0 & -c^{-3/2} \end{bmatrix}$$

which has three eigenvalues $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -c^{-3/2}$, with λ_1 and λ_2 being on the imaginary axis. Therefore, M_0 is normally hyperbolic. Consequently, it follows from Lemma 3.1 that for $\epsilon > 0$ sufficiently small, there exists a two-dimensional submanifold M_ϵ in $\mathbb{R}^3$, which is invariant under the flow of system (17), within the Hausdorff distance ϵ of M_0.

Let

$$M_\epsilon = \{(\mu, \nu, \omega) \in \mathbb{R}^3 : \omega = \mu^3 - \mu^4 + \eta(\mu, \nu, \epsilon)\},$$

where $\eta(\mu, \nu, \epsilon)$ is smooth in μ, ν and ϵ, satisfying $\eta(\mu, \nu, 0) = 0$, and expanded as

$$\eta(\mu, \nu, \epsilon) = \epsilon\eta_1(\mu, \nu) + O(\epsilon^2). \tag{19}$$

Substituting (19) into the last equation of (17) and comparing its coefficients yields

$$\eta_1(\mu, \nu) = c^{\frac{3}{2}} \left[(-3\mu^2 + 4\mu^3)\nu - c^{-3}\nu\right].$$

The dynamics of (17) on M_ϵ is determined by

$$\begin{aligned} \frac{d\mu}{d\tau} &= \nu, \\ \frac{d\nu}{d\tau} &= \mu^3 - \mu^4 + \epsilon c^{\frac{3}{2}}[(-3\mu^2 + 4\mu^3)\nu - c^{-3}\nu] + O(\epsilon^2). \end{aligned} \tag{20}$$

For any $h \in (-\frac{1}{20}, 0)$, $\mathcal{H}(\mu, \nu) = h$ defines a periodic orbit Γ_h of (12) (or the system (20) with $\epsilon = 0$). Let $(\alpha(h), 0)$ denote the intersection point of Γ_h and the positive μ-axis, T the period of Γ_h. Further, let $\Gamma_{h,\epsilon}$ be the positive orbit of (20) starting from the point $(\alpha(h), 0)$ at time $\tau = 0$, and $(\beta(h, \epsilon), 0)$ the first intersection point of $\Gamma_{h,\epsilon}$ with the positive μ-axis at time $\tau = \tau^*(\epsilon)$. Let $\mathcal{H}^*(\mu, \nu)$ denote the small perturbation of $\mathcal{H}(\mu,\nu)$. Then, the difference between the two points is given by

$$\begin{aligned}&\mathcal{H}^*((\beta(h,\epsilon),0))-\mathcal{H}^*((\alpha(h),0))=\int_{\Gamma_{h,\epsilon}} d\mathcal{H}^*=\int_{\Gamma_{h,\epsilon}}\big(-\mu^3+\mu^4\big)d\mu+\nu dv\\&=\int_0^{\tau^*(\epsilon)}\big\{(-\mu^3+\mu^4)\nu-\nu\big[(-\mu^3+\mu^4)-\epsilon c^{\frac{3}{2}}\big(-3\mu^2\nu+4\mu^3\nu-c^{-3}\nu\big)\big]\big\}d\tau\\&=\int_0^{\tau^*(\epsilon)}\epsilon\nu c^{\frac{3}{2}}\big[\big(-3\mu^2+4\mu^3\big)\nu-c^{-3}\nu\big]d\tau=\epsilon\int_0^{\tau^*(\epsilon)}c^{-\frac{3}{2}}\big[c^3\big(-3\mu^2+4\mu^3\big)\nu^2-\nu^2\big]d\tau\\&\triangleq\epsilon F(h,\epsilon).\end{aligned}$$

By continuousness theorem, we have

$$\lim_{\epsilon\to 0}\Gamma_{h,\epsilon}=\Gamma_h,\quad \lim_{\epsilon\to 0}b(h,\epsilon)=a(h),\quad \lim_{\epsilon\to 0}\tau^*(\epsilon)=T,$$

and thus,

$$\begin{aligned}F(h,\epsilon)&=\int_0^T c^{-\frac{3}{2}}[c^3(-3\mu^2+4\mu^3)\nu^2-\nu^2]d\tau+O(\epsilon),\\&=c^{-\frac{3}{2}}\int_{\Gamma_h}[c^3(-3\mu^2+4\mu^3)\nu-\nu]d\mu+O(\epsilon)\\&\triangleq c^{-\frac{3}{2}}M(h)+O(\epsilon),\end{aligned}\tag{21}$$

where $M(h)$ is called Abelian integral or Melnikov function, given by

$$M(h)=\int_{\Gamma_h}\big[c^3(-3\mu^2+4\mu^3)\nu-\nu\big]d\mu=\int_{\Gamma_h}\big[-1+c^3(-3\mu^2+4\mu^3)\big]\nu d\mu.\tag{22}$$

It has been noted that compared with the application of Picard-Focus equation method (eg. see [7]) which is often used to derive the Abelian integral, here our approach developed above is much simpler.

Similarly, for system (11), we take $\mu'(\tau)=\nu$ and follow the above procedure to obtain the following regular perturbation problem which is not restricted on a manifold,

$$\begin{aligned}\frac{d\mu}{d\tau}&=\nu,\\\frac{d\nu}{d\tau}&=\mu^3-\mu^4+\epsilon(a_0+a_1\mu+a_2\mu^2)\nu.\end{aligned}\tag{23}$$

Let $(\alpha^*(h),0)$ be the intersection point of Γ_h and the positive μ-axis, T the period of Γ_h, $\Gamma^*_{h,\epsilon}$ the positive orbit of (23) starting from the point $(\alpha^*(h),0)$ at time $\tau=0$, and $(\beta^*(h,\epsilon),0)$ the first intersection point of $\Gamma_{h,\epsilon}$ with the positive μ-axis at time $\tau=\tau^*(\epsilon)$. Then, the difference between the two points $(\alpha^*(h),0)$ and $(\beta^*(h,\epsilon),0)$ can be expressed as

$$\begin{aligned}&\mathcal{H}^*(\beta^*(h,\epsilon),0)-\mathcal{H}^*(\alpha^*(h),0)=\int_{\Gamma^*_{h,\epsilon}}d\mathcal{H}^*\\=\;&\epsilon\int_{\Gamma_h}(a_0+a_1\mu+\mu^2)\nu d\mu+O(\epsilon)\triangleq\epsilon\mathcal{M}(h)+O(\epsilon^2),\end{aligned}\tag{24}$$

where the Abelian integral $\mathcal{M}(h)$ is given by

$$\mathcal{M}(h)=\int_{\Gamma_h}(a_0+a_1\mu+\mu^2)\nu d\mu.\tag{25}$$

To investigate the existence of periodic and solitary waves for the two perturbation problems, we need study the zeros of the functions $\mathcal{H}^*(\beta(h,\epsilon),0)-\mathcal{H}^*(\alpha(h),0)$ and $\mathcal{H}^*(\beta^*(h,\epsilon),0)-\mathcal{H}^*(\alpha^*(h),0)$ and their distributions. It follows from (21) and (24) that it suffices to consider the Abelian integrals $M(h)$ and $\mathcal{M}(h)$.

4. **Analysis of system (6) with perturbation P_1.** In the section, we study system (6) with perturbation P_1. Based on the discussion in the previous sections, we need only study the Abelian integral $M(h)$. Let

$$J_n(h)=\oint_{\Gamma_h}\mu^n\nu d\mu. \tag{26}$$

Then,

$$M(h)=c^3(-3J_2+4J_3)-J_0.$$

Lemma 4.1. *For $h\in(-\frac{1}{20},0)$, $J_0'(h)>0$ and $J_0(h)>0$.*

Proof. It is easy to obtain

$$J_0'(h)=\oint_{\Gamma_h}\frac{d\mu}{\nu}=\oint_{\Gamma_h}\frac{\nu d\tau}{\nu}=\int_0^{T(h)}d\tau=T(h)>0,$$

where $T(h)$ denotes the period of Γ_h.

Since $\nu\to 0$ as $h\to-\frac{1}{20}$, we have

$$J_0(-\frac{1}{20})=\lim_{h\to-\frac{1}{20}}\oint_{\Gamma_h}\nu d\mu=\lim_{h\to-\frac{1}{20}}\int_0^T\nu^2 d\tau=0.$$

which, together with $J_0'(h)>0$, implies $J_0(h)>0$ for $h\in(-\frac{1}{20},0)$. ☐

It follows from Lemma 4.1 that the following ratio is well defined,

$$X(h)=\frac{-3J_2+4J_3}{J_0}. \tag{27}$$

Then,

$$M(h)=J_0(c^3X(h)-1). \tag{28}$$

In the remaining of this section we mainly prove the following proposition, which is needed for proving Theorem 2.1.

Proposition 1. *For $h\in(-\frac{1}{20},0)$, $X'(h)<0$. Moreover,*

$$\tfrac{25}{39}<X(h)<1,\quad \lim_{h\to-\frac{1}{20}}X(h)=1,\quad \lim_{h\to0}X(h)=\tfrac{25}{39}.$$

To prove Proposition 1, we need the following lemmas.

Lemma 4.2. *Suppose $B(p,q)=\int_0^1x^{p-1}(1-x)^{q-1}dx$ is the Beta function with $p>0$ and $q>0$. Then, the following holds:*

$$J_0(0)=\sqrt{2}\left(\tfrac{5}{4}\right)^3B\left(\tfrac{3}{2},3\right),\ J_2(0)=\sqrt{2}\left(\tfrac{5}{4}\right)^5B\left(\tfrac{3}{2},5\right),\ J_3(0)=\sqrt{2}\left(\tfrac{5}{4}\right)^6B\left(\tfrac{3}{2},6\right).$$

In addition,

$$\tfrac{J_2(0)}{J_0(0)}=\tfrac{25}{33},\quad \tfrac{J_3(0)}{J_0(0)}=\tfrac{625}{858},\quad \tfrac{J_3(0)}{J_2(0)}=\tfrac{25}{26}.$$

Proof. Let $\mathcal{H} - \frac{\nu^2}{2} = 0$. Then, we have

$$J_n(0) = \sqrt{2}\int_0^{\frac{5}{4}} \mu^{n+2}\sqrt{1-\tfrac{4}{5}\mu}\; d\mu.$$

Let $1 - \frac{4}{5}\mu = t$, and so $\mu = \frac{5}{4}(1-t)$, $d\mu = -\frac{5}{4}dt$. Then, we obtain

$$J_n(0) = \sqrt{2}\int_0^1 \Big(\frac{5}{4}\Big)^{n+3}(1-t)^{n+2}t^{\frac{1}{2}}\,dt = \sqrt{2}\Big(\frac{5}{4}\Big)^{n+3}B\Big(\frac{3}{2}, n+3\Big),$$

which proves the first part of the lemma.

Next, it follows from

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad \text{and} \quad \Gamma(s+1) = s\Gamma(s),$$

where $\Gamma(s) = \int_0^{+\infty} x^{s-1}e^{-x}dx$, $(s>0)$ is the Gamma function, that

$$\begin{aligned}\frac{J_2(0)}{J_0(0)} &= \frac{(\frac{5}{4})^5 B(\frac{3}{2},5)}{(\frac{5}{4})^3 B(\frac{3}{2},3)} = \Big(\frac{5}{4}\Big)^2 \times \frac{\Gamma(\frac{3}{2})\Gamma(5)}{\Gamma(\frac{13}{2})} \times \frac{\Gamma(\frac{9}{2})}{\Gamma(\frac{3}{2})\Gamma(3)} \\ &= \Big(\frac{5}{4}\Big)^2 \times \frac{4\times 3\times\Gamma(3)\times\Gamma(\frac{9}{2})}{\frac{11}{2}\times\frac{9}{2}\times\Gamma(\frac{9}{2})\times\Gamma(3)} = \frac{25}{33}.\end{aligned}$$

Similarly, we obtain

$$\frac{J_3(0)}{J_0(0)} = \frac{625}{858} \quad \text{and} \quad \frac{J_3(0)}{J_2(0)} = \frac{25}{26}.$$

□

Lemma 4.3. *The following rates at $h = -\frac{1}{20}$ hold:*

$$\begin{aligned}\frac{J_2(-\frac{1}{20})}{J_0(-\frac{1}{20})} &= \lim_{h\to-\frac{1}{20}}\frac{J_2(h)}{J_0(h)} = 1,\\ \frac{J_3(-\frac{1}{20})}{J_0(-\frac{1}{20})} &= \lim_{h\to-\frac{1}{20}}\frac{J_3(h)}{J_0(h)} = 1,\\ \frac{J_3(-\frac{1}{20})}{J_2(-\frac{1}{20})} &= \lim_{h\to-\frac{1}{20}}\frac{J_3(h)}{J_2(h)} = 1.\end{aligned}$$

Proof. Let $\mu = r\cos\theta + 1$, $\nu = r\sin\theta$. Then, $\mathcal{H} - h = 0$ becomes

$$\mathcal{F}(r,\rho) \triangleq \frac{r^5}{5}\cos^5\theta + \frac{3}{4}r^4\cos^4\theta + r^3\cos^3\theta + \frac{r^2}{2} - \rho^2 = 0,$$

where $\rho = (h+\frac{1}{20})^{\frac{1}{2}}$. Applying the implicit function theorem to $\mathcal{F}(r,\rho)$ at $(r,\rho) = (0,0)$, we can show that there exist a smooth function $r = \chi(\rho)$ and a small positive number δ, $0 < \rho < \delta \ll 1$ such that $\mathcal{F}(\chi(\rho),\rho) = 0$, and $\chi(\rho)$ can be expanded as

$$\begin{aligned}\chi(\rho) &= \sqrt{2}\rho - 2\rho^2\cos^3\theta + \sqrt{2}\Big(-\frac{3}{2}\cos^4\theta + 5\cos^6\theta\Big)\rho^3 \\ &\quad + \Big(-\frac{4}{5}\cos^5\theta + 18\cos^7\theta - 32\cos^9\theta\Big)\rho^4 + O(\rho^5).\end{aligned} \tag{29}$$

Therefore,

$$J_n(h) = \oint_{\Gamma_h}\mu^n\nu d\mu = \iint\limits_{int\Gamma_h}\mu^n d\mu d\nu = \int_0^{2\pi}d\theta\int_0^{\chi(\rho)} r^{n+1}\cos^n\theta dr. \tag{30}$$

Noticing $\rho = (h + \frac{1}{20})^{\frac{1}{2}}$ and substituting (29) into (30) yields

$$\begin{aligned} J_0(h) &= 2\pi(h + \tfrac{1}{20}) + \tfrac{21\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3), \\ J_1(h) &= 2\pi(h + \tfrac{1}{20}) + \tfrac{9\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3), \\ J_2(h) &= 2\pi(h + \tfrac{1}{20}) + \tfrac{\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3), \\ J_3(h) &= 2\pi(h + \tfrac{1}{20}) - \tfrac{3\pi}{4}(h + \tfrac{1}{20})^2 + O((h + \tfrac{1}{20})^3), \end{aligned} \tag{31}$$

for $0 < h + \frac{1}{20} \ll 1$. Therefore,

$$\frac{J_i(-\frac{1}{20})}{J_j(-\frac{1}{20})} = \lim_{h \to -\frac{1}{20}} \frac{J_i(h)}{J_j(h)} = 1, \ i, \ j = 0, \ 1, \ 2, 3.$$

This completes the proof of Lemma 4.3. □

Lemma 4.4. *$J_n(h) = \sum_{i=0}^{n}(-1)^i C_n^i I_i(h)$, where $I_i(h) = \oint_{\widetilde{\mathcal{H}}=h} \widetilde{\mu}^i \widetilde{\nu} d\widetilde{\mu}$, in which $\widetilde{\mu} = -\mu + 1$, $\widetilde{\nu} = -\nu$, and*

$$\widetilde{\mathcal{H}}(\widetilde{\mu}, \widetilde{\nu}) = \mathcal{H}(1 - \widetilde{\mu}, -\widetilde{\nu}). \tag{32}$$

In particular,

$$\begin{aligned} J_0(h) &= I_0(h), \\ J_2(h) &= I_2(h) - 2I_1(0) + I_0(h), \\ J_3(h) &= -I_3(h) + 3I_2(0) - 3I_1(h) + I_0(h). \end{aligned} \tag{33}$$

Proof. A direct computation shows that

$$\begin{aligned} J_n(h) &= \oint_{\Gamma_h} \mu^n \nu d\mu = \oint_{\widetilde{\mathcal{H}}=h} (1 - \widetilde{\mu})^n (-\widetilde{\nu}) d(1 - \widetilde{\mu}) \\ &= \oint_{\widetilde{\mathcal{H}}=h} \Big[\sum_{i=0}^{n} C_n^i (-1)^i \widetilde{\mu}^i \widetilde{\nu}\Big] d\widetilde{\mu} = \sum_{i=0}^{n} C_n^i (-1)^i \oint_{\widetilde{\mathcal{H}}=h} \widetilde{\mu}^i \widetilde{\nu} d\widetilde{\mu} \\ &= \sum_{i=0}^{n} (-1)^i C_n^i I_i(h). \end{aligned}$$

Then, substituting $n = 0, \ 2, \ 3$ respectively into $J_n(h)$ yields (33). □

Lemma 4.5. *On $(-\frac{1}{20}, 0)$, $\frac{J_2(h)}{J_0(h)}$ is decreasing monotonically from 1 to $\frac{25}{33}$, and $\frac{J_3(h)}{J_2(h)}$ is decreasing monotonically from 1 to $\frac{25}{26}$.*

Proof. By Lemmas 4.2 and 4.3, we need only prove that $\frac{J_2(h)}{J_0(h)}$ and $\frac{J_3(h)}{J_2(h)}$ are monotonic on the interval $(-\frac{1}{20}, 0)$, which implies that each of the linear combination $\alpha_1 J_0(h) + \alpha_2 J_2(h)$ and $\alpha_1^* J_2(h) + \alpha_2^* J_3(h)$ has at most one zero on $(-\frac{1}{20}, 0)$. Let

$$f_0(\widetilde{\mu}) = 1, \quad f_2(\widetilde{\mu}) = \widetilde{\mu}^2 - 2\widetilde{\mu} + 1, \quad f_3(\widetilde{\mu}) = -\widetilde{\mu}^3 + 3\widetilde{\mu}^2 - 3\widetilde{\mu} + 1.$$

By Lemma 4.4, we have

$$J_k(h) = \oint_{\widetilde{\mathcal{H}}=h} f_k(\widetilde{\mu}) \widetilde{\nu} d\widetilde{\mu}, \ k = 0, \ 1, \ 2, \ 3.$$

Then, for each $f_i(\widetilde{\mu})$, $i = 0, 2, 3$, setting

$$l_i(\widetilde{\mu}) = \left(\frac{f_i}{A'}\right)(\widetilde{\mu}) - \left(\frac{f_i}{A'}\right)(z(\widetilde{\mu})), \tag{34}$$

where $z(\widetilde{\mu})$ is an analytic involution defined by $A(\widetilde{\mu}) = A(z(\widetilde{\mu}))$ on $(-\frac{1}{4}, 1)$, and

$$A(\widetilde{\mu}) = \widetilde{\mathcal{H}}(\widetilde{\mu}, \widetilde{\nu}) - \frac{\widetilde{\nu}^2}{2}. \tag{35}$$

Factorizing $A(\widetilde{\mu}) - A(z)$ gives $-\frac{1}{20}(\widetilde{\mu} - z)q(\widetilde{\mu}, z)$, where

$$q(\widetilde{\mu}, z) = 4\sum_{i=0}^{4} \widetilde{\mu}^i z^{4-i} - 15\sum_{i=0}^{3} \widetilde{\mu}^i z^{3-i} + 20\sum_{i=0}^{2} \widetilde{\mu}^i z^{2-i} - 10(\widetilde{\mu} + z).$$

In fact, $z(\widetilde{\mu})$ is defined implicitly by $q(\widetilde{\mu}, z)$. Therefore,

$$\frac{d}{d\widetilde{\mu}} l_i(\widetilde{\mu}) = \frac{d}{d\widetilde{\mu}} \left(\frac{f_i}{A'}\right)(\widetilde{\mu}) + \frac{d}{dz}\left[\left(\frac{f_i}{A'}\right)(z(\widetilde{\mu}))\right]\frac{dz}{d\widetilde{\mu}}$$

with

$$\frac{dz}{d\widetilde{\mu}} = -\frac{\partial q(\widetilde{\mu}, z)}{\partial \widetilde{\mu}} \Bigg/ \frac{\partial q(\widetilde{\mu}, z)}{\partial z}.$$

Further, a direct computation shows that

$$l_0(\widetilde{\mu}) = \frac{(\widetilde{\mu} - z)\, P_1(\widetilde{\mu})}{\widetilde{\mu}\,(\widetilde{\mu} - 1)^3\, z\,(z - 1)^3},$$

$$W[l_0(\widetilde{\mu}), l_2(\widetilde{\mu})] = \begin{vmatrix} l_0(\widetilde{\mu}) & l_2(\widetilde{\mu}) \\ l_0'(\widetilde{\mu}) & l_2'(\widetilde{\mu}) \end{vmatrix} = \frac{-(\widetilde{\mu} - z)^3\, P_2(\widetilde{\mu})}{\widetilde{\mu}^2 z^2\,(z - 1)^5\,(\widetilde{\mu} - 1)^5\, P_0(\widetilde{\mu})},$$

and

$$l_2(\widetilde{\mu}) = \frac{(x - z)\,(z + x - 1)}{(z - 1)\, z\,(x - 1)\, x},$$

$$W[l_2(\widetilde{\mu}), l_3(\widetilde{\mu})] = \begin{vmatrix} l_2(\widetilde{\mu}) & l_3(\widetilde{\mu}) \\ l_2'(\widetilde{\mu}) & l_3'(\widetilde{\mu}) \end{vmatrix} = \frac{(\widetilde{\mu} - z)^3\, P_3(\widetilde{\mu})}{\widetilde{\mu}^2 z^2\,(z - 1)^2\,(\widetilde{\mu} - 1)^2\, P_0(\widetilde{\mu})},$$

where

$$\begin{aligned}
P_0(\widetilde{\mu}) =\; & 4\sum_{i=0}^{3}(4 - i)\widetilde{\mu}^i z^{3-i} - 15\sum_{i=0}^{2}(3 - i)\widetilde{\mu}^i z^{2-i} + 20\,\widetilde{\mu} + 40\,z - 10,\\
P_1(\widetilde{\mu}) =\; & \sum_{i=0}^{3}\widetilde{\mu}^i z^{3-i} - 3\sum_{i=0}^{2}\widetilde{\mu}^i z^{2-i} + 3(\widetilde{\mu} + z) - 1,\\
P_2(\widetilde{\mu}) =\; & 8\left(4\,\widetilde{\mu}^4 + 3\,\widetilde{\mu}^3 z + 6\,\widetilde{\mu}^2 z^2 + 3\,\widetilde{\mu} z^3 + 4\,z^4\right)(\widetilde{\mu} + z)^3\\
& + 20\,(\widetilde{\mu} + z)\left(83\,\widetilde{\mu}^2 + 97\,\widetilde{\mu} z + 83\,z^2\right) - (250\,\widetilde{\mu}^6 + 850\,\widetilde{\mu}^5 z\\
& + 1398\,\widetilde{\mu}^4 z^2 - 1604\,\widetilde{\mu}^3 z^3 + 1398\,\widetilde{\mu}^2 z^4 + 850\,\widetilde{\mu} z^5 + 250\,z^6)\\
& + (\widetilde{\mu} + z)\left(834\,\widetilde{\mu}^4 + 1683\,\widetilde{\mu}^3 z + 2026\,\widetilde{\mu}^2 z^2 + 1683\,\widetilde{\mu} z^3 + 834\,z^4\right)\\
& - (1531\,\widetilde{\mu}^4 + 3997\,\widetilde{\mu}^3 z + 5074\,\widetilde{\mu}^2 z^2 + 3997\,\widetilde{\mu} z^3 + 1531\,z^4)\\
& - 5(211\,\widetilde{\mu}^2 + 350\,\widetilde{\mu} z + 211\,z^2 + +360(\widetilde{\mu} + z - 50,\\
P_3(\widetilde{\mu}) =\; & 4\left(\widetilde{\mu}^2 + z^2\right)\left(4\,\widetilde{\mu}^2 + 7\,\widetilde{\mu} z + 4\,z^2\right) - (\widetilde{\mu} + z)\left(61\,\widetilde{\mu}^2 + 38\,\widetilde{\mu} z + 61\,z^2\right)\\
& + 5(17\,\widetilde{\mu}^2 + 24\,\widetilde{\mu} z + 17\,z^2) - 50(\widetilde{\mu} + z) + 10.
\end{aligned}$$

Now, computing the resultant of $z + \widetilde{\mu} - 1$ and $q(\widetilde{\mu}, z)$ with respect to z, we obtain

$$\mathrm{res}(z + \widetilde{\mu} - 1, q, z,) = 4\widetilde{\mu}^4 - 8\widetilde{\mu}^3 + 6\widetilde{\mu}^2 - 2\widetilde{\mu} - 1,$$

which has no real roots on $(0, 1)$. This implies that $l_2(\widetilde{\mu})$ does not vanish for $\widetilde{\mu} \in (0, 1)$.

Similarly, computing the resultant of $P_i(\widetilde{\mu}, z)$ and $q(\widetilde{\mu}, z)$ with respect to z and applying the Sturm's Theorem, we can show that $P_i(\widetilde{\mu}, z)$ $(i = 0, 1, 2, 3)$ does not vanish for $\widetilde{\mu} \in (0, 1)$. This means that $l_0(\widetilde{\mu})$, $W[l_0, l_2]$ and $W[l_2, l_3]$ do not vanish for $\widetilde{\mu} \in (0,1)$. By Lemma 3.3, we have shown that $\{J_0, J_2\}$ and $\{J_2, J_3\}$ are all Chebyshev systems, and therefore, $\frac{J_2(h)}{J_0(h)}$ and $\frac{J_3(h)}{J_0(h)}$ are all monotonic on $(-\frac{1}{20}, 0)$. □

Based on the above results, the proof for Proposition 1 is straightforward.

Proof. [For Proposition 1] Since $\frac{J_3(h)}{J_2(h)}$ is decreasing monotonically from 1 to $\frac{25}{26}$, implying that $-3+4\frac{J_3(h)}{J_2(h)}$ is positive and decreasing monotonically from 1 to $\frac{11}{13}$. Because $\frac{J_2(h)}{J_0(h)}$ is also positive and decreasing monotonically, we obtain that $X(h) = \frac{-3J_2+4J_3}{J_0} = \frac{J_2}{J_0}\big(-3+\frac{4J_3}{J_2}\big)$, which is decreasing monotonically on $(-\frac{1}{20}, 0)$. This implies that $X'(h) < 0$, and so

$$\frac{25}{39} = \lim_{h\to 0} X(h) < X(h) < \lim_{h\to -\frac{1}{20}} X(h) = 1.$$

□

Now, we are ready to prove Theorem 2.1.

Proof. [For Theorem 2.1] By (28), we choose $c = c(h) = (X(h))^{-\frac{1}{3}}$ for each $h \in (-\frac{1}{20}, 0)$, then $M(h) = 0$. The monotonicity of $X(h)$ means that the zero h is unique, and $c(h)$ satisfies $c'(h) > 0$. Thus,

$$1 < c(h) < \left(\tfrac{39}{25}\right)^{\frac{1}{3}}, \quad \lim_{h\to -\frac{1}{20}} c(h) = 1, \quad \lim_{h\to 0} c(h) = \left(\tfrac{39}{25}\right)^{\frac{1}{3}}.$$

The above results, together with the implicit function theorem imply that choosing $c = c(h) + O(\epsilon)$ leads to that $M(h) + O(\epsilon)$ has a unique zero near h. This proves the first part of Theorem 2.1 since $\mathcal{H}(\beta(h,\epsilon),0) - \mathcal{H}(\alpha(h),0) = \epsilon\big[c^{-\frac{3}{2}}M(h) + O(\epsilon)\big]$. The second part of the theorem is the limit case of $M(h)$ as $h \to 0$, which can be proved similarly. □

5. Analysis of system (6) with perturbation P_2.

In this section, we study the BBM equation (6) with perturbation P_2. As discussed in sections 2 and 3, we need only consider the Abelian integral $\mathcal{M}(h)$. Using the same notation in (26), we have

$$\mathcal{M}(h) = a_0 J_0(h) + a_1 J_1(h) + a_2 J_2(h).$$

5.1. Asymptotic expansion of the Abelian integral.

One efficient method for studying the weak Hilbert's 16th problem is to investigate the asymptotic expansions of Abelian integrals, see [18, 50, 19]. $\mathcal{M}(h)$ has the following expansion (see [50, 19]):

$$\begin{aligned}\mathcal{M}(h) = {} & c_0(\delta) + c_1(\delta)|h|^{\frac{3}{4}} + \big[c_2(\delta) + b_0^* c_1(\delta)\big] h \ln|h| \\ & + \big[c_3(\delta) + b_1^* c_1(\delta) + b_2^* c_2(\delta)\big] h + O((-h)^{\frac{5}{4}})\end{aligned} \tag{36}$$

for $0 < -h \ll 1$, where the coefficients $c_j(\delta)$ are obtained by using the methods and formulas developed in [19] as follows:

$$\begin{aligned}&c_0(\delta) = \tfrac{25\sqrt{2}}{72072}(715\,a_1 + 650\,a_2 + 858\,a_0), \quad c_1(\delta) = 4\widetilde{A}_0 a_0,\\ &c_2(\delta) = -\tfrac{\sqrt{2}}{2}a_1, \quad c_3(\delta) = 5\sqrt{2}a_2,\end{aligned}$$

with $\widetilde{A}_0 < 0$. Therefore, we obtain the expansions of J_0, J_1, J_2, J_3, J_0', J_1', J_2' and J_3' as follows:

$$\begin{aligned}
J_0(h) &= \tfrac{25\sqrt{2}}{84} + 4\widetilde{A}_0|h|^{\frac{3}{4}} + 4\widetilde{A}_0 b_0^* h\ln(-h) + 4b_1^*\widetilde{A}_0 h + O((-h)^{\frac{5}{4}}),\\
J_1(h) &= \tfrac{125\sqrt{2}}{504} - \tfrac{\sqrt{2}}{2} h\ln|h| - \tfrac{\sqrt{2}}{2} b_2^* h + O((-h)^{\frac{5}{4}}),\\
J_2(h) &= \tfrac{625\sqrt{2}}{2772} + 5\sqrt{2}h + O((-h)^{\frac{5}{4}}),\\
J_3(h) &= \tfrac{15625\sqrt{2}}{72072} + \tfrac{25\sqrt{2}}{6} h + O((-h)^{\frac{5}{4}}),
\end{aligned} \tag{37}$$

and

$$\begin{aligned}
J_0'(h) &= -3\widetilde{A}_0(-h)^{-\frac{1}{4}} + 4\widetilde{A}_0 b_0^* \ln(-h) + 4\widetilde{A}_0 b_0^* + 4\widetilde{A}_0 b_1^* + h.o.t.,\\
J_1'(h) &= -\tfrac{\sqrt{2}}{2}\ln(-h) - \tfrac{\sqrt{2}}{2} - \tfrac{\sqrt{2}}{2} b_2^* + h.o.t.,\\
J_2'(h) &= 5\sqrt{2}h + h.o.t.,\\
J_3'(h) &= \tfrac{25\sqrt{2}}{6} + h.o.t.,
\end{aligned} \tag{38}$$

where $h.o.t.$ denotes higher order terms.

5.2. **Existence of periodic waves.** It follows from Lemma 4.4 that

$$J_i(h) = \oint_{\widetilde{\mathcal{H}}=h} f_i(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu} = \sum_{j=0}^{i}(-1)^j C_i^j I_j(h),$$

where $f_i(\widetilde{\mu}) = (-\widetilde{\mu}+1)^i$, $I_j(h) = \oint_{\widetilde{\mathcal{H}}=h} \widetilde{\mu}^j \widetilde{\nu} d\widetilde{\mu}$. Then, we have

Lemma 5.1. $8\left(h+\frac{1}{20}\right)^3 I_j(h) = \oint_{\widetilde{\mathcal{H}}=h} f_j(\widetilde{\mu})\widetilde{\nu}^5 d\widetilde{\mu} \equiv \widetilde{I}_j(h)$, *where* $f_j(\widetilde{\mu}) = \widetilde{\mu}^j + G_j(\widetilde{\mu}) + \widetilde{G}_j(\widetilde{\mu})$, *with* $G_j(\widetilde{\mu}) = \frac{\widetilde{\mu}^j g_j(\widetilde{\mu})}{30(\widetilde{\mu}-1)^4}$ *and* $\widetilde{G}_j(\widetilde{\mu}) = \frac{\widetilde{\mu}^j \widetilde{g}_j(\widetilde{\mu})}{1500(\widetilde{\mu}-1)^8}$, *in which* $g_j(\widetilde{\mu})$ *and* $\widetilde{g}_j(\widetilde{\mu})$ *are polynomials in* $\widetilde{\mu}$.

Proof. Multiplying $I_j(h)$ by $\frac{\widetilde{\nu}^2+2A(\widetilde{\mu})}{2(h+\frac{1}{20})} = 1$ yields

$$\begin{aligned}
I_j(h) &= \oint_{\widetilde{\mathcal{H}}=h} \frac{2A(\widetilde{\mu})+\widetilde{\nu}^2}{2(h+\frac{1}{20})}\widetilde{\mu}^j\widetilde{\nu}d\widetilde{\mu}\\
&= \frac{1}{2(h+\frac{1}{20})}\Big(\oint_{\widetilde{\mathcal{H}}=h} 2\widetilde{\mu}^j A(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu} + \oint_{\widetilde{\mathcal{H}}=h}\widetilde{\mu}^j\widetilde{\nu}^3 d\widetilde{\mu}\Big), \quad i=0,1,2,3.
\end{aligned} \tag{39}$$

By Lemma 4.1 of [14] (with $k=3$ and $F(\widetilde{\mu}) = 2\widetilde{\mu}^j A(\widetilde{\mu})$), we have

$$\oint_{\widetilde{\mathcal{H}}=h} 2\widetilde{\mu}^j A(\widetilde{\mu})\widetilde{\nu}d\widetilde{\mu} = \oint_{\widetilde{\mathcal{H}}=h} G_j(\widetilde{\mu})\widetilde{\nu}^3 d\widetilde{\mu}, \tag{40}$$

where $G_j(\widetilde{\mu}) = \frac{d}{3d\widetilde{\mu}}\frac{2\widetilde{\mu}^j A(\widetilde{\mu})}{A'(\widetilde{\mu})} = \frac{\widetilde{\mu}^j g_j(\widetilde{\mu})}{30(\widetilde{\mu}-1)^4}$ with

$$g_j(\widetilde{\mu}) = 4j\widetilde{\mu}^4 - 19j\widetilde{\mu}^3 + 4\widetilde{\mu}^4 + 35j\widetilde{\mu}^2 - 16\widetilde{\mu}^3 - 30j\widetilde{\mu} + 25\widetilde{\mu}^2 + 10j - 20\widetilde{\mu} + 10.$$

Substituting (40) into (39) and multiplying $\frac{2A(\widetilde{\mu})+\widetilde{\nu}^2}{2(h+\frac{1}{20})} = 1$ gives

$$\begin{aligned}
I_j(h) &= \frac{1}{2(h+\frac{1}{20})}\oint_{\widetilde{\mathcal{H}}=h}(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu}\\
&= \frac{1}{4(h+\frac{1}{20})^2}\oint_{\widetilde{\mathcal{H}}=h}(2A(\widetilde{\mu})+\widetilde{\nu}^2)(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu}
\end{aligned} \tag{41}$$

$$= \frac{1}{4(h+\frac{1}{20})^2}\oint_{\widetilde{\mathcal{H}}=h} 2A(\widetilde{\mu})(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu}$$
$$+ \frac{1}{4(h+\frac{1}{20})^2}\oint_{\widetilde{\mathcal{H}}=h} (\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^5 d\widetilde{\mu}.$$

Again by Lemma 4.1 of [14] (here $k=5$ and $F(\widetilde{\mu}) = 2A(\widetilde{\mu})(\widetilde{\mu}^j + G_j(\widetilde{\mu}))$), we obtain

$$\int_{\widetilde{\mathcal{H}}=h} 2A(\widetilde{\mu})(\widetilde{\mu}^j + G_j(\widetilde{\mu}))\widetilde{\nu}^3 d\widetilde{\mu} = \int_{\widetilde{\mathcal{H}}=h} \widetilde{G}_j(\widetilde{\mu})\widetilde{\nu}^5 d\widetilde{\mu}, \tag{42}$$

where $\widetilde{G}_j(\widetilde{\mu}) = \frac{d}{5d\widetilde{\mu}}(\frac{2A(\widetilde{\mu})(\widetilde{\mu}^j+G_j(\widetilde{\mu}))}{A'(\widetilde{\mu})}) = \frac{\widetilde{\mu}^j\widetilde{g}_j(\widetilde{\mu})}{1500(\widetilde{\mu}-1)^8}$, and $\widetilde{g}_j(\widetilde{\mu})$ is a lengthy polynomial and omitted here for brevity. Substituting (42) into (41) proves Lemma 5.1. □

Without loss of generality, we assume that $a_1 = \lambda$ and $a_3 = 1$. Further, let

$$\mathcal{J}_1(h) = \oint_{\Gamma_h}\Big(\mu + \frac{1}{\lambda}\mu^2\Big)\nu d\mu. \tag{43}$$

Then, $\mathcal{M}(h) = \alpha_0 J_0(h) + \lambda\mathcal{J}_1(h)$. By Lemma 5.1, we have

Lemma 5.2.

$$8\Big(h+\frac{1}{20}\Big)^3 J_i(h) = \oint_{\widetilde{\mathcal{H}}=h} \widetilde{f}_i(\widetilde{\mu})\widetilde{\nu}^5 d\widetilde{\mu} \triangleq \widetilde{J}_i(h),$$

and

$$8\Big(h+\frac{1}{20}\Big)^3 \mathcal{J}_1(h) = \oint_{\widetilde{\mathcal{H}}=h} (\widetilde{f}_1(\widetilde{\mu}) + \frac{1}{\lambda}\widetilde{f}_2(\widetilde{\mu}))\widetilde{\nu}^5 d\widetilde{\mu} \triangleq \widetilde{\mathcal{J}}_1(h),$$

where $\widetilde{f}_i(\widetilde{\mu}) = \sum\limits_{j=0}^{i}(-1)^j C_i^j f_j(\widetilde{\mu})$.

Now, let

$$L_i(\widetilde{\mu}) = \Big(\frac{\widetilde{f}_i}{A'}\Big)(\widetilde{\mu}) - \Big(\frac{\widetilde{f}_i}{A'}\Big)(z(\widetilde{\mu})),$$
$$\mathcal{L}_1(\widetilde{\mu}) = \Big(\frac{\widetilde{f}_1 + \frac{1}{\lambda}\widetilde{f}_2}{A'}\Big)(\widetilde{\mu}) - \Big(\frac{\widetilde{f}_2 + \frac{1}{\lambda}\widetilde{f}_3}{A'}\Big)(z(\widetilde{\mu})).$$

Then,

$$\frac{d}{d\widetilde{\mu}}L_i(\widetilde{\mu}) = \frac{d}{d\widetilde{\mu}}\Big(\frac{f_i}{A'}\Big)(\widetilde{\mu}) - \frac{d}{dz}\Big[\Big(\frac{f_i}{A'}\Big)(z(\widetilde{\mu}))\Big] \times \frac{dz}{d\widetilde{\mu}},$$
$$\frac{d}{d\widetilde{\mu}}\mathcal{L}_i(\widetilde{\mu}) = \frac{\partial}{\partial\widetilde{\mu}}(\mathcal{L}_i(\widetilde{\mu})) + \frac{\partial}{\partial z}(\mathcal{L}_i(\widetilde{\mu})) \times \frac{dz}{d\widetilde{\mu}},$$

and we obtain

$$W[L_0](\widetilde{\mu}) = \frac{3(\widetilde{\mu}-z)Q_1(\widetilde{\mu},z)}{500\widetilde{\mu}z(\widetilde{\mu}-1)^{11}(z-1)^{11}},$$
$$W[L_0(\widetilde{\mu}),\mathcal{L}_1(\widetilde{\mu})] = \begin{vmatrix} L_0(\widetilde{\mu}) & \mathcal{L}_1(\widetilde{\mu}) \\ L_0'(\widetilde{\mu}) & \mathcal{L}_1'(\widetilde{\mu}) \end{vmatrix} = \frac{-(\widetilde{\mu}-z)^3 Q_2(\widetilde{\mu},z)}{250000\widetilde{\mu}^2 z^2\lambda(z-1)^{22}(\widetilde{\mu}-1)^{22}P_0(\widetilde{\mu},z)},$$
$$W[L_0(\widetilde{\mu}),L_1(\widetilde{\mu})] = \begin{vmatrix} L_0(\widetilde{\mu}) & L_1(\widetilde{\mu}) \\ L_0'(\widetilde{\mu}) & L_1'(\widetilde{\mu}) \end{vmatrix} = \frac{-(\widetilde{\mu}-z)^3 Q_3(\widetilde{\mu},z)}{250000\widetilde{\mu}^2 z^2(z-1)^{22}(\widetilde{\mu}-1)^{22}P_0(\widetilde{\mu},z)},$$

where $Q_1(\widetilde{\mu},z)$ is a two-variate polynomial of degree 19, $Q_2(\widetilde{\mu},z) = S_{12}(\widetilde{\mu},z)\lambda - S_{11}(\widetilde{\mu},z)$, in which $S_{11}(\widetilde{\mu},z)$ and $S_{12}(\widetilde{\mu},z)$ are two-variate polynomials of degree 41 and 40, respectively, and $Q_3(\widetilde{\mu},z)$ is of degree 40.

Computing the resultant of Q_1 and q with respect to z, and applying Sturm's Theorem, we can show that Q_1 and q have no common zeros for $\widetilde{\mu} \in (0,1)$. Therefore, $W[L_0]$ does not vanish for $\widetilde{\mu} \in (0,1)$.

Similarly, computing the resultant of S_{12} and q with respect to z, and applying Stuum's Theorem, we can prove that S_{12} and q have no common zeros for $\widetilde{\mu} \in (0,1)$. Therefore, S_{12} does not vanish for $\widetilde{\mu} \in (0,1)$. Solving $S_{12}(\widetilde{\mu},z)\lambda - S_{11}(\widetilde{\mu},z) = 0$ gives

$$\lambda(\widetilde{\mu},z) = \frac{S_{11}(\widetilde{\mu},z)}{S_{12}(\widetilde{\mu},z)}, \tag{44}$$

for which we have the following result.

Lemma 5.3. *$\lambda(\widetilde{\mu},z)$ is monotonic for $\widetilde{\mu} \in (0,1)$, and $\lambda(\widetilde{\mu},z) \in (-\frac{5}{3},0)$.*

Proof. A direct computation shows that

$$\lambda'(\widetilde{\mu},z) = \frac{\partial \lambda(\widetilde{\mu},z)}{\partial \widetilde{\mu}} + \frac{\partial \lambda(\widetilde{\mu},z)}{\partial z} \times \frac{dz}{d\widetilde{\mu}} = \frac{S_{21}(\widetilde{\mu},z)}{S_{22}(\widetilde{\mu},z)}.$$

Computing the corresponding resultant $\mathrm{res}(S_{22},q,z)$ and applying Sturm's theorem, we can show that $S_{22}(\widetilde{\mu},z)$ has no zeros for $\widetilde{\mu} \in (0,1)$. Similarly, computing the resultant $\mathrm{res}(S_{21},q,z)$ and applying Sturm's Theorem, we can prove that $\mathrm{res}(S_{21},q,z)$ has a unique zero in $\left[\frac{40125}{65536},\frac{80251}{131072}\right] \subseteq (0,1)$. Further, computing the resultant $\mathrm{res}(S_{21},q,x)$ and applying Sturm's Theory show that $\mathrm{res}(S_{21},q,x)$ has three zeros for z in the three intervals:

$$\left[-\tfrac{64373}{262144},-\tfrac{128745}{524288}\right], \ \left[-\tfrac{125503}{1048576},-\tfrac{62751}{524288}\right] \quad \text{and} \quad \left[-\tfrac{56117}{524288},-\tfrac{112233}{1048576}\right].$$

Therefore, if S_{21} and q have common roots on $(-\frac{1}{4},0)\times(0,1)$, the roots must lie in one of the following three domains:

$$\begin{aligned} D_1: &\quad \left[-\tfrac{64373}{262144},-\tfrac{128745}{524288}\right] \times \left[\tfrac{40125}{65536},\tfrac{80251}{131072}\right], \\ D_2: &\quad \left[-\tfrac{125503}{1048576},-\tfrac{62751}{524288}\right] \times \left[\tfrac{40125}{65536},\tfrac{80251}{131072}\right], \\ D_3: &\quad \left[-\tfrac{56117}{524288},-\tfrac{112233}{1048576}\right] \times \left[\tfrac{40125}{65536},\tfrac{80251}{131072}\right]. \end{aligned}$$

The resultant $\mathrm{res}(\frac{\partial q}{\partial \widetilde{\mu}},\frac{\partial q}{\partial z},z)$ has no zeros in $[\frac{40125}{65536},\frac{80251}{131072}]$, implying that q reaches its extreme values on the boundary of D_i. By Sturm's Theorem, we know that the derivatives of the four functions obtained by restricting $q(x,z)$ on the four boundaries of D_i ($i=1,2,3$) have no zeros. Therefore, $q(\widetilde{\mu},z)$ gets its maximal and minimum values at the four vertexes on each D_i. A direct computation yields

$$\begin{aligned} &\max_{D_1} q(\widetilde{\mu},z) = \tfrac{48507113359773182621}{118059162071741130342\,4}, &&\min_{D_1} q(\widetilde{\mu},z) = \tfrac{77559011637435494538\,1}{188894659314785808547\,84}, \\ &\max_{D_2} q(\widetilde{\mu},z) = -\tfrac{306099632527480469877479}{302231454903657293676544}, &&\min_{D_2} q(\widetilde{\mu},z) = -\tfrac{191313935307213917945\,19}{188894659314785808547\,84}, \\ &\max_{D_3} q(\widetilde{\mu},z) = -\tfrac{205079641482712115472\,59}{188894659314785808547\,84}, &&\min_{D_3} q(\widetilde{\mu},z) = -\tfrac{328130217324045232444719}{302231454903657293676544}. \end{aligned}$$

The minimum and maximum values have the same signs on each D_i. Hence, q and S_{21} have no common zeros on each D_i. Therefore, S_{21} does not vanish for $\widetilde{\mu} \in (0,1)$. This implies that $\lambda'(\widetilde{\mu},z) \neq 0$ for $\widetilde{\mu} \in (0,1)$. Thus, $\lambda(\widetilde{\mu},z)$ is monotonic for $\widetilde{\mu} \in (0,1)$, and so

$$\lim_{\widetilde{\mu}\to 0} = \lambda(\widetilde{\mu},z) = -\tfrac{5}{3}, \quad \lim_{\widetilde{\mu}\to 1} = \lambda(\widetilde{\mu},z) = 0.$$

This completes the proof of Lemma 5.3. ∎

Lemma 5.3 implies that when $\lambda \in (-\frac{5}{3},0)$, $W[L_0,\mathcal{L}_1]$ has a simple root for $\widetilde{\mu} \in (0,1)$, and $W[L_0,\mathcal{L}_1]$ has no roots for $\widetilde{\mu} \in (0,1)$ when $\lambda \in (-\infty,-\frac{5}{3}]\bigcup(0,+\infty)$. By Lemmas 3.3 and 3.4, we have the following result.

Lemma 5.4. *$\mathcal{M}(h)$ has at most two zeros (counting multiplicities) for $\lambda \in (-\frac{5}{3}, 0)$, and at most one zero (counting multiplicity) for $\lambda \in (-\infty, -\frac{5}{3}] \bigcup (0, +\infty)$.*

Let

$$\kappa(h) = \frac{\lambda J_1(h) + J_2(h)}{J_0(h)}. \tag{45}$$

Then,

$$\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h)).$$

Lemma 5.4 implies the following proposition.

Proposition 2. *The ratio $\kappa(h)$ is monotonic for $\lambda \in (-\infty, -\frac{5}{3}] \bigcup [0, +\infty)$.*

Lemma 5.5. *If $\kappa'(h)$ has zeros, they must be simple. Moreover, $\kappa'(h)$ has $2n+1$ simple zeros on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, and $2n$ simple zeros on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{10}{11}, 0)$.*

Proof. Firstly, we give a short proof for the first assertion by using an argument of contradiction. Let h^* be a zero of $\kappa'(h)$ with l multiplicities, $l \geq 2$. Then there must exist an a_0 such that $a_0 + \kappa(h)$ has a zero at $h = h^*$ with $l+1$ (≥ 3) multiplicities. Because $J_0(h) > 0$, the relationship between $\mathcal{M}(h)$ and $a_0 + \kappa(h)$ implies that $\mathcal{M}(h)$ has a zero at $h = h^*$ with $l+1$ (≥ 3) multiplicities. This contradicts Lemma 5.4.

With the expansion of $J_i(h)$ near $h = -\frac{1}{20}$, given in (31), a direct computation shows that

$$\kappa'(-\tfrac{1}{20}) = \lim_{h \to -\frac{1}{20}} \kappa'(h) = \lim_{h \to -\frac{1}{20}} \tfrac{(\lambda J_1'(h) + J_2'(h)) J_0(h) - (\lambda J_1(h) + J_2(h)) J_0'(h)}{J_0^2(h)} = -\tfrac{3}{2}\lambda - \tfrac{5}{2}.$$

Further, using the expansions of $J_i(h)$ and $J_i'(h)$ near $h = 0$ given in (37) and (38), we can prove that

$$\begin{aligned} \kappa'(0) &= \lim_{h \to 0} \kappa'(h) = \lim_{h \to 0} \tfrac{(\lambda J_1'(h) + J_2'(h)) J_0(h) - (\lambda J_1(h) + J_2(h)) J_0'(h)}{J_0^2(h)} \\ &= \operatorname{sign}(\widetilde{A}_0(11\lambda + 10))\, \infty. \end{aligned}$$

Because $\widetilde{A}_0 < 0$, it is obvious that $\kappa'(h)$ has different signs at the two endpoints of the interval $(-\frac{1}{20}, 0)$ if $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, and has the same sign at the two endpoints of $(-\frac{1}{20}, 0)$ if $\lambda \in (-\frac{10}{11}, 0)$. This completes the proof. □

Proposition 3. *$\kappa'(h)$ has a unique simple zero on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, namely, $\kappa(h)$ decreases from $\kappa(-\frac{1}{20})$ to a minimum value and then increases to $\kappa(0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$.*

Proof. By Lemma 5.5, for a fixed $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, if $\kappa'(h)$ has three or more than three simple zeros on $(-\frac{1}{20}, 0)$, then there must exist an a_0 such that $a_0 + \kappa(h)$ has at least three zeros. This implies that $\mathcal{M}(h)$ can have at least three zeros, which contradicts Lemma 5.4. Therefore, $\kappa'(h)$ has a unique zero on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$. The signs of $\kappa'(-\frac{1}{20})$ and $\kappa'(0)$ when $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$ determines the property of $\kappa(h)$. □

Proposition 4. *$\kappa'(h)$ has no zeros on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{10}{11}, 0)$, that is, $\kappa(h)$ is monotonic on $(-\frac{1}{20}, 0)$ for any $\lambda \in (-\frac{10}{11}, 0)$.*

Proof. By Lemma 5.5, for a fixed $\lambda \in (-\frac{10}{11}, 0)$, if $\kappa'(h)$ has four or more than four zeros on $(-\frac{1}{20}, 0)$, then there must exist an a_0 such that $a_0 + \kappa(h)$ has at least three zeros counting multiplicities. This contradicts Lemma 5.4.

Next, we prove that $\kappa'(h)$ does not have two zeros. Suppose otherwise $\kappa'(h)$ has two zeros for $\lambda \in (-\frac{10}{11}, 0)$. We have known that $\kappa'(-\frac{1}{20}) < 0$, and $\kappa'(0) < 0$, implies that $\kappa(h)$ is decreasing at the endpoints of the interval $(-\frac{1}{20}, 0)$. Further, for $\lambda \in (-\frac{10}{11}, 0)$, $\lambda + 1 = \kappa(-\frac{1}{20}) > \kappa(0) = \frac{5}{6}\lambda + \frac{25}{33}$. This clearly indicates that there must exist an a_0 such that $a_0 + \kappa(h)$ has at least three zeros. This contradicts Lemma 5.4, and so the proof is complete. □

5.3. **Coexistence of one solitary and one periodic wave.** In this section, we will investigate the condition for the existence of one solitary wave, for which we need study the condition satisfying $\mathcal{H}(\beta^*(h, \epsilon), 0) - \mathcal{H}(\alpha^*(h), 0) = \epsilon \mathcal{M}(h) + O(\epsilon^2) = 0$ at $h = 0$ (and so $\alpha^*(0) = 0$). Firstly, solving $\mathcal{M}(0) = \oint_{\Gamma_0} (a_0 + \lambda\mu + \mu^2)\nu d\mu = 0$ gives

$$\lambda = -\frac{10}{11} - \frac{6}{5}a_0, \tag{46}$$

under which

$$\kappa(-\frac{1}{20}) = \frac{1}{11} - \frac{6}{5}a_0, \quad \kappa(0) = -a_0. \tag{47}$$

We need discuss two cases for λ. If $\lambda \in (-\infty, -\frac{5}{3}] \bigcup [-\frac{10}{11}, +\infty)$, then (46) yields $a_0 \in (-\infty, 0] \bigcup [\frac{5}{9}, +\infty)$, and $\kappa(h)$ is monotonic by Propositions 2 and 4. Therefore, $a_0 + \kappa(h)$ increases from 0 to $\frac{1}{11} - \frac{6}{5}a_0$ for $a_0 \in (-\infty, 0)$, and decreases from 0 to $\frac{1}{11} - \frac{6}{5}a_0$ for $a_0 \in [\frac{5}{9}, +\infty)$. Hence, $\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h))$ has no zeros for $\lambda \in (-\infty, -\frac{5}{3}] \bigcup [-\frac{10}{11}, +\infty)$ under the condition (46).

If $\lambda \in (-\frac{5}{3}, -\frac{10}{11})$, then it follows from (46) that $a_0 \in (0, \frac{5}{9})$. Further, we divide the interval $(0, \frac{5}{9})$ into three parts by using $\kappa(-\frac{1}{20})$ and $\kappa(0)$.

(i) When $a_0 = \frac{5}{11}$, $\kappa(-\frac{1}{20}) = \kappa(0) = -a_0$. The property of $\kappa(h)$ given in Proposition 3 implies that $a_0 + \kappa(h) < a_0 + \kappa(-\frac{1}{20}) = a_0 + \kappa(0) = 0$ for $h \in (-\frac{1}{20}, 0)$. Thus, $\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h))$ has no zeros.

(ii) When $a_0 \in (\frac{5}{11}, \frac{5}{9})$, $\kappa(-\frac{1}{20}) \le \kappa(0) = -a_0$. The property of $\kappa(h)$ given in Proposition 3 shows that $a_0 + \kappa(h) < a_0 + \kappa(0) = 0$ for $h \in (-\frac{1}{20}, 0)$. Thus, $\mathcal{M}(h) = J_0(h)(a_0 + \kappa(h))$ has no zeros.

(iii) When $a_0 \in (0, \frac{5}{11})$, $\kappa(-\frac{1}{20}) > \kappa(0) = -a_0$ and $a_0 + \kappa(-\frac{1}{20}) > a_0 + \kappa(0) = 0$. The property of $\kappa(h)$ given in Proposition 3 shows that in the h-κ plane, the graph of $a_0 + \kappa(h)$ decreases form the point $(-\frac{1}{20}, a_0 + \kappa(-\frac{1}{20}))$, passing through the h-axis at some $h = h^* \in (-\frac{1}{20}, 0)$, then to a minimum point and then increases to $(0, 0)$. This implies that $a_0 + \kappa(h)$ has a unique zero $h = h^*$ in $(-\frac{1}{20}, 0)$. Summarizing the above results gives the following proposition.

Proposition 5. *$\mathcal{M}(h)$ has a zero at $h = 0$ and anther zero at $h = h^* \in (-\frac{1}{20}, 0)$ if and only if $a_0 \in (0, \frac{5}{11})$ with $\lambda = -\frac{10}{11} - \frac{6}{5}a_0$.*

Finally, we prove Theorem 2.2.

Proof. [For Theorem 2.2] With the above results, we have proved the first parts of Theorem 2.2 (i), (ii) and (iii), as Propositions 2 and 4 for (i), Proposition 3 for (ii) and Proposition 5 for (iii). The second parts of (i) (ii) and (iii) of Theorem 2.2 can be directly proved by applying the implicit function theorem. □

6. **Conclusion.** In this paper, we have used bifurcation theory to study the existence of periodic and solitary waves in a BBM equation under weak dissipative influences and Marangoni effect. A special transformation given in (32) is introduced so that the Chebyshev criteria can be applied to overcome the difficulty arising from higher-order degenerate singularities, and then the exact condition on the number of periodic waves is obtained for the case with regular multiple-parameter perturbations. Also, the condition on the co-existence of one solitary and one periodic waves is derived. The methodologies developed in this paper include the reduction of three generating elements to special two ones, asymptotic expansion of Abelian integrals, and asymptotic analysis on the dominating part of the Abelian integrals. Combination of these methods is not only useful in the study of other types of wave equations, but also has potential to be generalized to consider perturbations on hyper-elliptic Hamiltonian systems.

Acknowledgments. This work was supported by the Natural Science and Engineering Research Council of Canada (No. R2686A02), the National Natural Science Foundation of China (No. 11461001), the Natural Science Foundation of Guangxi Province (No. 2015jjBA10052) and the Youth Foundation Ability Promotion Plan in Guangxi in 2016 (No. KY2016YB404). The authors would like to thank the anonymous reviewer for providing useful comments and suggestions which help to strengthen the manuscript.

REFERENCES

[1] B. Amitabha, A geometric approach to singularly perturbed nonlocal reaction-diffusion equations, *SIAM J. Math. Anal.,* **31** (2000), 431–454.

[2] V. I. Arnold, Small denominators and problems of stability of motion in classical and celestial mechanics, *Uspehi Mat. Nauk.,* **18** (1963), 91–192.

[3] E. Benilov, R. Grimshaw and E. Kuznetsov, The generation of radiating waves in a singularly-perturbed Korteweg-de Vries equation, *Physica D,* **69** (1993), 270–278.

[4] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive system, *Phil. Trans. R. Soc. Lond. A,* **272** (1972), 47–78.

[5] F. H. Busse, Non-linear properties of thermal convection, *Reports on Progress in Physics,* **41** (1978), 1929–1967.

[6] R. Camassa and D. Holm, An integrable shallow wave equation with peaked solitons, *Phys. Rev. Lett.,* **71** (1993), 1661–1664.

[7] A. Chen, L. Guo and X. Deng, Existence of solitary waves and periodic waves for a perturbed generalized BBM equation, *J. Differential Equations,* **261** (2016), 5324–5349.

[8] C. I. Christov and M. G. Velarde, Dissipative solitons, *Physica D,* **86** (1995), 323–347.

[9] G. Derks and S. Gils, On the uniqueness of traveling waves in perturbed Korteweg-de Vries equations, *Japan J. Indust. Appl. Math.,* **10** (1993), 413–430.

[10] C. Elphick, G. R. Ierley, O. Regev and E. A. Spiegel, Interacting localized structures with Galilean invariance, *Phys. Rev. A*, **44** (1991), 1110–1122.

[11] N. Fenichel, Geometric singular perturbation theory for ordinary differential equations, *J. Differential Equations,* **31** (1979), 53–98.

[12] P. L. Garcia-Ybarra, J. L. Castillo and M. G. Velarde, Bénard-Marangoni convection with a deformable interface and poorly conducting boundaries, *Phys. Fluids,* **30** (1987), 2655–2661.

[13] A. Geyera and J. Villadelpratb. On the wave length of smooth periodic traveling waves of the Camassa-Holm equation, *J. Differential Equations,* **259** (2015), 2317–2332.

[14] M. Grau, F. Mañosas and J. Villadelprat, A Chebyshev criterion for Abelian integrals, *Trans. Amer. Math. Soc.,* **363** (2011), 109–129.

[15] A. E. Green and P. M. Naghdi, A derivation of equations for wave propagation in water of variable depth, *J. Fluid. Mech.,* **78** (1976), 237–246.

[16] W. Hai and Y. Xiao, Soliton solution of a singularly perturbed KdV equation, *Phys. Lett. A,* **208** (1995), 79–83.

[17] S. Hakkaev, I. D. Iliev and K. Kirchev, Stability of periodic travelling shallow-water waves determined by Newton's equation, *J. Phys. A,* **41** (2008), 085203, 31pp.

[18] M. Han, Asymptotic expansions of Abelian integrals and limit cycle bifurcations, *Int. J. Bifur. Chaos,* **22** (2012), 1250296 (30 pages).

[19] M. Han, J. Yang and D. Xiao, Limit cycle bifurcations near a double homoclinic loop with a nilpotent saddle, Int. J. Bifur. Chaos **22** (2012), 1250189 (33 pages).

[20] J. M. Hyman and B. Nicolaenko, The Kuramoto-Sivashinsky equation: A bridge between PDE's and dynamical systems, *Physica D,* **18** (1986), 113–126.

[21] J. M. Hyman and B. Nicolaenko, Coherence and chaos in Kuramoto-Velarde equation, *Directions in Partial Differential Equations (Madison, WI, 1985)*, 89–111, Publ. Math. Res. Center Univ. Wisconsin, 54, Academic Press, Boston, MA, 1987.

[22] E. R. Johnsona and D. E. Pelinovsky, Orbital stability of periodic waves in the class of reduced Ostrovsky equations, *J. Differential Equations,* **261** (2016), 3268–3304.

[23] V. I. Karpman, *Non-Linear Waves in Dispersive Media*, Pergamon Press, Oxford-New York-Toronto, Ont., 1975.

[24] T. Kawahara and S. Toh, Pulse interactions in an unstable dissipative-dispersion nonlinear system, *Phys. Fluids,* **31** (1988), 2103–2111.

[25] A. N. Kolmogorov. On the conservation of conditionally periodic motions under small perturbation of the Hamiltonian, *Dokl. Akad. Nauk SSSR,* **98** (1954), 527–530.

[26] D. J. Korteweg and F. de Vries, On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves, *Philosophical Magazine,* **39** (1895), 422–443.

[27] N. A. Kudryashev, Exact soliton solutions for a generalized equation of evolution for the wave dynamics, *Prikl. Math. Mekh.,* **52** (1988), 465–470.

[28] S. Y. Lou, G. X. Huang and X.-Y. Ruan, Exact solitary waves in a convecting fluid, *J. Phys. A: Math. Gen.,* **24** (1991), 587–590.

[29] F. Mañosas and J. Villadelprat, Bounding the number of zeros of certain Abelian integrals, *J. Differential Equations,* **251** (2011), 1656–1669.

[30] M. B. A. Mansour, Travelling wave solutions for a singularly perturbed Burgers-KdV equation, *Pramana J. Phys.,* **73** (2009), 799–806.

[31] M. B. A. Mansour, Traveling waves for a dissipative modified KdV equation, *J. Egypt. Math. Soc.,* **20** (2012), 134–138.

[32] M. B. A. Mansour, A geometric construction of traveling waves in a generalized nonlinear dispersive-dissipative equation, *J. Geom. Phy.,* **69** (2013), 116–122.

[33] S. Micu, On the controllability of the linearized Benjamin-Bona-Mahony equation, *SIAM J. Control. Optim.,* **39** (2011), 1677–1696.

[34] J. Moser, On invariant curves of area-preserving mappings of an annulus, *Nach. Akad. Wiss. Göttingen, Math. Phys. Kl.,* **II** (1962), 1–20.

[35] C. Normad, Y. Pomeau and M. G. Velarde, Convective instability: A physicist's approach, *Rev. Mod. Phys.,* **49** (1977), 581–624.

[36] T. Ogawa, Travelling wave solutions to a perturbed Korteweg-de Vries equation, *Hiroshima J. Math.,* **24** (1994), 401–422.

[37] T. Ogawa, Periodic travelling waves and their modulation, *Japan J. Indust. Appl. Math.,* **18** (2001), 521–542.

[38] T. Ogawa and S. Hiromasa. On the spectra of pulses in a nearly integrable system, *SIAM J. Appl. Math.,* **57** (1997), 485–500.

[39] K. Omrani, The convergence of fully discrete Galerkin approximations for the Benjamin-Bona-Mahony(BBM)equation, *Appl. Math. Comput.,* **180** (2006), 614–621.

[40] Y. Pomeau, A. Ramani and B. Grammaticos, Structural stability of the Korteweg-de Vries solitons under a singular perturbation, *Physica D,* **31** (1988), 127–134.

[41] J. Topper and T. Kawahara, Approximate equations for long nonlinear waves on a viscous fluid, *J. Phys. Soc. Japan,* **44** (1978), 663–666.

[42] J. Tyson and J. Keener, Singular perturbation theory of traveling waves in excitable media, *Physica D,* **32** (1988), 327–361.

[43] M. G. Velarde, *Physicochemical Hydrodynamics: Interfacial Phenomena*, Plenum, New York, 1987.

[44] M. G. Velarde and C. Normand, Natural convection, *Sci. Am.,* **243** (1980), 92–106.

[45] J. Von Zur Gathen and J. Gerhard, *Modern Computer Algebra*, Cambridge University Press, 2013.

[46] A. M. Wazwaz, Exact solution with compact and non-compact structures for the one-dimensional generalized Benjamin-Bona-Mahony equation, *Commun. Nonlinear. Sci. Numer. Simulat.,* **10** (2005), 855–867.
[47] W. Yan, Z. Liu and Y. Liang. Existence of solitary waves and periodic waves to a perturbed generalized KdV equation, *Math. Modell. Anal.,* **19** (2014), 537–555.
[48] J. Yang, A normal form for Hamiltonian-Hopf bifurcations in nonlinear Schrodinger equations with general external potentials, *SIAM J. Appl. Math.,* **76** (2016), 598–617.
[49] W. Yong, Singular perturbations of first-order hyperbolic systems with stiff source terms, *J. Differential Equations,* **155** (1988), 89–132.
[50] H. Zang, M. Han and D. Xiao, On Abelian integrals of a homoclinic loop through a nilpotent saddle for planar near-Hamiltonian systems, *J. Differential Equations,* **245** (2008), 1086–1111.
[51] X. Zhao, W. Xu, S. Li and J. Shen, Bifurcations of traveling wave solutions for a class of the generalized Benjamin-Bona-Mahony equation, *Appl. Math. Comput.,* **175** (2006), 1760–1774.
[52] Y. Zhou, Q. Liu and W. Zhang, Bounded traveling waves of the Burgers-Huxley equation, *Nonlinear Analysis (TMA),* **74** (2011), 1047–1060.
[53] Y. Zhou, Q. Liu and W. Zhang, Bounded traveling wave of the generalized Burgers-Fisher equation, *Int. J. Bifur. Chaos,* **23** (2013), 1350054 (11 pages).

Received May 2017; revised September 2017.

E-mail address: xsun244@uwo.ca (X. Sun)
E-mail address: pyu@uwo.ca (P. Yu)