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ZERO BIFURCATION DIAGRAMS FOR ABELIAN INTEGRALS: A STUDY ON HIGHER-ORDER HYPERELLIPTIC HAMILTONIAN SYSTEMS WITH THREE PERTURBATION PARAMETERS

Xianbo Sun[1,2] and Pei Yu[2,†]

Abstract In this paper, we present the zero bifurcation diagrams for the Abelian integrals of two hyperelliptic Hamiltonian systems with three perturbation parameters using an algebraic-geometric approach. The method can be used to study the bifurcation diagrams for higher-order Hamiltonian systems with polynomial perturbations of any degree.

Keywords Hyperelliptic Hamiltonian system, bifurcation diagram, Abelian integral.

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1. Introduction

The polynomial vector field

$$y\frac{\partial}{\partial x}+\big[-g(x)+\varepsilon f(x)y\big]\frac{\partial}{\partial y}=0$$

with $0<\varepsilon\ll 1$ is a result of perturbation on the Hamiltonian system,

$$\dot{x}=y,\quad \dot{y}=-g(x)+\varepsilon f(x)y, \tag{1.1}$$

which has the Hamiltonian $H(x,y)=\frac{y^2}{2}+\int g(y)dx$. When analyzing limit cycles arising from Bogdanov-Takens bifurcation, the normal form

$$\dot{x}=y,\quad \dot{y}=-1+x^2+\varepsilon(a_0+a_1x)y$$

is usually used for codimension two [18], and

$$\dot{x}=y,\quad \dot{y}=-1+x^2+\varepsilon(a_0+a_1x+a_3x^3)y$$

for codimension three of cusp case, see [7]. For codimension four of cusp case, the reduced perturbed Hamiltonian is given by (see [19]),

$$\dot{x}=y,\quad \dot{y}=-1+x^2+\varepsilon(a_0+a_1x+a_2x^3+a_3x^4)y.$$

†The corresponding author. Email address:pyu@uwo.ca(P. Yu)
1Department of Applied Mathematics, Guangxi University of Finance and Economics, West Mingxiu Road, 530003 Nanning, China
2Department of Applied Mathematics, Western University, London, Ontario, N6A 5B7, Canada

The bifurcation diagrams have been obtained for codimensions two, three and four by a sophisticated analysis. In [28], Xiao studied a codimension five bifurcation of cusp case, for which two different truncated systems were obtained by applying the principle re-scaling and central re-scaling. The first truncated system is a Hamiltonian system, and its bifurcation diagram was obtained. The second truncated system is the following 5th-degree perturbed Hamiltonian,

$$\dot{x} = y, \quad \dot{y} = x(x-1)(x-\alpha)(x-\beta) + \varepsilon(a_0 + a_1 x + a_2 x^2)y. \tag{1.2}$$

The unperturbed system (1.2) can be classified into 11 cases according to the values of α and β when it has at least one period annulus [28]. The Hopf bifurcation surface was obtained for all values of α and β. However, the limit cycle bifurcation surfaces on the whole period annulus are still unknown. The main difficulty of the analysis is due to the higher order of the Hamiltonian. The outer boundaries of the period annulus of the system $(1.2)_{\varepsilon=0}$ have two degenerate cases, one is the nilpotent-saddle loop when $\alpha = \beta = 1$, and the other is the cusp-saddle cycle when $\alpha = 0$ and $\beta = \frac{3}{5}$, as shown in Figure 1(a) and (b), respectively.

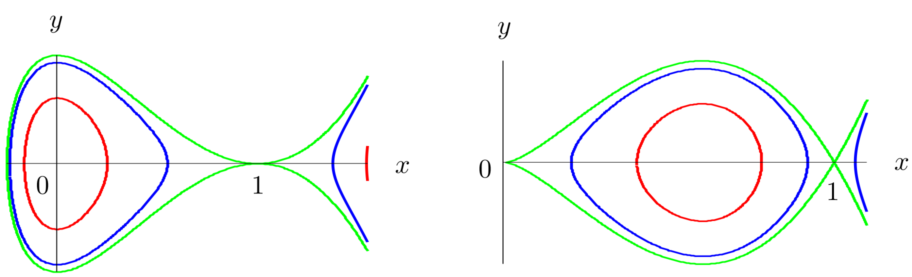


Figure 1. Phase portraits of system (1.2) showing (a) a nilpotent-saddle loop for $\alpha = \beta = 1$; and (b) a cusp-saddle cycle for $\alpha = 0$, $\beta = \frac{3}{5}$.

Perturbation on Hamiltonian systems is also a main topic in the study of the weak Hilbert's 16th problem, proposed by Arnold [1] for studying the maximal number of zeros of the Abelian integral, which is the first order approximation of the return map,

$$A(h, \delta) = \oint_{\Gamma_h} q(x, y)dx - p(x, y)dy, \quad h \in J,$$

for the perturbed Hamiltonian system,

$$\dot{x} = H_y(x, y) + \varepsilon p(x, y), \quad \dot{y} = -H_x(x, y) + \varepsilon q(x, y),$$

where $p(x, y)$ and $q(x, y)$ are polynomials of degree $n \geq 2$, $H(x, y)$ is of degree $n + 1$, Γ_h represents closed algebra curves of $H(x, y)$ which are parameterized by $\{(x, y)|H(x, y) = h,\ h \in J\}$, J is an open interval, δ is a set of coefficients of $p(x, y)$ and $q(x, y)$, and ε represents small perturbations satisfying $|\varepsilon| \ll 1$. Some relatively new estimated bounds for general n can be found in [23, 24] and references therein. More relative references on study of the number of zeros of Abelian integrals for perturbed integral systems can be found in [17]. A so called "much weaker" case is defined by $H = \frac{y^2}{2} + \int g(x)dx$, $p = 0$ and $q = f(x)y$, for which the perturbed

Hamiltonian is reduced to system (1.1), and the corresponding Abelian integral has a simple form,

$$A(h,\delta) = \oint_{\Gamma_h} f(x)y dx.$$

System (1.1) is called type (m,n) if $g(x)$ and $f(x)$ are polynomials of degree m and n, respectively. A lot of works on the study of $A(h,\delta)$ with various polynomials $f(x)$ and $g(x)$ were reported, for example, see [6] for type $(3,2)$, and [2–4,27,32] for type $(5,4)$ with symmetry. For type $(3,2)$, the authors not only deduced the sharp bounds for the number of zeros but also obtained the bifurcation diagrams. The main tools used in these studies are Picárd-Focus equations and Ricatti equations. Because of symmetry in type $(5,4)$, the perturbation terms are reduced to three, and the dimension of Picárd-Focus equations is thus the same as that of type $(3,2)$. The sharp bounds were obtained for most cases, however the bifurcation diagrams have not been obtained and an advanced analysis is needed. We also noticed that the existence and uniqueness of limit cycles in system (1.1) without restriction on the first order bifurcation have been reported in [5,10] and references therein, and the integrability of system (1.1) was summarized in [21]. The lower bounds on the number of system (1.2) with more higher order perturbation terms have been studied in [25,30,31].

When the Hamiltonian is of degree more than four, i.e., the degree of $g(x)$ is more than three without symmetry, it is more difficult to obtain the bifurcation diagrams with three perturbation parameters, since the dimensions of the Picárd-Focus equation and Ricatti equation become higher and some new perturbation terms will appear when analyzing the centroid curve (see the method in [6]). It is almost impossible to analyze the intersection of the lines (planes) and curves (surfaces) by a classical method. In this paper, we will propose a method to analyze the zero bifurcation diagrams of the Abelian integrals for the three parameter perturbation of higher order Hamiltonian systems. We use the method to obtain bifurcation diagrams for two perturbed systems which exhibit truncated codimension-5 bifurcations, and present the results for a higher order perturbation of a Hamiltonian. We will not only show this integral is Chebyshev, but also give the exact zero bifurcation diagram.

The rest of the paper is organized as follows. In section 2, we give the asymptotic expansions of the Abelian integrals $\mathcal{A}(h)$ and $\mathcal{M}(h)$ for two degenerate cases, which are obtained by using the methods developed in [13,15,16,26]. Then we present the Chebyshev criterion [9,22], which is the generalization of the method developed in [20] to determine the monotonicity of two Abelian integrals. It was efficiently used to bound the number of zeros of some Abelian integrals with more than two generation elements. In our work, we will apply it to "restrict" the perturbation parameters with the help of "combination" skill, which is essential in the bifurcation diagram analysis. In order to achieve this, we need a primary analysis on the unperturbed systems and the Abelian integrals $\mathcal{A}(h)$ and $\mathcal{M}(h)$. Especially, for the saddle-cusp cycle case, instead of directly studying the original system, we transform the system to an equivalent system with the same Abelian integral of the original one. The main aim of this step is to demonstrate that our method can deal with general polynomial perturbations involving three parameters. In section 3, we study the bifurcation diagram for $\mathcal{A}(h)$ using the asymptotic expansions and the combination skill. In section 4, we consider the bifurcation diagram for $\mathcal{M}(h)$. Finally, in section 5 we discuss the Hamiltonian systems with higher-degree perturbations and give a complete zero bifurcation diagram for this problem.

2. Asymptotic Expansions and Chebyshev Criterion

Consider system (1.2) with $\alpha = \beta = 1$. The Hamiltonian is

$$\mathcal{H}(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}, \tag{2.1}$$

satisfying $\mathcal{H} = h$ for $h \in (0, \frac{1}{20})$ and $x \in (-\frac{1}{4}, 1)$, as depicted in Figure 1(a), which shows a family of closed orbits Γ_h surrounded by a homoclinic loop $\Gamma_{\frac{1}{20}}$, with a nilpotent saddle of order one at $(1,0)$. Correspondingly, we have the Abelian integral,

$$\mathcal{A}(h) = a_0 I_0(h) + a_1 I_1(h) + a_2 I_2(h), \tag{2.2}$$

where

$$I_n(h) = \oint_{\Gamma_h} x^n y dx, \quad n = 0,\ 1,\ 2. \tag{2.3}$$

Next, consider system (1.2) with $\alpha = 0$ and $\beta = \frac{3}{5}$, yielding the Hamiltonian,

$$\overline{\mathcal{H}}(x,y) = \frac{y^2}{2} - \frac{x^3}{5} + \frac{2x^4}{5} - \frac{x^5}{5}, \tag{2.4}$$

satisfying $\overline{\mathcal{H}}(x,y) = h$ for $h \in (-\frac{108}{15625}, 0)$ and $x \in (0,1)$, as given in Figure 1(b), showing a family of closed orbits $\overline{\Gamma}_h$ surrounded by a saddle-cusp cycle $\overline{\Gamma}_0$, with a nilpotent cusp of order 1 at $(0,0)$ and a hyperbolic saddle at $(1,0)$. Correspondingly, we have the Abelian integral,

$$\overline{\mathcal{A}}(h) = a_0 \bar{I}_0(h) + a_1 \bar{I}_1(h) + a_2 \bar{I}_2(h), \tag{2.5}$$

where $\bar{I}_n(h) = \oint_{\overline{\Gamma}_h} x^n y dx,\ n = 0,\ 1,\ 2.$

Introducing the transformation $u = -\frac{5}{3}x + 1$, $v = -\frac{25\sqrt{15}y}{27}$ and the time scaling $d\tau = \frac{3\sqrt{15}}{25}dt$ into (1.2) yields the perturbed Hamiltonian system, (x and y are still used instead of u and v for convenience),

$$\frac{dx}{d\tau} = y, \quad \frac{dy}{d\tau} = -x\Big(x + \frac{2}{3}\Big)(x-1)^2 + \varepsilon\big[a_0 q_0(x) + a_1 q_1(x) + a_2 q_2(x)\big]y, \tag{2.6}$$

where

$$q_0(x) = \frac{5\sqrt{15}}{9}, \quad q_1(x) = \frac{\sqrt{15}}{3}(1-x) \text{ and } q_2(x) = \frac{\sqrt{15}}{5}(x^2 - 2x + 1).$$

The above perturbed Hamiltonian system is a cubic polynomial perturbation to the Hamiltonian,

$$\mathcal{H}^*(x,y) = -\frac{4}{45} + \frac{y^2}{2} + \frac{x^2}{3} - \frac{x^3}{9} - \frac{x^4}{3} + \frac{x^5}{5},$$

satisfying $\mathcal{H}^*(x,y) = h$ for $h \in (-\frac{4}{45}, 0)$ and $x \in (-\frac{2}{3}, 1)$, as depicted in Figure 2, showing a family of closed orbits Γ_h^* surrounded by a saddle-cusp cycle Γ_0^*, with a nilpotent cusp of order one at $(1,0)$ and a hyperbolic saddle at $(-\frac{2}{3}, 0)$.

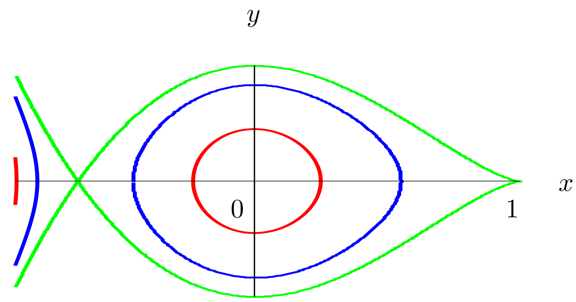


Figure 2. Phase portrait of system (1.2) showing a family of closed orbits.

The Abelian integral of the perturbed Hamiltonian system (2.6) is given by

$$\mathcal{M}(h) = a_0 J_0(h) + a_1 J_1(h) + a_2 J_2(h), \tag{2.7}$$

with $J_i(h) = \oint_{\Gamma_h^*} q_i(x) y dx$. By [14], we have

$$\mathcal{M}(h) = \frac{243}{3125}\,\overline{\mathcal{A}}\Big(\frac{3125}{243}h\Big),$$

which will be analyzed in the following sections.

The asymptotic expansions of Abelian integrals are proposed to study its zeros near the endpoints of the annuluses, and these zeros correspond to limit cycles near the centers, homoclinic loops and heteroclinic loops, see [11, 29], and the book [14]. In this work, we will apply the expansions to study the dynamics of the Abelian integrals on the whole period annulus.

2.1. Asymptotic Expansion of $\mathcal{A}(h)$

$\mathcal{A}(h)$ has the following expansion (see [12, 13]):

$$\begin{aligned}\mathcal{A}(h) =& c_0(\delta) + c_1(\delta)\Big|h - \frac{1}{20}\Big|^{\frac{3}{4}} + \big[c_2(\delta) + m_0 c_1(\delta)\big]\Big(h - \frac{1}{20}\Big)\ln\Big|h - \frac{1}{20}\Big| \\ &+ \big[c_3(\delta) + m_1 c_1(\delta) + m_2 c_2(\delta)\big](h - \tfrac{1}{20}) + O\Big(\Big(h - \frac{1}{20}\Big)^{\frac{5}{4}}\Big)\end{aligned} \tag{2.8}$$

for $0 < \frac{1}{20} - h \ll 1$, m_i $(i = 0, 1, 2)$ are some constants, and

$$\mathcal{A}(h) = \sum_{i \geq 0} b_i(\delta) h^{i+1}, \tag{2.9}$$

for $0 < h \ll 1$. The explicit expressions of the coefficients $c_i(\delta)$ are obtained by using the formulas in [12] as follows:

$$\begin{aligned}c_0(\delta) &= \frac{25\sqrt{2}}{72072}(858a_0 + 143a_1 + 78a_2),\\ c_1(\delta) &= 4\widetilde{A}_0(a_0 + a_1 + a_2),\\ c_2(\delta) &= \frac{\sqrt{2}}{2}(a_1 + 2a_2),\end{aligned}$$

with the constant $\widetilde{A}_0 < 0$, and the coefficients $b_i(\delta)$ can be obtained by applying the program in [13] as

$$b_0 = 2\,\pi\, a_0, \quad b_1 = \frac{\pi}{4}\,(21\, a_0 + 12\, a_1 + 4\, a_2)\,,$$
$$b_2 = \frac{\pi}{32}\,(1379\, a_0 + 872\, a_1 + 440\, a_2)\,.$$

Thus, using the expansions (2.8) and (2.9), we can easily obtain the expansions for $I_i(h)$ and its derivative $I_i'(h)$, which will be used in the following sections.

2.2. Asymptotic Expansion of $\mathcal{M}(h)$

By [13,15,16,26], we obtain the expansions of $\mathcal{M}(h)$ as follows:

$$\begin{aligned}\mathcal{M}(h) = {} & e_0(\delta) + B_{00} e_1(\delta)|h|^{\frac{5}{6}} + e_2(\delta) h \ln|h| \\ & + \big[e_3(\delta) + m_1^* e_1(\delta) + m_2^* e_2(\delta)\big] h + O\big(|h|^{\frac{7}{6}}\big),\end{aligned} \tag{2.10}$$

for $0 < -h \ll 1$, where m_1^*, m_2^* and B_{00} are constants with $B_{00} > 0$, and

$$\mathcal{M}(h) = \sum_{j\geq 0} d_j(\delta) \Big(h + \frac{4}{45}\Big)^{j+1}, \tag{2.11}$$

for $0 < h + \frac{4}{45} \ll 1$. The coefficients $e_i(\delta)$ can be computed by applying the methods developed in [15,16,26], given by

$$\begin{aligned} e_0 &= \frac{1000\sqrt{10}}{168399}\,(99\, a_0 + 55\, a_1 + 35\, a_2)\,, \\ e_1 &= -2\,\sqrt{2}\,\sqrt[3]{5}\ a_0, \\ e_2 &= -\frac{\sqrt{10}}{2}\,(a_0 + a_1 + a_2)\,, \end{aligned}$$

and $d_j(\delta)$ can be obtained by employing the program in [13],

$$\begin{aligned} d_0 &= \frac{\pi\,\sqrt{10}}{15}\,(25\, a_0 + 15\, a_1 + 9\, a_2)\,, \\ d_1 &= \frac{\pi\,\sqrt{10}}{480}\,(1025\, a_0 + 435\, a_1 + 369\, a_2)\,, \\ d_2 &= \frac{\pi\,\sqrt{10}}{9216}\,(85085\, a_0 + 33843\, a_1 + 28557\, a_2)\,. \end{aligned}$$

Similarly, we can easily obtain the expansions of $J_i(h)$ and its derivative $J_i'(h)$ by using the expansions (2.10) and (2.11).

2.3. Chebyshev criterion

For convenience of analysis in the next section, we introduce some preliminary results related to Chebyshev criterion.

Definition 2.1. Suppose $f_0(x), f_1(x), \ldots, f_{n-1}(x)$ are analytic functions on an real open interval J.

(i) The family of sets $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ is called a Chebyshev system (T-system for short) provided that any nontrivial linear combination,

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{n-1} f_{n-1}(x),$$

has at most $n-1$ isolated zeros on J.

(ii) An ordered set of n functions $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ is called complete Chebyshev system (CT-system for short) provided that any nontrivial linear combination,

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{i-1} f_{i-1}(x),$$

has at most $i-1$ zeros for all $i = 1, 2, \cdots, n$. Moreover it is called extended complete Chebyshev system (ECT-system for short) if the multiplicities of zeros are taken into account.

(iii) The continuous Wronskian of $\{f_0(x), f_1(x), \ldots, f_{n-1}(x)\}$ at $x \in R$ is defined by

$$W[f_0(x), f_1(x), \ldots, f_{k-1}(x)] = \begin{vmatrix} f_0(x) & f_1(x) & \cdots & f_{k-1} \\ f_0'(x) & f_1'(x) & \cdots & f_{k-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_0^{(k-1)}(x) & f_1^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix},$$

where $f_i'(x)$ is the first order derivative of $f_i(x)$, and $f_i^{(j)}(x)$ is the jth order derivative of $f_i(x)$, $j \geq 2$. The definitions imply that the function tuple $\{f_0(x), f_1(x), \cdots, f_{k-1}(x)\}$ is an ECT-system on J, so it is a CT-system on J, and thus a T-system on J. However the inverse is not necessary true.

Let $\text{res}(f_1, f_2)$ denote the resultant of $f_1(x)$ and $f_2(x)$, where $f_1(x)$ and $f_2(x)$ are two univariate polynomials of x on rational number field Q. As it is known, $\text{res}(f_1(x), f_2(x)) = 0$ if and only if $f_1(x)$ and $f_2(x)$ have at least one common root.

Let $\text{res}(f, g, x)$ and $\text{res}(f, g, z)$ denote respectively the resultants between $f(x, z)$ and $g(x, z)$ with respect to x and z, where $f(x, z)$ and $g(x, z)$ are two polynomials in $\{x, z\}$ on rational number field. $\text{res}(f, g, x)$ is a polynomial in z and $\text{res}(f, g, z)$ is a polynomial in x. Regarding the relation between the common roots of two polynomials and their resultants, the following result can be found in many works on polynomial algebra, such as [8]. For completeness we give a short proof.

Lemma 2.1 ([8])**.** (i) *Let (x_0, z_0) be a common root of $f(x, z)$ and $g(x, z)$. Then,* $\text{res}(f, g, x_0) = 0$ *and* $\text{res}(f, g, z_0) = 0$. *However, the inverse is not necessary true.*

(ii) *Let* $\text{res}(f, g, z)$ *have a unique real root on some open interval (α, β), and* $\text{res}(f, g, x)$ *have a unique real root on some open interval (γ, θ). Then, there exists at most one common real root of $f(x, z)$ and $g(x, z)$ on $(\alpha, \beta) \times (\gamma, \theta)$.*

Proof. (ii) is obvious if (i) is true. So we only prove (i). A two-variable polynomial can be treated as one univariate polynomial of one variable with the other treated as a parameter. Taking $f(x, z) = f_z(x)$ and $g(x, z) = g_z(x)$ which are polynomials of x with parameter z. Let x_0 be the common root of $f_{z_0}(x)$ and $g_{z_0}(x)$, where z_0 is the common root of $f_{x_0}(z)$ and $g_{x_0}(z)$. Then, $\text{res}(f_{z_0}(x), g_{z_0}(x)) = \text{res}(f, g, z_0) = 0$, and therefore, $\text{res}(f_{x_0}(z), g_{x_0}(z)) = \text{res}(f, g, x_0) = 0$. □

Let $H(x,y)=\mathbb{U}(x)+\frac{y^2}{2}$ be an analytic function. Assume there exists a punctured neighborhood $\mathcal{P}$ of the origin foliated by ovals $\Gamma_h\subseteq\{(x,y)|H(x,y)=h,\ h\in(0,h_0),\ h_0=H(\partial\mathcal{P})\}$. The projection of $\mathcal{P}$ on the x-axis is an interval (x_l,x_r) with $x_l<0<x_r$. Under these assumptions, it is easy to verify that $x\mathbb{U}'(x)>0$ for all $x\in(x_l,x_r)\backslash\{0\}$, and $U(x)$ has a zero of even multiplicity at $x=0$ and there exists an analytic involution $z(x)$, defined by $\mathbb{U}(x)=\mathbb{U}(z(x))$, for all $x\in(x_l,x_r)$. Let

$$\mathbb{I}_i(h)=\oint_{\Gamma_h} f_i(x)y^{2s-1}dx,\quad \text{for } h\in(0,h_0), \tag{2.12}$$

where $f_i(x)$, $i=0,1,\ldots,n-1$, are analytic functions on (x_l,x_r) and $s\in N$. Further, define

$$l_i(x):=\frac{f_i(x)}{\mathbb{U}'(x)}-\frac{f_i(z(x))}{\mathbb{U}'(z(x))}.$$

Then, we have

Lemma 2.2 ([9]). *Under the above assumption, $\{\mathbb{I}_0,\mathbb{I}_1,\cdots,\mathbb{I}_{n-1}\}$ is an* ECT *system on $(0,h_0)$ if $\{l_0,l_1,\cdots,l_{n-1}\}$ is an* ECT *system on $(x_l,0)$ or $(0,x_r)$ and $s>n-2$.*

Lemma 2.3 ([22]). *Under the above assumption, if the following conditions are satisfied:*
(i) *$W[l_0,l_1,\ldots,l_i]$ does not vanish on $(0,x_r)$ for $i=0,1,\cdots,n-2$,*
(ii) *$W[l_0,l_1,\ldots,l_{n-1}]$ has k zeros on $(0,x_r)$ with multiplicities counted, and*
(iii) *$s>n+k-2$,*
then, any nontrivial linear combination of $\{\mathbb{I}_0,\mathbb{I}_1,\cdots,\mathbb{I}_{n-1}\}$ has at most $n+k-1$ zeros on $(0,h_0)$ with multiplicities counted. In this case, $\{\mathbb{I}_0,\mathbb{I}_1,\cdots,\mathbb{I}_{n-1}\}$ is called a Chebyshev system with accuracy k on $(0,h_0)$, where $W[l_0,l_1,\ldots,l_i]$ denotes the Wronskian of $\{l_0,l_1,\ldots,l_i\}$.

3. Bifurcation Diagram of $\mathcal{A}(h)$

In this section, we study $\mathcal{A}(h)$. We write $\mathcal{H}(x,y)=\frac{y}{2}+U(x)$, and $U(x)-U(z)=(x-z)q(x,z)=0$, where

$$\begin{aligned} q(x,z)=&\ x^4+4\,x^3z+4\,x^2z^2+4\,x\,z^3+4\,z^4-15\,x^3-15\,x^2z-15\,x\,z^2\\ &-15\,z^3+20\,x^2+20\,x\,z+20\,z^2-10\,x-10\,z, \end{aligned}$$

which defines the involution $z(x)$ on the period annulus. We have the following result.

Lemma 3.1. $2hI_i(h)=\oint_{\Gamma_h}S_i(x)y^3dx\equiv\widetilde{I}_i(h)$, *where $S_i(x)=x^i+G_i(x)$, with $G_i(x)=\frac{x^jg_i(x)}{30(x-1)^4}$ and $\widetilde{G}_i(x)=\frac{x^i\widetilde{g}_i(x)}{1500(x-1)^8}$, in which $g_i(x)$ is a polynomial in x.*

Proof. Multiplying $I_i(h)$ by $\frac{y^2+2U(x)}{2h}=1$ yields

$$\begin{aligned} I_i(h)=&\oint_{\Gamma_h}\frac{2U(x)+y^2}{2h}x^iydx\\ =&\frac{1}{2h}\left(\oint_{\Gamma_h}2x^iU(x)ydx+\oint_{\Gamma_h}x^iy^3dx\right),\quad i=0,1,2,3. \end{aligned} \tag{3.1}$$

By Lemma 4.1 in [9] (with $k = 3$ and $F(x) = 2x^iU(x)$), we have

$$\oint_{\Gamma_h} 2x^iU(x)ydx = \oint_{\Gamma_h} G_i(x)y^3dx, \tag{3.2}$$

where $G_i(x) = \frac{d}{3dx}\frac{2x^iU(x)}{U'(x)} = \frac{x^ig_i(x)}{30(x-1)^4}$ with

$$g_i(x) = i(4\,x^4 - 19\,x^3 + 35\,x^2 - 30\,x + 10) + 4\,x^4 - 16\,x^3 + 25\,x^2 - 20\,x + 10.$$

Substituting (3.2) into (3.1) proves Lemma 3.1. □

Without loss of generality, we assume that $\eta = \frac{a_0}{a_2}$ and $\lambda = \frac{a_1}{a_2}$ if $a_2 \neq 0$. Further, let

$$\mathcal{I}_1(h) = \oint_{\Gamma_h} \Big(x + \frac{1}{\lambda}x^2\Big)ydx. \tag{3.3}$$

Then, $\mathcal{A}(h) = a_2(\eta I_0(h) + \lambda\mathcal{I}_1(h))$. By Lemma 3.1, we have

Lemma 3.2.

$$2h\mathcal{I}_1(h) = \oint_{\Gamma_h} \Big[S_1(x) + \frac{1}{\lambda}S_2(x)\Big]y^3dx \triangleq \widetilde{\mathcal{I}}_1(h).$$

Let

$$L_i(x) = \Big(\frac{S_i}{U'}\Big)(x) - \Big(\frac{S_i}{U'}\Big)(z(x)),$$

$$\mathcal{L}_1(x) = \Big(\frac{S_1 + \frac{1}{\lambda}S_2}{U'}\Big)(x) - \Big(\frac{S_1 + \frac{1}{\lambda}S_2}{U'}\Big)(z(x)).$$

Then,

$$\frac{d}{dx}L_i(x) = \frac{d}{dx}\Big(\frac{S_i}{U'}\Big)(x) - \frac{d}{dz}\Big[\Big(\frac{S_i}{U'}\Big)(z(x))\Big]\frac{dz}{dx},$$

$$\frac{d}{dx}\mathcal{L}_i(x) = \frac{\partial}{\partial x}(\mathcal{L}_i(x)) + \frac{\partial}{\partial z}(\mathcal{L}_i(x))\frac{dz}{dx},$$

where $\frac{dz}{dx} = -\frac{q_x(x,z)}{q_z(x,z)}$. So we obtain

$$W[L_0](x) = \frac{(x-z)Q_1(x,z)}{30xz(x-1)^7(z-1)^7},$$

$$W[L_0(x), L_1(x)] = \begin{vmatrix} L_0(x) & L_1(x) \\ L_0'(x) & L_1'(x) \end{vmatrix} = \frac{(x-z)^3\,Q_2(x,z)}{900x^2z^2(z-1)^{14}(x-1)^{14}P(x,z)},$$

$$W[L_0(x), \mathcal{L}_1(x)] = \begin{vmatrix} L_0(x) & \mathcal{L}_1(x) \\ L_0'(x) & \mathcal{L}_1'(x) \end{vmatrix} = \frac{-(x-z)^3\,Q_3(x,z)}{900x^2z^2\lambda(z-1)^{14}(x-1)^{14}P^3(x,z)},$$

where

$$P(x,z) = 4\,x^3 + 8\,x^2z + 12\,xz^2 + 16\,z^3 - 15\,x^2 - 30\,xz - 45\,z^2 + 20\,x + 40\,z - 10,$$

$Q_1(x,z)$ and $Q_2(x,z)$ are two-variate polynomials of degree 11 and 24, respectively. $Q_3(x,z) = \gamma_{12}(x,z)\lambda - \gamma_{11}(x,z)$, in which $\gamma_{11}(x,z)$ and $\gamma_{12}(x,z)$ are two-variate polynomials of degree 24 and 25, respectively, and $Q_3(x,z)$ is of degree 25.

Computing the resultant of Q_1 and q with respect to z, and applying Sturm's Theorem, we can show that Q_1 and q have no common zeros for $x \in (0,1)$. Therefore, $W[L_0]$ does not vanish for $x \in (0,1)$. Similarly, we can show that $P(x,z)$ and $W[L_0, L_1]$ do not vanish for $x \in (0,1)$.

Computing the resultant of γ_{12} and q with respect to z, and applying Sturm's Theorem, we can prove that γ_{12} and q have no common zeros for $x \in (0,1)$. Therefore, γ_{12} does not vanish for $x \in (0,1)$. Solving $\gamma_{12}(x,z)\lambda - \gamma_{11}(x,z) = 0$ gives

$$\lambda(x,z) = \frac{\gamma_{11}(x,z)}{\gamma_{12}(x,z)}, \tag{3.4}$$

for which we have the following result.

Lemma 3.3. *$\lambda(x,z)$ is monotonic for $x \in (0,1)$, and $\lambda(x,z) \in (-2, -\frac{1}{3})$.*

Proof. A direct computation shows that

$$\lambda'(x,z) = \frac{\partial \lambda(x,z)}{\partial x} + \frac{\partial \lambda(x,z)}{\partial z}\frac{dz}{dx} = \frac{\gamma_{21}(x,z)}{\gamma_{22}(x,z)}.$$

Computing the resultant $\mathrm{res}(\gamma_{22}, q, z)$ and applying Sturm's Theorem, we can prove that $\mathrm{res}(\gamma_{22}, q, z)$ has no zero for $x \in (0,1)$. Therefore, γ_{22} does not vanish for $x \in (0,1)$, and so the function $\lambda(x,z)$ is well defined.

Computing the resultant $\mathrm{res}(\gamma_{21}, q, z)$ and applying Sturm's Theorem, we can show that $\mathrm{res}(\gamma_{21}, q, z)$ has a unique zero in $\left[\frac{47393}{65536}, \frac{94787}{131072}\right] \subseteq (0,1)$. Further, computing the resultant $\mathrm{res}(\gamma_{21}, q, x)$ and applying Sturm's Theory show that $\mathrm{res}(\gamma_{21}, q, x)$ has three zeros for z in the three intervals:

$$\left[-\frac{130431}{524288}, -\frac{65215}{262144}\right], \left[-\frac{71233}{524288}, -\frac{1113}{8192}\right] \quad \text{and} \quad \left[-\frac{123841}{1048576}, -\frac{1935}{16384}\right].$$

Therefore, if γ_{21} and q have common roots on $(-\frac{1}{4}, 0) \times (0,1)$, the roots must lie in one of the following three domains:

$$\begin{aligned} D_1 &: \left[-\frac{130431}{524288}, -\frac{65215}{262144}\right] \times \left[\frac{47393}{65536}, \frac{94787}{131072}\right], \\ D_2 &: \left[-\frac{71233}{524288}, -\frac{1113}{8192}\right] \times \left[\frac{47393}{65536}, \frac{94787}{131072}\right], \\ D_3 &: \left[-\frac{123841}{1048576}, -\frac{1935}{16384}\right] \times \left[\frac{47393}{65536}, \frac{94787}{131072}\right]. \end{aligned}$$

The resultant $\mathrm{res}(\frac{\partial q}{\partial x}, \frac{\partial q}{\partial z}, z)$ has no zeros in $\left[\frac{47393}{65536}, \frac{94787}{131072}\right] \subseteq (0,1)$, implying that q reaches its extreme values on the boundary of D_i. By Sturm's Theorem, we know that the derivatives of the four functions obtained by restricting $q(x,z)$ on the four boundaries of D_i ($i = 1,\ 2$) have no zeros. Therefore, $q(x,z)$ gets its maximal and minimum values at the four vertexes on each D_i. A direct computation yields

$$\begin{aligned} &\max_{D_1} q(x,z) = \frac{-307447524594536353 99}{73786976294838206464}, &&\min_{D_1} q(x,z) = \frac{-787096256015976145823 9}{1888946593147858085478 4}, \\ &\max_{D_2} q(x,z) = -\frac{-1556098303244192066443 5}{1888946593147858085478 4}, &&\min_{D_2} q(x,z) = -\frac{-927524299079375}{1125899906842624}. \end{aligned}$$

The minimum and maximum values have the same signs on each D_i. Hence, q and γ_{21} have no common zeros on each D_i. Therefore, γ_{21} does not vanish for

$x \in (0,1)$. This implies that $\lambda'(x,z) \neq 0$ for $x \in (0,1)$. Thus, $\lambda(x,z)$ is monotonic for $x \in (0,1)$, and so

$$\lim_{x\to 0} = \lambda(x,z) = -\frac{1}{3}, \quad \lim_{x\to 1} = \lambda(x,z) = -2.$$

This completes the proof of Lemma 3.3. □

Lemma 3.3 implies that when $\lambda \in (-2, -\frac{1}{3})$, $W[L_0, \mathcal{L}_1]$ has a simple root for $x \in (0,1)$, and $W[L_0, \mathcal{L}_1]$ has no roots for $x \in (0,1)$ when

$$\lambda \in (-\infty, -2] \bigcup \Big[-\frac{1}{3}, +\infty\Big).$$

By Lemmas 2.1 and 2.2, we have the following result.

Lemma 3.4. *$\mathcal{A}(h)$ has at most two zeros (counting multiplicities) on $(0, \frac{1}{20})$ for $\lambda \in (-2, -\frac{1}{3})$, and at most one zero (counting multiplicity) for $\lambda \in (-\infty, -2] \bigcup [-\frac{1}{3}, +\infty)$.*

Let

$$\kappa(h) = \frac{\lambda I_1(h) + I_2(h)}{I_0(h)}. \tag{3.5}$$

Then,

$$\mathcal{A}(h) = I_0(h)(\eta + \kappa(h)).$$

The asymptotic expansions (2.8) and (2.9) yield the values of $\kappa(h)$ at the end of annulus,

$$\begin{aligned} \kappa(\tfrac{1}{20}) &= \lim_{h\to\frac{1}{20}} \frac{\lambda I_1(h) + I_2(h)}{I_0(h)} = \frac{\lambda}{6} + \frac{1}{11}, \\ \kappa(0) &= \lim_{h\to 0} \frac{\lambda I_1(h) + I_2(h)}{I_0(h)} \quad = 0. \end{aligned} \tag{3.6}$$

Lemma 3.4 together with the endpoint values (3.6) implies the following proposition.

Proposition 3.1. *The ratio $\kappa(h)$ is monotonic for $\lambda \in (-\infty, -2] \bigcup [-\frac{1}{3}, +\infty)$. More precisely, $\kappa(h)$ is degreasing on $(0, \frac{1}{20})$ when $\lambda \in (-\infty, -2]$, and increasing on $(0, \frac{1}{20})$ when $\lambda \in [-\frac{1}{3}, +\infty)$.*

Lemma 3.5. *If $\kappa'(h)$ has zeros, they must be simple. Moreover, $\kappa'(h)$ has $2n+1$ simple zeros on $(0, \frac{1}{20})$ for any $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$, and $2n$ simple zeros on $(0, \frac{1}{20})$ for any $\lambda \in (-2, -\frac{12}{11})$.*

Proof. Firstly, we give a short proof for the first assertion by using an argument of contradiction. Let h^* be a zero of $\kappa'(h)$ with l multiplicities, $l \geq 2$. Then there must exist an η such that $\eta + \kappa(h)$ has a zero at $h = h^*$ with $l+1$ (≥ 3) multiplicities. Because $J_0(h) > 0$, the relationship between $\mathcal{A}(h)$ and $\eta + \kappa(h)$ implies that $\mathcal{A}(h)$ has a zero at $h = h^*$ with $l+1$ (≥ 3) multiplicities. This contradicts Lemma 3.4.

With the expansion of $I_i(h)$ near $h = \frac{1}{20}$, given in (2.8), a direct computation shows that

$$\begin{aligned} \kappa'(\tfrac{1}{20}) = \lim_{h\to\frac{1}{20}} \kappa'(h) &= \lim_{h\to\frac{1}{20}} \frac{(\lambda I_1'(h) + I_2'(h))I_0(h) - (\lambda I_1(h) + I_2(h))I_0'(h)}{I_0^2(h)} \\ &= \mathrm{sign}\big[-\widetilde{A}_0(12 + 11\lambda)\big]\infty. \end{aligned}$$

In particular, when $\lambda = -\frac{12}{11}$, a further computation shows $\kappa'(\frac{1}{20}) = -\infty$. Moreover, using the expansions of $I_i(h)$ and $I_i'(h)$ near $h = 0$ in (2.9), we can prove that

$$\kappa'(0) = \lim_{h\to 0} \kappa'(h) = \lim_{h\to 0} \frac{[\lambda I_1'(h) + I_2'(h)]I_0(h) - [\lambda I_1(h) + I_2(h)]I_0'(h)}{I_0^2(h)} = \frac{3\lambda+1}{2}.$$

Because $\widetilde{A}_0 < 0$, it is obvious that $\kappa'(h)$ has different signs at the two endpoints of the interval $(0, \frac{1}{20})$ if $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$, and has the same sign at the two endpoints of $(0, \frac{1}{20})$ if $\lambda \in (-2, -\frac{12}{11}]$. This completes the proof. $\square$

Proposition 3.2. **(i)** *$\kappa'(h)$ has a unique simple zero on $(0, \frac{1}{20})$ for any $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$, namely, $\kappa(h)$ decreases from $\kappa(0)$ to a minimum value $\kappa(h^\star)$ at $h = h^\star$ and then increases to $\kappa(\frac{1}{20})$ for any $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$.*

(ii)

$$\kappa(0) = \begin{cases} > \kappa\Big(\dfrac{1}{20}\Big) & \text{for } \lambda \in \Big(-\dfrac{12}{11}, -\dfrac{6}{11}\Big), \\ = \kappa\Big(\dfrac{1}{20}\Big) & \text{for } \lambda = -\dfrac{6}{11}, \\ < \kappa\Big(\dfrac{1}{20}\Big) & \text{for } \lambda \in \Big(-\dfrac{6}{11}, -\dfrac{1}{3}\Big). \end{cases}$$

Proof. By Lemma 3.5, for a fixed $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$, if $\kappa'(h)$ has three or more than three simple zeros on $(0, \frac{1}{20})$, then there must exist an η such that $\eta + \kappa(h)$ has at least three zeros. This implies that $\mathcal{A}(h)$ can have at least three zeros, which contradicts Lemma 3.4. Therefore, $\kappa'(h)$ has a unique zero on $(0, \frac{1}{20})$ for any $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$. The signs of $\kappa'(0)$ and $\kappa'(\frac{1}{20})$ when $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$ determine the property of $\kappa(h)$. The proof of part (ii) directly follows the endpoint values in (3.6). $\square$

Proposition 3.3. *$\kappa'(h)$ has no zeros on $(0, \frac{1}{20})$ for any $\lambda \in (-2, -\frac{12}{11}]$, that is, $\kappa(h)$ is monotonic (decreasing) on $(0, \frac{1}{20})$ for any $\lambda \in (-2, -\frac{12}{11})$.*

Proof. By Lemma 3.5, for a fixed $\lambda \in (-2, -\frac{12}{11}]$, if $\kappa'(h)$ has four or more than four zeros on $(0, \frac{1}{20})$, then there must exist an η such that $\eta + \kappa(h)$ has at least three zeros counting multiplicities. This contradicts Lemma 3.4.

Next, we prove that $\kappa'(h)$ does not have two zeros. Suppose otherwise $\kappa'(h)$ has two zeros for $\lambda \in (-2, -\frac{12}{11}]$. We have known that $\kappa'(0) < 0$ and $\kappa'(\frac{1}{20}) < 0$, which implying that $\kappa(h)$ is decreasing at the endpoints of the interval $(0, \frac{1}{20})$. Further, for $\lambda \in (-2, -\frac{12}{11}]$,

$$0 = \kappa(0) > \kappa\Big(\frac{1}{20}\Big) = \frac{\lambda}{6} + \frac{1}{11},$$

where $\frac{\lambda}{6} + \frac{1}{11} \in (-\frac{8}{33}, -\frac{1}{11}]$. This clearly indicates that there must exist an η such that $\eta + \kappa(h)$ has at least three zeros. This contradicts Lemma 3.4. The endpoint values determine its decreasing, so the proof is complete. $\square$

Summarizing Propositions 3.1, 3.2 and 3.3, we have the following theorem.

Theorem 3.1. *For system (1.2) with $\alpha = \beta = 1$, the Abelian integral $\mathcal{A}(h)$ has the following properties:*

(i) *When $\lambda \in (-\infty, -\frac{12}{11}]$, $\mathcal{A}(h)$ has a unique zero on $(0, \frac{1}{20})$ if and only if $\eta \in (-\kappa(0), -\kappa(\frac{1}{20}))$. $\eta = -\kappa(h)$ corresponds to $\mathcal{A}(h) = 0$. In particular, when $\lambda = -\frac{12}{11}$, $\kappa'(\frac{1}{20}) = 0$ and $\eta = -\kappa(\frac{1}{20})$ imply that $\mathcal{A}(h)$ has a zero of multiplicity* *two at the endpoint* $h = \frac{1}{20}$*. So* $(\lambda, \eta) = (-\frac{12}{11}, \kappa(\frac{1}{20})) = (-\frac{12}{11}, \frac{1}{11})$ *is a double homoclinic loop bifurcation point, denoted by* DL.

(ii) *When* $\lambda \in [-\frac{1}{3}, +\infty)$*,* $\mathcal{A}(h)$ *has a unique zero on* $(0, \frac{1}{20})$*. if and only if* $\eta \in (-\kappa(\frac{1}{20}), -\kappa(0))$*.* $\eta = -\kappa(h)$ *corresponds to* $\mathcal{A}(h) = 0$*. In particular, when* $\lambda = -\frac{1}{3}$*,* $\kappa'(0) = 0$ *and* $\eta = -\kappa(0) = 0$ *indicate that* $\mathcal{A}(h)$ *has a zero of multiplicity two at the endpoint* $h = 0$*. So* $(\lambda, \eta) = (-\frac{1}{3}, -\kappa(0)) = (-\frac{1}{3}, 0)$ *is a double Hopf bifurcation point, denoted by* DH.

(iii) *When* $\lambda \in (-\frac{12}{11}, -\frac{6}{11})$*,* $\mathcal{A}(h)$ *has a unique zero on* $(0, \frac{1}{20})$ *if and only if* $\eta \in (-\kappa(0), -\kappa(\frac{1}{20}))$*, and* $\mathcal{A}(h)$ *has two zeros on* $(0, \frac{1}{20})$ *if and only if*

$$\eta \in \Big(-\kappa\Big(\frac{1}{20}\Big), -\min\kappa(h)\Big) = \Big(-\kappa\Big(\frac{1}{20}\Big), -\kappa(h^\star)\Big).$$

In particular, $\eta = -\kappa(\frac{1}{20})$ *corresponds to a zero at* $h = \frac{1}{20}$ *and another zero at* $h = h^l$ *satisfying* $\kappa(h^l) = \kappa(\frac{1}{20})$*.*

(iv) *When* $\lambda \in (-\frac{6}{11}, -\frac{1}{3})$*,* $\mathcal{A}(h)$ *has a unique zero on* $(0, \frac{1}{20})$ *if and only if* $\eta \in (-\kappa(\frac{1}{20}), -\kappa(0)) = (-\kappa(\frac{1}{20}), 0)$*, and* $\mathcal{A}(h)$ *has two zeros on* $(0, \frac{1}{20})$ *if and only if*

$$\eta \in \big(-\kappa(0), -\min\kappa(h)\big) = \big(0, -\kappa(h^\star)\big).$$

In particular, $\eta = -\kappa(0)$ *corresponds to a zero at* $h = 0$ *and another zero at* $h = h^r$ *satisfying* $\kappa(h^r) = \kappa(0)$*.*

(v) *When* $\lambda = -\frac{6}{11}$*,* $\mathcal{A}(h)$ *has two zeros on* $(0, \frac{1}{20})$ *if and only if*

$$\eta \in \big(-\kappa(0), -\min\kappa(h)\big) = \Big(-\kappa\Big(\frac{1}{20}\Big), -\min\kappa(h)\Big) = \big(0, -\kappa(h^\star)\big).$$

In particular, $\eta = -\kappa(0) = -\kappa(\frac{1}{20}) = 0$ *corresponds to the zeros at endpoints, one at* $h = 0$ *and another at* $h = \frac{1}{20}$*. So* $(\lambda, \eta) = (-\frac{6}{11}, 0)$ *is a homoclinic-hopf bifurcation point, denoted by* HL*.*

(vi) *For all* $\lambda \in (-\frac{12}{11}, -\frac{1}{3})$*,* $\eta = -\min\kappa(h)$ *defines the double limit cycle bifurcation curve* C.

The corresponding bifurcation diagram is shown in Figure 3. The Hopf bifurcation line is $L_1\colon\ \eta = 0$ derived from $\eta = -\kappa(0)$, and the homoclinic loop bifurcation line is $L_2\colon\ \eta + \frac{\lambda}{6} + \frac{1}{11} = 0$ obtained from $\eta = -\kappa(\frac{1}{20})$. The expressions of the degenerate bifurcation points DL, DH and HL are defined in (i), (ii) and (v), respectively. The curve C is defined by the analytic expression: $\eta = -\min\kappa(h) = -\kappa(h^\star)$ in (v), or $\mathcal{A}(h) = \mathcal{A}'(h)$. Note that

$$\kappa'(h^\star) = \big[\lambda I_1'(h^\star) + I_2'(h^\star)\big] I_0(h^\star) - I_0'(h^\star)\big[\lambda I_1(h^\star) + I_2(h^\star)\big] = 0$$

defines a monotonic function $\lambda = \mathbb{III}(h^\star)$. Thus, $\eta = -\kappa(\mathbb{III}^{-1}(\lambda))$. A direct computation with the asymptotic property of $I_i(h)$ and $I_i'(h)$ gives

$$\lim_{\lambda\to -\frac{12}{11}} \eta'(\lambda) = -\lim_{h^\star\to\frac{1}{20}} \kappa'(h^\star) = -\frac{1}{6} \quad \text{and} \quad \lim_{\lambda\to -\frac{1}{3}} \eta'(\lambda) = -\lim_{h^\star\to 0} \kappa'(h^\star) = 0.$$

Therefore, the double limit cycle curve C is tangent to the line L_1 at DH, and tangent to the line L_2 at DL. Table 1 shows the distribution of the number of zeros of $\mathcal{A}(h)$ and the corresponding phase portraits of system (1.2) with $\alpha = \beta = 1$ in different regions V_i, $i = 1, 2, \ldots, 5$. It should be noted that the singular point $(1, 0)$ is a nilpotent saddle when $\eta+\lambda+1 = 0$ and a degenerated saddle when $\eta+\lambda+1 \neq 0$.

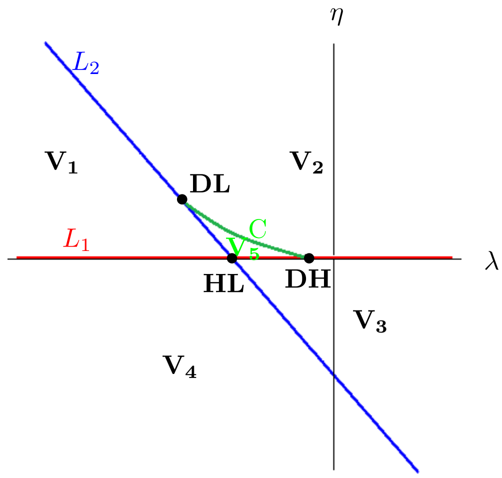
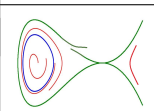
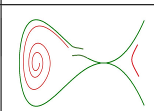
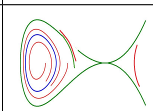
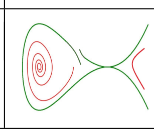
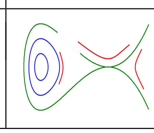


Figure 3. Bifurcation diagram of system (1.2) for $\alpha = \beta = 1$ with $\eta = \frac{a_0}{a_2}$ and $\lambda = \frac{a_1}{a_2}$, showing the Hopf bifurcation line L_1, the homoclinic bifurcation line L_2, the degenerate bifurcation points DL, DH and HL, and the curve C characterized by one zero of multiplicity two.

Table 1. Distribution of zeros with phase portraits.

Region	V_1	V_2	V_3	V_4	V_5
Zero	1	0	1	0	2
Portrait					

4. The Bifurcation Diagram of $\mathcal{M}(h)$

In this section, we study $\mathcal{M}(h)$. We write $\mathcal{H}^*(x,y) = \frac{y}{2} + U^*(x)$, and $U^*(x) - U^*(z) = (x-z)q^*(x,z) = 0$, where

$$q^*(x,z) = 9\,x^4 + 9\,x^3 z + 9\,x^2 z^2 + 9\,x\,z^3 + 9\,z^4 - 15\,x^3 - 15\,x^2 z$$
$$- 15\,x\,z^2 - 15\,z^3 - 5\,x^2 - 5\,x\,z - 5\,z^2 + 15\,x + 15\,z,$$

which defines the involution $z(x)$ on the period annulus of $\mathcal{H}^*(x,y)$. Introducing the notation $I_i^*(h) = \oint_{\Gamma_h^*} x^i y dx$ gives

$$J_0(h) = \frac{5\sqrt{15}}{9} I_0^*(h),$$
$$J_1(h) = \frac{\sqrt{15}}{3}\big[I_0^*(h) - I_1^*(h)\big],$$
$$J_2(h) = \frac{\sqrt{15}}{5}\big[I_2^*(h) - 2I_1^*(h) + I_0^*(h)\big].$$

We have the following result.

Lemma 4.1. **(i)** $2hI_i^*(h) = \oint_{\Gamma_h^*} S_i^*(x)y^3dx \equiv \widetilde{I}_i^*(h)$,

(ii) $2hJ_i(h) = \oint_{\Gamma_h^*} S_i^{**}(x)y^3dx \equiv \widetilde{J}_i(h)$,

where $S_i^*(x) = x^i + G_i^*(x)$, *with* $G_i^*(x) = \frac{x^j g_i^*(x)}{30(x-1)^4}$ *in which* $g_i^*(x)$ *is a polynomial in* x*; and*

$$\begin{aligned}
S_0^{**}(x) &= \frac{5\sqrt{15}}{9}S_0^*(x),\\
S_1^{**}(x) &= \frac{\sqrt{15}}{3}\big[S_0^*(x) - S_1^*(x)\big],\\
S_2^{**}(x) &= \frac{\sqrt{15}}{5}\big[S_2^*(x) - 2S_1^*(x) + S_0^*(x)\big].
\end{aligned}$$

The proof of part (i) is similar to that of proving Lemma 3.1, and the proof of part (ii) directly follows the relation between $I_i^*(h)$ and $J_i(h)$.

Similarly, we assume that $\eta = \frac{a_0}{a_2}$ and $\lambda = \frac{a_1}{a_2}$ if $a_2 \neq 0$, and introduce

$$\mathcal{J}_1(h) = J_1(h) + \frac{1}{\lambda}J_2(h) = \oint_{\Gamma_h^*}\Big[q_1(x) + \frac{1}{\lambda}q_2(x)\Big]ydx. \tag{4.1}$$

Then, $\mathcal{M}(h) = \eta J_0(h) + \lambda\mathcal{J}_1(h)$. By Lemma 4.1, we have

Lemma 4.2.

$$2h\mathcal{J}_1(h) = \oint_{\Gamma_h^*}\Big[S_1^*(x) + \frac{1}{\lambda}S_2^*(x)\Big]y^3dx \triangleq \widetilde{\mathcal{J}}_1(h).$$

Let

$$\begin{aligned}
L_i^*(x) &= \Big(\frac{S_i^{**}}{(U^*)'}\Big)(x) - \Big(\frac{S_i^{**}}{(U^*)'}\Big)(z(x)),\\
\mathcal{L}_1^*(x) &= \Big(\frac{S_1^{**} + \frac{1}{\lambda}S_2^{**}}{(U^*)'}\Big)(x) - \Big(\frac{S_1^{**} + \frac{1}{\lambda}S_2^{**}}{(U^*)'}\Big)(z(x)).
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d}{dx}L_i^*(x) &= \frac{d}{dx}\Big(\frac{S_i^{**}}{(U^*)'}\Big)(x) - \frac{d}{dz}\Big[\Big(\frac{S_i^{**}}{(U^*)'}\Big)(z(x))\Big]\frac{dz}{dx},\\
\frac{d}{dx}\mathcal{L}_i^*(x) &= \frac{\partial}{\partial x}(\mathcal{L}_i^*(x)) + \frac{\partial}{\partial z}(\mathcal{L}_i^*(x))\frac{dz}{dx},
\end{aligned}$$

where $\frac{dz}{dx} = -\frac{q_x^*(x,z)}{q_z^*(x,z)}$. Thus, we obtain

$$\begin{aligned}
W[L_0^*](x) &= \frac{\sqrt{15}(x-z)Q_1^*(x,z)}{27\,xz\,(3\,x+2)^3\,(x-1)^5\,(3\,z+2)^3\,(z-1)^5},\\
W[L_0^*(x), L_1^*(x)] &= \begin{vmatrix} L_0^*(x) & L_1^*(x)\\ (L_0^*)'(x) & (L_1^*)'(x)\end{vmatrix}\\
&= \frac{(x-z)^3\,Q_2^*(x,z)}{81\,x^2z^2\,(z-1)^{10}\,(3\,z+2)^5\,(x-1)^{10}\,(3\,x+2)^5\,P_*(x,z)},\\
W[L_0^*(x), \mathcal{L}_1^*(x)] &= \begin{vmatrix} L_0^*(x) & \mathcal{L}_1^*(x)\\ (L_0^*)'(x) & (\mathcal{L}_1^*)'(x)\end{vmatrix}\\
&= \frac{(x-z)^3\,Q_3^*(x,z)}{405x^2z^2(z-1)^{10}(2+3z)^5(x-1)^{10}(2+3x)^5P_*^3(x,z)},
\end{aligned}$$

where

$$P_*(x,z) = 9\,x^3 + 18\,x^2 z + 27\,xz^2 + 36\,z^3 - 15\,x^2 - 30\,xz - 45\,z^2 - 5\,x - 10\,z + 15,$$

$Q_1^*(x,z)$ and $Q_2^*(x,z)$ are two-variate polynomials of degree 13 and 26, respectively. $Q_3^*(x,z) = \gamma_{12}^*(x,z)\lambda - \gamma_{11}^*(x,z)$, in which $\gamma_{11}^*(x,z)$ and $\gamma_{12}^*(x,z)$ are two-variate polynomials of degree 27 and 26, respectively, and $Q_3^*(x,z)$ is of degree 27.

Computing the resultant of P_* and q^* with respect to z, and applying Sturm's Theorem, we can show that P_* and q^* have no common zeros for $x \in (0,1)$. Therefore, P_* does not vanish, and so $W[L_0^*, L_1^*]$ and $W[L_0^*, \mathcal{L}_1^*]$ are well defined. Similarly, $Q_1^*(x,z)$ does not vanish by Sturm's theorem, therefore $W[L_0^*]$ does not vanish for $x \in (0,1)$.

Using the similar method in proving Lemma 3.3, we can show that γ_{21} does not vanish, and further we can prove that $Q_2^*(x,z)$ and γ_{12}^* do not vanish for $x \in (0,1)$. Therefore, $W[L_0^*, L_1^*]$ does not vanish and the function

$$\lambda(x,z) = \frac{\gamma_{11}^*(x,z)}{\gamma_{12}^*(x,z)} \tag{4.2}$$

is well defined by $\gamma_{12}^*(x,z)\lambda - \gamma_{11}^*(x,z) = 0$.

We have the following result.

Lemma 4.3. *$\lambda(x,z)$ is monotonic for $x \in (0,1)$, and $\lambda(x,z) \in (-1,0)$.*

Proof. A direct computation shows that

$$\lambda'(x,z) = \frac{\partial\lambda(x,z)}{\partial x} + \frac{\partial\lambda(x,z)}{\partial z}\frac{dz}{dx} = \frac{\gamma_{21}^*(x,z)}{\gamma_{22}^*(x,z)}.$$

Applying Sturm's Theorem to the resultant $\mathrm{res}(\gamma_{21}^*, q^*, z)$, we can prove that γ_{21}^* does not vanish for $x \in (0,1)$. γ_{22}^* does not vanish for $x \in (0,1)$, which can be shown by applying the method used in proving Lemma 3.3. This implies that $\lambda'(x,z)$ is well defined and $\lambda'(x,z) \neq 0$ for $x \in (0,1)$. Thus, $\lambda(x,z)$ is monotonic for $x \in (0,1)$, and so

$$\lim_{x\to 0} = \lambda(x,z) = 0, \quad \lim_{x\to 1} = \lambda(x,z) = -1.$$

This completes the proof of Lemma 4.3. $\square$

Lemma 4.3 implies that when $\lambda \in (-1,0)$, $W[L_0^*, \mathcal{L}_1^*]$ has a simple root for $x \in (0,1)$, and $W[L_0^*, \mathcal{L}_1^*]$ has no roots for $x \in (0,1)$ when $\lambda \in (-\infty,-1] \bigcup [0,+\infty)$. By Lemmas 2.1 and 2.2, we have the following lemma.

Lemma 4.4. *$\mathcal{M}(h)$ has at most two zeros (counting multiplicities) on $(-\frac{4}{45}, 0)$ for $\lambda \in (-1,0)$, and at most one zero (counting multiplicity) for*

$$\lambda \in (-\infty,-1] \bigcup [0,+\infty).$$

Let

$$\vartheta(h) = \frac{\lambda J_1(h) + J_2(h)}{J_0(h)}. \tag{4.3}$$

Then,

$$\mathcal{M}(h) = J_0(h)(\eta + \vartheta(h)).$$

The asymptotic expansions (2.10) and (2.11) yield the values of $\vartheta(h)$ at the end of annulus,

$$\begin{aligned}\vartheta(0) &= \lim_{h\to 0}\frac{\lambda J_1(h)+J_2(h)}{J_0(h)} = \frac{35+55\lambda}{99},\\ \vartheta\Big(-\frac{4}{45}\Big) &= \lim_{h\to -\frac{4}{45}}\frac{\lambda J_1(h)+J_2(h)}{J_0(h)} = \frac{15\lambda+9}{25}.\end{aligned} \tag{4.4}$$

Lemma 4.4 together with the values at the endpoints implies the following proposition.

Proposition 4.1. *The ratio $\vartheta(h)$ is monotonic for $\lambda \in (-\infty,-1]\bigcup[0,+\infty)$. More precisely, $\vartheta(h)$ is increasing on $(-\frac{4}{45},0)$ for $\lambda \in (-\infty,-1]$ and decreasing on $(-\frac{4}{45},0)$ for $\lambda \in [0,+\infty)$.*

Moreover, we have the following result.

Lemma 4.5. *If $\vartheta'(h)$ has zeros, they must be simple. In fact, it has $2n+1$ simple zeros on $(-\frac{4}{45},0)$ for any $\lambda \in (-\frac{7}{11},0)$, and $2n$ simple zeros on $(-\frac{4}{45},0)$ for any $\lambda \in (-1,-\frac{7}{11})$.*

Proof. We first give a short proof for the first assertion by using an argument of contradiction. Let h^* be a zero of $\vartheta'(h)$ with l multiplicities, $l \geq 2$. Then there must exist an η such that $\eta+\vartheta(h)$ has a zero at $h=h^*$ with $l+1$ (≥ 3) multiplicities. Because $J_0(h)>0$, the relationship between $\mathcal{M}(h)$ and $\eta+\vartheta(h)$ implies that $\mathcal{M}(h)$ has a zero at $h=h^*$ with $l+1$ (≥ 3) multiplicities. This contradicts Lemma 4.4.

With the expansion of $J_i(h)$ near $h=0$, given in (2.10), a direct computation shows that

$$\begin{aligned}\vartheta'(0) = \lim_{h\to 0}\vartheta'(h) &= \lim_{h\to 0}\frac{[\lambda J_1'(h)+J_2'(h)]J_0(h)-[\lambda J_1(h)+J_2(h)]J_0'(h)}{J_0^2(h)}\\ &= -\operatorname{sign}\big[B_{00}(11\lambda+7)\big]\infty.\end{aligned}$$

Further, using the expansions of $J_i(h)$ and $J_i'(h)$ near $h=-\frac{4}{45}$ in (2.11), we can prove that

$$\begin{aligned}\vartheta'\Big(-\frac{108}{15625}\Big) &= \lim_{h\to -\frac{4}{45}}\vartheta'(h)\\ &= \lim_{h\to -\frac{4}{45}}\frac{[\lambda J_1'(h)+J_2'(h)]J_0(h)-[\lambda J_1(h)+J_2(h)]J_0'(h)}{J_0^2(h)}\\ &= -\frac{9}{40}\lambda.\end{aligned}$$

Because $B_{00}>0$, it is obvious that $\vartheta'(h)$ has different signs at the two endpoints of the interval $(-\frac{4}{45},0)$ if $\lambda \in (-\frac{7}{11},0)$, and has the same sign at the two endpoints if $\lambda \in (-1,-\frac{7}{11}]$. This completes the proof. □

Proposition 4.2. **(i)** *$\vartheta'(h)$ has a unique simple zero on $(-\frac{4}{45},0)$ for any $\lambda \in (-\frac{7}{11},0)$, namely, $\vartheta(h)$ increases from $\vartheta(-\frac{4}{45})$ to a maximum value $\vartheta(h^*)$ at $h=h^*$ and then decreases to $\vartheta(0)$ for any $\lambda \in (-\frac{7}{11},0)$.*

(ii) *$\vartheta(-\frac{4}{45}) < \vartheta(0)$ for $\lambda \in (-\frac{7}{11},-\frac{8}{55})$, $\vartheta(-\frac{4}{45}) = \vartheta(0)$ for $\lambda = -\frac{8}{55}$ and $\vartheta(-\frac{4}{45}) > \vartheta(0)$ for $\lambda \in (-\frac{8}{55},0)$.*

Proof. By Lemma 4.5, for a fixed $\lambda \in (-\frac{7}{11},0)$, if $\vartheta'(h)$ has three or more than three simple zeros on $(-\frac{4}{45},0)$, then there must exist an η such that $\eta+\vartheta(h)$ has at least three zeros. This implies that $\mathcal{M}(h)$ can have at least three zeros, which contradicts Lemma 4.4. Therefore, $\vartheta'(h)$ has a unique zero on $(-\frac{4}{45}, 0)$ for any $\lambda \in (-\frac{7}{11}, 0)$. The signs of $\vartheta'(-\frac{4}{45})$ and $\vartheta'(0)$ when $\lambda \in (-\frac{7}{11}, 0)$ determines the property of $\vartheta(h)$. A direct comparison on the values at the endpoints proves part (ii). $\square$

Proposition 4.3. *$\vartheta'(h)$ has no zeros on $(-\frac{4}{45}, 0)$ for any $\lambda \in (-1, -\frac{7}{11}]$, that is, $\vartheta(h)$ is monotonic (increasing) on $(-\frac{4}{45}, 0)$ for any $\lambda \in (-1, -\frac{7}{11}]$.*

Proof. By Lemma 4.5, for a fixed $\lambda \in (-1, -\frac{7}{11}]$, if $\vartheta'(h)$ has four or more than four zeros on $(-\frac{4}{45}, 0)$, then there must exist an η such that $\eta + \vartheta(h)$ has at least three zeros counting multiplicities. This contradicts Lemma 4.4.

Next, we prove that $\vartheta'(h)$ does not have two zeros. Suppose otherwise $\vartheta'(h)$ has two zeros for $\lambda \in (-1, -\frac{7}{11}]$. We have known that $\vartheta'(-\frac{4}{45}) > 0$, and $\vartheta'(0) > 0$, implies that $\vartheta(h)$ is increasing at the endpoints of the interval $(-\frac{4}{45}, 0)$. Further, for $\lambda \in (-1, -\frac{7}{11}]$,

$$\frac{3\lambda}{5} + \frac{9}{25} = \vartheta\Big(-\frac{4}{45}\Big) < \vartheta(0) = \frac{35}{99} + \frac{5\lambda}{9}.$$

This clearly indicates that there must exist an η such that $\eta + \vartheta(h)$ has at least three zeros. This contradicts Lemma 4.4, and so the proof is complete. $\square$

Similarly, summarizing the results in Propositions 4.1, 4.2 and 4.3, we obtain the following theorem, as illustrated in the bifurcation diagram given in Figure 4.

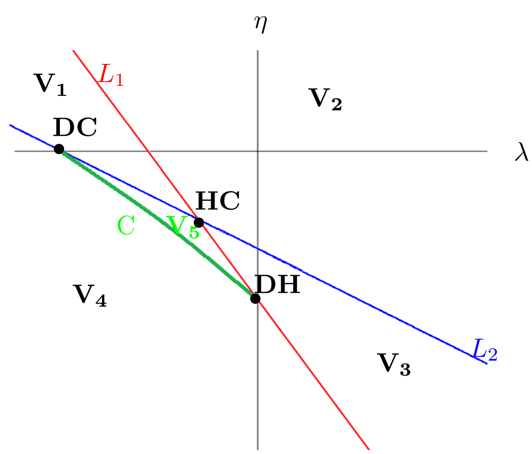


Figure 4. Bifurcation diagram of system (1.2) for $\alpha = 0$ and $\beta = \frac{3}{5}$ with $\eta = \frac{a_0}{a_2}$ and $\lambda = \frac{a_1}{a_2}$, showing the Hopf bifurcation line L_1, the heteroclinic bifurcation line L_2, the degenerate bifurcation points DC, DH and HC, and the curve C characterized by one zero of multiplicity two.

Theorem 4.1. *For system (1.2) with $\alpha = 0$, $\beta = \frac{3}{5}$, the Abelian integral $\mathcal{M}(h)$ has the following properties:*

(i) *When $\lambda \in (-\infty, -\frac{7}{11}]$, $\mathcal{M}(h)$ has a unique zero on $(-\frac{4}{45}, 0)$ if and only if $\eta \in (-\vartheta(0), -\vartheta(-\frac{4}{45}))$ with $\eta = -\vartheta(h)$ corresponding to $\mathcal{M}(h) = 0$. In particular, when $\lambda = -\frac{7}{11}$, $\vartheta'(0) = 0$ and $\eta = -\vartheta(0)$ imply that $\mathcal{M}(h)$ has a zero of* *multiplicity two at the endpoint* $h = 0$*. So* $(\lambda, \eta) = (-\frac{7}{11}, \vartheta(0)) = (-\frac{7}{11}, 0)$ *is a double heteroclinic cycle bifurcation point, denoted by* DC.

(ii) *When* $\lambda \in [0, +\infty)$*,* $\mathcal{M}(h)$ *has a unique zero on* $(-\frac{4}{45}, 0)$ *if and only if* $\eta \in (-\vartheta(-\frac{4}{45}), -\vartheta(0))$ *with* $\eta = -\vartheta(h)$ *corresponding to* $\mathcal{M}(h) = 0$*. In particular, when* $\lambda = 0$*,* $\vartheta'(-\frac{4}{45}) = 0$ *and* $\eta = -\vartheta(-\frac{4}{45})$ *yield that* $\mathcal{M}(h)$ *has a zero of multiplicity two at the endpoint* $h = -\frac{4}{45}$*. So* $(\lambda, \eta) = (0, -\vartheta(-\frac{4}{45})) = (0, -\frac{9}{25})$ *is a double Hopf bifurcation point, denoted by* DH.

(iii) *When* $\lambda \in (-\frac{7}{11}, -\frac{8}{55})$*,* $\mathcal{M}(h)$ *has a unique zero on* $(-\frac{4}{45}, 0)$ *if and only if* $\eta \in (-\vartheta(0), -\vartheta(-\frac{4}{45}))$*, and* $\mathcal{M}(h)$ *has two zeros on* $(-\frac{4}{45}, 0)$ *if and only if* $\eta \in (-\max \vartheta(h), -\vartheta(0))$*. In particular,* $\eta = -\vartheta(0)$ *corresponds to a zero at* $h = 0$ *and another zero at* $h = h^{**}$ *satisfying* $\vartheta(h^{**}) = \vartheta(0)$*.*

(iv) *When* $\lambda \in (-\frac{8}{55}, 0)$*,* $\mathcal{M}(h)$ *has a unique zero on* $(-\frac{4}{45}, 0)$ *if and only if* $\eta \in (-\vartheta(-\frac{4}{45}), -\vartheta(0))$*, and* $\mathcal{M}(h)$ *has two zeros on* $(-\frac{4}{45}, 0)$ *if and only if* $\eta \in (-\max \vartheta(h),\, -\vartheta(-\frac{4}{45}))$*. In particular,* $\eta = -\vartheta(-\frac{4}{45})$ *corresponds to a zero at* $h = -\frac{4}{45}$ *and another zero at* $h = h^{***}$ *satisfying* $\vartheta(h^{***}) = \vartheta(-\frac{4}{45})$*.*

(v) *When* $\lambda = -\frac{8}{55}$*,* $\mathcal{M}(h)$ *has two zeros on* $(-\frac{4}{45}, 0)$ *if and only if* $\eta \in (-\max \vartheta(h), -\vartheta(-\frac{4}{45})) = (-\max \vartheta(h), -\frac{3}{11})$*. In particular,* $\eta = -\vartheta(-\frac{4}{45}) = -\vartheta(0) = -\frac{3}{11}$ *corresponds to zeros at endpoints, one at* $h = -\frac{4}{45}$ *and another at* $h = 0$*. So* $(\lambda, \eta) = (-\frac{8}{55}, -\frac{3}{11})$ *is a heteroclinic-Hopf bifurcation point, denoted by* HC.

(vi) *For all* $\lambda \in (-\frac{7}{11}, 0)$*,* $\eta = -\max \vartheta(h)$ *defines the double limit cycle bifurcation curve.*

It is seen from Figure 4 that the Hopf bifurcation line is $L_1\colon\ \eta + \frac{3}{5}\lambda + \frac{9}{25} = 0$ derived from $\eta = -\vartheta(-\frac{4}{45})$, and the heteroclinic cycle bifurcation line is $L_2\colon\ \eta + \frac{5}{9}\lambda + \frac{35}{99} = 0$ obtained from $\eta = -\vartheta(0)$. The expressions of the degenerate bifurcation points DC, DH and HC are given in (i), (ii) and (v), respectively. The curve C is defined by the analytic expression, $\eta = -\max \vartheta(h) = -\vartheta(h^\star)$ in (v), which is equivalent to $\mathcal{M}(h^\star) = \mathcal{M}'(h^\star)$ and

$$\vartheta'(h^\star) = \big[\lambda J_1'(h^\star) + J_2'(h^\star)\big] J_0(h^\star) - J_0'(h^\star)\big[\lambda J_1(h^\star) + J_2(h^\star))\big] = 0.$$

Similarly, we can prove that the double limit cycle curve C is tangent to L_1 at DH, and tangent to L_2 at DC. Therefore, system (1.2) with $\alpha = 0$ and $\beta = \frac{3}{5}$ has one limit cycle in the regions V_1 and V_3, one limit cycle of multiplicity two on the curve C, two limit cycles in V_5, and no limit cycles in regions V_2 and V_4.

5. Conclusion and Discussion

In this paper, we have successfully analyzed the hyperelliptic Hamiltonian systems (1.2) with three perturbation parameters by using the algebraic criterion and asymptotic property with the help of combination of Abelian integrals. We obtained complete bifurcation diagrams for two cases: $\alpha = \beta = 1$ and $\alpha = 0$, $\beta = \frac{3}{5}$. The method we developed provides an efficient tool to study zero bifurcation diagrams of Abelian integrals for a wide class of perturbed Hamiltonian systems. The method can indeed deal with general polynomial perturbations, in particular for the systems with the degree of perturbations higher than that of the unperturbed Hamiltonian systems. As an example, consider the following perturbed Hamiltonian system,

$$\dot{x} = y, \quad \dot{y} = x(x-1)^3 + \varepsilon(a_0 + a_5 x^5 + a_7 x^7)y.$$

We can follow the procedure developed in this paper to obtain the bifurcation diagram, as shown in Figure 5.

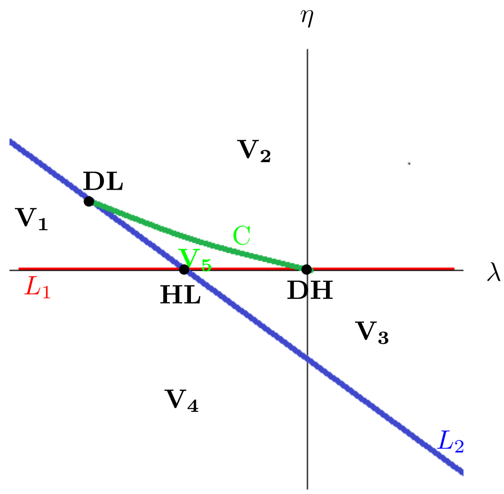


Figure 5. The zero bifurcation diagram of $\oint_{\Gamma_h}(a_0+a_5x^5+a_7x^7)ydx$ with $\eta=\frac{a_0}{a_7}$ and $\lambda=\frac{a_5}{a_7}$, showing the Hopf bifurcation line $L_1:\ \eta=0$ and the homoclinic bifurcation line $L_2:\ \eta+\frac{497\lambda}{29172}+\frac{117}{14212}=0$, and the degenerate bifurcation points $\mathrm{HL}=(-\frac{4563}{9443},0)$, $\mathrm{DL}=(-\frac{109941}{108965},\frac{976}{108965})$ and $\mathrm{DH}=(0,0)$. Zero bifurcation in the regions is as follows: one zero in V_1 and V_3, one zero of multiplicity two on C, two zeros in V_5, and no zeros in V_2 and V_4.

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