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Exact bound on the number of zeros of Abelian integrals for two hyper-elliptic Hamiltonian systems of degree 4

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Abstract

This paper concerns the exact bound on the number of zeros of Abelian integrals associated with two hyper-elliptic Hamiltonian systems of degree 4. The upper and lower bounds for the two systems have been obtained in several previous works, however the sharp bounds are still unknown. In this paper, we provide a proof to show that the exact bound is 3 for both systems. The basic idea of our method is to bound the parameter space in $\mathbb{R}^3$ to obtain two parameter sets, which might yield maximal 4 zeros of Abelian integrals corresponding to the two sets. Further, we show that the existence of 4 zeros on the two sets is not possible, and thus the sharp bound is 3.

MSC: 34C07; 34D10; 37G20

Keywords: Hyper-elliptic Hamiltonian system; Abelian integral; Sharp bound

1. Introduction

Hilbert's 16th problem [1] asks for the maximal number of limit cycles and their distribution for a polynomial planar vector field of degree n. It is extremely difficult and still unsolved even for $n = 2$. In order to reduce the difficulty, general polynomial systems are restricted to the following perturbed Hamiltonian systems,

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$$\dot{x} = H_y(x,y) + \varepsilon p(x,y), \qquad \dot{y} = -H_x(x,y) + \varepsilon q(x,y), \tag{1}$$

where $p(x,y)$ and $q(x,y)$ are polynomials of degree $n \geq 2$, ε is sufficiently small, $H(x,y)$ is a polynomial of degree $n+1$ which has at least one family of closed orbits denoted by Γ_h for the unperturbed system $(1)_{\varepsilon=0}$, parameterized by $\{(x,y)|H(x,y)=h,\ h \in J\}$, where J is an open interval. The perturbations destroy integrability and most periodic orbits of $(1)_{\varepsilon=0}$ become spirals. Only a finite number of isolated closed orbits with small deformation persist as limit cycles of (1). The main idea for studying the "persisting limit cycles" is to investigate the zeros of Poincaré map or return map on the period annulus. Hence, the "persisting limit cycles" generated by perturbation is usually called Poincaré bifurcation. When the perturbation parameter ε is close to zero, the return map is approximated by the following Abelian integral,

$$A(h) = \oint_{\Gamma_h} q(x,y)dx - p(x,y)dy, \quad h \in J. \tag{2}$$

The zeros of $A(h)$ correspond to the number of the persisting limit cycles of system (1) in the sense of first order Poincaré bifurcation, see [2]. Studying the maximal number of zeros of $A(h)$ is called weak Hilbert's 16th problem and was proposed by Arnold [3]. In fact, most of results on Hilbert's 16th problem were obtained from studying system (1).

However, the weak version is still very difficult, and up to now only the case $n = 2$ has been completely solved, see a unified proof in [4] and references therein. A much weaker case is defined by $H(x,y) = \frac{y^2}{2} + \int g(x)dx$, $p = 0$ and $q = f(x)y$, for which the perturbed Hamiltonian system is given by

$$\dot{x} = y, \qquad \dot{y} = -g(x) + \varepsilon f(x)y, \tag{3}$$

which has a simpler form of the Abelian integral,

$$\mathbb{I}(h) = \oint_{\Gamma_h} f(x)ydx.$$

Note that a special form of system (3) is the classical Liénard system (with $\varepsilon = -1,\ g(x) = x$),

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{4}$$

and Smale proposed to study the maximal number of limit cycles of system (4) as one of the mathematical problems for the 21th century [5].

Although system (3) has a simple form, it is important in studying the weak Hilbert's 16th problem and has many applications in the real word. System (3) has been used to generate a new perturbed Hamiltonian system by replacing the first equation in system (3) with $\dot{x} = y(y^2 - a^2)$, and the zeros of $\mathbb{I}(h)$ for system (3) plays an important role in studying the zeros of Abelian integral of the new system. In fact, the best result of 13 limit cycles for cubic systems was obtained by such a transform [6]. Also, it is noted that system (3) often appears in studying local bifurcations such as Bogdanov-Takens bifurcation with higher codimension [7], and often occurs in many application [8].

For convenience, we call system (3) type (m, n) if $g(x)$ and $f(x)$ are of degrees m and n, respectively. Type $(m, m-1)$ implies that the degree of the perturbation is the same as that of the unperturbed system. Dumortier and Li [9–12] obtained the sharp bounds on the number of zeros of the corresponding Abelian integrals for five cases of system (3) with type $(3, 2)$, for which 5 is the sharp bound on the number of isolated zeroes of Abelian integrals when the unperturbed system has a figure eight loop, while 2 is the sharp bound for the saddle-loop case. The main tools used in their study are Picárd-Focus equations and Riccati equations in algebraic geometry, which transferred the problem to studying the intersections of the related line with a curve. For type $(5, 4)$ of system (3) with symmetry, the perturbation still has three terms, and the dimension of Picárd-Focus equations is the same as that of type $(3, 2)$. It has been proved that 2 is the sharp bound for the cases of heteroclinic loop [13–16] and for double homoclinic loop (corresponding to each bounded period annulus) [17]. The methods used there include Picárd-Focus equations and Chebyshev criterion [18,19]. The later is a generalization of Li and Zhang's criterion [20] for determining the Chebyshev property of two Abelian integrals. The advantage of using the criterion is to change the complicated geometric study to a purely algebraic analysis.

A difficulty will arise if the Hamiltonian has degree more than 4 without symmetry, implying that there will be more than three generating elements in $\mathbb{I}(h)$. Thus, the Picárd-Focus equations and Riccati equations have higher dimensions, which increases difficulty in investigating the intersection of the related plane and surface. Many results have been obtained for the least upper bounds on the number of zeros of $\mathbb{I}(h)$ by Chebyshev criterion, see [21–28] for type $(5, 4)$ without symmetry and [29–31] for type $(7, 6)$ with symmetry. However, it is noted that the upper bounds, obtained in almost all above mentioned results, are not the exact upper bounds or sharp bounds. Therefore, the sharp bounds are still open, even for $\mathbb{I}(h)$ of type $(4, 3)$ except one case to be discussed below. The type $(4, 3)$ of system (3) is the following perturbed Hamiltonian system of degree 4,

$$\dot{x} = y, \quad \dot{y} = \mu x(x-1)(x-\alpha)(x-\beta) + \varepsilon(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)y, \tag{5}$$

where $\mu = \pm 1$, $(\alpha, \beta) \in \mathbb{R}^2$. The unperturbed system (5) has 11 cases according to the outside boundaries of the period annulus, determined by the values of α and β, see [7]. We would not list all 11 cases of the topological classification except the following cases that have results on zeros of Abelian integrals:

(I) a cusp-saddle cycle ($\alpha = 1$ and $\beta = -\frac{2}{3}$, $\mu = -1$),
(II) a nilpotent-saddle loop ($\alpha = \beta = 1$, $\mu = 1$),
(III) a saddle loop surrounding a nilpotent center ($\alpha = \beta = 0$, $\mu = 1$),
(IV) a saddle loop with a cusp outside but near the saddle ($\alpha = -1$, $\beta < -1$, $\mu = 1$).

For the cases (I), (II) and (III), it has been proved that 4, 4 and 5 are respectively their least upper bounds. However, only 3 zeros have been obtained for the three cases, see the reports in [21–26]. The bounds on the number of zeros of Abelian integrals for the cases (I) and (II) were first investigated in [21,24], in which the verification of Chebyshev property was based on numerical computation. It was pointed out in [23] that 3 was not reliable to be considered as the sharp bound and it was claimed in [23,25] that a least upper bound on the number of zeros of the Abelian integral is 4 based upon interval analysis and symbolic computation. Therefore, whether 3 or 4 is the sharp bound for the two cases (I) and (II) is still unknown.

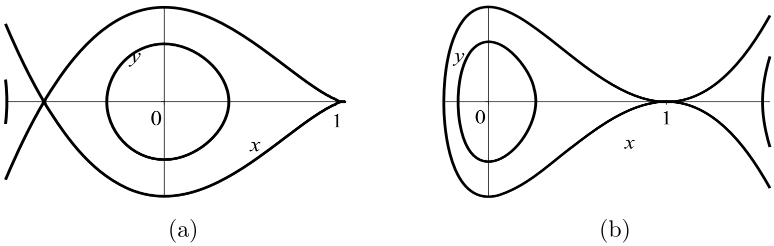

Fig. 1. Phase portraits of system (5) showing (a) a cusp-saddle cycle for $\alpha = 1$, $\beta = -\frac{2}{3}$, $\mu = -1$, and (b) a nilpotent-saddle loop for $\alpha = \beta = 1$, $\mu = 1$.

For case (I), the corresponding Hamiltonian of system $(5)_{\varepsilon=0}$ with $\alpha = 1$, $\beta = -\frac{2}{3}$ and $\mu = -1$ is

$$\mathcal{H}(x, y) = \frac{y^2}{2} + \frac{x^2}{3} - \frac{x^3}{9} - \frac{x^4}{3} + \frac{x^5}{5}. \tag{6}$$

The phase portrait corresponding to $\mathcal{H} = h$ for $h \in (0, \frac{4}{45})$ and $x \in (-\frac{2}{3}, 1)$, is given in Fig. 1(a), showing a family of closed orbits Γ_h surrounded by a heteroclinic cycle $\Gamma_{\frac{4}{45}}$, connecting a hyperbolic saddle at $(-\frac{2}{3}, 0)$ and a nilpotent cusp of order 1 at $(1, 0)$. The corresponding Abelian integral is given by

$$\mathcal{A}(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h) + \alpha_3 I_3(h), \tag{7}$$

where

$$I_i(h) = \oint_{\Gamma_h} x^i y dx, \quad i = 0,\ 1,\ 2,\ 3. \tag{8}$$

For case (II), the Hamiltonian of system $(5)_{\varepsilon=0}$ with $\alpha = \beta = 1$ and $\mu = 1$ is

$$\mathcal{H}^*(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - x^3 + \frac{3x^4}{4} - \frac{x^5}{5}. \tag{9}$$

The phase portrait corresponding to $\mathcal{H}^* = h$ for $h \in (0, \frac{1}{20})$ and $x \in (-\frac{1}{4}, 1)$, is depicted in Fig. 1(b), indicating a family of closed orbits L_h surrounded by a homoclinic loop $L_{\frac{1}{20}}$, with a nilpotent saddle of order 1 at $(1, 0)$. Similarly, we obtain the Abelian integral for this case,

$$\mathcal{M}(h) = \alpha_0 \overline{I}_0(h) + \alpha_1 \overline{I}_1(h) + \alpha_2 \overline{I}_2(h) + \alpha_3 \overline{I}_3(h), \tag{10}$$

where

$$\overline{I}_i(h) = \oint_{L_h} x^i y dx, \quad i = 0,\ 1,\ 2,\ 3. \tag{11}$$

In this work, we will prove that the sharp bound on the number of zeros of $\mathcal{A}(h)$ and $\mathcal{M}(h)$ is 3. The main results are stated in the following two theorems.

Theorem 1. *The Abelian integral $\mathcal{A}(h)$ for Case (I) of system (5) has at most 3 zeros on $(0, \frac{4}{45})$ for all possible $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in R^4$, and this is the sharp bound.*

Theorem 2. *The Abelian integral $\mathcal{M}(h)$ for Case (II) of system (5) has at most 3 zeros on $(0, \frac{1}{20})$ for all possible $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in R^4$, and this is the sharp bound.*

The main mathematical tools that we will apply to prove the two theorems are the Chebyshev criterion and asymptotic property of the Abelian integrals. However, we will not directly apply the Chebyshev criterion to $\{I_i(h),\ i = 0, 1, 2, 3\}$ and $\{\overline{I}_i(h),\ i = 0, 1, 2, 3\}$, since that leads to an upper bound 4, see [22–25]. Instead, we combine the generating elements $\{I_i(h)\}$ or $\{\overline{I}_i(h)\}$ $(i = 0, 1, 2, 3)$ and treat one perturbation parameter as a parameter in the algebraic Chebyshev systems. The range of this parameter is then bounded to yield a bounded 3-dimensional parameter set via three different combinations, on which a further analysis is given to exclude the possibility of 4 zeros of the Abelian integrals. Properly combining the generating elements plays a crucial role in obtaining the sharp bound. The detailed proof is only given for Theorem 1, since Theorem 2 can be similarly proved.

The rest of this paper is organized as follows. In section 2, we present expansions of Abelian integrals near the centers and briefly introduce the Chebyshev criterion. The proof of Theorem 1 is given in section 3, and an outline for proving Theorem 2 is given in section 4. Conclusion is drawn in section 5.

2. Asymptotic expansions and Chebyshev criterion

The asymptotic expansions of Abelian integrals are proposed to study its zeros near the endpoints of the annuluses, and these zeros correspond to limit cycles near the centers, homoclinic loops and heteroclinic loops, see a survey article [32]. In our work, we will use it to study the dynamics of the Abelian integrals on the whole period annulus.

2.1. Asymptotic expansions of $\mathcal{A}(h)$ and $\mathcal{M}(h)$ near the centers

Near the center $(x, y) = (0, 0)$, $\mathcal{A}(h)$ and $\mathcal{M}(h)$ have the following expansions (see [33]):

$$\mathcal{A}(h) = \sum_{i\ge 0} b_i h^{i+1} \qquad \text{and} \qquad \mathcal{M}(h) = \sum_{i\ge 0} \overline{b}_i h^{i+1}, \tag{12}$$

for $0 < h \ll 1$. The coefficients of b_i and $\overline{b}_i$ can be obtained by using the program developed in [33] as

$$b_0 = \sqrt{6}\pi\, \alpha_0,\ b_1 = \frac{\sqrt{6}\pi}{32}\,(41\,\alpha_0 + 12\,\alpha_1 + 24\,\alpha_2)$$
$$b_2 = \frac{\sqrt{6}\pi}{3072}(17017\alpha_0 + 5736\alpha_1 + 10320\alpha_2 + 2880\alpha_3)$$

and

$$\overline{b}_0 = 2\pi\,\alpha_0,\ \overline{b}_1 = \frac{\pi}{4}\,(21\,\alpha_0 + 12\,\alpha_1 + 4\,\alpha_2)\,,$$
$$\overline{b}_2 = \frac{\pi}{32}\,(1379\,\alpha_0 + 872\,\alpha_1 + 440\,\alpha_2 + 160\,\alpha_3)\,.$$

Using the expansions in (12), we can easily find the limit of the ratios of two integrals such as $\lim\limits_{h\to 0}\frac{I_2(h)}{I_0(h)}$. They will be used in our proof.

2.2. Chebyshev criterion

In this subsection, we present some results on Chebyshev criterion, which are needed for proving our main theorems.

Definition 3. Suppose the analytic functions $l_0(x), l_1(x)$ and $l_{m-1}(x)$ are defined on a real open interval J.

(A) The continuous Wronskian of $\{l_0(x), l_1(x), \dots, l_{i-1}(x)\}$ for $x \in J$ is

$$W[l_0(x), l_1(x), \dots, l_{i-1}(x)] = \begin{vmatrix} l_0(x) & l_1(x) & \cdots & l_{i-1}(x) \\ l_0'(x) & l_1'(x) & \cdots & l_{i-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ l_0^{(i-1)}(x) & l_1^{(i-1)}(x) & \cdots & l_{i-1}^{(i-1)}(x) \end{vmatrix},$$

where $l_i^{(j)}(x)$ is the jth order derivative of $l_i(x)$, $j \ge 2$.

(B) The set $\{l_0(x), l_1(x), \dots, l_{m-1}(x)\}$ is called a Chebyshev system if any nontrivial linear combination,

$$\kappa_0 l_0(x) + \kappa_1 l_1(x) + \cdots + \kappa_{m-1} l_{m-1}(x),$$

has at most $m-1$ isolated zeros on J, while $W[l_0(x), l_1(x), \dots, l_{m-1}(x)] \ne 0$ is one sufficient condition assuring $\{l_0(x), l_1(x), \dots, l_{m-1}(x)\}$ forms a Chebyshev system.

(C) The ordered set $\{l_0(x), l_1(x), \dots, l_{m-1}(x)\}$ is called extended complete Chebyshev system (ECT-system) if for each $i \in \{1, 2, \cdots, m\}$ any nontrivial linear combination,

$$\kappa_0 l_0(x) + \kappa_1 l_1(x) + \cdots + \kappa_{i-1} l_{i-1}(x),$$

has at most $i-1$ zeros with multiplicities accounted.

Let $H(x, y) = \mathbb{U}(x) + \frac{y^2}{2}$ be an analytic function. Assume there exists a punctured neighborhood $\mathcal{N}$ of the origin $(0, 0)$ foliated by closed curves $\Gamma_h \subseteq \{(x, y)|H(x, y) = h,\ \ h \in (0, h^*),\ h^* = H(\partial\mathcal{N})\}$. The projection of $\mathcal{N}$ on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$, and $x\mathbb{U}'(x) > 0$ for all $x \in (x_l, x_r)\backslash\{0\}$. $\mathbb{U}(x) = \mathbb{U}(z(x))$ defines an analytic involution $z = z(x)$ for all $x \in (x_l, x_r)$. Let

$$\mathbb{I}_i(h) = \oint_{\Gamma_h} \eta_i(x) y^{2s-1} dx, \ \text{for } h \in (0, h^*), \tag{13}$$

where $s \in \mathbb{N}$ and $\eta_i(x)$ is analytic functions on (x_l, x_r), $i = 0, 1, \dots, m-1$. Further, define

$$l_i(x) := \frac{\eta_i(x)}{\mathbb{U}'(x)} - \frac{\eta_i(z(x))}{\mathbb{U}'(z(x))}. \tag{14}$$

Then, we have

Lemma 4 *([18]). Consider the integrals* (13) *and the functions* (14). $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *is an ECT system on* $(0, h^*)$ *if* $s > m-2$ *and* $\{l_0, l_1, \cdots, l_{m-1}\}$ *is an ECT system on* $(x_l, 0)$ *or* $(0, x_r)$.

Lemma 5 *([19]). Consider the integrals* (13) *and the functions* (14). *If the following conditions hold:*

(a) $W[l_0, l_1, \ldots, l_i]$ *does not vanish on* $(0, x_r)$ *for* $i = 0, 1, \cdots, m-2$,
(b) $W[l_0, l_1, \ldots, l_{m-1}]$ *has* k *zeros on* $(0, x_r)$ *with multiplicities counted, and*
(c) $s > m + k - 2$,

then any nontrivial linear combination of $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *has at most* $m + k - 1$ *zeros on* $(0, h^*)$ *with multiplicities counted. In this case, we call* $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *a Chebyshev system with accuracy* k *on* $(0, h^*)$.

3. Proof of Theorem 1

3.1. Partition of the parameter space

In this section, we divide the parameter space for $\mathcal{A}(h)$ to obtain a subset which is the only set for $\mathcal{A}(h)$ to might have 4 zeros on $(0, \frac{4}{45})$. We write $\mathcal{H}(x, y) = \frac{y}{2} + U(x)$. Then, $q(x, z) = \frac{U(x)-U(z)}{x-z} = 0$ defines the involution $z(x)$, $x \in (0, 1)$ on the period annulus. We have the following result.

Lemma 6. *The following equations hold:*

$$8h^3 I_i(h) = \oint_{\Gamma_h} S_i(x) y^7 dx \equiv \widetilde{I}_i(h), \quad i = 0,\ 1,\ 2,\ 3,$$

where $S_i(x) = \frac{x^i g_i(x)}{354375(2+3x)^6(x-1)^9}$, *in which each polynomial* $g_i(x)$ *has degree* 15.

Proof. First, multiplying $I_i(h)$ by $\frac{y^2+2U(x)}{2h} = 1$ yields

$$\begin{aligned} 8h^3 I_i(h) &= \oint_{\Gamma_h} (2U(x) + y^2)^3 x^i y dx \\ &= \oint_{\Gamma_h} 8x^i U^3(x) y dx + \oint_{\Gamma_h} 12 x^i U^2(x) y^3 dx \\ &+ \oint_{\Gamma_h} 6x^i U(x) y^5 dx + \oint_{\Gamma_h} x^i y^7 dx,\ i = 0, 1, 2, 3. \end{aligned} \tag{15}$$

Then, applying Lemma 4.1 in [18] to (15) to increase the power of y in the first three integrals to 7 proves the lemma. □

Without loss of generality, we assume that $\alpha_3 = 1$ when $\alpha_3 \neq 0$. Further, introduce the following combinations:

$$\begin{aligned}\mathcal{I}_{23}(h) &= \oint_{\Gamma_h} \left(\alpha_2 x^2 + x^3\right) y dx,\\ \mathcal{I}_{13}(h) &= \oint_{\Gamma_h} \left(\alpha_1 x + x^3\right) y dx,\\ \mathcal{I}_{03}(h) &= \oint_{\Gamma_h} \left(\alpha_0 + x^3\right) y dx.\end{aligned} \tag{16}$$

Then,

$$\begin{aligned}\mathcal{A}(h) &= \alpha_0 I_0(h) + \alpha_1 I_1(h) + \mathcal{I}_{23}(h)\\ &= \alpha_0 I_0(h) + \alpha_2 I_2(h) + \mathcal{I}_{13}(h)\\ &= \alpha_1 I_1(h) + \alpha_2 I_2(h) + \mathcal{I}_{03}(h).\end{aligned}$$

The following lemma directly follows Lemma 6.

Lemma 7. *The following equations hold:*

$$\begin{aligned}8h^3\mathcal{I}_{23}(h) &= \oint_{\Gamma_h} (\alpha_2 S_2(x) + S_3(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{23}(h),\\ 8h^3\mathcal{I}_{13}(h) &= \oint_{\Gamma_h} (\alpha_1 S_1(x) + S_3(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{13}(h),\\ 8h^3\mathcal{I}_{03}(h) &= \oint_{\Gamma_h} (\alpha_0 S_0(x) + S_3(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{03}(h).\end{aligned}$$

Now, let

$$\begin{aligned}L_i(x) &= \left(\frac{S_i}{U'}\right)(x) - \left(\frac{S_i}{U'}\right)(z(x)),\\ \mathcal{L}_{i3}(x) &= \left(\frac{\alpha_i S_i + S_3}{U'}\right)(x) - \left(\frac{\alpha_i S_i + S_3}{U'}\right)(z(x)).\end{aligned} \tag{17}$$

Then,

$$\begin{aligned}\frac{d}{dx} L_i(x) &= \frac{d}{dx}\left(\frac{S_i}{U'}\right)(x) - \frac{d}{dz}\left[\left(\frac{S_i}{U'}\right)(z(x))\right] \times \frac{dz}{dx},\\ \frac{d}{dx}\mathcal{L}_{i3}(x) &= \frac{\partial}{\partial x}(\mathcal{L}_{i3}(x)) + \frac{\partial}{\partial z}(\mathcal{L}_{i3}(x)) \times \frac{dz}{dx},\end{aligned}$$

where $\frac{dz}{dx} = -\frac{q_x(x,z)}{q_z(x,z)}$. Direct computations yield

$$
\begin{aligned}
&W[L_0](x) = \frac{(x-z)\,Q_0(x,z)}{118125\,xz(2+3\,x)^7(x-1)^{11}(2+3\,z)^7(z-1)^{11}},\\
&W[L_1](x) = \frac{(x-z)\,Q_1(x,z)}{118125\,(2+3\,x)^7(x-1)^{11}(2+3\,z)^7(z-1)^{11}},\\
&W[L_0(x), L_1(x)] = \frac{(x-z)^3\,Q_{01}(x,z)}{13953515625x^2z^2(z-1)^{22}(3\,z+2)^{13}(x-1)^{22}(3\,x+2)^{13}\,P_0(x,z)},\\
&W[L_0(x), L_2(x)] = \frac{(x-z)^3\,Q_{02}(x,z)}{13953515625x^2z^2(z-1)^{22}(3\,z+2)^{13}(x-1)^{22}(3\,x+2)^{13}\,P_0(x,z)},\\
&W[L_1(x), L_2(x)] = \frac{(x-z)^3\,Q_{12}(x,z)}{13953515625x^2z^2(z-1)^{22}(3\,z+2)^{13}(x-1)^{22}(3\,x+2)^{13}\,P_0(x,z)},\\
&W[L_0(x), L_1(x), \mathcal{L}_{23}] = \frac{(x-z)^6M_1(x,z,\alpha_2)}{c^*x^3z^3(x-1)^{32}(3\,x+2)^{18}(z-1)^{32}(3\,z+2)^{18}\,P_0^3(x,z)},\\
&W[L_0(x), L_2(x), \mathcal{L}_{13}] = \frac{(x-z)^6M_2(x,z,\alpha_1)}{c^*x^3z^3(x-1)^{32}(3\,x+2)^{18}(z-1)^{32}(3\,z+2)^{18}\,P_0^3(x,z)\alpha_1},\\
&W[L_1(x), L_2(x), \mathcal{L}_{03}] = \frac{(x-z)^6M_3(x,z,\alpha_0)}{c^*x^3z^3(x-1)^{32}(3\,x+2)^{18}(z-1)^{32}(3\,z+2)^{18}\,P_0^3(x,z)\alpha_0},
\end{aligned}
\tag{18}
$$

where Q_0, Q_1, Q_{01}, Q_{02} and Q_{12} are polynomials of degree 34, 33, 66, 67 and 64, respectively, $c^* = 1648259033203125$, and $z = z(x)$ is determined by $q(x,z) = 0$, and

$$P_0(x,z) = 9\,x^3 + 18\,x^2z + 27\,xz^2 + 36\,z^3 - 15\,x^2 - 30\,xz - 45\,z^2 - 5\,x - 10\,z + 15.$$

Applying Sturm's Theory to the resultant between $q(x,z)$ and $P_0(x,z)$ with respect to z shows that the resultant has no roots for $x \in (0,1)$, which implies that $P_0(x,z)$ does not vanish for $x \in (0,1)$. Hence, the Wronskians are well defined.

The following result indicates that we only need to discuss the case when $\alpha_3 \neq 0$.

Proposition 8. *When $\alpha_3 = 0$, $\mathcal{A}(h)$ has at most 2 zeros on $(0, \frac{4}{45})$.*

The proof of Proposition 8 relies on computing and verifying the non-vanishment of Wronskians $W[L_0]$, $W[L_0, L_1]$ and $W[L_0, L_1, L_2]$, and then the application of Lemma 5. Since the computation and verification are straightforward, we omit the proof here for briefness.

To prove Theorem 1, we need to show non-vanishing of certain numerators and denominators of the related Wronskians in (18) for $x \in (0,1)$. Taking the numerator $Q_{01}(x,z)$ of the Wronskian $W[L_0, L_1]$ for example, we only need to prove that the two-dimensional system $\{Q_{01}(x,z), q(x,z)\}$ does not vanish on $\{(x,z) | -\frac{2}{3} < z < 0 < x < 1\}$, because z in $Q_{01}(x,z)$ is determined by $q(x,z) = 0$, and $z(x) \in (-\frac{2}{3}, 0)$ when $x \in (0,1)$. To do this, we apply triangular-decomposition and root isolating to $\{Q_{01}(x,z), q(x,z)\}$ to decompose the nonlinear system into several triangular systems, and then isolate the roots of each triangular-decomposed system. Since all roots of these triangular systems are the roots of the original system $\{Q_{01}(x,z), q(x,z)\}$, we only need to check if these decomposed systems have roots on $\{(x,z) | -\frac{2}{3} < z < 0 < x < 1\}$. This idea has been successfully applied to determine the zeros of Abelian integrals, see [25–27, 34]. Instead of the triangular-decomposition method, one may also use the interval analysis [23], which computes two resultants between $Q_{01}(x,z)$ and $q(x,z)$ with respect x and z, respectively, yielding several two dimensional regions. Finally, one verifies if $Q_{01}(x,z)$ vanishes on these regions by determining the intersection of the curves $Q_{01}(x,z)$ and $q(x,z)$, see [23] for details.

By applying the triangular-decomposition and root isolating to the numerators of the Wronskians, we obtain the following result.

Lemma 9. *Each of the Wronskians, $W[L_0]$, $W[L_1]$, $W[L_0, L_1]$, $W[L_0, L_2]$ and $W[L_1, L_2]$, does not vanish for $x \in (0, 1)$.*

Next, we investigate the last three Wronskians in (18). Their numerators have the forms,

$$\begin{aligned} M_1(x,z,\alpha_2) &= \alpha_2\beta_2(x,z) - \beta_1(x,z), \\ M_2(x,z,\alpha_1) &= \alpha_1\gamma_2(x,z) - \gamma_1(x,z), \\ M_3(x,z,\alpha_0) &= \alpha_0\delta_2(x,z) - \delta_1(x,z), \end{aligned}$$

where β_1, β_2, γ_1, γ_2, δ_1 and δ_2 are polynomials of degrees 98, 97, 99, 97, 100 and 97, respectively. Note that in addition to the variables x and z, M_i contains α_{3-i} which are arbitrary real constants. In order to satisfy $M_i = 0$, we treat α_{3-i} as a function in x and z determined from $M_i = 0$, and then investigate the property of these functions. Thus, $M_i(x, z, \alpha_{3-i}) = 0$ $(i = 1, 2, 3)$ defines three functions:

$$\alpha_2(x,z) = \frac{\beta_1(x,z)}{\beta_2(x,z)}, \quad \alpha_1(x,z) = \frac{\gamma_1(x,z)}{\gamma_2(x,z)}, \quad \alpha_0(x,z) = \frac{\delta_1(x,z)}{\delta_2(x,z)},$$

and their derivatives with respect to x are given by

$$\begin{aligned} \overline{\alpha}_2(x,z) &= \frac{\partial\alpha_2(x,z)}{\partial x} + \frac{\partial\alpha_2(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\overline{\beta}_1(x,z)}{\overline{\beta}_2(x,z)}, \\ \overline{\alpha}_1(x,z) &= \frac{\partial\alpha_1(x,z)}{\partial x} + \frac{\partial\alpha_1(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\overline{\gamma}_1(x,z)}{\overline{\gamma}_2(x,z)}, \\ \overline{\alpha}_0(x,z) &= \frac{\partial\alpha_0(x,z)}{\partial x} + \frac{\partial\alpha_0(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\overline{\delta}_1(x,z)}{\overline{\delta}_2(x,z)}. \end{aligned}$$

The denominators $\beta_2(x,z)$, $\gamma_2(x,z)$, $\delta_2(x,z)$, $\overline{\beta}_2(x,z)$, $\overline{\gamma}_2(x,z)$ and $\overline{\delta}_2(x,z)$ do not vanish for $x \in (0, 1)$, because they do not have common roots with $q(x,z)$ for $(x,z) \in (0,1) \times (-\frac{2}{3}, 0)$ by triangular-decomposition and root isolating. Hence, all of the functions $\alpha_i(x,z)$ and $\overline{\alpha}_i(x,z)$ $(i = 2, 1, 0)$ are well defined for $x \in (0, 1)$.

We have the following lemma.

Lemma 10.

(i) *$\alpha_2(x, z(x))$ is decreasing from $(0, \frac{5}{2})$ to a minimum (x^*, α_2^*) and then increasing to $(1, -\frac{4}{3})$;*

(ii) *$\alpha_1(x, z(x))$ is increasing from $(0, -5)$ to a maximum $(x^\dagger, \alpha_2^\dagger)$ and then decreasing to $(1, -\frac{1}{3})$;*

(iii) *$\alpha_0(x, z(x))$ is increasing from $(0, 0)$ to a maximum $(x^\ddagger, \alpha_2^\ddagger)$ and then decreasing to $(1, \frac{2}{3})$, where*

$$x^*, x^\dagger, x^\ddagger \in \Big[\underbrace{\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}}_{1/10^{10}}\Big],$$

and

$$\alpha_2^* \in \underbrace{\Big[-\frac{64307\ldots30528}{32922\ldots64125}, -\frac{44191\ldots26125}{22624\ldots29984}\Big]}_{1/10^6} \approx [-1.95327706, -1.95327692],$$

$$\alpha_1^\dagger \in \underbrace{\Big[\frac{11707\ldots14473}{17176\ldots36512}, \frac{84772\ldots81309}{12437\ldots49248}\Big]}_{1/10^7} \approx [0.06815643778, 0.06815645227],$$

$$\alpha_0^\ddagger \in \underbrace{\Big[\frac{68092\ldots44807}{73774\ldots11552}, \frac{78889\ldots10781}{85472\ldots94528}\Big]}_{1/10^7} \approx [0.9229843148, 0.9229843212].$$

Proof. We only prove case (i), since the cases (ii) and (iii) can be proved similarly. A direct computation shows that

$$\lim_{x\to 0} \alpha_2(x, z(x)) = \frac{5}{2}, \qquad \lim_{x\to 1} \alpha_2(x, z(x)) = -\frac{4}{3}.$$

On $\{(x,z)|-\frac{2}{3} < z < 0 < x < 1\}$, $\overline{\beta}_1(x,z)$ and $q(x,z)$ have a unique common root $(x^*, z^*) \in D_0$, where

$$D_0 = \Big[\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}\Big] \times \Big[-\frac{41095301255}{68719476736}, -\frac{82190602509}{137438953472}\Big].$$

x^* is the unique simple zero of $\overline{\alpha}_2(x, z(x))$ by verifying that $\frac{d}{dx}\overline{\alpha}_2(x, z(x))$ has no zeros in $[\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}]$. Therefore, x^* is the unique critical point of $\alpha_2(x, z(x))$, and thus the monotonicity of $\alpha_2(x, z(x))$ on $(0, x^*) \bigcup (x^*, 1)$ can be easily determined by comparing the values of $\alpha_2(x, z(x))$ at $x = 0$, x^*, 1 as $\frac{5}{2}$, -1.953277, $-\frac{4}{3}$. Alternatively, using

$$\lim_{x\to 0^+} \overline{\alpha}_2(x, z(x)) = 0^-, \quad \text{and} \quad \lim_{x\to 0^+} \frac{d}{dx}(\overline{\alpha}_2(x, z(x))) = -\frac{441}{40} < 0,$$

we know that $\alpha_2(x, z(x))$ is monotonically deceasing on $(0, x^*)$ and monotonically increasing on $(x^*, 1)$.

It can be further shown that the resultant between $\frac{\partial \beta_i(x,z)}{\partial x}$ (for $i = 1, 2$) and $q(x,z)$ with respect to z has no roots in the interval $[\frac{108804604063}{137438953472}, \frac{3400143877}{4294967296}]$ by Sturm's Theorem. Hence, $\beta_i(x,z)$ $(i = 1, 2)$ reaches its maximal and minimum values at the boundaries of D_0. Direct computation gives

$$\begin{aligned}
\min_{D_0} \beta_1(x,z) &= -\tfrac{44292\ldots44113}{16265\ldots68512} \approx -2.723190846 \times 10^{13},\\
\max_{D_0} \beta_1(x,z) &= -\tfrac{13257\ldots78375}{48683\ldots66176} \approx -2.723190757 \times 10^{13},\\
\min_{D_0} \beta_2(x,z) &= \tfrac{98768\ldots92375}{70844\ldots04416} \approx 1.394165169 \times 10^{13},\\
\max_{D_0} \beta_2(x,z) &= \tfrac{52797\ldots66369}{37870\ldots34272} \approx 1.394165228 \times 10^{13}.
\end{aligned}$$

Then, we obtain

$$\alpha_2^* = \alpha_2(x^*, z(x^*)) \in \left[\frac{\min\limits_{D_0}\beta_1(x,z)}{\min\limits_{D_0}\beta_2(x,z}, \frac{\max\limits_{D_0}\beta_1(x,z)}{\max\limits_{D_0}\beta_2(x,z)}\right]$$
$$= \underbrace{\left[-\frac{64307\ldots30528}{32922\ldots64125}, -\frac{44191\ldots26125}{22624\ldots29984}\right]}_{1/10^6}$$
$$\approx [-1.95327706, -1.95327692]. \quad \square$$

Note in the above proof that the exact rational numbers are obtained from symbolic computation, demonstrating the accuracy of computation. It is also noted that the critical point $(x^*, \alpha_2(x^*, z(x^*)))$ divides the curve $\{(x, \alpha_2(x, z(x)))|0 < x < 1\}$ into two simple segments (curves). The points on the two curves correspond to the simple root of $M_1(x, z(x), \alpha_2(x, z(x)))$, while x^* is a root of multiplicity 2. The following lemma follows from Lemma 10.

Lemma 11. *For* $x \in (0, 1)$*, when* α_2 *belongs to the intervals* $[\alpha_2^*, -\frac{4}{3})$, $[-\frac{4}{3}, \frac{5}{2})$ *and* $(-\infty, \alpha_2^*)\bigcup[\frac{5}{2}, +\infty)$, $W[L_0, L_1, \mathcal{L}_{23}]$ *has* 2, 1 *and* 0 *roots with multiplicities counted, respectively.*

Combining Lemmas 9 and 11 and applying Lemma 5, we have the following result.

Proposition 12. $\mathcal{A}(h)$ *has at most* 4, 3, 2 *zeros in* $(0, \frac{4}{45})$ *when* α_2 *is located in the intervals* $[\alpha_2^*, -\frac{4}{3})$, $[-\frac{4}{3}, \frac{5}{2})$, *and* $(-\infty, \alpha_2^*)\bigcup[\frac{5}{2}, +\infty)$, *respectively.*

Similarly, we have

Proposition 13. $\mathcal{A}(h)$ *has at most* 4, 3, 2 *zeros in* $(0, \frac{4}{45})$ *when* α_1 *belongs to the intervals* $(-\frac{1}{3}, \alpha_1^\dagger]$, $(-5, -\frac{1}{3}]$, *and* $(-\infty, -5]\bigcup(\alpha_1^\dagger, +\infty)$, *respectively.*

Proposition 14. $\mathcal{A}(h)$ *has at most* 4, 3, 2 *zeros in* $(0, \frac{4}{45})$ *when* α_0 *is located in the intervals* $(\frac{2}{3}, \alpha_0^\ddagger]$, $(0, \frac{2}{3}]$, *and* $(-\infty, 0]\bigcup(\alpha_0^\ddagger, +\infty)$, *respectively.*

Define

$$D^\ddagger = \left\{(\alpha_0, \alpha_1, \alpha_2)|\alpha_0 \in \left(\frac{2}{3}, \alpha_0^\ddagger\right], \ \alpha_1 \in \left(-\frac{1}{3}, \alpha_1^\dagger\right], \ \alpha_2 \in \left[\alpha_2^*, -\frac{4}{3}\right)\right\}.$$

Then, Propositions 12, 13 and 14 imply that

Proposition 15. $\mathcal{A}(h)$ *may have* 4 *zeros only if* $(\alpha_0, \alpha_1, \alpha_2) \in D^\ddagger$.

3.2. Non-existence of 4 *zeros of* $\mathcal{A}(h)$ *on* $D^\ddagger$

Finally, we prove that $\mathcal{A}(h)$ cannot have 4 zeros when $(\alpha_0, \alpha_1, \alpha_2) \in D^\ddagger$. First, we have

Lemma 16. *For* $h \in (0, \frac{4}{45})$*, the following holds:*

(1) *the generating element* $I_0(h)$ *is positive;*

(2) *the ratio* $\frac{I_1(h)}{I_0(h)}$ *is increasing from* 0 *to* $\frac{2}{27}$*;*
(3) *the ratio* $\frac{I_2(h)}{I_0(h)}$ *is increasing from* 0 *to* $\frac{116}{891}$*; and*
(4) *the ratio* $\frac{I_3(h)}{I_0(h)}$ *is increasing from* 0 *to* $\frac{136}{3861}$.

Proof. By Green formula, $I_0(h) = \oint_{\Gamma_h} ydx = \iint_{\mathcal{D}} dxdy$, where $\mathcal{D}$ is the region bounded by Γ_h (periodic annulus), and therefore, $I_0(h) > 0$. The non-vanishing property of $W[L_0]$, $W[L_0, L_1]$, $W[L_0, L_2]$ and $W[L_0, L_3]$ proved in Lemma 9 implies that $\frac{I_1(h)}{I_0(h)}$, $\frac{I_2(h)}{I_0(h)}$ and $\frac{I_3(h)}{I_0(h)}$ are monotonic on $(0, \frac{4}{45})$. By the expansion of $\mathcal{A}(h)$ near $h = 0$, we have

$$\lim_{h\to 0}\frac{I_1(h)}{I_0(h)} = \lim_{h\to 0}\frac{I_2(h)}{I_0(h)} = \lim_{h\to 0}\frac{I_3(h)}{I_0(h)} = 0.$$

Taking the limit as $h \to \frac{4}{45}$ yields

$$\lim_{h\to\frac{4}{45}}\frac{I_1(h)}{I_0(h)} = \lim_{h\to\frac{4}{45}}\frac{\oint_{\Gamma_h} xydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{4}{45}}} xydx}{\oint_{\Gamma_{\frac{4}{45}}} ydx} = \frac{2}{27},$$

$$\lim_{h\to\frac{4}{45}}\frac{I_2(h)}{I_0(h)} = \lim_{h\to\frac{4}{45}}\frac{\oint_{\Gamma_h} x^2ydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{4}{45}}} x^2ydx}{\oint_{\Gamma_{\frac{4}{45}}} ydx} = \frac{116}{891},$$

and

$$\lim_{h\to\frac{4}{45}}\frac{I_3(h)}{I_0(h)} = \lim_{h\to\frac{4}{45}}\frac{\oint_{\Gamma_h} x^3ydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{4}{45}}} x^3ydx}{\oint_{\Gamma_{\frac{4}{45}}} ydx} = \frac{136}{3861}. \qquad \square$$

Proposition 17. $\mathcal{A}(h) > 0$ *for* $(\alpha_0, \alpha_1, \alpha_2) \in D^{\ddagger}$.

Proof. When $(\alpha_0, \alpha_1, \alpha_2) \in D^{\ddagger}$, by the results obtained in Lemma 16, it is easy to show that

$$\alpha_0 + \alpha_2\frac{I_2(h)}{I_0(h)} > \frac{2}{3} + \frac{116}{891}\alpha_2^* \geq \frac{2}{3} + \frac{116}{891} \times \left(-\frac{64307...30528}{32922...64125}\right) > \frac{2}{3} + \frac{116}{891} \times (-2) = \frac{362}{891},$$

and

$$\alpha_1\frac{I_1(h)}{I_0(h)} > -\frac{1}{3} \times \frac{2}{27} = -\frac{2}{81}.$$

Then, using the results for $\frac{I_3(h)}{I_0(h)}$ and $I_0(h)$ in Lemma 16, we have, for $h \in (0, \frac{4}{45})$, that

$$\begin{aligned}
\mathcal{A}(h) &= \left[\left(\alpha_0 + \alpha_2\frac{I_2(h)}{I_0(h)}\right) + \alpha_1\frac{I_1(h)}{I_0(h)} + \frac{I_3(h)}{I_0(h)}\right]I_0(h) \\
&> \left(\frac{362}{891} - \frac{2}{81} + 0\right)I_0(h) \\
&= \frac{340}{891}I_0(h) > 0.
\end{aligned}$$

So $\mathcal{A}(h)$ has no zeros for $(\alpha_0, \alpha_1, \alpha_2) \in D^\ddagger$. □

Proof of Theorem 1. Combining Propositions 8, 15 and 17 proves Theorem 1.

4. An outline of the proof for Theorem 2

Theorem 2 can be similarly proved as that for Theorem 1. Hence, we give an outline of the proof for Theorem 2. Similar to Propositions 8, 15 and Lemma 16, we have the following results.

Proposition 18. *When $\alpha_3 = 0$, $\mathcal{M}(h)$ has at most 2 zeros on $(0, \frac{1}{20})$.*

Proposition 19. *$\mathcal{M}(h)$ may have 4 zeros only if $(\alpha_0, \alpha_1, \alpha_2) \in D^*$, where*

$$D^* = \Big\{(\alpha_0, \alpha_1, \alpha_2) | \alpha_0 \in [\alpha_0^\ddagger, -1),\ \alpha_1 \in (3, \alpha_1^\dagger],\ \alpha_2 \in [\alpha_2^*, -3)\Big\}$$

with

$$\alpha_2^* \in \underbrace{\Big[-\frac{60508\cdots90001}{17339\cdots00000}, -\frac{55031\cdots00000}{15770\cdots80367}\Big]}_{1/10^9} \approx [-3.4896007790, -3.4896007772],$$

$$\alpha_1^\dagger \in \underbrace{\Big[\frac{61748\cdots00000}{15770\cdots80367}, -\frac{23327\cdots84791}{59578\cdots00000}\Big]}_{1/10^9} \approx [3.9154855416, 3.9154855436],$$

$$\alpha_0^\ddagger \in \underbrace{\Big[-\frac{23405\cdots41729}{16376\cdots00000}, -\frac{22538\cdots00000}{15770\cdots80367}\Big]}_{1/10^9} \approx [-1.4291710471, -1.4291710464].$$

Lemma 20. *For $h \in (0, \frac{1}{20})$, the ratio $\frac{I_1(h)}{I_0(h)}$ increases from 0 to $\frac{1}{6}$, and the ratio $\frac{I_3(h)}{I_2(h)}$ increases from 0 to $\frac{19}{39}$.*

Proposition 21. *$\mathcal{M}(h) < 0$ when $(\alpha_0, \alpha_1, \alpha_2) \in D^*$.*

Proof. When $(\alpha_0, \alpha_1, \alpha_2) \in D^*$, considering the above intervals expressed by fractions, $\frac{I_1(h)}{I_0(h)} \in (0, \frac{1}{6})$ and $\frac{I_3(h)}{I_2(h)} \in (0, \frac{19}{39})$ in Lemma 20, it is obvious that

$$\alpha_0 + \alpha_1 \frac{I_1(h)}{I_0(h)} < 0 \ \text{ and } \alpha_2 + \frac{I_3(h)}{I_2(h)} < 0.$$

By Green formula, $I_i(h) = \oint_{\Gamma_h} y dx = \iint_{D^\dagger} x^i dxdy$, where $D^\dagger$ is the region (periodic annulus) surrounded by Γ_h, therefore, $I_0(h) > 0$ and $I_2(h) > 0$. Hence, we have

$$\mathcal{M}(h) = \Big(\alpha_0 + \alpha_1 \frac{I_1(h)}{I_0(h)}\Big) I_0(h) + \Big(\alpha_2 + \frac{I_3(h)}{I_2(h)}\Big) I_2(h) < 0.$$

So $\mathcal{M}(h)$ has no zeros when $(\alpha_0, \alpha_1, \alpha_2) \in D^*$. □

Combining Propositions 18, 19 and 21 proves Theorem 2.

5. Conclusion

In this work, we have given a further investigation on the works [21–25] and proved that the sharp bound is 3 on the number of zeros of the Abelian integrals of system (3) for the cases with a cusp-saddle and a nilpotent-saddle loop. Previous works have obtained 3 zeros and an upper bound 4. Our approach narrows the parameters to a set which is the only set to possibly have 4 zeros of the Abelian integrals. Then, we rule out the possibility of 4 zeros and thus proved the sharp bound to be 3. This completely solved the upper bound problem for the cases (I) and (II). For case (III), it has been shown in [23,26] that a least upper bound is 5 but only 3 zeros have been obtained. We can apply the method developed in this paper to investigate case (III) and to show that a least upper bound is 4. However, whether 4 is the sharp bound for case (III) needs a further study.

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