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Cyclicity of periodic annulus and Hopf cyclicity in perturbing a hyper-elliptic Hamiltonian system with a degenerate heteroclinic loop

Xianbo Sun, Pei Yu *

Department of Applied Mathematics, Western University, London, Ontario, N6A 5B7, Canada

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Abstract

In this paper, we study the cyclicity of periodic annulus and Hopf cyclicity in perturbing a quintic Hamiltonian system. The undamped system is hyper-elliptic, non-symmetric with a degenerate heteroclinic loop, which connects a hyperbolic saddle to a nilpotent saddle. We rigorously prove that the cyclicity is 3 for periodic annulus when the weak damping term has the same degree as that of the associated Hamiltonian system. This result provides a positive answer to the open question whether the annulus cyclicity is 3 or 4. When the smooth polynomial damping term has degree n, first, a transformation based on the involution of the Hamiltonian is introduced, and then we analyze the coefficients involved in the bifurcation function to show that the Hopf cyclicity is $\left[\frac{2n+1}{3}\right]$. Further, for piecewise smooth polynomial damping with a switching manifold at the y-axis, we consider the damping terms to have degrees l and n, respectively, and prove that the Hopf cyclicity of the origin is $\left[\frac{3l+2n+4}{3}\right]$ ($\left[\frac{3n+2l+4}{3}\right]$) when $l \geq n$ ($n \geq l$).

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* Corresponding author.
E-mail address: pyu@uwo.ca (P. Yu).

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1. Introduction

Periodic motions appear in almost all natural and engineering dynamical systems. Determing the number of periodic solutions and their locations plays an important role in the study of dynamical systems. For example, in chemical reactions [18], it is important to determine what may cause oscillation and what may destroy oscillation, and what affects the period and amplitude of oscillation. However, it is not easy to determine all possible locations, periods and amplitudes even for the oscillations in a two-dimensional reactor. The relative open problem in mathematics is the well-known Hilbert's 16th problem [24], which considers the maximal number of limit cycles, denoted by $\mathbb{H}(n)$, and their distribution in two-dimensional polynomial systems. This problem is still not completely solved even for quadratic polynomial systems (i.e., for the simplest case $n = 2$). Many theories and methodologies have been developed for solving the problem, and a lot of good results such as lower bounds on $\mathbb{H}(n)$ have been obtained, see a recent paper [1].

In order to overcome the difficulty in solving the Hilbert's 16th problem, researchers have tried to study the relative weakened problems or weaker versions of the problem, for example, studying limit cycles arising from certain special bifurcations, or focusing on systems with simper forms. Anorld's version of Hilbert's 16th problem [2] is equivalent to studying limit cycles by investigating the first-order Poincaré bifurcation of the following perturbed Hamiltonian system,

$$\dot{x} = H_y(x, y) + \varepsilon P(x, y), \qquad \dot{y} = -H_x(x, y) + \varepsilon Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials of degree $n \geq 2$, $\varepsilon > 0$ is sufficiently small, $H(x, y)$ is a polynomial of degree $n + 1$ and has at least one family of closed orbits. Suppose the ovals, parameterized by $\{(x, y)|H(x, y) = h,\ h \in J\}$ where J is an open interval, are periodic orbits of system $(1)_{\varepsilon=0}$, forming a periodic annulus denoted by $\{\Gamma_h\}$. The number of zeros of the Abelian integral,

$$A(h) = \oint_{\Gamma_h} Q(x, y)dx - P(x, y)dy, \quad h \in J,$$

estimates the zeros of the return map that is constructed on the periodic annulus $\{\Gamma_h\}$. Therefore, the zeros of $A(h)$ provide the information on the persisting limit cycles of system (1) in the sense of the first order Poincaré bifurcation when ε is sufficiently small, see [23]. Studying the zeros of $A(h)$ is so-called the weak Hilbert's 16th problem, which has produced most of results on the Hilbert's 16th problem. However, even the weak version of the Hilbert's 16th problem is still very difficult to solve, and so far only the case $n = 2$ has been completely solved, see [8] and references therein.

Smale [33] proposed a simple version of the Hilbert's 16th problem based on the classical polynomial Liénard system,

$$\ddot{x} + f(x)\dot{x} + x = 0, \tag{2}$$

where $f(x)$ is a polynomial of degree n. Lins et al. [27] proved that system (2) has at most $\left[\frac{n}{2}\right]$ limit cycles for $n = 1, 2$ and conjectured that the result is true for all $n \geq 1$. Li and Llibre [26] proved that the conjecture is true for $n = 3$. However, in 2007, Dumortier et al. [17] proved that there exist systems which have at least $[\frac{n}{2}]+1$ limit cycles for even $n \geq 6$. Four years later, Maesschalck and Dumortier [11] proved that there exist systems that have at least $[\frac{n}{2}]+2$ limit cycles for $n \geq 5$. Maesschalck and Huzak [12] proved the results to $n-2$ for $n \geq 5$, which improved $[\frac{n}{2}]+2$ for $n \geq 9$. In short, up to now, the sharp bound on the maximal number of limit cycles of (2) is still unknown and the conjecture of Lins et al. is still open for $n=4$.

Recently, more interests have focused on the generalized Liénard system, which includes many various types of oscillators,

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{3}$$

where $g(x)$ and $f(x)$ are polynomials with degrees m and n, respectively, usually called type (m,n). In Newtonian mechanics, $f(x)$ is the damping term and $g(x)$ is the restoring or potential term. The sharp bound on the number of limit cycles of system (3) depends on the degrees m and n denoted by $\mathbb{H}_L(m,n)$, where L represents Liénard system. It is more difficult to determine $\mathbb{H}_L(m,n)$ for the generalized Liénard system (3) than that for system (2), because of the nonlinear restoring term $g(x)$. However, even when system (3) has a simple form, it still plays a very important role in studying limit cycles of general planar systems obtained from modifying (3), see [25,32]. Further, system (3) can be applied to model real world oscillating phenomena, see [10].

There are two different ways to study $\mathbb{H}_L(m,n)$. One way is to consider the limit cycles via Poincaré bifurcation by assuming the damping term in the form of $-\varepsilon f(x)y$. Then, system (3) becomes a special form of (1),

$$\dot{x} = y, \qquad \dot{y} = -g(x) + \varepsilon f(x)y, \tag{4}$$

with $P(x,y)=0$, $Q(x,y)=f(x)y$ and the Hamiltonian $H(x,y)=\frac{y^2}{2}+\int g(x)dx$. The corresponding Abelian integral is in the simple form,

$$\mathbb{I}(h) = \oint_{\Gamma_h} f(x)y\,dx.$$

It is known that the sharp bound on the maximal number of zeros of $\mathbb{I}(h)$ on a periodic annulus is the annulus cyclicity (for a concrete system) by Poincaré-Pontryagin-Andronov Theorem [23]. The cyclicity is denoted by $\mathbb{Z}_L(m,n)$ with L representing the Liénard system. However, even for the simple form, it is not easy to determine the cyclicity $\mathbb{Z}_L(m,n)$, which was only completely determined for type $(m,m-1)$ with $m=2,3$, see [13–16]. Type $(m,m-1)$ means that the perturbation term $-\varepsilon f(x)y$ and the restoring term $g(x)$ have the same degree. For type $(5,4)$ of system (4) with symmetry, since there are only three perturbation terms, the Picárd-Fuchs equation method can be applied. It has been proved that 2 is the sharp bound if the unperturbed system has a heteroclinic loop [3,4,37,44]. It becomes much more difficult when system (4) is non-symmetric or has degree equal to or larger than 4, implying that $\mathbb{I}(h)$ has more than 3 generating elements. Thus, the dimensions of the Picárd-Fuchs equation system and Ricatti equations are higher, which makes it trouble in determining the intersection of the related planes and surfaces. On the other hand, it has been shown that the Chybeshev criterion [19,31] can be applied to bound $\mathbb{Z}_L(m,n)$ for Abelian integrals with more than 3 elements, see [6,34,35, 40–42,45] for type $(4,3)$, but only an upper bound of $\mathbb{Z}_L(4,3)$ was obtained for each system investigated in these papers. Recently, Sun and Yu [38] improved the results by introducing a combination technique for two systems with a nilpotent singularity. There is no sharp bound reported for non-symmetric type $(5, 4)$ systems.

Another way to study limit cycles of the generalized Liénard system (3) is to investigate the small limit cycles bifurcating from Hopf singularities. The exact bound on the maximal number of small limit cycles due to Hopf bifurcation is usually called Hopf cyclicity. For convenience, we denote the Hopf cyclicity of system (3) by $\mathbb{H}_L^s(m,n)$, where s represents small limit cycles. There are lots of results on $\mathbb{H}_L^s(m,n)$ which were obtained by computing Lyapunov coefficients. In 2006, it was proved by Yu and Han [43] that $\mathbb{H}_L^s(4,n) = \mathbb{H}_L^s(n,4)$ and $\mathbb{H}_L^s(5,n) = \mathbb{H}^s(n,5)$ for $n = 10, 11, 12, 13$, and $\mathbb{H}_L^s(6,n) = \mathbb{H}_L^s(n,6)$ for $n = 5, 6$. Other exact values of $\mathbb{H}_L^s(m,n)$ for some fixed values of m and n were summarized in Table 1 of [29]. We have noticed that these results were mainly obtained by studying $g(x)$ in the form of $g(x) = -x + \varepsilon g_m(x)$ with $\deg g_m(x) = m$. In other words, they perturb a linear center. When the degrees m and n are not fixed, two better lower bounds on $\mathbb{H}_L^s(m,n)$ were estimated in [29] and [22]. The averaging method of order 1, 2 and 3 was applied in [29] to system (3) by assuming

$$(g(x), f(x)) = \Big(\sum_{k\ge1} \varepsilon^k g_m^k(x), \sum_{k\ge1} \varepsilon^k f_n^k(x)\Big),$$

while $g(x) = \bar{g}_m(x) + \varepsilon g_m(x)$ was taken in [22]. However, fewer results were reported on $\mathbb{H}_L^s(m,n)$ for arbitrary values of m or n. Up to now, we only know that, see [7,30],

$$\begin{aligned}
&\mathbb{H}_L^s(m,n) = \frac{n}{2}, \text{ if } g(x) \text{ is an odd degree polynomial;}\\
&\mathbb{H}_L^s(m,n) = \frac{n}{2}, \text{ if } f(x) \text{ is an even degree polynomial;}\\
&\mathbb{H}_L^s(m,2n+1) = \Big[\frac{m-2}{2}\Big] + n, \text{ if } f(x) \text{ is an odd degree polynomial;}\\
&\mathbb{H}_L^s(2m,2) = m, \text{ if } g(x) = x + g_e(x) \text{ with } g_e(x) \text{ being an even degree polynomial.}
\end{aligned}$$

The above results were obtained with a strict assumption on $f(x)$ or $g(x)$. Moreover, in the last three decades, few results were obtained with similar restrictions on the damping and restoring terms. The main difficulty comes from analyzing the dimension of the related algebraic variety of the set of the Lyapunov coefficients.

For fixed $m = 2$, it was proved respectively by Han [20,21], and Christopher and Lynch [9] that

$$\mathbb{H}_L^s(2,n) = \mathbb{H}_L^s(n,2) = \Big[\frac{2n+1}{3}\Big]$$

for all $n \ge 1$. When $m = 3$, it was proved in [9] that system (3) with $g(x) = -x(2 + 3x + 4bx^2)$ has the Hopf cyclicity at the origin,

$$\mathbb{H}_L^s(3,n) = \mathbb{H}_L^s(n,3) = 2\Big[\frac{3n+2}{8}\Big], \quad 1 \le n \le 50.$$

Recently, Tian et al. [39] studied an equivalent system to the one in [9] by taking $g(x) = -x(x - 2x + ax^2)$, and proved that the Hopf cyclicity near the origin is

$$\mathbb{H}_L^s(3,n) = \left[\frac{3n+2}{4}\right] \text{ for } n \geq 1 \text{ if } a = \frac{8}{9}.$$

It should be noted that when $m \geq 3$, system (3) may have richful topological phase portraits due to the complicated topological phase portraits of the system $\ddot{x} + g(x) = 0$, for example, there may exist more than one singularity of focus type except the singularity at the origin. Therefore, $\mathbb{H}_L^s(3,n)$ $(n \geq 1)$ only includes the number of small limit cycles bifurcating from the origin.

For fixed $m \geq 4$, however, there are no results reported on the Hopf cyclicity for any type (m,n) of system (3) with arbitrary $n \geq 1$ due to the difficulty arising from stronger nonlinear restoring term $g(x)$.

In this paper, we study a non-symmetric system (3) with $m = 5$. It has a unique singularity of center-focus type and the undamped system ($f(x) \equiv 0$) has a unique periodic annulus. This periodic annulus is bounded by a non-symmetric heteroclinic loop connecting a degenerate singularity. We study the annulus cyclicity for $f(x) = \varepsilon \sum_{i=0}^{4} \alpha_i x^i$ with $\varepsilon > 0$ sufficiently small, for which the damping term $\varepsilon f(x)y$ has the same degree as that of the undamped system. We also study the small limit cycles near the origin and determine the Hopf cyclicity when the damping term is an nth degree smooth polynomial or piecewise smooth polynomials. The system is given in the form of

$$\dot{x} = y, \quad \dot{y} = x(x-1)^3\left(x + \frac{1}{2}\right) + f(x)y, \tag{5}$$

where $f(x) = f_i(x)$ $(i = 1, 2, 3)$ with

$$f_1(x) = \varepsilon(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4),$$

$$f_2(x) = \sum_{i=0}^{n} \alpha_i x^i,$$

$$f_3(x) = \begin{cases} \sum_{i=0}^{n} \alpha_i^+ x^i, & x > 0, \\ \sum_{i=0}^{l} \alpha_i^- x^i, & x < 0, \end{cases}$$

where $\varepsilon > 0$ is sufficiently small, α_i and $\alpha_i^{\pm}$ are bounded parameters. The undamped system (5) is a Hamiltonian system with the Hamiltonian,

$$\mathcal{H}(x,y) = \frac{y^2}{2} + \frac{x^2}{4} - \frac{x^3}{6} - \frac{3x^4}{8} + \frac{x^5}{2} - \frac{x^6}{6}.$$

There is a family of closed orbits $\Gamma_h = \{(x,y) | \mathcal{H}(x,y) = h, h \in (0, \frac{1}{24})\}$, which forms a unique periodic annulus $\{\Gamma_h\}$ bounded by a degenerate heteroclinic loop, denoted by Γ^*. The hetero-

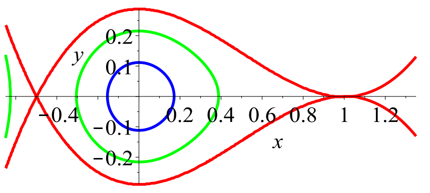

Fig. 1. The phase portrait of the undamped system (5).

clinic loop connects a hyperbolic saddle $(-\frac{1}{2}, 0)$ and a nilpotent saddle $(1, 0)$, see Fig. 1. For $f(x) = f_1(x)$, the associated Abelian integral is given by

$$\mathcal{A}(h) = \alpha_0 I_0(h) + \alpha_1 I_1(h) + \alpha_2 I_2(h) + \alpha_3 I_3(h) + \alpha_4 I_4(h), \tag{6}$$

where

$$I_i(h) = \oint_{\Gamma_h} x^i y dx, \quad i = 0,\ 1,\ 2,\ 3,\ 4. \tag{7}$$

As discussed above, the difficulty in studying the bifurcation of limit cycles of system (5) arises from the non-symmetry, stronger nonlinearity of $g(x)$ and the degeneracy, implying that one has to consider more than 3 generating elements in the Abelian integral for studying the annulus cyclicity, which requires more efficient computation in dealing with the damping terms and their independence for studying the Hopf cyclicity. In fact, system (5) with $f(x) = f_1(x)$ was first studied by Ashegh et al. [5], who claimed that there are at most three limit cycles bifurcating from the periodic annulus by analyzing the first order Poincaré bifurcation, and the three limit cycles can be obtained near the boundary of the annulus. The result implies that the cyclicity of the periodic annulus is three when $f(x) = f_1(x)$. However, the result was questionable because there exists a discrepancy between the symbolic computation and numerical analysis. As a matter of fact, two years later, Sun et al. [36] reconsidered the problem and provided a rigorous proof, but only an upper bound was obtained. For convenience, the results obtained in [5,36] are summarized in the following theorem.

Theorem 1 *([5,36]). For system* (5) *with* $f(x) = f_1(x)$*,*

(i) *there exist no closed orbits enclosing three singularities* $(-\frac{1}{2}, 0)$, $(0, 0)$ *and* $(1, 0)$ *for all possible bounded parameters* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^5$*;*
(ii) $\mathcal{A}(h)$ *has at most four zeros in* $(0, \frac{1}{24})$*, and three zeros can be reached near the endpoints of the interval* $(0, \frac{1}{24})$*, implying that there are at most four limit cycles bifurcating from the periodic annulus for sufficiently small* $\varepsilon > 0$*, and three limit cycles can be obtained either near the singularity* $(0, 0)$ *or near the heteroclinic loop.*

Therefore, it is still unknown whether the annulus cyclicity is three or four for system (5) when $f(x) = f_1(x)$. In this paper, we will provide a rigorous proof to give a positive answer on the exact cyclicity. This result is stated in the following theorem.

Theorem 2. *For system* (5) *with* $f(x) = f_1(x)$*, the Abelian integral* $\mathcal{A}(h)$ *has at most three zeros in* $(0, \frac{1}{24})$ *for all possible* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^5$*, and this is the sharp bound, i.e., the cyclicity of the periodic annulus is three.*

In addition, we study the Hopf cyclicity of the unique center-focus singularity at the origin for different types of damping terms. In particular, for smooth dampings, we have the result on the Hopf cyclicity of the origin as follows.

Theorem 3. *For system* (5) *with* $f(x) = f_2(x)$*, the Hopf cyclicity of the origin is* $\left[\frac{2n+1}{3}\right]$.

When the damping term is a piecewise smooth polynomial in x of degree l and n, we have the following result.

Theorem 4. *For system* (5) *with* $f(x) = f_3(x)$*, the Hopf cyclicity of the origin is* $\left[\frac{3n+2l+4}{3}\right]$ *if* $n \geq l$ *or* $\left[\frac{3l+2n+4}{3}\right]$ *if* $l \geq n$.

The main mathematical tools that we will apply to prove Theorem 2 are asymptotic property and Chebyshev criterion of the Abelian integrals $\{I_i(h)\}_{i=0}^4$. We will introduce three combinations of related two Abelian integrals to obtain three new integral systems including a parameter, and then apply the Chebyshev criterion to the new systems. The range of each parameter in the integral system is then bounded via the algebraic property of the curves and number of zeros of the algebraic system. The algebraic system is derived from the ratio of two Wronskians. The ranges of three parameters give a bounded 3-dimensional parameter set on which the full Abelian integral may have four zeros. A further analysis is carried out to exclude the possibility of 4 zeros of the Abelian integrals. Properly combing the generating elements plays a crucial role in obtaining the sharp bound, since directly applying Chebyshev criterion fails [5,36]. To prove Theorems 3 and 4, we properly utilize the potential in the undamped system (5) to define an involution, and then to introduce two transformations composed of trigonometric functions for the two components of the involution. This makes it possible to analyze the independence of the elements in algebraic variety, finally yielding the Hopf cyclicity for the smooth and non-smooth damping system (5).

The rest of this paper is organized as follows. In section 2, we present some preliminaries, which contain certain new theories and methods on Poincaré bifurcation and Hopf bifurcation, and an extended Chebyshev criterion. We prove Theorems 2, 3 and 4 in sections 3, 4 and 5, respectively. Conclusion is drawn in section 6.

2. Chebyshev criterion and local bifurcation theory

2.1. Chebyshev criterion

In this subsection, we briefly present the Chebyshev criterion developed in [19,31], which are one of the basic tools for proving our main results.

Definition 5. Suppose $s_0(x), s_1(x), \cdots$ and $s_{m-1}(x)$ are analytic functions on a real open interval Ω.

(A) The continuous Wronskian of $\{s_0(x), s_1(x), \ldots, s_{i-1}(x)\}$ for $x \in \Omega$ is

$$W[s_0(x), s_1(x), \ldots, s_{i-1}(x)] = \begin{vmatrix} s_0(x) & s_1(x) & \cdots & s_{i-1}(x) \\ s_0'(x) & s_1'(x) & \cdots & s_{i-1}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ s_0^{(i-1)}(x) & s_1^{(i-1)}(x) & \cdots & s_{i-1}^{(i-1)}(x) \end{vmatrix},$$

where $s_i^{(j)}(x)$ is the jth order derivative of $s_i(x)$, $j \geq 2$.

(B) The set $\{s_0(x), s_1(x), \ldots, s_{m-1}(x)\}$ is called a Chebyshev system if any nontrivial linear combination,

$$\kappa_0 s_0(x) + \kappa_1 s_1(x) + \cdots + \kappa_{m-1} s_{m-1}(x),$$

has at most $m-1$ isolated zeros on Ω, while $W[s_0(x), s_1(x), \ldots, s_{m-1}(x)] \neq 0$ is one sufficient condition assuring $\{s_0(x), s_1(x), \ldots, s_{m-1}(x)\}$ to form a Chebyshev system.

(C) The ordered set $\{s_0(x), s_1(x), \ldots, s_{m-1}(x)\}$ is called extended complete Chebyshev (ECT) system if for each $i \in \{1, 2, \cdots, m\}$ any nontrivial linear combination,

$$\kappa_0 s_0(x) + \kappa_1 s_1(x) + \cdots + \kappa_{i-1} s_{i-1}(x),$$

has at most $i-1$ zeros with multiplicities accounted.

Let $H(x, y) = V(x) + \frac{y^2}{2}$ be an analytic function with $xV'(x) > 0$ and $V(0) = 0$. There exists a family of closed ovals $\{\Gamma_h\} \subseteq \{(x, y) | H(x, y) = h,\ h \in (0, h^*)\}$ surrounding the origin $(0, 0)$, where $h^* = H(\partial\{\Gamma_h\})$. The projection of $\{\Gamma_h\}$ on the x-axis is an interval (x_l, x_r) with $x_l < 0 < x_r$, for all $x \in (x_l, x_r) \backslash \{0\}$. $V(x) = V(z(x))$ defines an analytic involution $z = z(x)$ for all $x \in (x_l, x_r)$. Let

$$\mathbb{I}_i(h) = \oint_{\Gamma_h} \xi_i(x) y^{2n^*-1} dx, \quad \text{for } h \in (0, h^*), \tag{8}$$

where $n^* \in \mathbb{N}$ and $\xi_i(x)$ is analytic in (x_l, x_r), $i = 0, 1, \ldots, m-1$. Further, define

$$s_i(x) := \frac{\xi_i(x)}{V'(x)} - \frac{\xi_i(z(x))}{V'(z(x))}. \tag{9}$$

Then, we have

Lemma 6 *([19]). Consider the integrals* $\mathbb{I}_i$ *in* (8) *and the functions* s_i *in* (9). $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *is an* ECT *system in* $(0, h^*)$ *if* $n^* > m-2$ *and* $\{s_0, s_1, \cdots, s_{m-1}\}$ *is an* ECT *system in* $(x_l, 0)$ *or* $(0, x_r)$.

Lemma 7 *([31]). Consider the integrals* (8) *and the functions* (9). *If the following conditions hold:*

(a) $W[s_0, s_1, \ldots, s_i]$ *does not vanish in* $(0, x_r)$ *for* $i = 0, 1, \cdots, m-2$,

(b) $W[s_0, s_1, \ldots, s_{m-1}]$ *has* k *zeros in* $(0, x_r)$ *with multiplicities counted, and*
(c) $n^* > m + k - 2$,

then, any nontrivial linear combination of $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *has at most* $m + k - 1$ *zeros in* $(0, h^*)$ *with multiplicities counted. In this case, we call* $\{\mathbb{I}_0, \mathbb{I}_1, \cdots, \mathbb{I}_{m-1}\}$ *a Chebyshev system with accuracy* k *in* $(0, h^*)$.

2.2. Hopf bifurcation theory for Liénard system

Computing and analyzing the Lyapunov coefficients of Poincaré map, which is locally constructed around a foci, is the classical method to study Hopf bifurcation of general planar differential systems. However, it is not an easy task for computing the Lyapunov coefficients, which are needed to analyze the algebraic varieties in order to determine Hopf cyclicity. For Liénard type system, one equivalent method to computing the Lyapunov coefficients was developed in [20,21], which is summarized as follows.

Consider the system of the form,

$$\dot{x} = P(y) - F(x, \eta), \quad \dot{y} = -g(x), \tag{10}$$

where η is an n-dimensional parameter vector, $P(y)$, $F(x, \eta)$ and $g(x)$ are analytic satisfying $P'(0)g'(0) > 0$, $F(0, \eta) = g(0) = P(0) = 0$ and $F_x(0, \eta^*) = 0$ for some $\eta^* \in \mathbb{R}^n$. These assumptions assure that the origin is a center or focus of system (10) for η chosen from a very small neighborhood of η^*. Then, one can construct the Poincaré map locally around the origin, which has the following expansion,

$$\mathcal{P}(r, \delta) = \sum_{i=1}^{\infty} v_i(\eta) r^i, \quad \text{for } |r| \ll 1 \text{ and } |\eta - \eta^*| \ll 1, \tag{11}$$

where $v_i(\eta) \in C^\infty$. An isolated positive zero of $\mathcal{P}(r, \eta)$ near $r = 0$ corresponds to a small limit cycle of system (10) due to Hopf bifurcation. Therefore, it only needs to study the sharp upper bound on the maximal number of isolated positive zeros of $\mathcal{P}(r, \eta)$ for studying the Hopf cyclicity of a focus or a center. Particularly, we have the following expansion of the bifurcation function for the generalized Liénard system (10),

$$F(z(x), \eta) - F(x, \eta) = \sum_{i=1}^{\infty} B_i(\eta) x^i, \quad \text{for } 0 < x \ll 1, \tag{12}$$

where $z(x)$ is the involution defined by the potential $G(x) = \int g(x)dx$ with $G(z(x)) = G(x)$. It was proved in [20,21] that,

$$\begin{aligned}
v_1 &= N_1(B_1)B_1, \\
v_{2j} &= O(B_1, B_3, \cdots, B_{2j-1}), \\
v_{2j+1} &= N_{2j+1}(B_1)B_{2j+1} + O(B_1, B_3, \cdots, B_{2j-1}),
\end{aligned}$$

where $N_{2j+1}(B_1) \in C^\infty$. Therefore, we only need compute B_i and analyze its algebraic variety $\{B_i = 0\}$ for all $i \geq 1$ to study the Hopf cyclicity. The following Lemma [21] states the criterion.

Lemma 8. *Consider system (10) and the expansion (12). Suppose there exists $k \geq 1$ such that*

$$F(z(x), \eta) \equiv F(x, \eta), \;\; B_{2j+1} = 0,$$

for $j = 0, 1, \cdots, k$ and there exists some $\eta^* \in \mathbb{R}^n$ such that

$$\begin{aligned} &B_{2j+1}(\eta^*) = 0, \;\; j = 0, 1, \cdots, k, \\ &\mathrm{rank}\frac{\partial(B_1, B_3, \cdots, B_{2k+1})}{\partial \eta}\bigg|_{\eta=\eta^*} = k+1. \end{aligned}$$

Then, the Hopf cyclicity of system (10) at the origin is k.

Liu and Han [28] extended the theory to study Hopf bifurcation of the following piecewise nonsmooth Liénard system,

$$(\dot{x}, \dot{y}) = \begin{cases} (P(y) - F^+(x, \eta), -g^+(x)), & x > 0, \\ (P(y) - F^-(x, \eta), -g^-(x)), & x < 0, \end{cases} \tag{13}$$

where η is an n dimensional parameter vector, $P(y)$, $g^\pm(x)$ and $F^\pm(x)$ are analytic and satisfy

$$P(0) = F^\pm(0, \eta) = g^\pm(0) = 0,$$

and

$$(F_x^\pm(0, \eta^*))^2 - 4P'(0)(g^\pm)'(0) < 0,$$

for some $\eta^* \in \mathbb{R}^n$. Similarly, one can construct a Poincaré map expanded for η near η^* as

$$d(\rho, \eta) = v_1(\eta)\rho + v_2(\eta)\rho^2 + \cdots + v_j(\eta)\rho^j + \cdots, \;\; 0 < \rho \ll 1,$$

and the bifurcation function similar to (12) is given by

$$F^-(z(x), \eta) - F^+(x, \eta) = \sum_{j=1}^{\infty} B_j(\eta) x^j, \tag{14}$$

for $0 < x \ll 1$, where $z(x)$ is the involution defined by $G^+(z(x)) = G^-(x)$ with $G^\pm(s) = \int g^\pm(s)ds$. Liu and Han [28] gave that,

$$\begin{aligned} &v_1(\eta) = W_1(\eta)B_1, \\ &v_j(\eta) = W_j(\eta)B_j + O(|B_1, B_2, \cdots, B_{j-1}|), \end{aligned}$$

where $W_j \in C^\infty$ and $W_j > 0$ for B_1 small. The following lemma gives the Hopf cyclicity.

Lemma 9 *([39]). Let k positive integers satisfy $r_1 < r_2 < \cdots < r_k$ and form the ordered sequence $\{r_i\}_{i=1}^k$. If the following items are verified:*

(i) $B_j(\eta) \equiv 0$ *for* $0 \le j < r_1$*;*
(ii) $B_j(\eta) = O(B_{r_1}, B_{r_2}, \cdots, B_{r_{s(j)}})$ *where* $r_{s(j)} = \max\{r_s < j\}$*;*
(iii) *there exists some* η^* *such that* $B_{r_j}(\eta^*) = 0$ *for* $0 \le j \le k$ *and*

$$\operatorname{rank} \frac{\partial(B_{r_1}, B_{r_2}, \cdots, B_{r_k})}{\partial \eta}\Big|_{\eta=\eta^*} = k,$$

then the Hopf cyclicity of system (13) *at the origin is* $k-1$.

3. Proof of Theorem 2

In this section, we prove Theorem 2. We divide the parameter space for $\mathcal{A}(h)$ to obtain a cube which is the only set for $\mathcal{A}(h)$ to might have 4 zeros on $h \in (0, \frac{1}{24})$. As it was shown in [36] that $I_1(h) \equiv I_2(h)$, therefore, $\mathcal{A}(h)$ is spanned by

$$\{I_0(h), I_1(h), I_3(h), I_4(h)\}.$$

Hence, it only needs to analyze the set $\{I_0(h), I_1(h), I_3(h), I_4(h)\}$.

We write $V(x) = \mathcal{H}(x, y) - \frac{y^2}{2}$. Then,

$$v(x,z) := \frac{V(x) - V(z)}{x - z} = 0$$

defines the involution $z(x)$, $x \in (0, 1)$ on the period annulus. We have the following result.

Lemma 10. *The following equations hold:*

$$8h^3 I_i(h) = \oint_{\Gamma_h} \rho_i(x) y^7 dx \equiv \widetilde{I}_i(h), \quad i = 0,\ 1,\ 3, 4,$$

where $\rho_i(x) = \frac{x^i g_i(x)}{22680(1+2x)^6(x-1)^{12}}$, *in which each polynomial* $g_i(x)$ *has degree* 18.

Proof. First, multiplying $I_i(h)$ by $\frac{y^2+2V(x)}{2h} = 1$ yields

$$\begin{aligned} 8h^3 I_i(h) &= \oint_{\Gamma_h} (2V(x) + y^2)^3 x^i y dx \\ &= \oint_{\Gamma_h} 8x^i V^3(x) y dx + \oint_{\Gamma_h} 12 x^i V^2(x) y^3 dx \\ &\quad + \oint_{\Gamma_h} 6x^i V(x) y^5 dx + \oint_{\Gamma_h} x^i y^7 dx,\ i = 0, 1, 3, 4. \end{aligned} \tag{15}$$

Then, applying Lemma 4.1 in [19] to (15) to increase the power of y in the first three integrals to 7 proves the lemma. □

Without loss of generality, we assume that $\alpha_4 = 1$ when $\alpha_4 \neq 0$. Further, introduce the following combinations:

$$\begin{aligned}\mathcal{I}_{34}(h) &= \oint_{\Gamma_h} \left(\alpha_3 x^3 + x^4\right) y dx, \\ \mathcal{I}_{14}(h) &= \oint_{\Gamma_h} \left(\alpha_1 x + x^4\right) y dx, \\ \mathcal{I}_{04}(h) &= \oint_{\Gamma_h} \left(\alpha_0 + x^4\right) y dx.\end{aligned} \tag{16}$$

Then,

$$\begin{aligned}\mathcal{A}(h) &= \alpha_0 I_0(h) + \alpha_1 I_1(h) + \mathcal{I}_{34}(h) \\ &= \alpha_0 I_0(h) + \alpha_3 I_3(h) + \mathcal{I}_{14}(h) \\ &= \alpha_1 I_1(h) + \alpha_3 I_3(h) + \mathcal{I}_{04}(h).\end{aligned}$$

The following lemma directly follows Lemma 10.

Lemma 11. *The following equations hold:*

$$\begin{aligned}8h^3\mathcal{I}_{34}(h) &= \oint_{\Gamma_h} (\alpha_3\rho_3(x) + \rho_4(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{34}(h), \\ 8h^3\mathcal{I}_{14}(h) &= \oint_{\Gamma_h} (\alpha_1\rho_1(x) + \rho_4(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{14}(h), \\ 8h^3\mathcal{I}_{04}(h) &= \oint_{\Gamma_h} (\alpha_0\rho_0(x) + \rho_4(x)) y^7 dx \triangleq \widetilde{\mathcal{I}}_{04}(h).\end{aligned}$$

Now, let

$$\begin{aligned}l_i(x) &= \left(\frac{\rho_i}{V'}\right)(x) - \left(\frac{\rho_i}{V'}\right)(z(x)), \ i = 0, 1, 3, 4. \\ \mathcal{L}_{i4}(x) &= \left(\frac{\alpha_i\rho_i + \rho_4}{V'}\right)(x) - \left(\frac{\alpha_i\rho_i + \rho_4}{V'}\right)(z(x)), \ i = 0, 1, 3.\end{aligned} \tag{17}$$

Then,

$$\frac{d}{dx}l_i(x) = \frac{d}{dx}\Big(\frac{\rho_i}{V'}\Big)(x) - \frac{d}{dz}\Big[\Big(\frac{\rho_i}{V'}\Big)(z(x))\Big] \times \frac{dz}{dx},$$
$$\frac{d}{dx}\mathcal{L}_{i4}(x) = \frac{\partial}{\partial x}(\mathcal{L}_{i4}(x)) + \frac{\partial}{\partial z}(\mathcal{L}_{i4}(x)) \times \frac{dz}{dx},\ i = 0, 1, 3,$$

where $\frac{dz}{dx} = -\frac{v_x(x,z)}{v_z(x,z)}$. A direct computation yields

$$\begin{aligned}
W[l_0] &= \frac{(x-z)W_0(x,z)}{11340\,xz(2x+1)^7(x-1)^{15}(2z+1)^7(z-1)^{15}},\\
W[l_1] &= \frac{(x-z)W_1(x,z)}{11340\,(2x+1)^7(x-1)^{15}(2z+1)^7(z-1)^{15}},\\
W[l_0,l_1] &= \frac{(x-z)^3W_{01}(x,z)}{128595600\,x^2z^2(z-1)^{30}(2z+1)^{13}(x-1)^{30}(2x+1)^{13}w_0(x,z)},\\
W[l_0,l_3] &= \frac{(x-z)^3W_{03}(x,z)}{128595600\,x^2z^2(z-1)^{30}(2z+1)^{13}(x-1)^{30}(2x+1)^{13}w_0(x,z)},\\
W[l_0,l_4] &= \frac{(x-z)^3W_{04}(x,z)}{128595600\,x^2z^2(z-1)^{30}(2z+1)^{13}(x-1)^{30}(2x+1)^{13}w_0(x,z)},\\
W[l_1,l_3] &= \frac{(x-z)^3W_{13}(x,z)}{128595600\,(z-1)^{30}(2z+1)^{13}(x-1)^{30}(2x+1)^{13}w_0(x,z)},\\
W[l_0,l_1,l_3] &= \frac{(x-z)^6W_{013}(x,z)}{w^*x^3z^3(x-1)^{45}(2x+1)^{18}(z-1)^{45}(2z+1)^{18}w_0^3(x,z)},\\
W[l_0,l_1,\mathcal{L}_{34}] &= \frac{(x-z)^6\mathcal{W}_3(x,z,\alpha_3)}{w^*x^3z^3(x-1)^{45}(2x+1)^{18}(z-1)^{45}(2z+1)^{18}w_0^3(x,z)},\\
W[l_0,l_3,\mathcal{L}_{14}] &= \frac{(x-z)^6\mathcal{W}_1(x,z,\alpha_1)}{w^*x^3z^3(x-1)^{45}(2x+1)^{18}(z-1)^{45}(2z+1)^{18}w_0^3(x,z)},\\
W[l_1,l_3,\mathcal{L}_{04}] &= \frac{(x-z)^6\mathcal{W}_0(x,z,\alpha_0)}{w^*x^3z^3(x-1)^{45}(2x+1)^{18}(z-1)^{45}(2z+1)^{18}w_0^3(x,z)},
\end{aligned} \tag{18}$$

where $z = z(x)$ is the involution as defined by $v(x,z) = 0$, $w^* = 729137052000$, $w_0(x,z) = 2x + 4z - 3$, W_0, W_1, W_{01}, W_{03}, W_{04}, W_{13} and W_{013} are polynomials of degrees 40, 39, 78, 80, 81, 77 and 115, respectively, and the polynomials $\mathcal{W}_3$, $\mathcal{W}_2$ and $\mathcal{W}_1$ have the degrees 116, 118 and 119, respectively.

We claim that the Wronskians are well defined for $x \in (0,1)$, because $w_0(x,z)$ does not vanish for $x \in (0,1)$ by showing that the resultant between $v(x,z)$ and $w_0(x,z)$ with respect to z has no roots for $x \in (0,1)$.

The following result indicates that we only need to discuss the case when $\alpha_4 \neq 0$.

Proposition 12. *When* $\alpha_4 = 0$, $\mathcal{A}(h)$ *has at most* 2 *zeros in* $(0, \frac{1}{24})$.

The proof of Proposition 12 relies on symbolic computation for verifying the non-vanishment of Wronskians $W[l_0]$, $W[l_0,l_1]$ and $W[l_0,l_1,l_3]$ according to Lemma 7. Since the symbolic computation and verification are straightforward, we omit the proof here for briefness.

To prove Theorem 2, we need to show non-vanishing of certain denominators and numerators of the related Wronskians in (18) for $x \in (0,1)$. Taking the numerator $W_{01}(x,z)$ of the Wronskian $W[l_0,l_1]$, for example, we only need to prove that the 2-dimensional system $\{W_{01}(x,z), v(x,z)\}$

does not vanish on $\{(x,z)| -\frac{1}{2} < z < 0 < x < 1\}$, because z in $W_{01}(x,z)$ is determined by $v(x,z)=0$, and $z(x) \in (-\frac{1}{2}, 0)$ when $x \in (0,1)$. To do this, we apply triangular-decomposition and root isolating to $\{W_{01}(x,z), v(x,z)\}$ to decompose the nonlinear system into several triangular systems, and then isolate the roots of each triangular-decomposed system. Since all roots of these triangular systems are the roots of the original system $\{W_{01}(x,z), v(x,z)\}$, we only need to check if these decomposed systems have roots on $\{(x,z)| -\frac{1}{2} < z < 0 < x < 1\}$. This idea has been successfully applied to determine the zeros of Abelian integrals, see [34–36,38,45]. Instead of the triangular-decomposition method, one may also use the interval analysis [41], which computes two resultants between $W_{01}(x,z)$ and $v(x,z)$ with respect to x and z, respectively, yielding several two dimensional regions. Finally, one verifies if $W_{01}(x,z)$ vanishes on these regions by determining the intersection points of the curves $W_{01}(x,z)$ and $v(x,z)$, see [41] for details.

By applying the triangular-decomposition and root isolating to the numerators of the Wronskians, we obtain the following result.

Lemma 13. *All of the Wronskians, $W[l_0]$, $W[l_1]$, $W[l_0,l_1]$, $W[l_0,l_3]$, $W[l_0,l_4]$ and $W[l_1,l_3]$, do not vanish for $x \in (0,1)$.*

Next, we investigate the last three Wronskians in (18). Their numerators have the forms,

$$\begin{aligned}\mathcal{W}_3(x,z,\alpha_3) &= \alpha_3 S_2(x,z) - S_1(x,z),\\ \mathcal{W}_1(x,z,\alpha_1) &= \alpha_1 S_2^{\ddagger}(x,z) - S_1^{\ddagger}(x,z),\\ \mathcal{W}_0(x,z,\alpha_0) &= \alpha_0 S_2^{*}(x,z) - S_1^{*}(x,z),\end{aligned}$$

where S_1, S_2, $S_1^{\ddagger}$, $S_2^{\ddagger}$, S_1^{*} and S_2^{*}, are polynomials of degrees 116, 115, 118, 115, 119 and 115, respectively. $\mathcal{W}_i(x,z,\alpha_i)=0$ (for $i=3,1,0$) defines three functions,

$$\alpha_3(x,z) = \frac{S_1(x,z)}{S_2(x,z)}, \quad \alpha_1(x,z) = \frac{S_1^{\ddagger}(x,z)}{S_2^{\ddagger}(x,z)}, \quad \alpha_0(x,z) = \frac{S_1^{*}(x,z)}{S_2^{*}(x,z)},$$

and their derivatives,

$$\begin{aligned}\widetilde{\alpha}_3(x,z) &= \frac{\partial \alpha_3(x,z)}{\partial x} + \frac{\partial \alpha_3(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\widetilde{S}_1(x,z)}{\widetilde{S}_2(x,z)},\\ \widetilde{\alpha}_1(x,z) &= \frac{\partial \alpha_1(x,z)}{\partial x} + \frac{\partial \alpha_1(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\widetilde{S}_1^{\ddagger}(x,z)}{\widetilde{S}_2^{\ddagger}(x,z)},\\ \widetilde{\alpha}_0(x,z) &= \frac{\partial \alpha_0(x,z)}{\partial x} + \frac{\partial \alpha_0(x,z)}{\partial z} \times \frac{dz}{dx} = \frac{\widetilde{S}_1^{*}(x,z)}{\widetilde{S}_2^{*}(x,z)}.\end{aligned}$$

Each of the denominators $S_2(x,z)$, $S_2^{\ddagger}(x,z)$, $S_2^{*}(x,z)$, $\widetilde{S}_2(x,z)$, $\widetilde{S}_2^{\ddagger}(x,z)$ and $\widetilde{S}_2^{*}(x,z)$ does not vanish for $x \in (0,1)$, because they do not have common roots with $v(x,z)$ for $(x,z) \in (0,1) \times (-\frac{1}{2}, 0)$, verified by triangular-decomposition and root isolating. Hence, all of the functions $\alpha_i(x,z)$ and $\widetilde{\alpha}_i(x,z)$ $(i=3,1,0)$ are well defined for $x \in (0,1)$.

We have the following lemma.

Lemma 14.

(i) $\alpha_3(x, z(x))$ *is decreasing from* $(0, -\frac{3}{5})$ *to a minimum* $(x^\dagger, \alpha_3^\dagger)$ *and then increasing to* $(1, -\frac{5}{2})$*;*
(ii) $\alpha_1(x, z(x))$ *is increasing from* $(0, 0)$ *to a maximum* $(x^\ddagger, \alpha_1^\ddagger)$ *and then decreasing to* $(1, 2)$*;*
(iii) $\alpha_0(x, z(x))$ *is decreasing from* $(0, 0)$ *to a minimum* (x^*, α_0^*) *and then increasing to* $(1, -\frac{1}{2})$*, where*

$$x^\dagger, x^\ddagger, x^* \in \underbrace{\Big[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}\Big]}_{1/10^{13}},$$

and

$$\alpha_3^\dagger \in \underbrace{\Big[-\frac{44214\cdots40352}{12885\cdots98125}, -\frac{55559\cdots02475}{16191\cdots11584}\Big]}_{1/10^{10}} \approx [-3.4312932408, -3.4312932406],$$

$$\alpha_1^\ddagger \in \underbrace{\Big[\frac{31388\cdots59375}{94249\cdots82976}, \frac{16836\cdots18911}{50552\cdots20000}\Big]}_{1/10^{10}} \approx [3.3303719012, 3.3303719013],$$

$$\alpha_0^* \in \underbrace{\Big[-\frac{66176\cdots98433}{72854\cdots59040}, -\frac{53977\cdots21875}{59424\cdots65344}\Big]}_{1/10^{11}} \approx [-0.90833169736, -0.90833169731].$$

Proof. We only prove case (i), since other two cases (ii) and (iii) can be similarly proved. A direct computation shows that

$$\lim_{x\to 0} \alpha_3(x, z(x)) = -\frac{3}{5}, \qquad \lim_{x\to 1} \alpha_3(x, z(x)) = -\frac{5}{2}.$$

On $\{(x, z) | -\frac{1}{2} < z < 0 < x < 1\}$, $\widetilde{S}_1(x, z)$ and $v(x, z)$ have a unique common root $(x^\dagger, z^\dagger) \in D^\dagger$, where

$$D^\dagger = \Big[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}\Big] \times \Big[-\frac{32492637936074023}{72057594037927936}, -\frac{129970551744296065}{288230376151711744}\Big].$$

$x^\dagger$ is the unique simple zero of $\widetilde{\alpha}_3(x, z(x))$ by verifying that $\frac{d}{dx}\widetilde{\alpha}_3(x, z(x))$ has no zeros on $[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}]$. Therefore, $x^\dagger$ is the unique critical point of $\alpha_3(x, z(x))$, and thus the monotonicity of $\alpha_3(x, z(x))$ in $(0, x^\dagger) \bigcup (x^\dagger, 1)$ can be easily determined by comparing the values of $\alpha_3(x, z(x))$ at $x = 0$, $x^\dagger$ and 1 as $-\frac{3}{5}$, $-3.4312932408\cdots$ and $-\frac{5}{2}$, respectively. Alternatively, using

$$\lim_{x\to 0^+} \widetilde{\alpha}_3(x, z(x)) = 0^-, \quad \text{and} \quad \lim_{x\to 0^+} \frac{d}{dx}(\widetilde{\alpha}_3(x, z(x))) = -\frac{147}{25} < 0,$$

we know that $\alpha_3(x, z(x))$ is monotonically deceasing in $(0, x^\dagger)$ and monotonically increasing in $(x^\dagger, 1)$

It can be further shown that the resultant between $\frac{\partial S_i(x,z)}{\partial x}$ $(i = 1, 2)$ and $v(x, z)$ with respect to z has no roots over the interval

$$\left[\frac{99571576491449}{140737488355328}, \frac{49785788245729}{70368744177664}\right]$$

by Sturm's Theorem. Hence, $S_i(x, z)$ $(i = 1, 2)$ reaches its maximal and minimal values on the boundaries of $D^\dagger$. A direct computation yields

$$\begin{aligned}
\min_{D^\dagger} S_1(x,z) &= -\frac{21275\cdots15705}{13764\ldots10976} \approx -0.154565249238,\\
\max_{D^\dagger} S_1(x,z) &= -\frac{20094\ldots34375}{13000\ldots54592} \approx -0.154565249236,\\
\min_{D^\dagger} S_2(x,z) &= \frac{26631\ldots09375}{59119\ldots40896} \approx 0.045045770906,\\
\max_{D^\dagger} S_2(x,z) &= \frac{14098\ldots51625}{31297\ldots78144} \approx 0.045045770908.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\alpha_3^\dagger = \alpha_3(x^\dagger, z(x^\dagger)) &\in \left[\frac{\min_{D^\dagger} S_1(x,z)}{\min_{D^\dagger} S_2(x,z)}, \frac{\max_{D^\dagger} S_1(x,z)}{\max_{D^\dagger} S_2(x,z)}\right]\\
&= \underbrace{\left[-\frac{44214\cdots40352}{12885\cdots98125}, -\frac{55559\cdots02475}{16191\cdots11584}\right]}_{1/10^{10}}\\
&\approx [-3.4312932408, -3.4312932406]. \quad \square
\end{aligned}$$

Note in the above proof we have used symbolic computation which gives the exact expressions using rational numbers, demonstrating the accuracy of our analysis. It is also noted that the critical point $(x^\dagger, \alpha_3(x^\dagger, z(x^\dagger)))$ divides the curve $\{(x, \alpha_3(x, z(x)))|0 < x < 1\}$ into two simple segments (curves). Each point on the two segments corresponds to a simple root of $\mathcal{W}_3(x, z(x), \alpha_3(x, z(x)))$, while $x^\dagger$ is a root of multiplicity 2. The following lemma follows from Lemma 14.

Lemma 15. *For $x \in (0, 1)$, when α_3 is located in the intervals $[\alpha_3^\dagger, -\frac{5}{2})$, $[-\frac{5}{2}, -\frac{3}{5})$ and $(-\infty, \alpha_3^\dagger) \bigcup [-\frac{3}{5}, +\infty)$, $W[l_0, l_1, \mathcal{L}_{34}]$ has 2, 1 and 0 roots with multiplicities counted, respectively.*

Combining Lemmas 13 and 15 and applying Lemma 7, we have the following result.

Proposition 16. *$\mathcal{A}(h)$ has at most 4, 3 and 2 zeros in $(0, \frac{1}{24})$ when α_3 belongs to the intervals $[\alpha_3^\dagger, -\frac{5}{2})$, $[-\frac{5}{2}, -\frac{3}{5})$, and $(-\infty, \alpha_3^\dagger) \bigcup [-\frac{3}{5}, +\infty)$, respectively.*

Similarly, we have

Proposition 17. *$\mathcal{A}(h)$ has at most 4, 3 and 2 zeros in $(0, \frac{1}{24})$ when α_1 is located in the intervals $(2, \alpha_1^{\ddagger}]$, $(0, 2]$, and $(-\infty, 0]\bigcup(\alpha_1^{\ddagger}, +\infty)$, respectively.*

Proposition 18. *$\mathcal{A}(h)$ has at most 4, 3 and 2 zeros in $(0, \frac{1}{24})$ when α_0 belongs to the intervals $[\alpha_0^*, -\frac{1}{2})$, $[-\frac{1}{2}, 0)$, and $(-\infty, \alpha_0^*)\bigcup[0, +\infty)$, respectively.*

Define

$$\mathcal{D} = \Big\{(\alpha_0, \alpha_1, \alpha_3)|\alpha_0 \in \Big[\alpha_0^*, -\frac{1}{2}\Big),\ \alpha_1 \in \Big(2, \alpha_1^{\ddagger}\Big],\ \alpha_3 \in \Big[\alpha_3^{\dagger}, -\frac{5}{2}\Big)\Big\}.$$

Then, Propositions 16, 17 and 18 imply that

Proposition 19. *$\mathcal{A}(h)$ may have 4 zeros only if $(\alpha_0, \alpha_1, \alpha_3) \in \mathcal{D}$.*

Finally, we prove that $\mathcal{A}(h)$ cannot have 4 zeros when $(\alpha_0, \alpha_1, \alpha_3) \in \mathcal{D}$. First, we have

Lemma 20. *For $h \in (0, \frac{1}{24})$, the following holds:*

(1) *the generating element $I_0(h)$ is positive;*
(2) *the ratio $\frac{I_1(h)}{I_0(h)}$ is increasing from 0 to $\frac{1}{10}$;*
(3) *the ratio $\frac{I_3(h)}{I_0(h)}$ is increasing from 0 to $\frac{1}{28}$; and*
(4) *the ratio $\frac{I_4(h)}{I_0(h)}$ is increasing from 0 to $\frac{31}{1120}$.*

Proof. By Green formula, $I_0(h) = \oint_{\Gamma_h} ydx = \iint_{\mathcal{O}} dxdy$, where $\mathcal{O}$ represents the region bounded by Γ_h (a periodic annulus), and therefore, $I_0(h) > 0$. The non-vanishing property of $W[l_0]$, $W[l_0, l_1]$, $W[l_0, l_3]$ and $W[l_0, l_4]$ proved in Lemma 13 implies that $\frac{I_1(h)}{I_0(h)}$, $\frac{I_3(h)}{I_0(h)}$ and $\frac{I_4(h)}{I_0(h)}$ are monotonic in $(0, \frac{1}{24})$. By the expansion of $\mathcal{A}(h)$ near $h = 0$ (see the formulas given in [36]), we have

$$\lim_{h\to 0} \frac{I_1(h)}{I_0(h)} = \lim_{h\to 0} \frac{I_3(h)}{I_0(h)} = \lim_{h\to 0} \frac{I_4(h)}{I_0(h)} = 0.$$

Taking the limit as $h \to \frac{1}{24}$ yields

$$\lim_{h\to \frac{1}{24}} \frac{I_1(h)}{I_0(h)} = \lim_{h\to \frac{1}{24}} \frac{\oint_{\Gamma_h} xydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{1}{24}}} xydx}{\oint_{\Gamma_{\frac{1}{24}}} ydx} = \frac{1}{10},$$

$$\lim_{h\to \frac{1}{24}} \frac{I_3(h)}{I_0(h)} = \lim_{h\to \frac{1}{24}} \frac{\oint_{\Gamma_h} x^3ydx}{\oint_{\Gamma_h} ydx} = \frac{\oint_{\Gamma_{\frac{1}{24}}} x^3ydx}{\oint_{\Gamma_{\frac{1}{24}}} ydx} = \frac{1}{28},$$

and

$$\lim_{h\to\frac{1}{24}}\frac{I_4(h)}{I_0(h)}=\lim_{h\to\frac{1}{24}}\frac{\oint_{\Gamma_h}x^4ydx}{\oint_{\Gamma_h}ydx}=\frac{\oint_{\Gamma_{\frac{1}{24}}}x^4ydx}{\oint_{\Gamma_{\frac{1}{24}}}ydx}=\frac{31}{1120}. \quad \square$$

Proposition 21. $\mathcal{A}(h)<0$ *for* $(\alpha_0,\alpha_1,\alpha_3)\in\mathcal{D}$.

Proof. When $(\alpha_0,\alpha_1,\alpha_3)\in\mathcal{D}$, by the results obtained in Lemma 20, it is easy to show that for $h\in(0,\frac{1}{24})$,

$$\begin{aligned}\mathcal{A}(h)&=\Big(\alpha_0+\alpha_1\frac{I_1(h)}{I_0(h)}+\alpha_3\frac{I_3(h)}{I_0(h)}+\frac{I_4(h)}{I_0(h)}\Big)I_0(h)\\&<\Big(\alpha_0+\alpha_1\frac{I_1(h)}{I_0(h)}+\frac{I_4(h)}{I_0(h)}\Big)I_0(h)\\&<\Big(-\frac{1}{2}+\alpha_1^{\ddagger}\times\frac{1}{10}+\frac{31}{1120}\Big)I_0(h)\\&<0.\end{aligned}$$

So $\mathcal{A}(h)$ has no zeros for $(\alpha_0,\alpha_1,\alpha_3)\in\mathcal{D}$. $\square$

Proof of Theorem 2. Combining Propositions 12, 19 and 21 proves Theorem 2. $\square$

4. Proof of Theorem 3

Taking the transformation, $\tilde{y}=y-\int_0^x f_2(s)ds$, $\tilde{x}=x$, reduces system (5) to the following form, after dropping the tilde,

$$\dot{x}=y-F(x,\delta),\quad y=x(x-1)^3\Big(x+\frac{1}{2}\Big), \tag{19}$$

where

$$F(x,\delta)=-\int_0^x f_2(s)ds=-\sum_{i=0}^{n}\frac{\alpha_i}{i+1}x^{i+1}:=\sum_{i=1}^{N}\gamma_i x^i$$

with $N=n+1$, $\gamma_i=-\frac{1}{i}\alpha_{i-1}$ for $1\le i\le N$, and $\delta=(\gamma_1,\cdots,\gamma_N)\in\mathbb{R}^N$.

In order to prove our main result on the Hopf cyclicity, we first introduce some known results.

Lemma 22 *([39]). The following equalities hold for any constant* $\nu\in\mathbb{R}$*,*

(i)

$$\int_{-\pi}^{\pi}\sin^k(\theta+\nu)\sin(i\theta)=0,\ \text{if } i>k;$$

(ii)

$$\int_{-\pi}^{\pi} \sin^k(\theta+\nu)\cos(i\theta) = 0, \text{ if } i > k;$$

(iii)

$$\int_{-\pi}^{\pi} \sin^k(\theta+\nu)\sin(k\theta) = \frac{\pi}{2^{k-1}}\cos\left(k\nu - \frac{k-1}{2}\pi\right), \text{ if } k \in \mathbb{N}^+;$$

(iv)

$$\int_{-\pi}^{\pi} \cos^k(\theta+\nu)\sin(k\theta) = \frac{\pi}{2^{k-1}}\sin\left(k\nu - \frac{k-1}{2}\pi\right), \text{ if } k \in \mathbb{N}^+.$$

As discussed above, there exists an analytic involution $z(x)$ for the potential of the undamped system (5), defined on the periodic annulus by $v(x,z) = 2x^2 + 2xz + 2z^2 - 3x - 3z = 0$. Next, we introduce

$$x = \Theta(\theta) = \frac{1}{2} + \frac{\sqrt{3}}{2}\sin(\theta) + \frac{1}{2}\cos(\theta), \tag{20}$$

then

$$z = \Theta(-\theta) = \frac{1}{2} - \frac{\sqrt{3}}{2}\sin(\theta) + \frac{1}{2}\cos(\theta). \tag{21}$$

Let $J_k = \Theta^k(-\theta) - \Theta^k(\theta)$, and $K_k = \Theta^k(-\theta) + \Theta^k(\theta)$ for $k \in \mathbb{N}^+$. Then, we have the following lemma, which establishes a key success step in deriving the Hopf cyclicity.

Lemma 23. *For any $k \in \mathbb{N}^+$, we have*

$$J_k(\theta) = \sum_{i=1}^{k} c_{ki}\sin(i\theta), \quad K_k(\theta) = \sum_{i=1}^{k} \tilde{c}_{ki}\cos(i\theta), \tag{22}$$

where $c_{ki} = 0$ for i satisfying i mod $3 = 0$, $c_{kk} = -\frac{1}{2^{k-2}}\cos\left(\frac{k\pi}{6} - \frac{k-1}{2}\pi\right)$, $\tilde{c}_{kk} = \frac{1}{2^{k-2}}\sin\left(\frac{k\pi}{6} - \frac{k-1}{2}\pi\right)$.

Proof. It is obvious that $\Theta(\theta)$ can be expressed as a linear combination of $\sin(i\theta)$ and $\cos(i\theta)$, $i = 1, 2, \cdots, k$. Then, we have the formula (22) because $J_k(\theta)$ is an odd function and $K(\theta)$ an even one. By (20), we have

$$x = \Theta(\theta) = \frac{1}{2} + \sin(\theta + \frac{\pi}{6}).$$

By theory of Fourier series, we have

$$
\begin{aligned}
c_{kk} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(-\theta)\sin(k\theta)d\theta - \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(\theta)\sin(k\theta)d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(-\theta)\sin(-k\theta)d(-\theta) - \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(\theta)\sin(k\theta)d\theta \\
&= -\frac{2}{\pi}\int_{-\pi}^{\pi} \Theta^k(\theta)\sin(k\theta)d\theta \\
&= -\frac{2}{\pi}\int_{-\pi}^{\pi} \Big(\sum_{s=0}^{k} C_k^s \frac{1}{2^{k-s}}\sin^s(\theta+\frac{\pi}{6})\Big)\sin(k\theta)d\theta.
\end{aligned}
$$

Applying Lemma (22), we obtain

$$
c_{kk} = -\frac{2}{\pi}\int_{-\pi}^{\pi} \sin^k(\theta+\frac{\pi}{6})\sin(k\theta)d\theta = -\frac{1}{2^{k-2}}\cos\Big(\frac{k\pi}{6}-\frac{k-1}{2}\pi\Big).
$$

Similarly, we have

$$
\begin{aligned}
\widetilde{c}_{kk} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(-\theta)\cos(k\theta)d\theta + \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(\theta)\cos(k\theta)d\theta \\
&= -\frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(-\theta)\cos(-k\theta)d(-\theta) + \frac{1}{\pi}\int_{-\pi}^{\pi} \Theta^k(\theta)\cos(k\theta)d\theta \\
&= \frac{2}{\pi}\int_{-\pi}^{\pi} \Theta^k(\theta)\cos(k\theta)d\theta \\
&= \frac{2}{\pi}\int_{-\pi}^{\pi} \Big(\sum_{s=0}^{k} C_k^s \frac{1}{2^{k-s}}\sin^s(\theta+\frac{\pi}{6})\Big)\cos(k\theta)d\theta.
\end{aligned}
$$

It follows from Lemma (22) that

$$
\widetilde{c}_{kk} = \frac{2}{\pi}\int_{-\pi}^{\pi} \sin^k(\theta+\frac{\pi}{6})\cos(k\theta)d\theta = \frac{1}{2^{k-2}}\sin\Big(\frac{k\pi}{6}-\frac{k-1}{2}\pi\Big).
$$

A direct computation shows that

$$
\Theta\big(u+\frac{2\pi}{3}\big) = \Theta(-u).
$$

Then,

$$\Theta^k\Big(u+\frac{2\pi}{3}\Big)=\Theta^k(-u)\ \text{ for } k\in\mathbb{N}^+.$$

Further, for $i=3j<k,\ j=1,2,\cdots,\left[\frac{k}{3}\right]$, we can show that

$$\begin{aligned}\int_{-\pi}^{\pi}\Theta^k(\theta)\sin(3j\theta)d\theta &= \int_{-\frac{5\pi}{3}}^{\frac{\pi}{3}}\Theta^k\Big(u+\frac{2\pi}{3}\Big)\sin\Big(3j(u+\frac{2\pi}{3})\Big)du\\ &= \int_{-\frac{5\pi}{3}}^{\frac{\pi}{3}}\Theta^k\Big(u+\frac{2\pi}{3}\Big)\sin\big(3ju\big)du\\ &= \int_{-\frac{5\pi}{3}}^{\frac{\pi}{3}}\Theta^k(-u)\sin\big(3ju\big)du\\ &= \int_{-\pi}^{\pi}\Theta^k(-u)\sin\big(3ju\big)du.\end{aligned}$$

Therefore,

$$c_{k,3j}=\frac{1}{\pi}\int_{-\pi}^{\pi}\big(\Theta^k(-u)-\Theta^k(\theta)\big)\sin\big(3ju\big)du=0.$$

The proof is complete. □

Proof of Theorem 3. It only needs to prove that the cyclicity of system (19) at the origin is $\left[\frac{2N-1}{3}\right]$.

By Lemma 8, we construct the following power series

$$F(z(x))-F(x)=\sum_{k=1}^{N}\gamma_k(z^k-x^k)=\sum_{i\geq 1}B_i x^i, \tag{23}$$

for $|x|\ll 1$, where $z(x)$ is the involution defined by $v(x,z)=0$. We have two goals aiming at proving our result on the Hopf cyclicity. One goal is to prove the following relationship between the coefficients in $F(z(x))-F(x)$,

$$B_{2j+1}=O(|B_1,B_3,\cdots,B_{2N^*+1}|),\quad j\geq N^*+1, \tag{24}$$

where

$$N^* = N - \left[\frac{N}{3}\right] - 1 = \left[\frac{2N-1}{3}\right].$$

Another one is to show that

$$\text{rank}\frac{\partial(B_1, B_3, \cdots, B_{2N^*+1})}{\partial(\gamma_1, \gamma_2, \cdots, \gamma_N)} = N^* + 1. \tag{25}$$

First, substituting the trigonometric transformations (20) and (21) into $F(z(x)) - F(x)$, we have

$$F(z(x)) - F(x) = \sum_{k=1}^{N} \gamma_k(\Theta^k(-\theta) - \Theta^k(\theta)) = \sum_{k=1}^{N} \gamma_k J_k(\theta) := \widetilde{F}(\theta). \tag{26}$$

By Lemma 23,

$$\widetilde{F}(\theta) = \sum_{k=1}^{N} \gamma_k J_k(\theta) = \sum_{k=1}^{N} \gamma_k \sum_{i=1}^{k} c_{ki} \sin(i\theta) = \sum_{i=1}^{N} \tilde{b}_i \sin(i\theta), \tag{27}$$

where

$$\tilde{b}_i = \sum_{k=1}^{N} \gamma_k c_{ki}, \tag{28}$$

and

$$\tilde{b}_i = 0 \;\text{ for } i \text{ satisfying } i \bmod 3 = 0. \tag{29}$$

We have the following expansion of $\widetilde{F}(\theta)$ for θ near π,

$$\begin{aligned}
\widetilde{F}(\theta) &= \sum_{i=1}^{N} \tilde{b}_i \sin(i\theta) = \sum_{i=1}^{N} \tilde{b}_i \sum_{j\geq 0} \cos(i\pi)\frac{(-1)^j}{(2j+1)!} i^{2j+1}(\theta-\pi)^{2j+1} \\
&= \sum_{i=1}^{N} \tilde{b}_i (-1)^i \sum_{j\geq 0} \frac{(-1)^j}{(2j+1)!} i^{2j+1}(\theta-\pi)^{2j+1} \\
&= \sum_{j\geq 0} \frac{(-1)^j}{(2j+1)!} \widetilde{B}_{2j+1}(\theta-\pi)^{2j+1},
\end{aligned} \tag{30}$$

where

$$\widetilde{B}_{2j+1} = \sum_{i=1}^{N} (-1)^i i^{2j+1} \tilde{b}_i. \tag{31}$$

By (20), $\theta - \pi = x + O(x^2)$ for $|x| \ll 1$. The equalities (23) and (30) show that

$$B_{2j+1} = \frac{(-1)^j}{(2j+1)!}\widetilde{B}_{2j+1} + O(|\widetilde{B}_1, \widetilde{B}_3, \cdots, \widetilde{B}_{2j-1}|), \quad j \geq 0, \tag{32}$$

which implies that

$$\widetilde{B}_{2j+1} = \frac{(-1)^j}{(2j+1)!}B_{2j+1} + O(|B_1, B_3, \cdots, B_{2j-1}|), \quad j \geq 0. \tag{33}$$

Therefore, it only needs to prove the following result in order to reach our first goal.

$$\widetilde{B}_{2j+1} = O(|\widetilde{B}_1, \widetilde{B}_3, \cdots, \widetilde{B}_{2N^*+1}|), \quad \text{for } j \geq N^* + 1. \tag{34}$$

Note from (27) and (29) that $\widetilde{F}(\theta)$ should be a linear collection of $N - \left[\frac{N}{3}\right]$ functions $\sin(i\theta)$, where $i \in \mathbb{S}$, which is the ordered sequence,

$$\mathbb{S} = \{1, 2, \cdots, N\}/\{i \bmod 3 = 0\} = \{m_1, m_2, \cdots, m_{N^*+1}\}.$$

Note $N - \left[\frac{N}{3}\right] = N^* + 1$, and so the equality (31) guarantees the matrix equation

$$(\widetilde{B}_1, \widetilde{B}_3, \cdots, \widetilde{B}_{2N^*+1}) = ((-1)^{m_1}\tilde{b}_{m_1}, (-1)^{m_2}\tilde{b}_{m_2}, \cdots, (-1)^{m_{N^*+1}}\tilde{b}_{m_{N^*+1}})\mathbb{M}_0, \tag{35}$$

where $\mathbb{M}_0$ is an $(N^*+1) \times (N^*+1)$ matrix with $\mathbb{M}_0[i, j] = m_i^{2j-1}$. A direct computation shows that

$$\det \mathbb{M}_0 = \prod_{i=1}^{N^*+1} m_i \prod_{1 \leq i < j \leq N^*+1} (m_j^2 - m_i^2) \neq 0.$$

Then, we have that

$$\widetilde{B}_{2j+1} = 0, \quad \text{for } 0 \leq j \leq N^*$$

if and only if $\tilde{b}_i = 0$ for $i = m_1, m_2, \cdots, m_{N^*+1}$, which implies that $\widetilde{F}(\theta) = 0$ if and only if $\widetilde{B}_{2j+1} = 0$ for all $0 \leq j \leq N^*$. Thus, (34) holds.

Finally, we need to prove that

$$\operatorname{rank} \frac{\partial(B_1, B_3, \cdots, B_{2N^*+1})}{\partial(\gamma_1, \gamma_2, \cdots, \gamma_N)} = N^* + 1. \tag{36}$$

The equation (32) gives the following matrix equation,

$$(B_1, B_3, \cdots, B_{2N^*+1}) = (\widetilde{B}_1, \widetilde{B}_3, \cdots, \widetilde{B}_{2N^*+1})\mathbb{M}_1, \tag{37}$$

where

$$\mathbb{M}_1 = \begin{pmatrix} \frac{(-1)^0}{1!} & * & \cdots & * \\ & \frac{(-1)^1}{3!} & \cdots & * \\ & & \ddots & \vdots \\ \mathbf{0} & & & \frac{(-1)^{N^*}}{(2N^*+1)!} \end{pmatrix}$$

and

$$\det \mathbb{M}_1 = \prod_{j=0}^{N^*} \frac{(-1)^j}{(2j+1)!} \neq 0.$$

It follows from (28) and Lemma 23 that

$$(\tilde{b}_1, \tilde{b}_2, 0, \tilde{b}_3, \tilde{b}_4, 0, \cdots, \tilde{b}_N) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \cdots, \gamma_N)\mathbb{M}_2,$$

where

$$\mathbb{M}_2 = \begin{pmatrix} c_{11} & 0 & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & 0 & 0 & \cdots & 0 \\ c_{31} & c_{32} & 0 & 0 & \cdots & 0 \\ c_{41} & c_{42} & 0 & c_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ c_{N1} & c_{N2} & 0 & c_{N4} & \cdots & c_{NN} \end{pmatrix},$$

which is an $N \times N$ triangular matrix with the $3j$th column being zero for $j = 1, 2, \cdots, [\frac{N}{3}]$. Then, we can delete all $3j$th columns and rows in $\mathbb{M}_2$ and assume $\alpha_{3j} = 0$ for $j = 1, 2, \cdots, [\frac{N}{3}]$. Therefore, we have the following result, by similarly using the ordered sequence $\mathbb{S}$,

$$(\tilde{b}_{m_1}, \tilde{b}_{m_2}, \cdots, \tilde{b}_{m_{N^*+1}}) = (\gamma_{m_1}, \gamma_{m_2}, \cdots, \gamma_{m_{N^*+1}})\mathbb{M}_3, \tag{38}$$

where

$$\mathbb{M}_3 = \begin{pmatrix} c_{11} & 0 & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & 0 & 0 & \cdots & 0 \\ c_{41} & c_{42} & c_{44} & 0 & \cdots & 0 \\ c_{51} & c_{52} & c_{54} & c_{55} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ c_{m_{N^*+1},1} & c_{m_{N^*+1},2} & c_{m_{N^*+1},4} & c_{m_{N^*+1},5} & \cdots & c_{m_{N^*+1},m_{N^*+1}} \end{pmatrix},$$

with

$$\begin{aligned} \det \mathbb{M}_3 &= \prod_{k=m_1,m_2,\cdots,m_{N^*+1}} c_{kk} = \prod_{k=m_1,m_2,\cdots,m_{N^*+1}} -\frac{1}{2^{k-2}} \cos\left(\frac{k\pi}{6} - \frac{k-1}{2}\pi\right) \\ &= 3^{\frac{N^*}{2}} 2^{-N^*} \prod_{k=m_1,m_2,\cdots,m_{N^*+1}} -\frac{1}{2^{k-2}} \neq 0, \end{aligned}$$

by Lemma 23.

Combining (35), (37) and (38) completes the proof for (36). With the reached two goals, we have proved our result on the Hopf cyclicity by Lemma 8. Therefore, system (19) has the Hopf cyclicity $\left[\frac{2N-1}{3}\right]$, so it is $\left[\frac{2n+1}{3}\right]$ for system (5) with $f(x) = f_2(x)$. This completes the proof of Theorem 3. □

5. Proof of Theorem 4

We only prove Theorem 4 for the case $l \ge n$, and the other case $l \le n$ can be similarly proved. Like (19), we can rewrite system (5) as

$$\dot{x} = y - F^{\pm}(x, \delta), \quad y = x(x-1)^3\Big(x + \frac{1}{2}\Big), \tag{39}$$

where

$$F^{-}(x,\delta) = -\int_0^x \sum_{i=0}^{l} \alpha_i^- s^i \, ds = -\sum_{i=0}^{l} \frac{1}{i+1} \alpha_i^- x^{i+1} := \sum_{i=1}^{L} \gamma_i^- x^i,$$

and

$$F^{+}(x,\delta) = -\int_0^x \sum_{i=0}^{n} \alpha_i^+ s^i \, ds = -\sum_{i=0}^{n} \frac{1}{i+1} \alpha_i^+ x^{i+1} := \sum_{i=1}^{N} \gamma_i^+ x^i$$

with $L = l+1$, $N = n+1$, $\gamma_i^{\pm} = -\frac{1}{i}\alpha_{i-1}^{\pm}$ for $i \ge 1$, and

$$\delta = (\gamma_1^-, \cdots, \gamma_L^-, \gamma_1^+, \cdots, \gamma_N^+) \in \mathbb{R}^{N+L}.$$

Proving Theorem 4 is equivalent to showing that the Hopf cyclicity of system (39) is $\left[\frac{3L+2N-1}{3}\right]$.

We have the bifurcation function of system (39) for $0 < x \ll 1$, given by

$$\begin{aligned} F(z(x)) - F(x) = F^-(z(x)) - F^+(x) &= \sum_{k=1}^{L} \gamma_k^- z^k(x) - \sum_{k=1}^{N} \gamma_k^+ x^k \\ &= \sum_{k=1}^{L} \big[b_k (z^k(x) - x^k) + e_k (z^k(x) + x^k) \big], \end{aligned} \tag{40}$$

where

$$b_k = \begin{cases} \frac{\gamma_k^- + \gamma_k^+}{2}, & 1 \le k \le N, \\ \frac{\gamma_k^-}{2}, & N < k \le L, \end{cases} \qquad e_k = \begin{cases} \frac{\gamma_k^- - \gamma_k^+}{2}, & 1 \le k \le N, \\ \frac{\gamma_k^-}{2}, & N < k \le L. \end{cases}$$

We again use the transformations (20) and (21) to obtain with Lemma 23,

$$
\begin{aligned}
F(z(x)) - F(x) &= \sum_{k=1}^{L} b_k J_k(\theta) + e_k K_k(\theta) \\
&= \sum_{k=1}^{L} \big(b_k \sum_{i=1}^{k} c_{ki} \sin(i\theta) + e_k \sum_{i=0}^{k} \tilde{c}_{ki} \cos(i\theta) \big), \\
&= \sum_{i=1}^{N} \tilde{b}_i \sin(i\theta) + \sum_{i=0}^{N} \tilde{e}_i \cos(i\theta) + \sum_{i=N+1}^{L} \gamma_i^- R_i(\theta) := \widetilde{F}(\theta),
\end{aligned}
\tag{41}
$$

where

$$
\tilde{b}_i = \sum_{k=i}^{L} b_k c_{ki}, \quad \tilde{e}_0 = \sum_{k=1}^{L} e_k \tilde{c}_{k0}, \quad \tilde{e}_i = \sum_{k=i}^{L} e_k \tilde{c}_{ki},
$$

for $1 \le i \le N$ and

$$
R_i(\theta) = \frac{1}{2} \sum_{j=N+1}^{i} \big(c_{ij} \sin(j\theta) + \tilde{c}_{ij} \cos(j\theta) \big),
$$

for $N+1 \le i \le L$, with $\tilde{b}_i = 0$ if i mod $3 = 0$ for $1 \le i \le N$. We have

$$
\tilde{e}_0 = -\sum_{i=1}^{N} \tilde{e}_i - \sum_{i=N+1}^{L} \frac{\gamma_i^-}{2} \sum_{j=N+1}^{i} \tilde{c}_{ij}
$$

by $\widetilde{F}(\pi) = 0$. Then,

$$
\widetilde{F}(\theta) = \sum_{i \ mod \ 3 \neq 0}^{1 \le i \le N} \tilde{b}_i \sin(i\theta) + \sum_{i=1}^{N} \tilde{e}_i (\cos(i\theta) - 1) + \sum_{i=N+1}^{L} \gamma_i^- \widetilde{R}_i(\theta), \tag{42}
$$

with

$$
\widetilde{R}_i(\theta) = \frac{1}{2} \sum_{j=N+1}^{i} \big(c_{ij} \sin(j\theta) + \tilde{c}_{ij} (\cos(j\theta) - 1) \big), \ N+1 \le i \le L.
$$

We define the parameter vector

$$
\mathbf{v} = (\tilde{b}_1, \tilde{b}_2, \tilde{b}_4, \tilde{b}_5, \tilde{b}_7, \cdots, \tilde{e}_1, \cdots, \tilde{e}_N, \gamma_{N+1}^-, \gamma_{N+2}^-, \cdots, \gamma_L^-),
$$

which has the dimension $L + N - \left[\frac{N}{3}\right]$. Thus, we can show that

$$
\text{rank} \frac{\partial \mathbf{v}}{\partial (b_1, b_2, \cdots, b_N, e_1, e_2, \cdots, e_N, \gamma_{N+1}^-, \cdots, \gamma_L^-)} = L + N - \left[\frac{N}{3}\right]
$$

by a similar proof for (38). Then,

$$\mathrm{rank}\frac{\partial \mathbf{v}}{\partial(\gamma_1^+,\cdots,\gamma_N^+,\gamma_1^-,\cdots,\cdots,\gamma_L^-)}=L+N-\left[\frac{N}{3}\right]. \tag{43}$$

$\widetilde{F}(\theta)$ can be expanded near $\theta=\pi$ as below,

$$\widetilde{F}(\theta)=\widetilde{B}_1(\theta-\pi)+\widetilde{B}_2(\theta-\pi)^2+\widetilde{B}_3(\theta-\pi)^3+\cdots, \tag{44}$$

where each coefficient $\widetilde{B}_i$ is a linear collection of the entries of $\mathbf{v}$. Let s be the maximal number of linear independent coefficients in (44), $S=\{r_1,r_2,\cdots,r_s\}$ with $r_i<r_{i+1}$ and these coefficients be denoted by

$$\widetilde{B}_{r_1},\widetilde{B}_{r_2},\cdots,\widetilde{B}_{r_s}.$$

Obviously, these independent coefficients can be determined one by one by taking $\widetilde{B}_{r_1}$ to be the first nonzero coefficient in (44) and $\widetilde{B}_{r_j}$ the first one that independent of $\widetilde{B}_{r_1},\cdots,\widetilde{B}_{r_{j-1}}$, up to the sth one. Then, $s\le L+N-\left[\frac{N}{3}\right]$ and

$$\widetilde{B}_j=LC_j(\widetilde{B}_{r_1},\widetilde{B}_{r_2},\cdots,\widetilde{B}_{r_j^*}), \tag{45}$$

where $j\notin S$ and $j<r_s$ with $r_j^*=\max\{r_i\in S|r_i<j\}$, LC_j denotes a linear combination, and $\widetilde{B}_j=O(\widetilde{B}_{r_1},\cdots,\widetilde{B}_{r_s})$ for $j>r_s$. There exists an $s\times\left(L+N-\left[\frac{N}{3}\right]\right)$ matrix $\mathcal{M}_1$ such that,

$$\left(\widetilde{B}_{r_1},\widetilde{B}_{r_2},\cdots,\widetilde{B}_{r_s}\right)^T=\mathcal{M}_1\mathbf{v}^T. \tag{46}$$

In the following, we prove the claim,

$$\mathrm{rank}\mathcal{M}_1=L+N-\left[\frac{N}{3}\right],$$

which only need prove all $\widetilde{B}_{r_j}=0$, $j=1,2,\cdots,s$, if and only if $\mathbf{v}=0$. By definition of $\widetilde{B}_{r_j}$, $j=1,2,\cdots,s$, it only need to prove $\widetilde{F}(\theta)\equiv 0$ if and only if $\mathbf{v}=0$.

The elements in (42), $\{\sin i\theta,\cos s\theta-(-1)^s,\widetilde{R}_u(\theta)\}$ with $1\le i\le N$, $i \bmod 3\ne 0$, $1\le j\le N$ and $N+1\le u\le L$ is a Chebyshev system of dimension $L+N-\left[\frac{N}{3}\right]$. Then, $\widetilde{F}(\theta)\equiv 0$ if and only if $\mathbf{v}=0$.

Suppose

$$F^-(z(x))-F^+(x)=B_1x+B_2x^2+\cdots+B_jx^j+\cdots,\quad 0<x\ll 1. \tag{47}$$

By (20), $\theta-\pi=x+O(x^2)$ for $|x|\ll 1$. We substitute it into (47) and compare the coefficients to get

$$B_j=\widetilde{B}_j+\widetilde{E}_i(\widetilde{B}_1,\widetilde{B}_2,\cdots,\widetilde{B}_{j-1}),\quad j=1,2,\cdots, \tag{48}$$

where $\widetilde{E}_i$ is a linear function. Then, by (45) and (48), we have

$$B_j = O(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_j^*}), \quad \text{for } j \notin S \tag{49}$$

and

$$B_{r_j} = \widetilde{B}_{r_j} + \widetilde{E}^*_{r_j}(\widetilde{B}_1, \widetilde{B}_2, \cdots, \widetilde{B}_{r_{j-1}}), \quad \text{for } r_j \in S, \tag{50}$$

which further implies

$$\widetilde{B}_{r_j} = B_{r_j} + E_{r_j}(B_{r_1}, B_{r_2}, \cdots, B_{r_{j-1}}), \quad \text{for } r_j \in S, \tag{51}$$

where $\widetilde{E}^*_{r_j}$ and E_{r_j} are linear functions. Then, combining (49) and (51) yields

$$B_j = O(B_{r_1}, B_{r_2}, \cdots, B_{r_j^*})$$

for $j \notin S$. The reminder of the proof is to show

$$\operatorname{rank} \frac{\partial(B_{r_1}, B_{r_2}, \cdots, B_{r_s})}{\partial(\gamma_1^+, \cdots, \gamma_N^+, \gamma_1^- \cdots, \gamma_L^-)} = L + N - \left[\frac{N}{3}\right]. \tag{52}$$

By (50), we have

$$\operatorname{rank} \frac{\partial(B_{r_1}, B_{r_2}, \cdots, B_{r_s})}{\partial(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_s})} = L + N - \left[\frac{N}{3}\right], \tag{53}$$

and so

$$\operatorname{rank} \frac{\partial(\widetilde{B}_{r_1}, \widetilde{B}_{r_2}, \cdots, \widetilde{B}_{r_s})}{\partial \mathbf{v}} = \operatorname{rank} \mathcal{M}_1 = L + N - \left[\frac{N}{3}\right] \tag{54}$$

by (46). Combining (43), (53) and (54), we have shown that (52) holds. Therefore, the Hopf cyclicity of system (39) is $L + N - \left[\frac{N}{3}\right] - 1 = \left[\frac{3L+2N-1}{3}\right]$. This completes the proof of Theorem 4.

6. Conclusion

In this work, we conduct a further study on a Liénard system and give a rigorous proof to the open question remained in [5,36]. We prove that the cyclicity of the periodic annulus of the Hamiltonian is 3 by showing the sharp bound to be 3 on the maximal number of zeros of the associated Abelian integral. The annulus cyclicity can be extended to the elementary center because the displacement map is analytic for $h = 0$. The non-symmetry and degeneracy of the system causes much difficulty in the computation and analysis for the Poincaré bifurcation, as well as in the study of Hopf bifurcation. We have obtained the Hopf cyclicity as $\left[\frac{2n+1}{3}\right]$ when the damping term is a smooth polynomial with an arbitrary degree n. The involution determined by the annulus is well utilized, based on which a transformation composed of trigonometric functions is introduced, which provides a tool to overcome the difficulty in analysis and computation. However, it is not easy to find such kind of a transformation for the involution in a general undamped Liénard system. It is even unknown if there exists such a transformation for the involution of a Hamiltonian. In this paper, we find such a transform for our system that belongs to hyper-elliptic Hamiltonian. We have also studied the Hopf cyclicity of the origin when the damping term is a non-smooth polynomial with the switching manifold at the y-axis, having respectively degrees l and n, and proved that the Hopf cyclicity is $\left[\frac{3l+2n+4}{3}\right]$ ($\left[\frac{3n+2l+4}{3}\right]$) when $l \geq n$ ($n \geq l$).

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