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Chaos induced by regular snap-back repellers ☆

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Abstract

This paper is concerned with chaos induced by regular snap-back repellers. One new criterion of chaos induced by strictly coupled-expanding maps in compact sets of metric spaces is established. By employing this criterion, the nondegenerateness assumption in the Marotto theorem established in 1978 is weakened. In addition, it is proved that a regular snap-back repeller and a regular homoclinic orbit to a regular expanding fixed point in finite-dimensional spaces imply chaos in the sense of Li–Yorke. An illustrative example is provided with computer simulations.

Keywords: Chaos; Snap-back repeller; Coupled-expanding map

1. Introduction

During the late 1970s and the decade of the 1980s, chaos of one-dimensional maps has been extensively studied, and many elegant results have been obtained (cf. [1,3,8,10,15,20] and the references cited therein). In 1992, Block and Coppel introduced the concept of turbulence for continuous interval maps [3]; that is, a continuous interval map $f: I \to I$ is said to be turbulent if there exist compact subintervals J and K with at most one common point such that

$$f(J) \supset J \cup K, \qquad f(K) \supset J \cup K.$$

Further, it is said to be strictly turbulent if J and K can be chosen disjoint. It has been proved that a turbulent interval map has a positive topological entropy and is chaotic in the sense of both Li–Yorke and Devaney (see Definitions 2.2 and 2.3).

Although higher-dimensional chaos problems are difficult to study, some important progresses have been made. In 1978, Marotto [16] introduced the concept of snap-back repeller for continuously differentiable d-dimensional maps and established a criterion of chaos: snap-back repellers imply chaos in the sense of Li–Yorke (see [17] and [22] for correction of a certain error found in [16]). This criterion plays an important role in the study of chaos for higher-

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but finite-dimensional noninvertible maps (cf. [9,19] and some references cited in [17]). Recently, it was employed to study the occurrence of chaos in partial differential equations, neural networks, and financial markets [4–6,13].

Now, we briefly recall some recent results on the chaos theory for maps in metric spaces. In 2001, Kennedy and Yorke [14] proved that a continuous map in a compact invariant set of a metric space is topologically semiconjugate to a one-sided symbolic dynamical system under the horseshoe hypotheses. Recently, we captured the essential meanings of the concept of turbulence for interval maps and extended it to maps in metric spaces [26], where the maps were still called turbulent. Since the term turbulence is well established in fluid mechanics, we changed the term "turbulent map" to "coupled-expanding map" in the conference paper [24] (see Definition 2.1 in Section 2). The new name is more intuitive in reflecting the conditions that the map satisfies. In 2004, Yang and Tang extended the one-dimensional turbulence result [3, Chapter II, Proposition 15] to maps in metric spaces and showed that if a continuous map is strictly coupled-expanding in mutually disjoint compact sets of a metric space, then the map in a compact invariant set is semiconjugate to a one-sided symbolic dynamical system [31, Theorem 1] (the term "strictly coupled-expanding" is used here for briefness). Hence, the map has a positive topological entropy and is chaotic in the sense of Li–Yorke by [2, Corollary 2.4] in [14] and [31], respectively. However, a higher-dimensional map, which is strictly coupled-expanding in compact sets, is not necessarily chaotic in the sense of Devaney (see Example 2.1). We proved that under an expanding condition in distance, a strictly coupled-expanding continuous map in disjoint compact sets of a metric space is topologically conjugate to a one-sided symbolic dynamical system and consequently, is chaotic in the sense of both Devaney and Li–Yorke [21, Theorem 3.2]. In addition, several criteria of chaos induced by coupled-expanding maps in bounded and closed sets (may be noncompact) of complete metric spaces were established [21,26]. We further developed the snap-back repeller theory by the coupled-expansion theory. We extended the concept of snap-back repeller for continuously differentiable finite-dimensional maps by Marotto to maps in general metric spaces, and divided it into two classifications: regular and singular, nondegenerate and degenerate [21] (see Definition 2.5 in Section 2). In the Marotto paper [16], a snap-back repeller is regular and nondegenerate. We proved that a nondegenerate and regular snap-back repeller or a nondegenerate and regular homoclinic orbit to an expanding fixed point can generate a strict coupled-expansion and so generate chaos in the sense of both Devaney and Li–Yorke (cf. [21,26] for maps in complete metric spaces and [22] for Banach spaces). Consequently, the snap-back repeller in the Marotto theorem [16] implies chaos in the sense of Devaney as well as Li–Yorke. This is analogous to the Smale–Birkhoff homoclinic theorem, which claims, in brief, that a transversal homoclinic orbit can generate a horseshoe and so generate chaos for diffeomorphisms. These criteria of chaos induced by coupled-expanding maps and snap-back repellers were applied to study chaotification (or anti-control) problems for maps in higher-dimensional and general Banach spaces [23,25,29]. We refer to [27] for a survey of chaos criteria induced by snap-back repellers and their applications to anti-control of chaos.

It is noted that snap-back repellers are always required to be nondegenerate and regular in the literature [4–6,9,13, 16,17,19,21,22,25,29]. We study regular snap-back repellers, which may be degenerate, in the present paper. To do so, we will first establish a generalized inverse function theorem and a criterion of chaos induced by coupled-expanding maps in compact sets of metric spaces.

The rest of the paper is organized as follows. In Section 2, some basic concepts and lemmas are introduced. A generalized inverse function theorem is established, which is very useful in studying degenerate snap-back repellers. In Section 3, a criterion of chaos induced by strictly coupled-expanding maps in compact sets is established, in which the maps are proved to be chaotic in the sense of both Li–Yorke and Wiggins. By applying these results, the other two criteria of chaos characterized by regular snap-back repellers in finite-dimensional spaces are obtained in Section 4. These snap-back repellers may be degenerate. The assumptions in the Marotto theorem [16] are weakened. In order to illustrate that the assumptions given in the present paper are weaker than those in the relative existing results, an example is provided in Section 5 with computer simulations.

Remark 1.1. The criteria of chaos induced by regular snap-back repellers will be applied to study chaos in partial difference equations in our another paper [28].

2. Preliminaries

In this section, some basic concepts and lemmas are introduced. This section is divided into two subsections.

2.1. Some basic concepts

Three definitions of chaos in the sense of Li–Yorke, Devaney, and Wiggins are given in this subsection. For convenience, the concepts of coupled-expanding map, expanding fixed point, snap-back repeller, and homoclinic orbit are introduced.

Definition 2.1. Let (X,d) be a metric space and $f: D \subset X \to X$ a map. If there exist m $(\geqslant 2)$ subsets V_i, $1 \leqslant i \leqslant m$, of D with $V_i \cap V_j = \partial_D V_i \cap \partial_D V_j$ for each pair of (i,j), $1 \leqslant i \neq j \leqslant m$, such that

$$f(V_i) \supset \bigcup_{j=1}^{m} V_j, \quad 1 \leqslant i \leqslant m, \tag{2.1}$$

where $\partial_D V_i$ is the relative boundary of V_i with respect to D, then f is said to be coupled-expanding in V_i, $1 \leqslant i \leqslant m$. Further, the map f is said to be strictly coupled-expanding in V_i, $1 \leqslant i \leqslant m$, if $d(V_i, V_j) > 0$ for all $1 \leqslant i \neq j \leqslant m$.

Remark 2.1. In the case of $X = \mathbf{R}$, when f maps an interval I into itself, and V_1 and V_2 are closed and bounded subintervals of I, the definitions of coupled-expansion and strict coupled-expansion are the same as that of turbulence and strict turbulence for interval maps by Block and Coppel [3] (see the first part in Section 1). It is noted that in Definition 2.1, the sets V_i, $1 \leqslant i \leqslant m$, may not be connected and compact, and the number m of the sets V_i may be larger than 2. These differences make the definition more convenient and more universal. The coupled-expansion for a transitive matrix will be studied in our forthcoming paper.

Remark 2.2. The concept of coupled-expanding map has some similar idea to Markov partitions for a diffeomorphism (cf. [11,20] and some references cited therein).

Definition 2.2. Let (X,d) be a metric space and $f: X \to X$ a map. Then, $S \subset X$ is called a scrambled set of f if, for any two distinct points $x, y \in S$,

$$\liminf_{n\to\infty} d\big(f^n(x), f^n(y)\big) = 0; \qquad \limsup_{n\to\infty} d\big(f^n(x), f^n(y)\big) > 0.$$

The map f is said to be chaotic in the sense of Li–Yorke if it has an uncountable scrambled set S.

Definition 2.3. (See [8].) Let X be a metric space. A map $f: V \subset X \to V$ is said to be chaotic on V in the sense of Devaney if

(i) the set of the periodic points of f in V is dense in V;
(ii) f is topologically transitive in V;
(iii) f has sensitive dependence on initial conditions in V.

It is noted that condition (iii) is redundant in the above definition if f is continuous in V by the result of [1].

Definition 2.4. Let X be a metric space. A map $f: V \subset X \to V$ is said to be chaotic on V in the sense of Wiggins if f satisfies conditions (ii) and (iii) in Definition 2.3.

Remark 2.3. Definition 2.4 is the same as [20, p. 86, Definition]. Wiggins gave this definition in the special case that $X = \mathbf{R}^n$ and V is a compact set of $\mathbf{R}^n$ [30, Definition 4.11.2]. Since this definition was first introduced by Wiggins, we use here the term "chaos in the sense of Wiggins."

Obviously, chaos in the sense of Devaney is stronger than that in the sense of Wiggins. But the converse is not true in general (see a counterexample in [18]). Under some conditions, chaos in the sense of Devaney and Wiggins is stronger than that in the sense of Li–Yorke by [12, Theorem 4.1]. However, chaos in the sense of Li–Yorke does not necessarily imply chaos in the sense of Devaney in general, shown by the following example, which also shows that a strict coupled-expansion in disjoint compact sets of metric space does not necessarily imply chaos in the sense of Devaney.

Example 2.1. Consider the following three-dimensional map:

$$F := f \times g : [0,\ 1] \times S^1 \to \mathbf{R}^3,$$

where $f : [0,\ 1] \to \mathbf{R}$ with $f(x) = 5x(1-x)$ and $g : S^1 \to S^1$ with $g(e^{i\theta}) = e^{i(\theta+\theta_0)}$, where θ_0/π is irrational. It is evident that F is continuous in $D := [0,\ 1] \times S^1$. Let

$$V_1 = [0,\ x_1] \times S^1, \qquad V_2 = [x_2,\ 1] \times S^1$$

with $x_1 = (5-\sqrt{5})/10$ and $x_2 = (5+\sqrt{5})/10$. Then V_1 and V_2 are disjoint compact subsets of D and

$$F(V_1) = F(V_2) = D \supset V_1 \cup V_2.$$

So, F is strictly coupled-expanding in V_i, $i = 1, 2$, and consequently, it has a positive topological entropy and chaotic in the sense of Li–Yorke. However, F has no periodic points in D since g is an irrational rotation. Hence, F is not chaotic in the sense of Devaney on any invariant subset of D.

In the following, by $B_r(x)$ and $\bar{B}_r(x)$ denote the open and closed balls of radius r centered at $x \in X$, respectively.

Definition 2.5. (See [21, Definitions 2.1–2.4].) Let (X, d) be a metric space and $f : X \to X$ a map.

(i) A point $z \in X$ is called an expanding fixed point (or a repeller) of f in $\bar{B}_{r_0}(z)$ for some constant $r_0 > 0$, if $f(z) = z$ and

$$d\big(f(x), f(y)\big) \geqslant \lambda\, d(x, y), \quad \forall x, y \in \bar{B}_{r_0}(z)$$

for some constant $\lambda > 1$. Furthermore, z is called a regular expanding fixed point of f in $\bar{B}_{r_0}(z)$ if z is an interior point of $f(B_{r_0}(z))$.

(ii) Assume that z is an expanding fixed point of f in $\bar{B}_{r_0}(z)$ for some $r_0 > 0$. Then z is said to be a snap-back repeller of f if there exists a point $x_0 \in B_{r_0}(z)$ with $x_0 \neq z$ and $f^m(x_0) = z$ for some positive integer $m \geqslant 2$. Furthermore, z is said to be a nondegenerate snap-back repeller of f if there exist positive constants μ and δ_0 such that $B_{\delta_0}(x_0) \subset B_{r_0}(z)$ and

$$d\big(f^m(x), f^m(y)\big) \geqslant \mu\, d(x, y), \quad \forall x, y \in \bar{B}_{\delta_0}(x_0);$$

z is called a regular snap-back repeller of f if $f(B_{r_0}(z))$ is open and there exists a positive constant δ_0^* such that $B_{\delta_0^*}(x_0) \subset B_{r_0}(z)$ and z is an interior point of $f^m(B_\delta(x_0))$ for any positive constant $\delta \leqslant \delta_0^*$.

(iii) Assume that $z \in X$ is a regular expanding fixed point of f. Let U be the maximal open neighborhood of z in the sense that for any $x \in U$ with $x \neq z$, there exists $k \geqslant 1$ with $f^k(x) \notin U$, $f^{-n}(x)$ is uniquely defined in U for all $n \geqslant 1$, and $f^{-n}(x) \to z$ as $n \to \infty$. U is called the local unstable set of F at z and is denoted by $W^u_{\mathrm{loc}}(z)$.

(iv) Assume that $z \in X$ is a regular expanding fixed point of f. A point $x \in X$ is called homoclinic to z if $x \in W^u_{\mathrm{loc}}(z)$, $x \neq z$, and there exists an integer $n \geqslant 1$ such that $f^n(x) = z$. A homoclinic orbit to z, consisting of a homoclinic point x with $f^n(x) = z$, its backward orbit $\{f^{-j}(x)\}_{j=1}^{\infty}$, and its finite forward orbit $\{f^j(x)\}_{j=1}^{n-1}$, is called nondegenerate if for each point x_0 on the homoclinic orbit there exist positive constants r_1 and μ_1 such that

$$d\big(f(x), f(y)\big) \geqslant \mu_1\, d(x, y), \quad \forall x, y \in \bar{B}_{r_1}(x_0).$$

A homoclinic orbit is called regular if for each point x_0 on the orbit, there exists a positive constant r_2 such that for any positive constant $r \leqslant r_2$, $f(x_0)$ is an interior point of $f(B_r(x_0))$.

2.2. Several lemmas

In this subsection, several lemmas are given. Especially, a generalized inverse function theorem in finite-dimensional spaces is established.

Lemma 2.1. *Let (X, d) be a metric space, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of compact sets of X and satisfy the nestedness condition $A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset \cdots$. Then $\bigcap_{n=1}^{\infty} A_n$ contains a single point if and only if $d(A_n) \to 0$ as $n \to \infty$, where $d(A_n)$ is the diameter of A_n.*

Proof. The sufficiency is well known and the proof of the necessity is elementary. So the details are omitted. □

Since the one-sided symbolic dynamical system (Σ_N^+, σ) is used in this paper, we first briefly recall its properties for convenience. Let $S := \{1, 2, \ldots, N\}$ and

$$\Sigma_N^+ := \{\alpha = (a_0, a_1, a_2, \ldots)\colon a_i \in S,\ i \geqslant 0\}$$

with the distance

$$\rho(\alpha, \beta) := \sum_{i=0}^{\infty} \frac{d_1(a_i, b_i)}{2^i}, \quad d_1(a_i, b_i) = \begin{cases} 0, & \text{if } a_i = b_i, \\ 1, & \text{if } a_i \neq b_i, \end{cases}$$

where $\alpha = (a_0, a_1, a_2, \ldots)$ and $\beta = (b_0, b_1, b_2, \ldots)$. Then (Σ_N^+, ρ) is a complete metric space and a Cantor set. The shift map $\sigma : \Sigma_N^+ \to \Sigma_N^+$ defined by $\sigma(a_0, a_1, a_2 \ldots) = (a_1, a_2, \ldots)$ is continuous. The system governed by σ is called the one-sided symbolic dynamical system on N symbols. It has plentiful dynamical behaviors (cf. e.g., [8,10,20]). Especially, it is chaotic in the sense of both Devaney and Li–Yorke, and has a positive topological entropy.

Lemma 2.2. *Let (X, d) be a metric space and $f : D \subset X \to X$ a map. Assume that a map f is coupled-expanding in V_j, $1 \leqslant j \leqslant N (N \geqslant 2)$.*

(i) *If V_α is a singleton set for some $\alpha = (a_0, a_1, a_2, \ldots) \in \Sigma_N^+$, then $V_{\sigma(\alpha)}$ is also a singleton set, where $V_\alpha := \bigcap_{n=0}^{\infty} f^{-n}(V_{a_n})$.*
(ii) *If f is injective in V_j, $1 \leqslant j \leqslant N$, respectively, and $V_{\sigma(\alpha)}$ is a singleton set for some $\alpha \in \Sigma_N^+$, then V_α is also a singleton set.*

Proof. Since the proof is elementary, its details are omitted. □

Lemma 2.3. *Let (X, d) be a metric space, $f : D \subset X \to X$ a map, and V a subset of D with $\bar{V} \subset D$.*

(i) *If the set of the periodic points of f in V is dense in V, then the set of the periodic points of f in $\bar{V}$ is dense in $\bar{V}$.*
(ii) *If f is topologically transitive in V, so is f in $\bar{V}$.*
(iii) *If f has sensitive dependence on initial conditions in V, so does f in $\bar{V}$.*

Proof. The proofs of results (i) and (ii) are trivial. So their details are omitted.

We only show result (iii). Suppose that f has sensitive dependence on initial conditions in V with a sensitivity constant δ. Now, suppose that f does not have sensitive dependence on initial conditions in $\bar{V}$. Then, there exists a relatively nonempty open set U with respect to $\bar{V}$ such that

$$d\big(f^n(x), f^n(y)\big) < \delta \quad \forall x, y \in U \text{ and } \forall n \geqslant 1. \tag{2.2}$$

It is evident that $U \cap V \neq \phi$ and consequently, $U \cap V$ is a relatively nonempty open set with respect to V. So, by the assumption, there exist $x_1, x_2 \in U \cap V$ and a positive integer k such that $d(f^k(x_1), f^k(x_2)) \geqslant \delta$, which contradicts (2.2). Hence, f has sensitive dependence on initial conditions in $\bar{V}$. The proof is complete. □

Lemma 2.4. *Let (X, d) be a metric space, $f : Y \subset X \to X$ a map, and D a nonempty subset of Y. Assume that f^j is continuous in D with $f^j(D) \subset Y$, $1 \leqslant j \leqslant n - 1$, and $f^n(D) = D$ for some positive integer n. Then $f(E) = E$ with $E = \bigcup_{j=0}^{n-1} f^j(D)$. Furthermore,*

(i) *if f^n has sensitive dependence on initial conditions in D, so does f in E;*
(ii) *if f^n is topologically transitive in D, so is f in E.*

Proof. By the assumption that $f^n(D) = D$, it can be easily verified that $f(E) = E$.

(1) Suppose that f^n has sensitive dependence on initial conditions in D with a sensitivity constant δ. Let U be any relatively nonempty open set with respect to E. There exists a j, $0 \leqslant j \leqslant n - 1$, such that $U \cap f^j(D) \neq \phi$. So, $D \cap$

$f^{-j}(U)$ is a relatively nonempty open set with respect to D by using the continuity of f and $f(E)=E$. Further, by the assumption, there exist two points $x, y \in D \cap f^{-j}(U)$ and a positive integer m such that $d(f^{mn}(x), f^{mn}(y)) \geqslant \delta$, which implies that $d(f^{mn-j}(f^j(x)), f^{mn-j}(f^j(y))) \geqslant \delta$. It is clear that $mn-j \geqslant 1$ and $f^j(x), f^j(y) \in U$. Hence, f also has sensitive dependence on initial conditions in E and δ is also a sensitivity constant of f in E.

(2) Suppose that f^n is topologically transitive in D. Let U and V be any two relatively nonempty open sets with respect to E. There exist i, j, $0 \leqslant i, j \leqslant n-1$, such that $U \cap f^i(D) \neq \phi$ and $V \cap f^j(D) \neq \phi$. So, $D \cap f^{-i}(U)$ and $D \cap f^{-j}(V)$ are relatively nonempty open sets with respect to D by using the continuity of f and $f(E)=E$. Since f^n is topologically transitive in D, there exists a positive integer k such that

$$f^{kn}\big(D \cap f^{-i}(U)\big) \cap \big(D \cap f^{-j}(V)\big) \neq \phi,$$

which implies that there exists a point $x \in D \cap f^{-i}(U)$ such that $f^{kn}(x) \in D \cap f^{-j}(V)$. This yields that $f^i(x) \in U$ and $f^{kn+j-i}(f^i(x)) = f^{kn+j}(x) \in V$. It is evident that $kn+j-i \geqslant 1$. Hence, $f^{kn+j-i}(U) \cap V \neq \phi$ and consequently, f is topologically transitive in E. The entire proof of the lemma is finished. □

Finally, we establish a generalized inverse function theorem in finite-dimensional spaces.

By the well-known inverse function theorem (cf. [7,20]), if a map $f: E \subset \mathbf{R}^d \to \mathbf{R}^d$ is continuously differentiable with the Jacobian $\det Df(x_0) \neq 0$, where E is an open set and contains x_0, then there exist two open sets $U \subset E$ and $V \subset \mathbf{R}^d$ and a unique map g such that

(i) $x_0 \in U$, $f(x_0) \in V$, and $Df(x)$ is invertible in U;
(ii) $f: U \to V$ is bijective;
(iii) $g: V \to U$ is continuously differentiable and $f(g(y)) = y$ for all $y \in V$.

This implies that g is the inverse of the map f and also continuously differentiable in some neighborhood of $f(x_0)$. In this theorem, the condition $\det Df(x_0) \neq 0$ is very important. We shall remark that the condition $\det Df^m(x_0) \neq 0$ is also important in the existing results that snap-back repellers imply chaos [16,17,22], where x_0 and m are specified in Definition 2.5. It guarantees that the snap-back repeller is regular and nondegenerate. It is very interesting how to weaken this condition such that the inverse of the map f still exists. It is noted that if the inverse map exists and is differentiable at $f(x_0)$, then $\det Df(x_0) \neq 0$. So, if this condition does not hold and the inverse map exists, then the inverse map may be continuous in some neighborhood of $f(x_0)$, but is not differentiable at $f(x_0)$.

Lemma 2.5 *(A generalized inverse function theorem). Let E be an open set of $\mathbf{R}^d$ and a map $f: E \to \mathbf{R}^d$ continuously differentiable with $x_0 \in E$ and* $\det Df(x_0) = 0$. *If* $\operatorname{rank} Df(x_0) = d-1$ *and* $\det Df(x) \neq 0$ *does not change the sign in $B_{r_0}(x_0) \setminus \{x_0\}$ for some constant $r_0 > 0$, then there exist two open sets $U \subset B_{r_0}(x_0)$ and $V \subset \mathbf{R}^d$ and a unique map g such that*

(i) *$x_0 \in U$, $f(x_0) \in V$, and* $\det Df(x) \neq 0$ *for all $x \in U \setminus \{x_0\}$*;
(ii) *$f: U \to V$ is bijective*;
(iii) *$g: V \to U$ is continuous and $f(g(y)) = y$ for all $y \in V$. Furthermore, g is continuously differentiable in $V \setminus \{f(x_0)\}$.*

Proof. Without loss of generality, suppose that $\det D_{x'} f'(x', x_d)|_{x=x_0} \neq 0$, where $f = (f', f_d)$, $x = (x', x_d)$, f' is a $(d-1)$-dimensional vector-valued function, and $x' \in \mathbf{R}^{d-1}$. By the inverse function theorem with parameters, there exist open sets $U_1, V_1 \subset \mathbf{R}^{d-1}$ and $U_2 \subset \mathbf{R}$ with $U_1 \times U_2 \subset E$, and a unique map g' such that

(1) $x'_0 \in U_1$, $y'_0 \in V_1$, $x_d^0 \in U_2$, and $D_{x'} f'(x', x_d)$ is invertible in $U_1 \times U_2$;
(2) $f'(\cdot, x_d): U_1 \to \mathbf{R}^{d-1}$ is injective for any $x_d \in U_2$;
(3) $g': V_1 \times U_2 \to U_1$ is continuously differentiable and $f'(g'(y', x_d), x_d) = y'$ for all $y' \in V_1$ and for all $x_d \in U_2$;

where $x_0 = (x'_0, x_d^0)$, $x'_0 \in \mathbf{R}^{d-1}$, $y_0 = f(x_0)$, and y'_0 is similarly defined to x'_0. Further, from the equation in (3), we get that for all $y' \in V_1$, $x_d \in U_2$,

$$D_{y'}g'(y',x_d) = \big(D_{x'}f'(x',x_d)\big)^{-1}\big|_{x'=g'(y',x_d)},$$
$$D_{x_d}g'(y',x_d) = -\big(D_{x'}f'(x',x_d)\big)^{-1}D_{x_d}f'(x',x_d)\big|_{x'=g'(y',x_d)}. \tag{2.3}$$

Now, we consider the following scalar map with the parameter $y' \in V_1$:

$$h(y',\cdot) := f_d\big(g'(y',\cdot),\cdot\big): U_2 \to \mathbf{R}.$$

It is clear that $h(y'_0, x^0_d) = y^0_d$, $h(y',x_d)$ is continuously differentiable in $V_1 \times U_2$, and

$$D_{x_d}h(y',x_d) = D_{x'}f_d\big(g'(y',x_d),x_d\big)D_{x_d}g'(y',x_d) + D_{x_d}f_d\big(g'(y',x_d),x_d\big),$$

which, together with (2.3), implies that for any $y' \in V_1$ and for any $x_d \in U_2$,

$$D_{x_d}h(y',x_d) = \gamma(x',x_d), \tag{2.4}$$

where $x' = g'(y',x_d)$ and

$$\gamma(x',x_d) = -D_{x'}f_d(x',x_d)\big(D_{x'}f'(x',x_d)\big)^{-1}D_{x_d}f'(x',x_d) + D_{x_d}f_d(x',x_d).$$

On the other hand, we have

$$\det Df(x) = \det\begin{pmatrix} D_{x'}f'(x',x_d) & D_{x_d}f'(x',x_d) \\ D_{x'}f_d(x',x_d) & D_{x_d}f_d(x',x_d)\end{pmatrix} = \det\big(D_{x'}f'(x',x_d)\big)\gamma(x',x_d). \tag{2.5}$$

It follows from (1) that $\det D_{x'}f'(x',x_d) \neq 0$ for all $(x',x_d) \in U_1 \times U_2$. By the continuity of $D_{x'}f'(x',x_d)$ in $U_1 \times U_2$, $\det D_{x'}f'(x',x_d)$ does not change the sign in $U_1 \times U_2$. In addition, it follows from (2.4) and (2.5) that for all $y' \in V_1$ and $x_d \in U_2$,

$$D_{x_d}h(y',x_d) = \big(\det D_{x'}f'(x',x_d)\big)^{-1}\det Df(x',x_d), \tag{2.6}$$

where $x' = g'(y',x_d)$. Since $\det Df(x) \neq 0$ does not change the sign in $B_{r_0}(x_0) \setminus \{x_0\}$, there exist open sets $U'_1 \subset U_1$ and $U'_2 \subset U_2$ with $x'_0 \in U'_1$ and $x^0_d \in U'_2$ such that $U'_1 \times U'_2 \subset B_{r_0}(x_0)$ and $V'_1 = f'(U'_1 \times U'_2) := \{f'(x',x_d)\colon x' \in U'_1,\ x_d \in U'_2\}$ is an open subset of V_1. It follows from (2.6) that $D_{x_d}h(y'_0,x^0_d) = 0$ and $D_{x_d}h(y',x_d) \neq 0$ does not change the sign in $V'_1 \times U'_2 \setminus \{(y'_0,x^0_d)\}$. Consequently, $h(y',\cdot)$ is strictly monotonic in U'_2 for any $y' \in V'_1$. Further, by using the fact that $h(y',x_d)$ is continuously differentiable in $V'_1 \times U'_2$, there exist open sets $U''_2 \subset U'_2$, $V''_1 \subset V'_1$, and $V_2 \subset \mathbf{R}$, and a unique map g_d such that

(4) $x^0_d \in U''_2$, $y'_0 \in V''_1$, $y^0_d \in V_2$, and $D_{x_d}h(y',x_d) \neq 0$ does not change the sign in $V''_1 \times U''_2 \setminus \{(y'_0,x^0_d)\}$;
(5) $h(y',\cdot): U''_2 \to \mathbf{R}$ is injective for any $y' \in V''_1$;
(6) $g_d: V''_1 \times V_2 \to U''_2$ is continuous and $h(y', g_d(y',y_d)) = y_d$ for all $y' \in V''_1$ and for all $y_d \in V_2$. Furthermore, g_d is continuously differentiable in $V''_1 \times V_2 \setminus \{y_0\}$.

Set $g(y) := (g'(y', g_d(y)), g_d(y))$ and $U''_1 := g'(V''_1 \times U''_2)$. It follows from (1)–(6) that

(7) U''_1 is an open subset of U'_1, $x_0 \in U''_1 \times U''_2$, $y_0 \in V''_1 \times V_2$, and $\det D_x f(x) \neq 0$ does not change the sign in $U''_1 \times U''_2 \setminus \{x_0\}$;
(8) $f: U''_1 \times U''_2 \to \mathbf{R}^d$ is injective;
(9) $g: V''_1 \times V_2 \to U''_1 \times U''_2$ is continuous and $f(g(y)) = y$ for all $y \in V''_1 \times V_2$. Furthermore, g is continuously differentiable in $V''_1 \times V_2 \setminus \{y_0\}$.

Therefore, g is the continuous inverse of f. By setting $V = V''_1 \times V_2$ and $U = g(V) \subset U''_1 \times U''_2$, (i)–(iii) hold. This completes the proof. □

3. Chaos induced by coupled-expanding maps in compact sets

In this section, we establish a criterion of chaos in the sense of both Li–Yorke and Wiggins, induced by strictly coupled-expanding maps in compact sets of metric spaces.

Lemma 3.1. *Let (X,d) be a metric space and V_j, $1 \leqslant j \leqslant N$ $(N \geqslant 2)$, disjoint compact sets of X. If a continuous map $f: \bigcup_{j=1}^{N} V_j \to X$ satisfies*

(i) *f is strictly coupled-expanding in V_j, $1 \leqslant j \leqslant N$;*
(ii) *there exists $\alpha = (a_0, a_1, a_2, \ldots) \in \Sigma_N^+$ such that V_α is a singleton set, where*

$$V_\alpha := \bigcap_{n=0}^{\infty} f^{-n}(V_{a_n});$$

(iii) *f is injective in V_j, $1 \leqslant j \leqslant N$, respectively,*

then

(1) *f has a positive topological entropy and is chaotic in the sense of Li–Yorke;*
(2) *there exists a perfect and compact subset D, which contains a Cantor set, of $\bigcup_{j=1}^{N} V_j$ such that $f(D) = D$ and f is chaotic in the sense of Wiggins on D.*

Proof. Result (1) can directly follow from [31, Theorem 1] and [2, Corollary 2.4]. So it only needs to show result (2). We only present the proof of result (2) in the case $N = 2$ for briefness. The proof in the other case is similar. The proof is divided into four parts.

Let

$$\Omega := \{\beta \in \Sigma_2^+ : V_\beta \text{ is a singleton set}\}, \qquad V := \bigcup_{\beta \in \Omega} V_\beta.$$

By assumption (ii), $\alpha \in \Omega$. So, Ω and V are both nonempty.

(1) The set Ω is perfect and contains infinitely many points. In fact, for any given $\beta = (b_0, b_1, \ldots) \in \Omega$ and for any $\varepsilon > 0$, there exists a positive integer n such that $2^{-n} < \varepsilon$. Set $\gamma = (b_0, b_1, \ldots, b_{n-1}, c_n, b_{n+1}, b_{n+2}, \ldots) \in \Sigma_2^+$, where $c_n \neq b_n$. Then $\rho(\gamma, \beta) = 2^{-n} < \varepsilon$ and $\sigma^{n+1}(\gamma) = \sigma^{n+1}(\beta)$. By repeatedly using Lemma 2.2, it can be concluded that V_γ is also a singleton set and consequently, $\gamma \in \Omega$. Hence, Ω is perfect and contains infinitely many points.

In addition, it can be easily verified that $\sigma(\Omega) = \Omega$ again by using Lemma 2.2, and consequently, $f(V) = V$.

(2) The map $f: V \to V$ is topologically conjugate to $\sigma: \Omega \to \Omega$. Define a map $h: V \to \Omega$ as follows. Note that $V_\beta \cap V_\gamma = \phi$ for any $\beta, \gamma \in \Omega$ with $\beta \neq \gamma$. So, for any $x \in V$ there exists a unique point $\beta \in \Omega$ such that $\{x\} = V_\beta$. Set $h(x) = \beta$. Then h is well defined in V. Since $f(x) \in V_{\sigma(h(x))}$ for any $x \in V$, we get that $h(f(x)) = \sigma(h(x))$, i.e., $h \circ f = \sigma \circ h$.

The rest is to show that h is homeomorphic. Obviously, h is bijective in V. We now show that h is continuous in V. To do so, for any given $x \in V$ and for any $\{x_n\}_{n=1}^{\infty} \subset V$ with $x_n \to x$ as $n \to \infty$, let $h(x) = \beta = (b_0, b_1, \ldots)$ and $h(x_n) = \beta_n = (b_0^{(n)}, b_1^{(n)}, \ldots)$, $n \geqslant 1$. Then $\beta, \beta_n \in \Omega$, $\{x\} = V_\beta$, and $\{x_n\} = V_{\beta_n}$ for each $n \geqslant 1$. For any $\varepsilon > 0$, there exists a positive integer k such that $2^{-k} < \varepsilon$. By the continuity of f, we have that $f^j(x_n) \to f^j(x)$ as $n \to \infty$ for each $j \geqslant 0$. Since $f^j(x) \in V_{b_j}$ and $f^j(x_n) \in V_{b_j^{(n)}}$, $n \geqslant 1$, for each $0 \leqslant j \leqslant k$, and $d(V_1, V_2) > 0$, it follows that there exists a positive integer N_1 such that $b_j^{(n)} = b_j$, $0 \leqslant j \leqslant k$, for all $n \geqslant N_1$. Hence, $\rho(\beta_n, \beta) \leqslant 2^{-k} < \varepsilon$ for all $n \geqslant N_1$. This yields that h is continuous in V. We turn to show that h^{-1} is continuous in Ω. Fix any $\beta = (b_0, b_1, \ldots) \in \Omega$ and any $\{\beta_n\}_{n=1}^{\infty} \subset \Omega$ with $\beta_n = (b_0^{(n)}, b_1^{(n)}, \ldots) \to \beta$ as $n \to \infty$. Set $x = h^{-1}(\beta)$ and $x_n = h^{-1}(\beta_n)$, $n \geqslant 1$. Since V_β is a singleton set, it follows from Lemma 2.1 that $d(V_\beta^n) \to 0$ as $n \to \infty$, where $V_\beta^n = \bigcap_{j=1}^{n} f^{-j}(V_{b_j})$. So, for any $\varepsilon > 0$, there exists a positive integer N_2 such that

$$d(V_\beta^n) < \varepsilon \quad \forall n \geqslant N_2. \tag{3.1}$$

In addition, since $\beta_n \to \beta$ as $n \to \infty$, there exists a positive integer $N_3 \geqslant N_2$ such that $\rho(\beta_n, \beta) < 2^{-N_2}$ for all $n \geqslant N_3$, which implies that for all $n \geqslant N_3$,

$$b_j^n = b_j, \quad 0 \leqslant j \leqslant N_2. \tag{3.2}$$

By using the fact that $\{x\} = \bigcap_{j=0}^{\infty} V_\beta^j$ and $\{x_n\} = \bigcap_{j=0}^{\infty} V_{\beta_n}^j$, $n \geqslant 1$, and from (3.2), it follows that $x, x_n \in V_\beta^{N_2}$ for each $n \geqslant N_3$. This, together with (3.1), implies that $d(x_n, x) \leqslant d(V_\beta^{N_2}) < \varepsilon$ $\forall n \geqslant N_3$, which yields that $x_n \to x$ as $n \to \infty$. Hence, h^{-1} is continuous in Ω. Therefore, $h: V \to \Omega$ is homeomorphic and consequently, $f: V \to V$ and $\sigma: \Omega \to \Omega$ is topologically conjugate.

(3) Based on the discussions in parts (1) and (2) and by using the fact that Σ_2^+ is totally disconnected, V is perfect and totally disconnected, and contains infinitely many points. We turn to show that f is chaotic on V in the sense of Wiggins. To do so, this part is divided into two subparts.

(3a) The map f has sensitive dependence on initial conditions in V. For any given $x \in V$ and for any relatively open neighborhood U of x with respect to V, $h(U)$ is a relative open neighborhood of β to Ω by using the homeomorphism of h, where $\beta = h(x) = (b_0, b_1, \ldots)$. Then there exists a positive integer n such that $B_{2^{-n}}(\beta) \cap \Omega \subset h(U)$. Set $\gamma = (b_0, b_1, \ldots, b_n, c_{n+1}, b_{n+2}, \ldots) \in \Sigma_2^+$ with $b_{n+1} \neq c_{n+1}$. It is evident that $\rho(\gamma, \beta) = 2^{-(n+1)} < 2^{-n}$ and $\sigma^{n+2}(\gamma) = \sigma^{n+2}(\beta)$. Since $\beta \in \Omega$, it follows from Lemma 2.2 that $\gamma \in \Omega$. So, we get that $\gamma \in B_{2^{-n}}(\beta) \cap \Omega$. Denoting $\{y\} = V_\gamma$, one has that $h(y) = \gamma$ and consequently, $y \in U$. Since $f^{n+1}(x) \in V_{b_{n+1}}$ and $f^{n+1}(y) \in V_{c_{n+1}}$, we have

$$d\big(f^{n+1}(x), f^{n+1}(y)\big) \geqslant d(V_{b_{n+1}}, V_{c_{n+1}}) = d(V_1, V_2) > 0.$$

Hence, f has sensitive dependence on initial conditions in V.

(3b) The map f is topologically transitive in V. Suppose that U and W are any two relatively nonempty open subsets with respect to V. Fix a point $x \in U$ and a point $y \in W$, and denote $h(x) = \beta = (b_0, b_1, \ldots)$ and $h(y) = \gamma = (c_0, c_1, \ldots)$. By the homeomorphism of h, $h(U)$ and $h(W)$ are relatively nonempty open subsets with respect to Ω. It is evident that $\beta \in h(U)$ and $\gamma \in h(W)$. So, there exists a sufficiently large integer n such that $B_{2^{-n}}(\beta) \cap \Omega \subset h(U)$. Set $\beta' = (b_0, b_1, \ldots, b_{n+1}, c_0, c_1, \ldots)$. Then $\sigma^{n+2}(\beta') = \gamma$ and consequently, $\beta' \in \Omega$ again by Lemma 2.2. It is evident that $\rho(\beta, \beta') \leqslant 2^{-(n+1)}$. This implies that $\beta' \in B_{2^{-n}}(\beta) \cap \Omega$. Hence, $z = h^{-1}(\beta') \in U$. In addition, we have that $h(f^{n+2}(z)) = \sigma^{n+2}(h(z)) = \sigma^{n+2}(\beta') = \gamma = h(y)$. By the injectivity of h, we get that $f^{n+2}(z) = y$, which yields that $y \in f^{n+2}(U) \cap W$ and consequently, $f^{n+2}(U) \cap W \neq \phi$. Therefore, f is topologically transitive in V.

Based on the discussions in (3a) and (3b), we have proved that f is chaotic on V in the sense of Wiggins.

(4) By the continuity of f, it follows that $f(\bar{V}) \subset \bar{V}$. Now, consider the converse inclusion. For any given $y \in \bar{V}$, there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset V$ such that $y_n \to y$ as $n \to \infty$. Since $f(V) = V$, there exists $x_n \in V$ such that $f(x_n) = y_n$ for each $n \geqslant 1$. From the compactness of V_1 and V_2, it follows that there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Suppose that $x_{n_k} \to x$ as $n_k \to \infty$. It is clear that $x \in \bar{V}$. So, by the continuity of f, we have that $y_{n_k} = f(x_{n_k}) \to f(x)$ as $n_k \to \infty$. This yields that $y = f(x) \in f(\bar{V})$ and consequently, $\bar{V} \subset f(\bar{V})$. It then follows that $f(\bar{V}) = \bar{V}$. In addition, since V is perfect and contains infinitely many points, and V_1 and V_2 are compact, it can be easily concluded that $\bar{V}$ is perfect and compact, and contains a Cantor set. Based on the discussion in part (3), f is chaotic on V in the sense of Wiggins. So, it follows from (ii) and (iii) in Lemma 2.3 that f is chaotic on $\bar{V}$ in the sense of Wiggins.

By setting $D = \bar{V}$, result (2) has been proved. The entire proof of this theorem is complete. □

Remark 3.1. From the proof of Lemma 3.1, V is totally disconnected. But the set $D = \bar{V}$ in result (2) in Lemma 3.1 may have a nonsingleton connected component. So, D contains a Cantor set, but may not be a Cantor set.

The following result is a direct consequence of Lemma 3.1.

Theorem 3.1. *Let (X, d) be a metric space and V_j, $1 \leqslant j \leqslant N$ $(N \geqslant 2)$, disjoint compact sets of X. If a continuous map $f: \bigcup_{j=1}^{N} V_j \to X$ satisfies*

(i) *f is strictly coupled-expanding in V_j, $1 \leqslant j \leqslant N$;*
(ii) *there exist a j_0, $1 \leqslant j_0 \leqslant N$, and a constant $\lambda > 1$ such that*

$$d\big(f(x), f(y)\big) \geqslant \lambda\, d(x, y) \quad \forall x, y \in V_{j_0},$$

and f is injective in V_j, $1 \leqslant j \neq j_0 \leqslant N$, respectively,

then all the results in Lemma 3.1 hold.

4. Chaos induced by regular snap-back repellers

If a map f in $\mathbf{R}^d$ is continuously differentiable in some neighborhood of $x_j = f^j(x_0)$ $(0 \leqslant j \leqslant m-1)$, where x_0 and m are specified in Definition 2.5, then the snap-back repeller is regular and nondegenerate if and only if $Df(x_j)$ is invertible for $0 \leqslant j \leqslant m-1$. In this section, this condition is weakened.

Theorem 4.1. *Let a map $f:\mathbf{R}^d \to \mathbf{R}^d$ have a snap-back repeller z in some norm $\|\cdot\|$ with $f^m(x_0) = z$ for some $x_0 \in B_{r_0}(z)$, $x_0 \neq z$, and some positive integer $m \geqslant 2$, where r_0 is specified in Definition 2.5. Assume that f is continuous in $\bar{B}_{r_0}(z)$ and in $\bar{B}_{\delta_j}(x_j)$, respectively, for some positive constants δ_j with $x_j = f^j(x_0)$, $1 \leqslant j \leqslant m-1$. If there exists a positive constant $\delta_0 < \|x_0 - z\|$ such that $\bar{B}_{\delta_0}(x_0) \subset \bar{B}_{r_0}(z)$, $f^j(\bar{B}_{\delta_0}(x_0)) \subset \bar{B}_{\delta_j}(x_j)$ for $1 \leqslant j \leqslant m-1$, and z is an interior point of $f^m(B_{\delta_0}(x_0))$, then f has a positive topological entropy and is chaotic in the sense of Li–Yorke.*

Proof. We shall show that there exist a positive integer $n \geqslant m$ and two disjoint compact sets V_1 and V_2 such that f^n is continuous in $V_1 \cup V_2$ and strictly coupled-expanding in V_1 and V_2.

By Definition 2.5 and by the assumptions, there exists a constant $\lambda > 1$ such that f satisfies

$$\big\|f(x) - f(y)\big\| \geqslant \lambda \|x - y\| \quad \forall x, y \in \bar{B}_{r_0}(z). \tag{4.1}$$

From (4.1), Lemma 2.2 in [21], and the continuity of f in $\bar{B}_{r_0}(z)$, it follows that, for any positive constant $r \leqslant r_0$, $f(\bar{B}_r(z))$ is closed, $f(B_r(z))$ is open, and

$$f\big(\bar{B}_r(z)\big) \supset \bar{B}_r(z), \qquad f\big(B_r(z)\big) \supset B_r(z). \tag{4.2}$$

This implies that z is a regular expanding fixed point of f and $\partial f(B_r(z)) = f(\partial B_r(z))$. Again from (4.1), for any positive constant $r \leqslant r_0$ and for any $x \in \partial B_r(z)$, we get that $\|f(x) - z\| \geqslant \lambda\|x - z\| = \lambda r$, which implies that

$$f\big(\bar{B}_r(z)\big) \supset \bar{B}_{\lambda r}(z). \tag{4.3}$$

Since z is an interior point of $f^m(\bar{B}_{\delta_0}(x_0))$ with x_0 and δ_0 specified in the assumptions in the theorem, there exists a positive constant $r_1 < r_0$ such that

$$\bar{B}_{r_1}(z) \subset f^m\big(\bar{B}_{\delta_0}(x_0)\big). \tag{4.4}$$

By noting $\lambda > 1$, there exists a positive integer k such that $\lambda^{k-1} r_1 < r_0$ and $\lambda^k r_1 \geqslant r_0$. So, it follows from (4.3) and (4.4) that

$$f^{m+k}\big(\bar{B}_{\delta_0}(x_0)\big) \supset \bar{B}_{r_0}(z) \supset \bar{B}_{\lambda^{k-1} r_1}(z). \tag{4.5}$$

Set

$$D_1 = f^{-1}\big(\bar{B}_{r_0}(z)\big) \cap \bar{B}_{r_0}(z), \qquad D_i = f^{-1}(D_{i-1}) \cap D_{i-1}, \quad i \geqslant 2.$$

Then D_1 contains z and is a closed subset of $\bar{B}_{r_0}(z)$, and D_i contains z and is a closed subset of D_{i-1} for each $i \geqslant 2$. Further, it follows from (4.2) that $f(D_1) = \bar{B}_{r_0}(z)$. It is easy to verify that $f(D_{i-1}) \supset D_{i-1}$ and $f(D_i) = D_{i-1}$ for $i \geqslant 2$ by induction. For any $x \in D_i$, $f^j(x) \in \bar{B}_{r_0}(z)$ for $0 \leqslant j \leqslant i$. Consequently, using (4.1), we have

$$\|x - z\| \leqslant \lambda^{-i}\big\|f^i(x) - z\big\| \leqslant \lambda^{-i} r_0, \quad \forall x \in D_i,$$

which implies that $D_i \subset \bar{B}_{\lambda^{-i} r_0}(z)$ for $i \geqslant 1$. By referring to $\lambda > 1$ and $\delta_0 < \|x_0 - z\|$, there exists a positive integer $n \geqslant m + k$ such that

$$D_n \cap \bar{B}_{\delta_0}(x_0) = \phi. \tag{4.6}$$

Hence, $V_1 := D_n$ contains z and is a closed subset of $\bar{B}_{r_0}(z)$, $f^n(V_1) = \bar{B}_{r_0}(z)$, and f^n is continuous in V_1. Next, set

$$E_1 = \bar{B}_{\delta_0}(x_0) \cap f^{-m}\big(\bar{B}_{r_1}(z)\big), \qquad E_i = E_{i-1} \cap f^{-(m+i-1)}\big(\bar{B}_{\lambda^{i-1} r_1}(z)\big), \quad 2 \leqslant i \leqslant k,$$
$$E_j = E_{j-1} \cap f^{-(m+j-1)}\big(\bar{B}_{r_0}(z)\big), \quad k+1 \leqslant j \leqslant n+1-m.$$

By the continuity of f and from (4.3)–(4.5), it can be concluded that $f^m(E_1) = \bar{B}_{r_1}(z)$, $f^{m+i-1}(E_i) = \bar{B}_{\lambda^{i-1}r_1}(z)$ $(2 \leqslant i \leqslant k)$, $f^{m+j-1}(E_j) = \bar{B}_{r_0}(z)$ $(k+1 \leqslant j \leqslant n+1-m)$; E_1 contains x_0 and is a closed subset of $\bar{B}_{\delta_0}(x_0)$ and E_i contains x_0 and is a closed subset of E_{i-1} for $2 \leqslant i \leqslant n+1-m$; and f^{m+j} is continuous in E_j for $1 \leqslant j \leqslant n+1-m$. Hence, $V_2 := E_{n+1-m}$ contains x_0 and is a closed subset of $\bar{B}_{\delta_0}(x_0)$, f^n is continuous in V_2, and $f^n(V_2) = \bar{B}_{r_0}(z)$. In addition, it follows from (4.6) that $V_1 \cap V_2 = \phi$.

In summary, V_1 and V_2 are disjoint compact sets, f^n is continuous in $V_1 \cup V_2$, and $f^n(V_j) = \bar{B}_{r_0}(z) \supset V_1 \cup V_2$, $j = 1, 2$. By [31, Theorem 1] and its proof, there exists a compact subset $\Lambda \subset V_1 \cup V_2$ with $f^n(\Lambda) = \Lambda$ such that $f^n : \Lambda \to \Lambda$ is topologically semi-conjugate $\sigma : \Sigma_2^+ \to \Sigma_2^+$. Since σ has a positive entropy, f^n has a positive topological entropy, and consequently, f^n is chaotic in the sense of Li–Yorke by [2, Corollary 2.4]. This yields that f has a positive topological entropy by Theorem 1.2 in [20, Chapter IX], and is chaotic in the sense of Li–Yorke. The proof is complete. □

The following two results are direct consequences of Theorem 4.1.

Corollary 4.1. *Assume that a map $f : \mathbf{R}^d \to \mathbf{R}^d$ has a regular snap-back repeller z with x_0, r_0, and m specified in Definition 2.5. If f is continuous in $\bar{B}_{r_0}(z)$ and in some neighborhood of $f^j(x_0)$ for $1 \leqslant j \leqslant m-1$, then f has a positive topological entropy and is chaotic in the sense of Li–Yorke.*

Corollary 4.2. *Assume that a map $f : \mathbf{R}^d \to \mathbf{R}^d$ has a regular homoclinic orbit Γ to a regular expanding fixed point z. If f is continuous in some neighborhood of each point on Γ and in some neighborhood of z, then f has a positive topological entropy and is chaotic in the sense of Li–Yorke.*

Remark 4.1. From the proof of Theorem 4.1, it follows that if f has a regular snap-back repeller or a regular homoclinic orbit to a regular expanding fixed point, and is continuous in the interested domains, then f^n is strictly coupled-expanding in certain two disjoint compact sets for some integer $n \geqslant 2$.

Remark 4.2. Theorem 4.1 only requires the following two assumptions:

(1) The first assumption: z is an expanding fixed point of f in $\bar{B}_{r_0}(z)$ in some norm $\|\cdot\|$ in $\mathbf{R}^d$ and f is continuous in $\bar{B}_{r_0}(z)$. Its sufficient condition is that f is continuously differentiable in a neighborhood of z and all eigenvalues of $Df(z)$ are larger than 1 in norm by Theorem 4.3 in [22].
(2) The second assumption: there exists $x_0 \in B_{r_0}(z)$, $x_0 \neq z$, such that $f^m(x_0) = z$ for some positive integer $m \geqslant 2$, and there exist positive constants δ_j $(0 \leqslant j \leqslant m-1)$ with $\delta_0 < \|x_0 - z\|$ such that $\bar{B}_{\delta_0}(x_0) \subset \bar{B}_{r_0}(z)$, $f^j(\bar{B}_{\delta_0}(x_0)) \subset \bar{B}_{\delta_j}(x_j)$, f is continuous in $\bar{B}_{\delta_j}(x_j)$, and z is an interior point of $f^m(\bar{B}_{\delta_0}(x_0))$, where $x_j = f^j(x_0)$, $1 \leqslant j \leqslant m-1$. Obviously, if x_{j+1} is an interior point of $f(\bar{B}_{\delta_j}(x_j))$ for each j, $0 \leqslant j \leqslant m-2$, and z is an interior point of $f(\bar{B}_{\delta_{m-1}}(x_{m-1}))$, then z is an interior point of $f^m(\bar{B}_{\delta_0}(x_0))$. Therefore, if f is continuously differentiable in some neighborhood of x_j and $\det Df(x_j) \neq 0$, $0 \leqslant j \leqslant m-1$, then the second assumption is satisfied by the classical inverse function theorem. In this special case, z is a nondegenerate and regular snap-back repeller and we have obtained a better result than Theorem 4.1; that is, f is chaotic in the sense of both Devaney and Li–Yorke (see [22, Theorem 4.4] and [25, Theorem 2.1]).

It is noted that the assumption, z is an interior point of $f^m(\bar{B}_{\delta_0}(x_0))$, is difficult to verify for higher-dimensional maps in the case that $\det Df(x_j) = 0$ or f is not differentiable in any neighborhood of x_j for some j, $1 \leqslant j \leqslant m-1$. We give some verifiable conditions about it in the following theorem.

Theorem 4.2. *Let a map $f : \mathbf{R}^d \to \mathbf{R}^d$ have a fixed point z. Assume that*

(i) *f is continuously differentiable in some neighborhood of z and all eigenvalues of $Df(z)$ are larger than* 1 *in norm, which implies that there exist a positive constant r_0 and a norm $\|\cdot\|$ in $\mathbf{R}^d$ such that z is an expanding fixed point of f in $\bar{B}_{r_0}(z)$ in the norm $\|\cdot\|$, and f is continuously differentiable in $\bar{B}_{r_0}(z)$;*

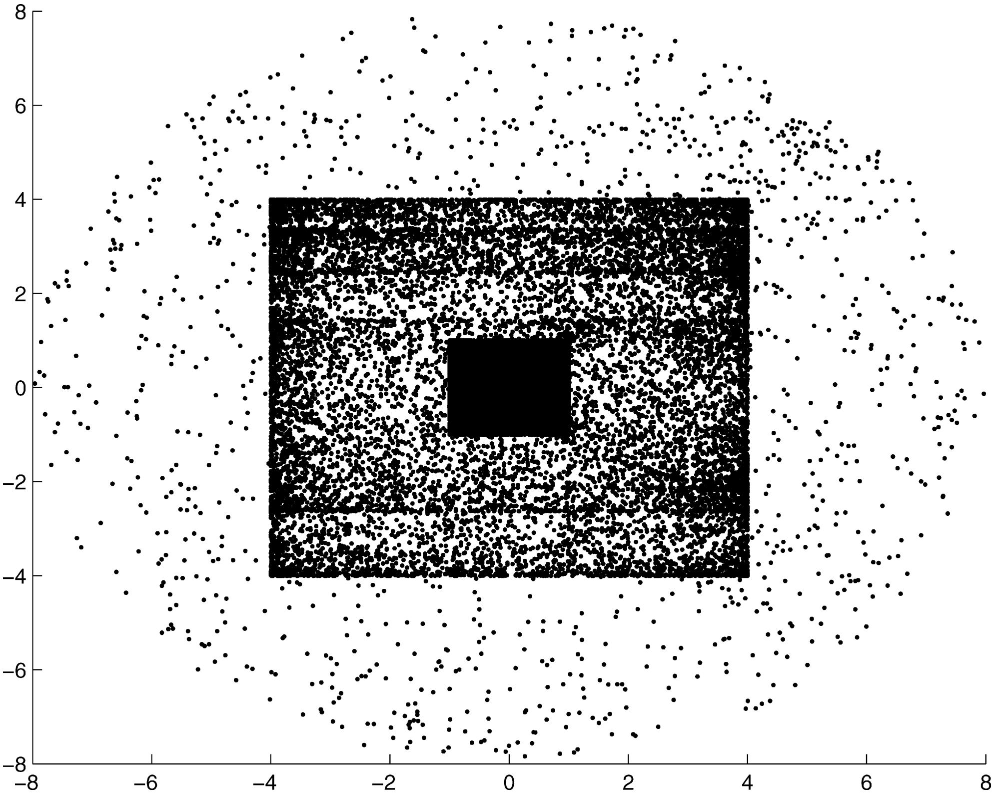

Fig. 1. 2D computer simulation result shows chaos in the (x, y) space in the rectangular box $[-8, 8] \times [-8, 8]$.

(ii) *there exists* $x_0 \in B_{r_0}(z)$, $x_0 \neq z$, *such that* $f^m(x_0) = z$ *for some positive integer* $m \geqslant 2$. *Furthermore,* f *is continuously differentiable in* $\bar{B}_{\delta_j}(x_j)$ *and satisfies that* $\det Df(x) \neq 0$ *does not change sign in* $\bar{B}_{\delta_j}(x_j) \setminus \{x_j\}$, *and* $\operatorname{rank} Df(x_j) \geqslant d-1$ *for* $1 \leqslant j \leqslant m-1$, *where* $x_j = f^j(x_0)$ *and* δ_j *is some positive constant.*

Then

(1) f *has a positive topological entropy and is chaotic in the sense of Li–Yorke*;
(2) *there exists a compact and perfect set* E *containing a Cantor set such that* $f(E) = E$ *and* f *is chaotic in the sense of Wiggins on* E.

Proof. We will apply Theorem 3.1 with $N = 2$ to prove this theorem. For each j, $1 \leqslant j \leqslant m-1$, by assumption (ii), and by Lemma 2.5 in the case of $\operatorname{rank} Df(x_j) = d-1$ and by the classical inverse function theorem in the case of $\operatorname{rank} Df(x_j) = d$, there exists a positive constant $\delta_j^0 \leqslant \delta_j$ such that $f : \bar{B}_{\delta_j^0}(x_j) \to f(\bar{B}_{\delta_j^0}(x_j))$ is homeomorphic. By assumption (i), z is an expanding fixed point of f in $\bar{B}_{r_0}(z)$ and f is continuous in $\bar{B}_{r_0}(z)$. It can be easily verified that $f : \bar{B}_{r_0}(z) \to f(\bar{B}_{r_0}(z))$ is homeomorphic by using the compactness of $\bar{B}_{r_0}(z)$, and the continuity and the expansion of f in $\bar{B}_{r_0}(z)$. Since f is continuous in $\bar{B}_{\delta_j}(x_j)$ for $0 \leqslant j \leqslant m-1$, there exists a positive constant δ_0 such that $\bar{B}_{\delta_0}(x_0) \subset \bar{B}_{r_0}(z)$, $z \notin \bar{B}_{\delta_0}(x_0)$, and $f^j(\bar{B}_{\delta_0}(x_0)) \subset \bar{B}_{\delta_j^0}(x_j)$, $1 \leqslant j \leqslant m-1$. So, $f^m : \bar{B}_{\delta_0}(x_0) \to f^m(\bar{B}_{\delta_0}(x_0))$ is homeomorphic.

Constructing the sets V_1 and V_2 and choosing an integer n similarly to those in the proof of Theorem 4.1, where z, x_0, r_0, δ_0, and m satisfy all the conditions in the above paragraph, one can conclude that V_1 is a closed subset of $\bar{B}_{r_0}(z)$; V_2 is a closed subset of $\bar{B}_{\delta_0}(x_0)$; $V_1 \cap V_2 = \phi$; $f^n(V_1) = f^n(V_2) = \bar{B}_{r_0}(z) \supset V_1 \cup V_2$; f^n is continuous in V_1;

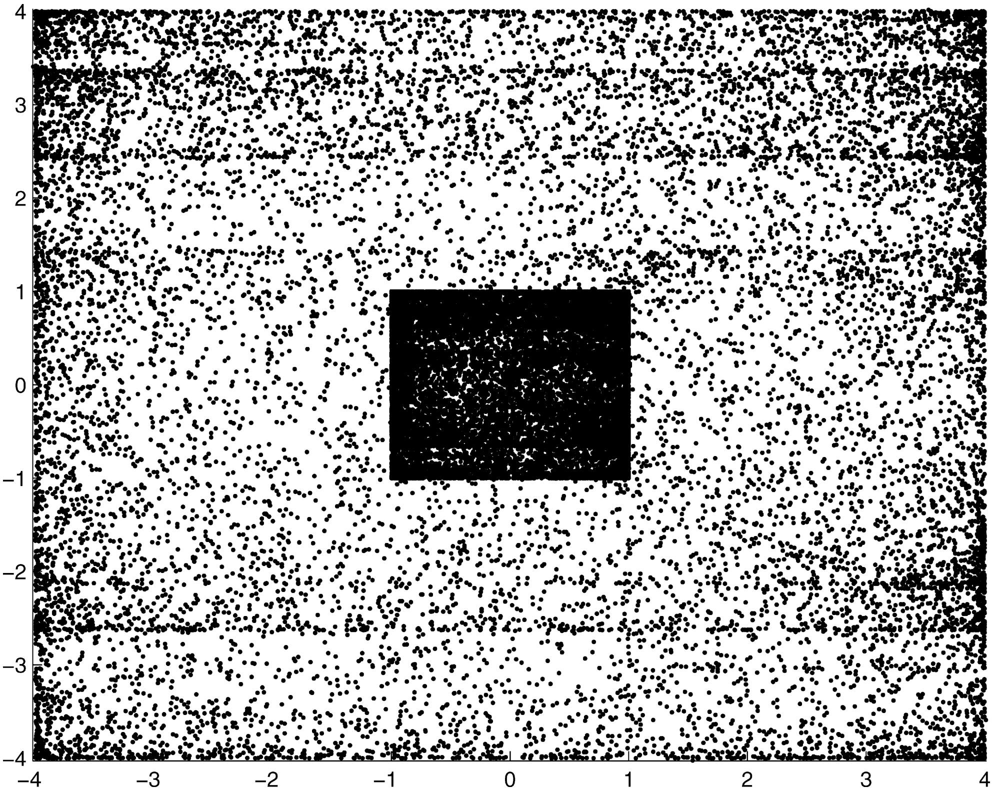

Fig. 2. Zoom area of the rectangular box $[-4, 4] \times [-4, 4]$ in Fig. 1.

$f^n : V_2 \to f^n(V_2)$ is homeomorphic; and f^j is continuous in V_1 and V_2 for each j, $1 \leqslant j \leqslant n-1$. Furthermore, for any $x, y \in V_1$, $f^i(x), f^i(y) \in \bar{B}_{r_0}(z)$, $0 \leqslant i \leqslant n$, and consequently, it follows that

$$\left\| f^n(x) - f^n(y) \right\| \geqslant \lambda^n \|x - y\|.$$

Hence, f^n satisfies all the assumptions in Theorem 3.1 for V_1 and V_2 with $j_0 = 1$. By Theorem 3.1, f^n has a positive topological entropy and is chaotic in the sense of Li–Yorke, and there exists a perfect and compact subset D, which contains a Cantor set, of $V_1 \cup V_2$ such that $f^n(D) = D$ and f^n is chaotic in the sense of Wiggins on D. With a similar argument to the last paragraph in the proof of Theorem 4.1, one can conclude that f has a positive topological entropy and is chaotic in the sense of Li–Yorke. Let $E = \bigcup_{j=0}^{n-1} f^j(D)$. It can be easily verified that E is also perfect and compact, contains a Cantor set, and satisfies $f(E) = E$. Hence, f is chaotic in the sense of Wiggins on E by Lemma 2.4. The proof is complete. □

Remark 4.3. From the proof of Theorem 4.2, z is a regular snap-back repeller under the assumptions in Theorem 4.2. Clearly, assumption (ii) in Theorem 4.2 is much easier to verify than the second assumption in Theorem 4.1 (see Remark 4.2). So it is more convenient in applications.

Remark 4.4. By Theorem 4.2, there exists a compact and perfect set E containing a Cantor set such that $f(E) = E$ and f is chaotic in the sense of Wiggins on E. Obviously, it is useful to know where the set E lies in its application. From the constructions of V_1 and V_2 in the proof of Theorem 4.2, we can figure out that

$$E \subset \bar{B}_{r_0}(z) \cup \bar{B}_{\delta_1^0}(x_1) \cup \cdots \cup \bar{B}_{\delta_{m-1}^0}(x_{m-1}),$$

where $z, r_0, m, x_1, \ldots, x_{m-1}$ are specified in assumptions (i) and (ii) in Theorem 4.2; and δ_j^0, $1 \leqslant j \leqslant m-1$, satisfy that $f : \bar{B}_{\delta_j^0}(x_j) \to f(\bar{B}_{\delta_j^0}(x_j))$ is homeomorphic.

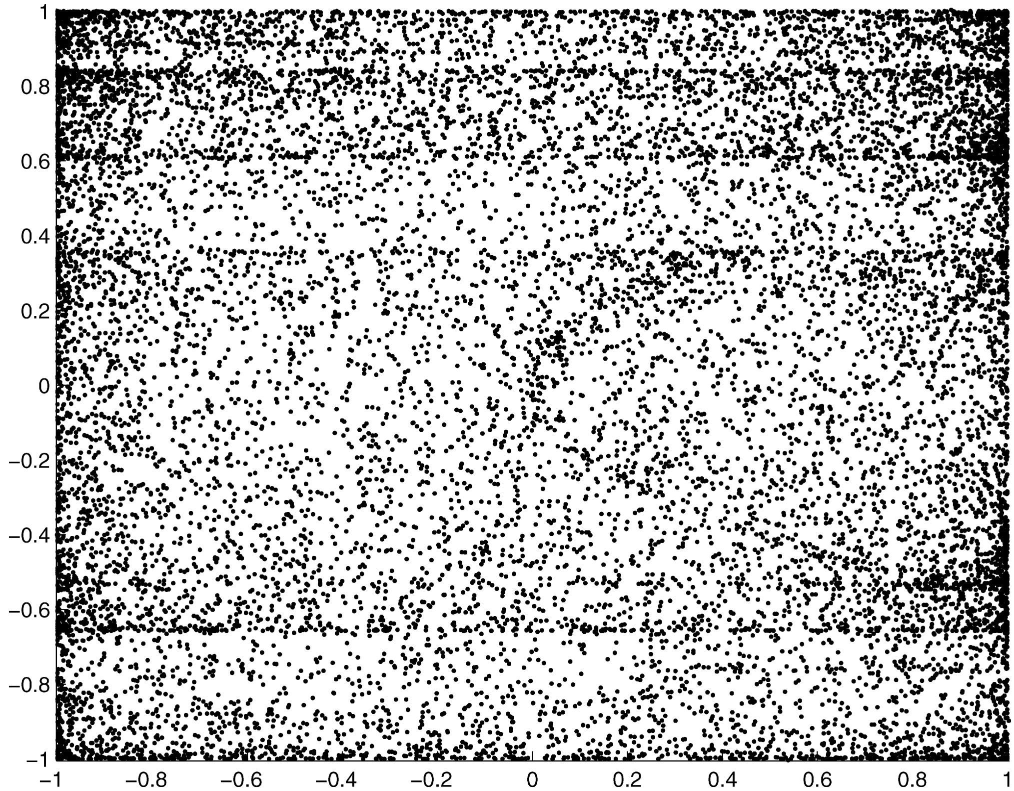

Fig. 3. Zoom area of the rectangular box $[-1, 1] \times [-1, 1]$ in Fig. 2.

5. An example

In this section, we discuss an example of two-dimensional map, which shows that the assumptions in Theorem 4.2 are indeed weaker than those in the relative existing results.

Example 5.1. Consider the following two-dimensional map:

$$f(x,y) = \begin{cases} 4(x,y), & \text{if } (x,y) \in \bar{B}_{\frac{3}{2}}(0), \\ \text{arbitrary}, & \text{if } (x,y) \in B_2(0) \setminus \bar{B}_{\frac{3}{2}}(0), \\ (\sin((x-3)^3 + x(y-3)^2), \sin(y-3)), & \text{if } (x,y) \notin B_2(0), \end{cases}$$

where $\bar{B}_{\frac{3}{2}}(0)$ is a closed ball of radius $\frac{3}{2}$, centered at the origin $(0,0)$, and $B_2(0)$ is an open ball of radius 2, centered at $(0,0)$, with the classical Euclidean norm $\|\cdot\|$ in $\mathbf{R}^2$. It is clear that $(0,0)$ is a fixed point of f and f is continuously differentiable in the open ball $B_{\frac{3}{2}}(0)$, satisfying $Df(0,0) = 4I_2$ and

$$\|f(x_1,y_1) - f(x_2,y_2)\| = 4\|(x_1,y_1) - (x_2,y_2)\|, \quad \forall (x_1,y_1), (x_2,y_2) \in \bar{B}_{\frac{3}{2}}(0).$$

This implies that $(0,0)$ is a regular expanding fixed point of f in $\bar{B}_{\frac{3}{2}}(0)$. Setting $x_0 = y_0 = \frac{3}{4}$, we see that $(x_0, y_0) \in B_{\frac{3}{2}}(0)$, $f(x_0, y_0) = (3,3)$, and $f(3,3) = (0,0)$. So $(0,0)$ is a snap-back repeller of f with $m = 2$. In addition, f is continuously differentiable in $B_1((3,3))$ with Jacobian matrix

$$Df(x,y) = \begin{pmatrix} (3(x-3)^2 + (y-3)^2)g(x,y) & 2x(y-3)g(x,y) \\ 0 & \cos(y-3) \end{pmatrix},$$

which yields that

$$\det Df(x,y) = \big(3(x-3)^2 + (y-3)^2\big)g(x,y)\cos(y-3), \quad (x,y) \in B_1\big((3,3)\big),$$

where $g(x, y) = \cos((x-3)^3 + x(y-3)^2)$. Clearly, $\operatorname{rank} Df(3,3) = 1$, and $\det Df(x, y) > 0$ for all $(x, y) \in \bar{B}_{\frac{\pi}{10}}((3,3)) \setminus \{(3,3)\}$. Hence, all the assumptions in Theorem 4.2 are satisfied. Therefore, f is chaotic in the sense of both Li–Yorke and Wiggins by Theorem 4.2.

Since $\det Df(3,3) = 0$, the origin is a degenerate snap-back repeller. So the relative existing results in [16,17,22] are not applicable to this map.

In order to help better visualize the theoretical results, three computer simulations are done. For it, take $f(x, y) = 4(x, y)$ for $(x, y) \in B_2(0) \setminus \bar{B}_1(0)$. In this case, $f(\mathbf{R}^2) = B_8(0)$. The simulation results for this 2D map in the (x, y) space are shown in the rectangular boxes $[-8, 8] \times [-8, 8]$, $[-4, 4] \times [-4, 4]$, and $[-1, 1] \times [-1, 1]$, respectively, see Figs. 1–3. They clearly show that the dynamical behaviors of the map are complicated.

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