International Journal of Bifurcation and Chaos, Vol. 24, No. 6 (2014) 1450083 (10 pages)

DOI: 10.1142/S0218127414500837

Seven Limit Cycles Around a Focus Point in a Simple Three-Dimensional Quadratic Vector Field

Yun Tian* and Pei Yu†
Department of Applied Mathematics, Western University,
London, Ontario N6A 5B7, Canada
**ytian56@uwo.ca*
†pyu@uwo.ca

Received December 16, 2013

In this paper, we show that a simple three-dimensional quadratic vector field can have at least seven small-amplitude limit cycles, bifurcating from a Hopf critical point. This result is surprisingly higher than the Bautin's result for quadratic planar vector fields which can only have three small-amplitude limit cycles bifurcating from an elementary focus or an elementary center. The methods used in this paper include computing focus values, and solving multivariate polynomial systems using modular regular chains. In order to obtain higher-order focus values for nonplanar dynamical systems, computationally efficient approaches combined with center manifold computation must be adopted. A recently developed explicit, recursive formula and Maple program for computing the normal form and center manifold of general n-dimensional systems is applied to compute the focus values of the three-dimensional vector field.

Keywords: Three-dimensional quadratic vector field; limit cycle; Hopf bifurcation; center manifold; normal form; focus value; Maple.

1. Introduction

Limit cycle theory has been playing a very important role in the study of dynamical behavior of nonlinear systems, emerging from many physical and engineering models, and recently even from financial systems and social systems. In mathematics, for a two-dimensional phase space, a limit cycle is a closed trajectory in the phase space having the property that at least one other trajectory spirals into it either as time approaches infinity or as time approaches negative infinity. Higher-dimensional vector fields often exhibit limit cycles which may coexist with more complex dynamical behaviors such as chaos.

The study of limit cycles was initiated by Poincaré [1881–1886]. He built a new branch of mathematics, called "qualitative theory of differential equations", and introduced the concept of limit cycles. Later, in the past more than 100 years, the development of limit cycle theory was perhaps motivated by the well-known Hilbert's 16th problem. The second part of this problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that planar polynomial systems of degree n can have. In early 1990s, Ilyashenko and Yakoveko [1991], and Écalle [1992] proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, the best result is 4 with $(3, 1)$ distribution, obtained more than 30 years ago [Shi, 1980; Chen & Wang, 1979]. Recently, this result was also obtained for near-integrable quadratic systems [Yu & Han, 2012], whether $H(2) = 4$ is still open. In other words, the finiteness problem remains unsolved even

†Author for correspondence

for quadratic polynomial systems. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geq 13$ [Li & Liu, 2010; Li *et al.*, 2009]. Note that the 13 limit cycles are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems.

Suppose we consider Hilbert's 16th problem with limit cycles bifurcating from isolated fixed points, then the question becomes how to study degenerate Hopf bifurcations, giving rise to weak (fine) focus points. This local problem has been completely solved only for generic quadratic systems [Bautin, 1952], which can have three limit cycles in the vicinity of such a singular point. For cubic systems, James and Lloyd [1991] obtained a formal construction, via symbolic computation, of a special cubic system with eight limit cycles. In 2009, Yu and Corless [2009] showed the existence of nine limit cycles with the help of a numerical method for another special cubic system. Recently, this special system was reconsidered using purely symbolic computation with the modular regular chains method to confirm the existence of nine limit cycles, and find all the possible real solutions [Chen *et al.*, 2013].

Due to the importance of limit cycle theory and frequent appearance in higher-order dynamical systems, we wish to study the bifurcation of limit cycles in higher-order vector fields. In this paper, particular attention will be focused on three-dimensional systems with a Hopf singular point. We would like to investigate the maximal number of limit cycles which may exist in the vicinity of a singular point of three-dimensional systems. This is certainly a very challenging problem. There are very few results in the literature. Over the last 20 years, a three-dimensional competitive Lotka–Volterra model has been studied extensively. The model is described by a three-dimensional differential system:

$$\dot{x}_i = x_i\left(b_i - \sum_{j=1}^{3} a_{ij}x_j\right), \quad i = 1, 2, 3, \tag{1}$$

where the dot indicates differentiation with respect to time t, x_i represents the population of ith species, and the coefficients take positive real values, $b_i > 0$, $a_{ij} > 0$, $i, j = 1, 2, 3$. This is a special case of general three-dimensional quadratic systems. In the past two decades, several researchers have paid attention to system (1) and particularly studied bifurcation of limit cycles (e.g. see [Hofbauer & So, 1994; Lu & Luo, 2002; Gyllenberg *et al.*, 2006; Gyllenberg & Yan, 2009]). So far, the best result is four limit cycles, obtained by Gyllenberg and Yan [2009], using appropriate parameter values. These four limit cycles include three small-amplitude limit cycles, proved by using focus value computation, and one large limit cycle, shown by constructing a heteroclinic cycle. Recently, Tian and Yu revisited this problem [Tian & Yu, 2013] and showed that this system might have maximal eight limit cycles, but it is very difficult to prove this result by using existing methodologies.

In this paper, we turn to consider a general three-dimensional quadratic system, given by

$$\begin{aligned}
\dot{x}_1 &= \alpha x_1 + x_2 + f_1(x_1, x_2, x_3),\\
\dot{x}_2 &= -x_1 + \alpha x_2 + f_2(x_1, x_2, x_3),\\
\dot{x}_3 &= -\beta x_3 + f_3(x_1, x_2, x_3),
\end{aligned} \tag{2}$$

where α and $\beta > 0$ are real parameters, and f_i's are quadratic homogeneous polynomials. This system has a Hopf singularity at the origin when $\alpha = 0$. For general quadratic polynomials f_i and $\beta \neq 1$, the highest order of the focus value obtained from a desktop machine with CPU 3.4 GHZ and 32G RAM memory is four [Tian & Yu, 2013]. Moreover, even just solving these four polynomial equations is not an easy job. Therefore, we make a number of simplifications in (2) so that we can manage to obtain higher-order focus values, at least up to seventh order, and then try to apply the modular regular chains [Chen *et al.*, 2013] to obtain seven limit cycles in the vicinity of the origin. Compared to the Bautin's result for quadratic planar vector fields which can only have three small-amplitude limit cycles bifurcating from an elementary focus or an elementary center, this result is quite surprising. The description of the simple three-dimensional quadratic vector field and proof of the existence of seven limit cycles around the origin will be given in the next section.

2. Main Result

We start from the general three-dimensional quadratic systems (2). Without loss of generality, the system can be written in the following form, with its linear part in Jordan canonical form,

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 + x_2 + a_{11}x_1^2 + (2b_{11} + a_{12})x_1x_2 \\ &\quad + a_{22}x_2^2 + a_{33}x_3^2 + a_{13}x_1x_3 + a_{23}x_2x_3, \\ \dot{x}_2 &= -x_1 + \alpha x_2 + b_{11}x_1^2 + (2a_{11} + b_{12})x_1x_2 \\ &\quad - b_{11}x_2^2 + b_{33}x_3^2 + b_{13}x_1x_3 + b_{23}x_2x_3, \\ \dot{x}_3 &= -\beta x_3 + c_{11}x_1^2 + c_{12}x_1x_2 + c_{22}x_2^2 \\ &\quad + c_{33}x_3^2 + c_{13}x_1x_3 + c_{23}x_2x_3,\end{aligned} \tag{3}$$

where $\alpha, \beta > 0$ and a_{ij}, b_{ij}, c_{ij} are parameters, and the formula in Bautin's equation [Bautin, 1952] has been used in the first two equations of (3), which can be achieved by a proper rotation around the x_3-axis. It is easy to see that the origin is an equilibrium point for any value of parameters, and a Hopf bifurcation occurs from the origin when α crosses the critical value $\alpha = \alpha_c = 0$.

Thus, we can use the formulas and Maple program developed in [Tian & Yu, 2013] to compute the normal form, which can then be used to determine small-amplitude limit cycles bifurcating from the origin. It is obvious that the zero-order focus value $v_0 = \alpha$, and at the critical point: $\alpha = \alpha_c = 0$, $v_0 = 0$. Then under the condition $\alpha = \alpha_c = 0$, the Maple program is executed on the desktop machine to obtain the focus values $v_1, v_2, \ldots$. It should be noted that for the general system (2), the computation of the higher-order normal form is very time consuming and memory demanding. Moreover, even if we can obtain higher-order normal forms by using the Maple program, it is almost impossible to find the solutions of the multivariate polynomial system of focus values. Thus, in order to simplify the computation, we make some simplifications. First, we suppose $b_{11} \neq 0$ and $c_{12} \neq 0$. Then, we can use parameter scaling and state variable scaling in (2) so that $b_{11} = c_{12} = 1$. In order to make the computation of focus values manageable, we further set $a_{13} = a_{23} = a_{33} = b_{13} = b_{23} = b_{12} = c_{11} = c_{22} = c_{23} = 0$ and $\beta = 1$, resulting in the following simple three-dimensional quadratic system,

$$\begin{aligned}\dot{x}_1 &= x_2 + a_{11}x_1^2 + (2 + a_{12})x_1x_2 + a_{22}x_2^2, \\ \dot{x}_2 &= -x_1 + x_1^2 + 2a_{11}x_1x_2 - x_2^2 + b_{33}x_3^2, \\ \dot{x}_3 &= -x_3 + x_1x_2 + c_{33}x_3^2 + c_{13}x_1x_3.\end{aligned} \tag{4}$$

This is perhaps the simplest three-dimensional quadratic system since it has only one coupling coefficient b_{33} between the first two equations and the third equation. When $b_{33} = 0$, the first two equations are decoupled from the third equation, and the problem becomes finding the limit cycles of the planar system, described by the first two equations of (4), and it is easy to show that this planar system has three small limit cycles around the origin, as expected. In fact, when $b_{33} = 0$, we can use the Maple program to find the first focus value v_1, given by $v_1 = -\frac{1}{8}a_{12}(a_{11} + a_{22})$. Letting $a_{12} = 0$ yields $v_1 = 0$ and then executing the Maple program produces $v_2 = -\frac{1}{12}a_{11}(a_{11} + a_{22})(a_{11} + 5a_{22})$. Further, letting $a_{11} = -5a_{22}$ results in $v_2 = 0$ and finally executing the Maple program yields

$$\begin{aligned}v_3 &= 25a_{22}^3(1 - 3a_{22}^2), \\ v_4 &= \frac{140}{9}a_{22}^3(1 - 3a_{22}^2)(7 - 38a_{22}^2), \\ v_5 &= \cdots,\end{aligned}$$

and all the v_i's contain the factor $a_{22}^3(1 - 3a_{22}^2)$, clearly indicating that at most three small-amplitude limit cycles can be obtained around the origin when $b_{33} = 0$.

Now, suppose $b_{33} \neq 0$. We have the following main result.

Theorem 1. *Suppose the parameters*, a_{11}, a_{12}, a_{22}, b_{33}, c_{33} *and* c_{13}, *in system (3) are arbitrary nonzero constants. Then system (3) can have at least seven small-amplitude limit cycles around the origin.*

In order to prove Theorem 1, we need the following lemma [Yu & Han, 2005].

Lemma 1. *Suppose the focus values obtained from a general dynamical system are functions of k independent system parameters $p_1, p_2, \ldots, p_k$. Further, assume that at a critical point, p_c defined by $(p_1, p_2, \ldots, p_k) = (p_{1c}, p_{2c}, \ldots, p_{kc})$, the focus values satisfy*

$$v_j(p_c) = 0, \quad j = 0, 1, \ldots, k-1, \qquad v_k(p_c) \neq 0;$$

and

$$\det\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(p_1, p_2, \ldots, p_k)}\right]_{p_c} \neq 0.$$

Then, proper perturbations can be made to the parameters $p_1, p_2, \ldots, p_k$ around the critical point p_c to generate k small-amplitude limit cycles in the vicinity of the Hopf critical point (the origin).

Proof. By using the Maple program [Tian & Yu, 2013], we can obtain the first seven focus values in terms of the system coefficients:

$$\begin{aligned} v_1 &= v_1(a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33}), \\ v_2 &= v_2(a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33}), \\ &\vdots \\ v_7 &= v_7(a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33}), \end{aligned} \tag{5}$$

and via them we can estimate the number of small-amplitude limit cycles around the origin, which are embedded in the center manifold (which is also obtained from the Maple program), described by

$$\begin{aligned} x_3 = {} & \frac{1}{5}(x_1^2 + x_1x_2 - x_2^2) - \frac{1}{5}(2a_{11} - c_{13} + 1)x_1^3 - \frac{1}{5}(3a_{11} + 2a_{12} - c_{13} + 2)x_1^2x_2 \\ & + \frac{1}{5}(4a_{11} - a_{12} - 2a_{22} - c_{13} - 1)x_1x_2^2 - \frac{1}{5}(a_{22} + 2)x_2^3 - \frac{1}{5}\left(c_{13} - \frac{1}{5}c_{33} + 2a_{11}c_{13} - c_{13}^2\right)x_1^4 \\ & - \frac{1}{5}\left(2c_{13} - \frac{2}{5}c_{33} + 3a_{11}c_{13} + 2a_{12}c_{13} - c_{13}^2\right)x_1^3x_2 \\ & - \frac{1}{5}\left(c_{13} + \frac{1}{5}c_{33} - 4a_{11}c_{13} + a_{12}c_{13} + 2a_{22}c_{13} + c_{13}^2\right)x_1^2x_2^2 \\ & + \frac{1}{5}\left(-\frac{2}{5}c_{33} - 2c_{13} - c_{13}a_{22}\right)x_1x_2^3 + \frac{1}{25}c_{33}x_2^4 + \cdots. \end{aligned} \tag{6}$$

It is obvious to see from (6) that the center manifold near the origin is approximated by a *hyperbolic paraboloid*, as shown in Fig. 1.

To obtain the maximal number of small-amplitude limit cycles bifurcating from the origin, we solve the parameters a_{11}, a_{12}, a_{22}, b_{33}, c_{13}, c_{33} from the six polynomial equations $v_1 = v_2 = \cdots = v_6 = 0$. Alternatively, we may solve these six polynomial equations one by one, with one parameter at each time. We start from the first focus value v_1, which is the same as that for the case $b_{33} = 0$, i.e.

$$v_1 = -\frac{1}{8}a_{12}(a_{11} + a_{22}).$$

Letting

$$a_{22} = -a_{11} \tag{7}$$

yields $v_1 = 0$, and then executing the Maple program we have

$$v_2 = \frac{1}{1200}b_{33}(a_{12} + 3c_{13} + 18a_{11} + 10).$$

Setting

$$a_{12} = -(3c_{13} + 18a_{11} + 10) \tag{8}$$

results in $v_2 = 0$ and then executing the Maple program gives

$$\begin{aligned} v_3 = {} & \frac{1}{272000}b_{33}[-187b_{33} + (695c_{13} + 2070a_{11} \\ & - 790)c_{33} + (92290a_{11}^2 + 74582a_{11} + 15384)c_{13} \\ & + (3342a_{11} - 45)c_{13}^2 - 666c_{13}^3 + 228172a_{11}^3 \\ & + 220428a_{11}^2 + 57028a_{11} + 2020]. \end{aligned}$$

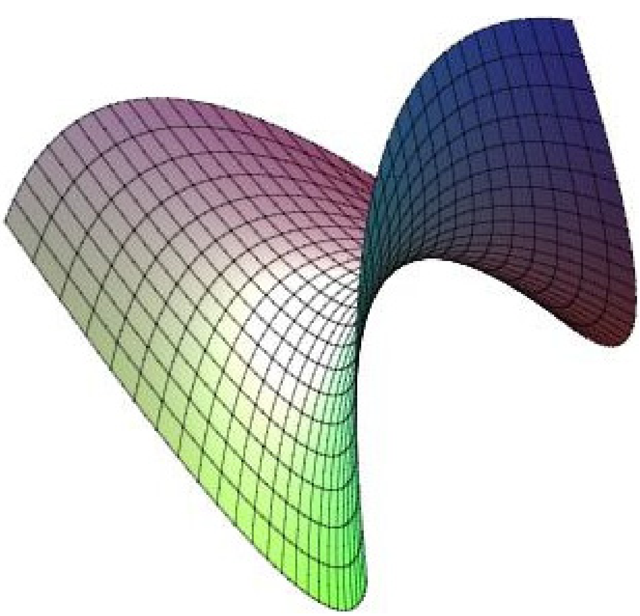

Fig. 1. The second-order approximation of the center manifold described by (6).

Thus, we may solve for b_{33} from the equation $v_3 = 0$ to obtain

$$b_{33} = \frac{1}{187}[(695c_{13} + 2070a_{11} - 790)c_{33} + (92290a_{11}^2 + 74582a_{11} + 15384)c_{13} + (3342a_{11} - 45)c_{13}^2 - 666c_{13}^3 + 228172a_{11}^3 + 220428a_{11}^2 + 57028a_{11} + 2020]. \tag{9}$$

Now, under the conditions given in (7)–(9), we have $v_1 = v_2 = v_3 = 0$, and further execute the Maple program to obtain

$$v_4 = \frac{F_0F_1}{1483804608000}, \quad v_5 = \frac{F_0F_2}{16015652769093120000},$$

$$v_6 = \frac{F_0F_3}{1613427109976213128396800000}, \quad v_7 = \frac{F_0F_4}{15831592173930593701080760320000000},$$

where

$$F_0 = 5(414a_{11} + 139c_{13} - 158)c_{33} + (15384 + 74582a_{11} + 92290a_{11}^2)c_{13} + 3(1114a_{11} - 15)c_{13}^2 - 666c_{13}^3 + 4(505 + 14257a_{11} + 55107a_{11}^2 + 57043a_{11}^3)$$

and

$$\begin{aligned} F_1 = {} & 4(4078333c_{13} + 14139153a_{11} - 2787647)c_{33}^2 + 2[2(373041446a_{11}^2 + 500749565a_{11} + 111353261)c_{13} \\ & - (98445579a_{11} + 52751465)c_{13}^2 - 16677015c_{13}^3 + 4(856767634a_{11}^3 + 967186323a_{11}^2 \\ & + 181154724a_{11} - 11713817)]c_{33} + 29601792c_{13}^5 + 9(86303536a_{11} + 25802705)c_{13}^4 \\ & + 6(882002754a_{11}^2 + 437785405a_{11} + 14608506)c_{13}^3 + 4(5370668262a_{11}^3 + 4204559671a_{11}^2 \\ & + 34389389a_{11} - 252727446)c_{13}^2 + 8(13079993487a_{11}^4 + 24728477022a_{11}^3 + 15158560637a_{11}^2 \\ & + 3657980072a_{11} + 287618390)c_{13} + 16(16499286495a_{11}^5 + 37837627784a_{11}^4 + 29685004857a_{11}^3 \\ & + 9784662107a_{11}^2 + 1218699212a_{11} + 2203887), \end{aligned}$$

and F_2 and F_3 are given in Appendix A, while the lengthy expression of F_4 is not listed here. Now in order to obtain limit cycles bifurcating from the origin (the Hopf critical point) as many as possible, we need to find critical parameter values of a_{11}, c_{13} and c_{33} such that $v_4 = v_5 = v_6 = 0$, but $v_7 \neq 0$ (i.e. $F_1 = F_2 = F_3 = 0$, but $F_4 \neq 0$). In this case, we can conclude that there exist at most seven small-amplitude limit cycles bifurcating from the origin. Then, proper perturbations may be applied to the seven parameters, α, a_{11}, a_{12}, a_{22}, b_{33}, c_{13} and c_{33}, to generate seven small-amplitude limit cycles, or we can apply Lemma 1 to prove the existence of seven limit cycles. Since we set $\alpha = 0$ to get $v_0 = 0$, $a_{22} = -a_{11}$ to get $v_1 = 0$, $a_{12} = -(18a_{11}+3c_{13}+c_{23}+10)$ to obtain $v_2 = 0$, and $b_{33} = \frac{1}{187}[(2070a_{11} + 695c_{13} - 95c_{23} - 790)c_{33} + \cdots]$ [given in (9)] to obtain $v_3 = 0$, perturbations on b_{33}, a_{12}, a_{22} and α can be made one by one. Thus, we only need to consider $v_4 = v_5 = v_6 = 0$, i.e. $F_1 = F_2 = F_3 = 0$, but $v_7 \neq 0$, at some critical values $a_{11c}, c_{13c}, c_{33c}$, and further

$$\det\left[\frac{\partial(v_4, v_5, v_6)}{\partial(a_{11}, c_{13}, c_{33})}\right]_{(a_{11c}, c_{13c}, c_{33c})} \neq 0.$$

To find the critical values $a_{11c}, c_{13c}, c_{33c}$ such that $F_1 = F_2 = F_3 = 0$, we apply the *Regular Chain* method [Chen *et al.*, 2013]. We use (7)–(9) to simplify v_4 to v_6 to obtain polynomial equations $F_1 = F_2 = F_3 = 0$. Then execute the Maple program (see [Chen *et al.*, 2013]) on the same desktop machine to obtain the following results by using the modular regular chains method: the formulas of c_{13} and c_{33} expressed in terms of a_{11},

$$c_{13} = -\frac{c_{13N}(a_{11})}{12c_{13D}(a_{11})}, \quad c_{33} = -\frac{c_{33N}(a_{11})}{Nc_{33D}(a_{11})}, \tag{10}$$

where N is an integer, and $c_{13N}(a_{11})$, $c_{13D}(a_{11})$, $c_{33N}(a_{11})$ and $c_{33D}(a_{11})$ are 156th-degree polynomials of a_{11}; and a resultant equation, given by a

157th-degree polynomial $g(a_{11}) = 0$, which in turn gives a total of 19 real solutions. We solve a_{11} from this polynomial equation up to 1000 digit points, with the results listed below (only showing the first 50 digits).

$$a_{11}^1 = -4.11276888495705654624708808345078873211396249503460\ldots,$$

$$a_{11}^2 = -1.82010942866258004577090611868371605998764973794356\ldots,$$

$$a_{11}^3 = -0.76440311387403968219929953842967589771581114232615\ldots,$$

$$a_{11}^4 = -0.75410520646463776589886974547597729068673851680993\ldots,$$

$$a_{11}^5 = -0.46061934131364857055550286413352190906564989377128\ldots,$$

$$a_{11}^6 = -0.44754772090870942476035011043695763950789559075632\ldots,$$

$$a_{11}^7 = -0.38187937918219584496343813228246930627419322177798\ldots,$$

$$a_{11}^8 = -0.31428920280160160469525336903289260600817103833470\ldots,$$

$$a_{11}^9 = -0.28314729830779529882213773988148784486261517488513\ldots,$$

$$a_{11}^{10} = -0.13330838515576413592119147947119785761283975388044\ldots,$$

$$a_{11}^{11} = -0.02861803346154083192185648912224434468926816974799\ldots,$$

$$a_{11}^{12} = -0.01129618883353299940696424356394530075959246381228\ldots,$$

$$a_{11}^{13} = 0.00003261862103285667320075873891685629773802493465\ldots,$$

$$a_{11}^{14} = 0.01557965760324882734099653888501403592680477722409\ldots,$$

$$a_{11}^{15} = 0.02629936725348609926921580980768242470782868685459\ldots,$$

$$a_{11}^{16} = 0.04674224356461493450786328894470060987403146438352\ldots,$$

$$a_{11}^{17} = 0.56032275926806357270588556057116717906044592783859\ldots,$$

$$a_{11}^{18} = 5.38438918903427504185594454194797573037902064705802\ldots,$$

$$a_{11}^{19} = 26.01492173704774508843595793963653547777807547320274\ldots.$$

We take $a_{11} = a_{11}^7$, which yields

$$c_{13} = -0.41261102816606685288914232443213702004650348278544\ldots,$$

$$c_{33} = -0.33160576682318949987643286719692488957369961896560\ldots,$$

and use Eqs. (7)–(9) to obtain $a_{22} = -a_{11}^7$ and

$$a_{12} = -1.88833809022227423199068664561914142692501155964000\ldots,$$

$$b_{33} = -0.14679339349579488722266912282493720766001218127019\ldots.$$

For these critical parameter values, the focus values become

$$v_1 = 0.0, \quad v_2 = -0.1 \times 10^{-1000}, \quad v_3 = -0.6847 \times 10^{-1000},$$

$$v_4 = -0.13219256310383786756022068742997222535380219931004\ldots \times 10^{-942},$$

$$v_5 = -0.31762418358601926533300695679923261099352343009257\ldots \times 10^{-942},$$
$$v_6 = -0.46950935768785676094927172098325782856331763210221\ldots \times 10^{-942},$$
$$v_7 = -0.83776339081446765262795751469808290872469085804425\ldots \times 10^{-5}.$$

The errors on v_2–v_6 are due to numerical computation in the final step of solving the 157th-degree polynomial of a_{11}. In fact, we can perform the interval computation in Maple to identify the interval for each of the parameters up to any accuracy, which proves that there exist solutions such that $v_1 = v_2 = \cdots = v_6 = 0$, but $v_7 \neq 0$. Therefore, we can conclude that there exist at most seven small-amplitude limit cycles around the origin. Moreover, a direct calculation shows that

$$\det\left[\frac{\partial(v_4, v_5, v_6)}{\partial(a_{11}, c_{13}, c_{33})}\right]_{(a_{11c}, c_{13c}, c_{33c})} \approx -0.0000000333723796304 \neq 0,$$

implying that there exist seven small-amplitude limit cycles around the origin. ■

3. Conclusion

In this paper, we have shown that a simple three-dimensional quadratic vector field can exhibit seven small-amplitude limit cycles in the vicinity of a Hopf critical point. The method of normal forms is applied to compute the focus values associated with Hopf bifurcation, while the modular regular chains method is used to solve higher-degree multivariate polynomial equations. This result may be further improved in future by developing more powerful computational tools.

Acknowledgment

This work was supported by the Natural Science and Engineering Research Council of Canada (NSERC).

References

Bautin, N. N. [1952] "On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type," *Mat. Sbornik* (*NS*) **30**, 181–196.

Chen, L. S. & Wang, M. S. [1979] "The relative position, and the number, of limit cycles of a quadratic differential system," *Acta. Math. Sin.* **22**, 751–758.

Chen, C., Corless, R., Maza, M., Yu, P. & Zhang, Y. [2013] "A modular regular chains method and its application to dynamical systems," *Int. J. Bifurcation and Chaos* **23**, 1350154.

Écalle, J. [1992] *Introduction aux Fonctions Analysables et Preuve Constructive de la Conjecture de Dulac* (Hermann, Paris) (in French).

Gyllenberg, M., Yan, P. & Wang, Y. [2006] "A 3-D competitive Lotka–Volterra system with three limit cycles: A classification of a conjecture by Hofbauer and So," *Appl. Math. Lett.* **19**, 1–7.

Gyllenberg, M. & Yan, P. [2009] "Four limit cycles for a three-dimensional competitive Lotka–Volterra system with a heteroclinic cycle," *Comput. Math. Appl.* **58**, 649–669.

Hofbauer, J. & So, J. W.-H. [1994] "Multiple limit cycles for three-dimensional Lotka–Volterra equations," *Appl. Math. Lett.* **7**, 65–70.

Ilyashenko, Yu. S. & Yakovenko, S. [1991] "Finitely smooth normal forms of local families of diffeomorphismes and vector fields," *Russ. Math. Surv.* **46**, 3–19.

James, E. & Lloyd, N. [1991] "A cubic system with eight small-amplitude limit cycles," *IMA J. Appl. Math.* **47**, 163–171.

Li, C., Liu, C. & Yang, J. [2009] "A cubic system with thirteen limit cycles," *J. Diff. Eqs.* **246**, 3609–3619.

Li, J. & Liu, Y. [2010] "New results on the study of Z_q-equivariant planar polynomial vector fields," *Qualit. Th. Dyn. Syst.* **9**, 167–219.

Lu, Z. & Luo, Y. [2002] "Two limit cycles in three-dimensional Lotka–Volterra systems," *Comput. Math. Appl.* **44**, 51–66.

Poincaré, H. [1881–1886] "Mémoire sur les courbes définies par une équation différentielle. I, II," *J. Math. Pures Appl.* [1881] **7**, 375–422; [1882] **8**, 251–296. [1885] Sur les courbes définies par les équations différentielles. III, IV, *ibid.* **1**, 167–224; [1886] **2**, 155–217.

Shi, S. [1980] "A concrete example of the existence of four limit cycles for plane quadratic systems," *Sci. Sin.* **23**, 153–158.

Tian, Y. & Yu, P. [2013] "An explicit recursive formula for computing the normal form and center manifold of n-dimensional differential systems associated with Hopf bifurcation," *Int. J. Bifurcation and Chaos* **23**, 1350104.

Yu, P. & Han, M. [2005] "Small limit cycles bifurcating from fine focus points in cubic order Z_2-equivariant vector fields," *Chaos Solit. Fract.* **24**, 329–348.
Yu, P. & Corless, R. [2009] "Symbolic computation of limit cycles associated with Hilbert's 16th problem," *Commun. Nonlin. Sci. Numer. Simul.* **14**, 4041–4056.
Yu, P. & Han, M. [2012] "Four limit cycles from perturbing quadratic integrable systems by quadratic polynomials," *Int. J. Bifurcation and Chaos* **22**, 1250254.

Appendix A

The functions F_2 and F_3 are listed in this Appendix.

$$\begin{aligned}
F_2 = {} & -350064(144902698c_{13} + 366576733a_{11} - 142697703)c_{33}^3 + 4[1322875486074\,92c_{13}^3 \\
& + (1587431095048589a_{11} + 371135538912053)c_{13}^2 + 2(1985595066771294a_{11}^2 + 332568076619189a_{11} \\
& - 167986412567563)c_{13} + 4(832557288593889a_{11}^3 + 343973807357514a_{11}^2 - 129381390471427a_{11} \\
& - 22505970396248)]c_{33}^2 - 4[213711424998672c_{13}^5 + 9(311807824390159a_{11} + 148318218680889)c_{13}^4 \\
& - (11809249798049\,06a_{11}^2 + 5452054185589939a_{11} + 352322443563555)c_{13}^3 \\
& - 4(427884779353644\,25a_{11}^3 + 67363587215176597a_{11}^2 + 24044781463740170a_{11} \\
& + 2314226039107142)c_{13}^2 - 4(27591776267013391a_{11} + 173646715268323199a_{11}^4 \\
& + 321001980070626737a_{11}^3 + 162732256648660003a_{11}^2 + 879991334108382)c_{13} \\
& - 8(116707360013076057a_{11}^5 + 257051474918548297a_{11}^4 + 171010776019582988a_{11}^3 \\
& + 42694054908035380a_{11}^2 + 2402049457670883a_{11} - 250745335036453)]c_{33} \\
& + 630912865266000c_{13}^7 + 9(2964712669290231a_{11} + 782392810580879)c_{13}^6 \\
& + 6(73541596962729056a_{11}^2 + 38427258756511039a_{11} + 3606485585632344)c_{13}^5 \\
& + 12(322411353706968259a_{11}^3 + 280026912237356567a_{11}^2 + 58525009028373887a_{11} \\
& + 366306838658389)c_{13}^4 + 8(2731040674800736927a_{11}^4 + 3810289896228085498a_{11}^3 \\
& + 1688659169110635940a_{11}^2 + 239657009603682009a_{11} - 731762510421390)c_{13}^3 \\
& + 16(5442580446106842105a_{11}^5 + 11332299759177700061a_{11}^4 + 8435369673472740199a_{11}^3 \\
& + 2768383347832399342a_{11}^2 + 372489286490816132a_{11} + 11783278257257817)c_{13}^2 \\
& + 32(6397426172395771554a_{11}^6 + 17484633288012047816a_{11}^5 + 17729148621051130479a_{11}^4 \\
& + 8645141995434202821a_{11}^3 + 2101633085727469205a_{11}^2 + 225196488860266879a_{11} \\
& + 6078244989652798)c_{13} + 64(3335635549859104292a_{11}^7 + 11004977896310477071a_{11}^6 \\
& + 13758987325223239501a_{11}^5 + 8649165820899777993a_{11}^4 + 2937263004217804984a_{11}^3 \\
& + 518248503098891291a_{11}^2 + 38482349044241143a_{11} + 320434896140845), \\
F_3 = {} & 15197445120(1295313405565c_{13} + 4841990370990a_{11} - 974241807866)c_{33}^4 \\
& - 1664[24752566511047772643c_{13}^3 + 2(81606533150962260337a_{11} + 20663006500550475\,8855)c_{13}^2 \\
& + 4(23784869920766047140a_{11}^2 + 45624768768634124\,5921a_{11} - 32067190881129442123)c_{13}
\end{aligned}$$

$$
\begin{aligned}
&- 4(718015199110540951179a_{11}^3 + 164083276954411890501a_{11}^2 + 497187713917569014161a_{11} \\
&+ 116882338879114660895)]c_{33}^3 + 4[1355162014974629674719 77c_{13}^5 \\
&+ (3652599114285837536632677a_{11} + 1372118809839451388158717)c_{13}^4 \\
&+ 8(4521694474249202984154340a_{11}^2 + 3422357624141089273121095a_{11} \\
&+ 5517728236829125351191 27)c_{13}^3 + 8(2300145412700326708770 1629a_{11}^3 \\
&+ 2263265776841173715485621 7a_{11}^2 + 4175449582577115578189071a_{11} \\
&- 1445926031691291791520 61)c_{13}^2 + 16(2519727776220478000500 4103a_{11}^4 \\
&+ 2359185569610347657804581 4a_{11}^3 - 1339991489632296089393848a_{11}^2 \\
&- 3368285960479851576060286a_{11} - 423709293782930794542471)c_{13} \\
&+ 16(2383326155782548297280640 1a_{11}^5 + 2360508826050202260603167 3a_{11}^4 \\
&- 1929258359888813266977358a_{11}^3 - 3852803639369438088651366a_{11}^2 \\
&- 2733547863390182491644 67a_{11} + 721353053524863394330 05)]c_{33}^2 \\
&- [708959378346946555776981 4c_{13}^7 + 6(166455480917751063916663 7a_{11} \\
&+ 1468549573192142753107527)c_{13}^6 - 4(1014247288861177169539700 6a_{11}^2 \\
&- 8093351702071968321732515a_{11} - 8797678108504067116393611)c_{13}^5 \\
&- 8(2703181381903094362111644 04a_{11}^3 + 3748669147306872500306944 33a_{11}^2 \\
&+ 9485772663550439383207171 8a_{11} + 1611714501224953912649461)c_{13}^4 \\
&- 32(7270208874166607484114171 49a_{11}^4 + 1495840229622491117284675 129a_{11}^3 \\
&+ 8428496599753182668165581 99a_{11}^2 + 1756200201442747651614963 11a_{11} \\
&+ 111579422390362695275257 24)c_{13}^3 - 32(3815035076500245513656904 529a_{11}^5 \\
&+ 9507605546464302764041140 761a_{11}^4 + 7368167729908292445287343 410a_{11}^3 \\
&+ 2417690260166455114448121 506a_{11}^2 + 3279249240883862890708806 21a_{11} \\
&+ 1248140371022022348160584 5)c_{13}^2 - 64(4623365964881386828206541 934a_{11}^6 \\
&+ 1287327421008990554939742 8201a_{11}^5 + 1184772196138938222445016 2225a_{11}^4 \\
&+ 4848749975836708041755541 250a_{11}^3 + 8796797588442719101692315 60a_{11}^2 \\
&+ 4735978129094717177149773 3a_{11} - 1993655132477530259763543)c_{13} \\
&- 128(2182688317704910216101782 722a_{11}^7 + 6650367630130223948338676 851a_{11}^6 \\
&+ 7055523579922110839931549 040a_{11}^5 + 3531962564650152984423497 445a_{11}^4 \\
&+ 8761583435747232640998307 30a_{11}^3 + 9186782203040847943076434 9a_{11}^2 \\
&+ 930748811756786870470 12a_{11} - 404530676080894666989045)]c_{33} \\
&+ 52377983042949992837606 4c_{13}^9 + 9(471187852575883096190846 4a_{11}
\end{aligned}
$$

$$
\begin{aligned}
&+ 9993453508777669019781 41)c_{13}^8 + 6(225876452729015835356342234a_{11}^2 \\
&+ 1108465238850563579499775 33a_{11} + 1025330569165384009179889 8)c_{13}^7 \\
&+ 12(183334526481947516593662777 8a_{11}^3 + 156597106455660053711006716 3a_{11}^2 \\
&+ 366397091618786153338784017 1a_{11} + 160515902146917838031493 52)c_{13}^6 \\
&+ 8(26086580583552734554940584951a_{11}^4 + 33532903248572391712065771052a_{11}^3 \\
&+ 1423801815068393498129794570 9a_{11}^2 + 207454022439345106906110852 6a_{11} \\
&+ 278616861424037602517465 46)c_{13}^5 + 16(785486597773263087873366716 59a_{11}^5 \\
&+ 1411383635024539839946922975 39a_{11}^4 + 928139099717044793312265255 67a_{11}^3 \\
&+ 2671732057040389304903993525 1a_{11}^2 + 289010667492522556599919221 8a_{11} \\
&+ 146305425565464676844367 42)c_{13}^4 + 32(158056462633238785803151049168a_{11}^6 \\
&+ 37926101370434927233922419665 5a_{11}^5 + 351155334849581032033712611590a_{11}^4 \\
&+ 159226588640254021019516857610a_{11}^3 + 36006057259210021488914610840a_{11}^2 \\
&+ 3443427334420574893921441871a_{11} + 5334871166607269684570593 0)c_{13}^3 \\
&+ 64(213934489231668821736726105776a_{11}^7 + 653860206996441311651858729543a_{11}^6 \\
&+ 787930912766941335180880197069a_{11}^5 + 489789605387889313586662661710a_{11}^4 \\
&+ 168372810485883167250927940970a_{11}^3 + 31052460989227661080767094867a_{11}^2 \\
&+ 2595121581266887210220232361a_{11} + 4767786406674403659391168 8)c_{13}^2 \\
&+ 128(175795180541454878327994323627a_{11}^8 + 646069392177506577189809704414a_{11}^7 \\
&+ 943974632974823926628515523427a_{11}^6 + 730811516906412079242497471216a_{11}^5 \\
&+ 328721576130872118296329634775a_{11}^4 + 86861820610588450810846403654a_{11}^3 \\
&+ 12669250522915877666530948173a_{11}^2 + 828317972353063952969598924a_{11} \\
&+ 843048559358415481020030 2)c_{13} + 256(65489078114939885475729056623a_{11}^9 \\
&+ 276207783896081023026631595440a_{11}^8 + 467148550007619228316050778351a_{11}^7 \\
&+ 427474964552370509561272719651a_{11}^6 + 235261518648158313822481113745a_{11}^5 \\
&+ 80473535394907792966346350991a_{11}^4 + 16803980816127953612017002885a_{11}^3 \\
&+ 1967923782290897527897320937a_{11}^2 + 100256044517078058643496652a_{11} \\
&+ 235738171481448869001845).
\end{aligned}
$$