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Center conditions in a switching Bautin system

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Abstract

A new method with an efficient algorithm is developed for computing the Lyapunov constants of planar switching systems, and then applied to study bifurcation of limit cycles in a switching Bautin system. A complete classification on the conditions of a singular point being a center in this Bautin system is obtained. Further, an example of switching systems is constructed to show the existence of 10 small-amplitude limit cycles bifurcating from a center. This is a new lower bound of the maximal number of small-amplitude limit cycles obtained in quadratic switching systems near a singular point.

MSC: 34C07; 34C23

Keywords: Switching system; Bautin switching system; Lyapunov constant; Center; Bifurcation; Limit cycle

1. Introduction

Many problems arising in science and engineering are modeled by dynamical systems whose vector fields (i.e., the right-hand sides of the equations) are not continuous or not differentiable. These systems are indistinctly called discontinuous or non-smooth systems. A full discussion on this subject can be found in the classical books [1,2].

During the past few decades, increasing interest has been attracted to the qualitative analysis of non-smooth systems, because non-smooth systems describe some real problems more accurately and display rich complex dynamical phenomena, which must not be disregarded in applications,

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for instance the squealing noise in car brakes [3,4], or the absence of a thermal equilibrium in gases modeled by scattering billiards [5,6]. Because of various forms of non-smoothness, non-smooth systems can exhibit not only the classical bifurcations, like Hopf bifurcation, homoclinic bifurcation, but also more complicated bifurcations that only non-smooth systems can have, such as border-collision bifurcation [7–9], grazing bifurcation [10,11] and so on. A great deal of work has been done to generalize the classical bifurcation methods for smooth systems to non-smooth ones, see for instance [12–17].

One class of planar non-smooth dynamical systems is the so-called switching system, which has different definitions of the continuous vector fields in two different regions divided by a line (or a curve). Our attention in this paper is focused on the switching systems, given in the form of

$$(\dot{x}, \dot{y}) = \begin{cases} \big(\delta x - y + f^{+}(x, y, \boldsymbol{\mu}),\ x + \delta y + g^{+}(x, y, \boldsymbol{\mu})\big), & \text{if } y > 0, \\ \big(\delta x - y + f^{-}(x, y, \boldsymbol{\mu}),\ x + \delta y + g^{-}(x, y, \boldsymbol{\mu})\big), & \text{if } y < 0, \end{cases} \tag{1}$$

where $\boldsymbol{\mu} \in \mathbf{R}^m$ is a parameter vector and $\delta = \mu_1$, $f^{\pm}(x, y, \boldsymbol{\mu})$ and $g^{\pm}(x, y, \boldsymbol{\mu})$ are analytic functions in x and y starting at least from second-order terms. Obviously, the origin is an equilibrium of system (1). There are two systems in (1): the system defined in the upper half-plane for $y > 0$, called the upper system, and the system defined in the lower half-plane for $y < 0$, called the lower system.

Many contributions have been made to the study of Hopf bifurcation in switching systems, see for example [12,13,16,18–20]. As in the study of smooth dynamical systems, the center problem, determining the center conditions of a singular point being a center, and the cyclicity problem, finding the maximal number of small-amplitude limit cycles around a singular point, are fundamental in the analysis of Hopf bifurcation in switching systems. These two problems in switching systems can be investigated by computing the Lyapunov constants [12,15,16]. Gasull and Torregrosa [12] applied a suitable decomposition of certain one-forms and developed a new method for computing the Lyapunov constants of switching systems.

For the center problem, it is well-known that a singular point is a center in planar smooth systems if and only if there exists a local first integral around the singular point. However, the situation is quite complicated in switching systems. The origin of system (1) can be a center even if it is not a center of either the upper system or the lower system. On the other hand, if the origin is a center for both the upper system and the lower system of (1), one can not ensure that system (1) has a center at the origin. It also requires that their first integrals of the upper and lower systems coincide on the line $y = 0$. So far, some center conditions have been obtained for some switching Kukles systems [12], switching Liénard systems [13,18] and switching Bautin systems [16].

It is well known that planar linear systems can not produce limit cycles. For general planar quadratic systems with a focus or center, Bautin [21] obtained the following form:

$$\begin{aligned} \dot{x} &= \delta x - y - a_3 x^2 + (a_5 + 2a_2)xy + a_6 y^2, \\ \dot{y} &= x + \delta y + a_2 x^2 + (a_4 + 2a_3)xy - a_2 y^2, \end{aligned} \tag{2}$$

with a focus or center at the origin, which is now called Bautin system, and proved that system (2) can have 3 small-amplitude limit cycles around the origin. Note that Bautin system has one less parameter. For cubic systems, it is only proved that 12 small-amplitude limit cycles can appear around a center [22]. With the same degrees, switching polynomial systems can exhibit more limit cycles. For example, Han and Zhang [20] proved that 2 limit cycles can appear near a focus in linear switching systems. Without loss of generality, quadratic switching systems can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y - a_3x^2 + (a_5+a_2)xy + (a_6+a_3)y^2 \\ x + \delta y + a_2x^2 + (a_4-a_3)xy + (a_1-a_2)y^2 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} \delta x - y - b_3x^2 + (b_5+b_2)xy + (b_6+b_3)y^2 \\ x + \delta y + b_2x^2 + (b_4-b_3)xy + (b_1-b_2)y^2 \end{pmatrix}, & \text{if } y < 0. \end{cases} \tag{3}$$

The number of small-amplitude limit cycles bifurcating from a focus in system (3) was investigated in [12,15–17]. Among them, it was shown in [12] that system (3) can have at most 5 small-amplitude limit cycles when its lower system is linear. Recently, 9 small-amplitude limit cycles were obtained in [15] from a concrete example of switching Bautin systems through perturbations, in which the upper and lower systems are both Bautin systems.

In this paper, we develop a recursive procedure to compute the Lyapunov constants of the general system (1), which only involves algebraic computations, and then apply this method to study bifurcation of limit cycles in the following switching Bautin system, obtained by setting $a_1 = b_1 = 0$ in (3),

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y - a_3x^2 + (a_5+a_2)xy + (a_6+a_3)y^2 \\ x + \delta y + a_2x^2 + (a_4-a_3)xy - a_2y^2 \end{pmatrix}, & \text{if } y > 0, \\ \begin{pmatrix} \delta x - y - b_3x^2 + (b_5+b_2)xy + (b_6+b_3)y^2 \\ x + \delta y + b_2x^2 + (b_4-b_3)xy - b_2y^2 \end{pmatrix}, & \text{if } y < 0. \end{cases} \tag{4}$$

Note that the upper and lower systems in (4) are not exactly in the form of Bautin's system (2), but a simple transformation on the parameters can make them equivalent. For system (4) we obtain a complete center classification under the condition $a_6b_6 = 0$. Moreover, we introduce perturbations into system (4) with a center, and obtain 10 small-amplitude limit cycles.

Denote by $\mathcal{E}$ the interchange of parameters $(a_2, a_3, a_4, a_5, a_6) \leftrightarrow (b_2, -b_3, -b_4, b_4, b_5, -b_6)$. Note that by the change of variables $(x, y, t) \to (x, -y, -t)$, the upper system and the lower system in (4) exchange their equations, which can be derived equivalently by the interchange $\mathcal{E}$ in (4).

Theorem 1. *Assume $a_6b_6 = 0$. Then, system (4) has a center at the origin if and only if $\delta = b_6 = 0$, $b_5 = a_5$, and one of the following conditions or the corresponding one under the interchange of parameters $\mathcal{E}$ holds:*

$$\begin{aligned}
\mathrm{I}:&\quad a_2 = a_5 = b_2b_3 = 0,\\
\mathrm{II}:&\quad a_2 - a_5 = (b_2 - a_2)(b_2 + 2a_2) = b_3 = a_4 - 3a_3 = 0,\\
\mathrm{III}:&\quad a_2 - a_5 = b_2 = a_4 - 3a_3 = b_4b_3 - 2a_5^2 = 0,\\
\mathrm{IV}:&\quad a_2 - a_5 = a_2 - b_2 = a_4 - 3a_3 = b_4 - 3b_3 = 0,\\
\mathrm{V}:&\quad a_6 = b_3 = a_3a_4 - b_2(a_5 + b_2) = a_2 = 0,\\
\mathrm{VI}:&\quad a_6 = b_3 = a_3 + a_4 = 3a_2 + a_5 = 0,\ (a_2 - b_2)(2a_2 - b_2) = 0,
\end{aligned}$$

$$
\begin{aligned}
\text{VII}: &\quad a_6 = a_3 = a_5 = b_3 = 0, \\
\text{VIII}: &\quad a_6 = b_3 = a_3 = (b_2 - a_2)(a_2 + b_2 + a_5) = 0, \\
\text{IX}: &\quad a_6 = b_3 b_4 - a_3 a_4 = b_2 = a_2 = 0, \\
\text{X}: &\quad a_6 = b_4 - a_4 = b_2 - a_2 = b_3 - a_3 = 0, \\
\text{XI}: &\quad a_6 = b_4 + a_4 = b_2 - a_2 = b_3 + a_3 = 0, \\
\text{XII}: &\quad a_6 = 9b_3 b_4 + 2a_5^2 = a_4 + a_3 = 3a_2 + b_5 = b_2 = 0, \\
\text{XIII}: &\quad a_6 = b_4 + b_3 = a_4 + a_3 = 3a_2 + a_5 = a_2 - b_2 = 0, \\
\text{XIV}: &\quad a_6 = a_2 + b_2 - a_5 = 0,\ (2b_2 - a_2)a_3^2 = (b_2 - a_2)^2 a_5 = (2a_2 - b_2)b_3^2, \\
&\quad (2b_2 - a_2)a_4^2 = (3a_2 - 4b_2)^2 a_5,\ (2a_2 - b_2)b_4^2 = (3b_2 - 4a_2)^2 a_5.
\end{aligned}
$$

It is important to determine the maximal number of small-amplitude limit cycles bifurcating from the origin of system (4). One approach to get these small-amplitude limit cycles is via perturbations on the parameters with one of the conditions I–XIV, and thus limit cycles bifurcate from a center. In fact, we have obtained the following new result, which is the best so far for quadratic switching systems.

Theorem 2. *For system (3) with the conditions $a_1 = b_1 = 0$, $\delta = b_6 = 0$, $b_5 = a_5$, and that given in the item* X *of Theorem 1,* 10 *limit cycles can appear near the origin under small perturbations.*

Remark 3. Note that in Theorem 2 when $a_1 = b_1 = 0$, the general quadratic switching system (3) becomes the switching Bautin system (4). Further, when other conditions are satisfied, the origin of the switching Bautin system becomes a center. Then, perturbing the special system (3) with a center at the origin yields 10 small-amplitude limit cycles.

The proofs for the above two theorems will be given later in Section 4.

2. Preliminary

Using $x = r\cos(\theta)$ and $y = r\sin(\theta)$, and treating time t as a parameter, we obtain the equations describing the orbits of system (1) on the phase plane,

$$
\frac{dr}{d\theta} = \begin{cases} \dfrac{\delta r + R^+(r,\theta)}{1 + \Theta^+(r,\theta)}, & \text{if } \theta \in (0,\pi), \\[2ex] \dfrac{\delta r + R^-(r,\theta)}{1 + \Theta^-(r,\theta)}, & \text{if } \theta \in (\pi, 2\pi), \end{cases} \tag{5}
$$

where

$$
\begin{aligned}
R^{\pm}(r,\theta) &= \cos(\theta) f^{\pm}(r\cos(\theta), r\sin(\theta), \boldsymbol{\mu}) + \sin(\theta) g^{\pm}(r\cos(\theta), r\sin(\theta), \boldsymbol{\mu}), \\
\Theta^{\pm}(r,\theta) &= \frac{1}{r}\big(\cos(\theta) g^{\pm}(r\cos(\theta), r\sin(\theta), \boldsymbol{\mu}) - \sin(\theta) f^{\pm}(r\cos(\theta), r\sin(\theta), \boldsymbol{\mu})\big).
\end{aligned}
$$

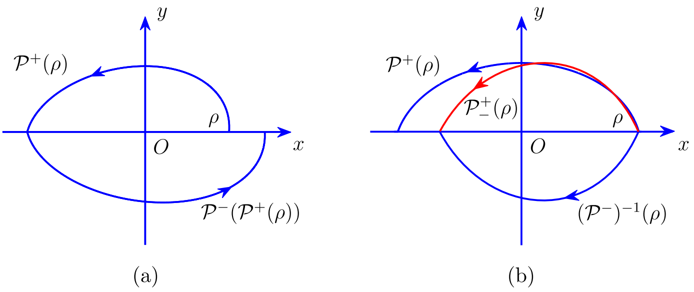

Fig. 1. (a) Poincaré map for system (1), and (b) Half-return maps $\mathcal{P}^+$ and $(\mathcal{P}^-)^{-1}$.

Let $r^+(\theta,\rho)$ and $r^-(\theta,\rho)$ be the solutions of the upper and lower systems of (5), respectively, with $r^+(0,\rho)=r^-(\pi,\rho)=\rho$. Then, through the positive half-return map $\mathcal{P}^+(\rho)=r^+(\pi,\rho)$ and the negative half-return map $\mathcal{P}^-(\rho)=r^-(2\pi,\rho)$, we can define the Poincaré map $\mathcal{P}(\rho)=\mathcal{P}^-(\mathcal{P}^+(\rho))$, as illustrated in Fig. 1(a).

Suppose the displacement function $d(\rho)=\mathcal{P}(\rho)-\rho$ can be expanded as

$$d(\rho)=V_1\rho+V_2\rho^2+V_3\rho^3+\cdots, \tag{6}$$

where V_k is called the kth-order Lyapunov constant of the switching system (1). It is easy to see that the origin is a center of system (1) if and only if $d(\rho)\equiv 0$ for $0<\rho\ll 1$, which means that all the Lyapunov constants in (6) vanish. The isolated zeros of $d(\rho)=0$ near $\rho=0$ correspond to the limit cycles around the origin. It is not difficult to get $V_1=e^{2\delta\pi}-1$ since $\mathcal{P}^\pm(\rho)=e^{\delta\pi}\rho+O(\rho^2)$. Thus, $V_1=0$ if and only if $\delta=0$. It is well known that for the first nonzero Lyapunov constant V_k in a smooth system, k must be an odd number [23, Lemma 2.1.1]. While if V_k is the first nonzero term in (6), k could be any positive integer. Because of this small difference, the theorem used to determine the number of limit cycles by Lyapunov constants should take some corresponding changes. We have the following lemma.

Lemma 4. *Assume that there exists a sequence of Lyapunov constants of system (1), $V_{i_0}, V_{i_1}, \ldots, V_{i_k}$, with $1=i_0<i_1<\cdots<i_k$, such that $V_j=O(|V_{i_0},\ldots,V_{i_l}|)$ for any $i_l<j<i_{l+1}$. If for system (1) at the critical point $\boldsymbol{\mu}=\boldsymbol{\mu}_0$, $V_{i_0}=V_{i_1}=\cdots=V_{i_k-1}=0$, $V_{i_k}\neq 0$, and*

$$\operatorname{rank}\left[\frac{\partial(V_{i_0},V_{i_1},\ldots,V_{i_{k-1}})}{\partial(\mu_1,\mu_2,\ldots,\mu_m)}(\boldsymbol{\mu}_0)\right]=k,$$

then k limit cycles can appear near the origin of system (1) for some $\boldsymbol{\mu}$ near $\boldsymbol{\mu}_0$.

Lemma 4 is based on Theorem 2.3.2 in [23]. So we give a brief proof here. By the assumption of Lemma 4, the displacement function $d(\rho)$ in (6) can be rewritten in the form

$$d(\rho)=V_{i_0}\rho^{i_0}(1+P_0(\rho))+\cdots+V_{i_{k-1}}\rho^{i_{k-1}}(1+P_{k-1}(\rho))+V_{i_k}\rho^{i_k}+O(\rho^{i_k+1}),$$

where $P_l(\rho)=O(\rho^{j_l})$, $l=0,\ldots,k-1$, j_l is the smallest positive integer satisfying $i_l+j_l<i_{l+j_l}$, otherwise $P_l(\rho)=0$. Since $V_{i_0},V_{i_1},\ldots,V_{i_{k-1}}$ are independent with respect to $\boldsymbol{\mu}$, we can vary $\boldsymbol{\mu}$ around $\boldsymbol{\mu}_0$ such that

$$0<|V_{i_0}|\ll|V_{i_1}|\ll\cdots\ll|V_{i_{k-1}}|\ll 1,\quad V_{i_j}V_{i_{j+1}}<0,\quad j=0,\dots,k-1,$$

which ensures the existence of k positive zeros of $d(\rho)$ in ρ around $\rho=0$.

Based on Lemma 4, we remark that the expressions in this paper for V_k, $k=2,3,\dots$, are obtained by setting $V_1=V_2=\cdots=V_{k-1}=0$. Then, for any $i_l<j<i_{l+1}$, $V_j=O(|V_{i_0},\dots,V_{i_l}|)$ in Lemma 4 becomes $V_j\equiv 0$.

From now on, we assume that $\delta=0$ in system (1) and so $V_1=0$. It is very difficult to compute the remaining Lyapunov constants by using (6), since it involves the composition of two maps $\mathcal{P}^+(\rho)$ and $\mathcal{P}^-(\rho)$. To simplify the computation of Lyapunov constants, the authors of [12] introduced a new function,

$$\mathcal{P}^+(\rho)-(\mathcal{P}^-)^{-1}(\rho)=W_1\rho+W_2\rho^2+W_3\rho^3+\cdots, \tag{7}$$

where $(\mathcal{P}^-)^{-1}(\rho)$ is the inverse map of $\mathcal{P}^-(\rho)$. For $(\mathcal{P}^-)^{-1}(\rho)$, we have $(\mathcal{P}^-)^{-1}(\rho)=\mathcal{P}^+_-(\rho)$, where $\mathcal{P}^+_-(\rho)$ is the positive half-return map of the system obtained from the lower system with the change of variables $(x,y,t)\to(x,-y,-t)$ (see Fig. 1(b)). Thus, to get (7) we only need to compute the two positive half-return maps $\mathcal{P}^+(\rho)$ and $\mathcal{P}^+_-(\rho)$. It is proved [12] that for (6) and (7), the conditions $V_k\neq 0$, $V_j=0$, $1\le j\le k-1$, are equivalent to $W_k\neq 0$, $W_j=0$, $1\le j\le k-1$. In Section 3, we shall present a new method to compute W_k's in (7). Because of the equivalence of V_k and W_k, we still use V_k instead of W_k in the rest of the paper for simplicity.

Note that any Lyapunov constant V_k is a polynomial in terms of the coefficients of system (1). Thus, having obtained the Lyapunov constants, we need to solve a system of multivariate polynomial equations, and to find the center conditions. We shall use the Maple built-in command "resultant" to solve these polynomial equations and find their common zeros.

Denote by $\mathbf{R}[x_1,x_2,\dots,x_r]$ the polynomial ring of multivariate polynomials in $x_1,x_2,\dots,x_r$ with coefficients in $\mathbf{R}$. Let

$$\begin{aligned}p(x_1,x_2,\dots,x_r)&=\sum_{i=0}^{m}p_i(x_1,\dots,x_{r-1})x_r^i,\\ q(x_1,x_2,\dots,x_r)&=\sum_{i=0}^{n}q_i(x_1,\dots,x_{r-1})x_r^i\end{aligned} \tag{8}$$

be two polynomials in $\mathbf{R}[x_1,x_2,\dots,x_r]$ of respective positive degrees m and n in x_r. The following matrix is called the Sylvester matrix of p and q with respect to x_r,

$$\mathrm{Syl}(p,q,x_r)=\begin{pmatrix} p_m & p_{m-1} & \cdots & p_0 & & & \\ & p_m & p_{m-1} & \cdots & p_0 & & \\ & & \ddots & \ddots & & \ddots & \\ & & & p_m & p_{m-1} & \cdots & p_0 \\ q_n & q_{n-1} & \cdots & q_0 & & & \\ & q_n & q_{n-1} & \cdots & q_0 & & \\ & & \ddots & \ddots & & \ddots & \\ & & & q_n & q_{n-1} & \cdots & q_0 \end{pmatrix}\begin{array}{l}\left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}n\\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\}m\end{array},$$

whose determinant is called the resultant of p and q with respect to x_r, denoted by $\mathrm{Res}(p, q, x_r)$. We have the following lemma.

Lemma 5. *(See [24, Chapter 7].) Consider two multivariate polynomials $p(x_1, x_2, \ldots, x_r)$ and $q(x_1, x_2, \ldots, x_r)$ in $\mathbf{R}[x_1, x_2, \ldots, x_r]$ given by (8). Let $\mathrm{Res}(p, q, x_r) = h(x_1, \ldots, x_{r-1})$. Then:*

1. *If the real vector $\langle\alpha_1, \alpha_2, \ldots, \alpha_r\rangle \in \mathbf{R}^r$ is a common zero of the two equations $p(x_1, x_2, \ldots, x_r) = q(x_1, x_2, \ldots, x_r) = 0$, then $h(\alpha_1, \ldots, \alpha_{r-1}) = 0$.*
2. *Conversely, if $h(\alpha_1, \ldots, \alpha_{r-1}) = 0$, then at least one of the following four conditions holds:*
   (a) $p_m(\alpha_1, \ldots, \alpha_{r-1}) = \cdots = p_0(\alpha_1, \ldots, \alpha_{r-1}) = 0$, *or*
   (b) $q_n(\alpha_1, \ldots, \alpha_{r-1}) = \cdots = q_0(\alpha_1, \ldots, \alpha_{r-1}) = 0$, *or*
   (c) $p_m(\alpha_1, \ldots, \alpha_{r-1}) = q_n(\alpha_1, \ldots, \alpha_{r-1}) = 0$, *or*
   (d) *for some $\alpha_r \in \mathbf{R}$, $\langle\alpha_1, \ldots, \alpha_r\rangle$ is a common zero of both $p(x_1, \ldots, x_r)$ and $q(x_1, \ldots, x_r)$.*

From the first statement of Lemma 5, we know that if the resultant h does not have zeros on the region $D \subset \mathbf{R}^{r-1}$, then polynomials p and q do not have common zeros in $D \times \mathbf{R}$. According to the second statement, in order to solve $p = q = 0$, we first find the zeros of $h = 0$, and then substitute them back into p and q to solve for x_r. In this way, no zeros should be missed. For m multivariate polynomials with m variables, we can apply the command "resultant" repeatedly. For instance, take $m = 3$. To solve $F_j(x_1, x_2, x_3) = 0$, $j = 1, 2, 3$, suppose we compute $\mathrm{Res}(F_1, F_j, x_1)$ to obtain $\mathrm{Res}(F_1, F_j, x_1) = F_a(x_2, x_3)E_j(x_2, x_3)$, $j = 2, 3$. Then, we need to find the solutions for $F_a(x_2, x_3) = 0$ and $E_2(x_2, x_3) = E_3(x_2, x_3) = 0$. For $E_2 = E_3 = 0$, we can apply the command "resultant" again, like solving $\mathrm{Res}(E_2, E_3, x_2) = 0$.

3. Computation of Lyapunov constants

Since $W_1 = 0$ (or $V_1 = 0$) yields $\delta = 0$, to compute higher-order W_k's for the upper and lower systems in (1), we only need to consider a differential system of the form,

$$\dot{x} = -y + \sum_{i=2}^{+\infty} P_i(x, y), \quad \dot{y} = x + \sum_{i=2}^{+\infty} Q_i(x, y), \tag{9}$$

where $P_i(x, y)$ and $Q_i(x, y)$ are homogeneous polynomials in x and y of degree i. Obviously, system (9) has a Hopf singular point at the origin. Introducing the transformation $x = r\cos(\theta)$ and $y = r\sin(\theta)$ into (9) yields

$$\begin{aligned} \dot{r} &= \sum_{i=2}^{+\infty}(\cos(\theta)P_i + \sin(\theta)Q_i) = \sum_{i=2}^{+\infty} A_i(\theta)r^i, \\ \dot{\theta} &= 1 + \sum_{i=2}^{+\infty}(\cos(\theta)Q_i - \sin(\theta)P_i)/r = 1 + \sum_{i=2}^{+\infty} B_i(\theta)r^{i-1}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} A_i(\theta) &= \cos(\theta)P_i(\cos(\theta), \sin(\theta)) + \sin(\theta)Q_i(\cos(\theta), \sin(\theta)), \\ B_i(\theta) &= \cos(\theta)Q_i(\cos(\theta), \sin(\theta)) - \sin(\theta)P_i(\cos(\theta), \sin(\theta)). \end{aligned} \tag{11}$$

Let $r(\theta, \rho)$ be the solution of system (10) with $r(0, \rho) = \rho$. Suppose that $r(\theta, \rho)$ can be expressed as the power series of ρ in the form of

$$r(\theta, \rho) = r_1(\theta)\rho + r_2(\theta)\rho^2 + r_3(\theta)\rho^3 + \cdots, \quad |\rho| \ll 1, \tag{12}$$

where $r_1(0) = 1$, $r_i(0) = 0$, $i \geq 2$. Then, we have the positive half-return map of system (9), given by

$$\mathcal{P}^+(\rho) = r(\pi, \rho) = r_1(\pi)\rho + r_2(\pi)\rho^2 + r_3(\pi)\rho^3 + \cdots, \quad \text{for } |\rho| \ll 1.$$

Hence, we need to compute $r_j(\theta)$ in order to obtain the Lyapunov constants. To achieve this, eliminating the time t from (10) we have

$$\frac{dr}{d\theta} = \frac{\sum_{i=2}^{+\infty} A_i(\theta) r^i}{1 + \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1}}, \tag{13}$$

which can be rewritten in the power series of r as

$$\frac{dr}{d\theta} = R_2(\theta) r^2 + R_3(\theta) r^3 + R_4(\theta) r^4 + \cdots, \tag{14}$$

where $R_i(\theta)$ is a polynomial in $\sin(\theta)$ and $\cos(\theta)$.

Lemma 6. *For system (13), let (11) and (14) hold. Then* $\deg(R_i(\theta), \{\sin(\theta), \cos(\theta)\}) = 3(i-1)$ *and $R_i(\theta)$ is odd (even) in* $\sin(\theta)$ *and* $\cos(\theta)$ *if i is even (odd).*

Proof. It follows from (11) that $A_i(\theta)$ and $B_i(\theta)$ are homogeneous polynomials of $\sin(\theta)$ and $\cos(\theta)$ of degree $i+1$. Also note that

$$\frac{1}{1 + \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1}} = 1 + \sum_{j=1}^{+\infty} \Big(- \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1} \Big)^j = 1 + \sum_{i=1}^{+\infty} \widetilde{B}_i(\theta) r^i, \quad |r| \ll 1.$$

Thus, $\widetilde{B}_i(\theta) r^i$ is a linear combination of the products of $B_2 r, B_3 r^2, \ldots, B_{i+1} r^i$. Suppose that $\widetilde{B}_i(\theta) = \sum B_{i_1} B_{i_2} \cdots B_{i_m}$. Then $\sum_{j=1}^{m} (i_j - 1) = i$. Since $i_j \geq 2$, the largest value for m should be i. Further, we have

$$\deg(B_{i_1} B_{i_2} \cdots B_{i_m}, \{\sin(\theta), \cos(\theta)\}) = \sum_{j=1}^{m} (i_j + 1) = i + 2m \leq 3i. \tag{15}$$

Therefore, $\deg(\widetilde{B}_i, \{\sin(\theta), \cos(\theta)\}) = 3i$, and further it follows from (15) that $\widetilde{R}_i(\theta)$ is odd (even) in $\sin(\theta)$ and $\cos(\theta)$ if i is odd (even).

Clearly, we have

$$\frac{\sum_{i=2}^{+\infty} A_i(\theta) r^i}{1 + \sum_{i=2}^{+\infty} B_i(\theta) r^{i-1}} = \Big(\sum_{i=2}^{+\infty} A_i(\theta) r^i \Big) \Big(1 + \sum_{i=1}^{+\infty} \widetilde{B}_i(\theta) r^i \Big).$$

Combining the above equation with (13) and (14) yields

$$R_i(\theta) = \sum_{j=2}^{i-1} A_j(\theta)\widetilde{B}_{i-j}(\theta) + A_i(\theta).$$

Finally, taking into account that $A_j(\theta)$ is a homogeneous polynomial in $\sin(\theta)$ and $\cos(\theta)$ of degree $j+1$ for any $j \geq 2$, the proof is complete. □

Further, assume that $r^j(\theta,\rho) = \sum_{i=j}^{+\infty} r_{j,i}(\theta)\rho^i$ for any $j \geq 2$. Substituting Eq. (12) into system (14) and comparing the coefficients yields $r_1'(\theta) = 0$ and

$$r_i'(\theta) = R_i(\theta) + R_{i-1}(\theta)r_{i-1,i}(\theta) + \cdots + R_2(\theta)r_{2,i}(\theta), \quad i \geq 2. \tag{16}$$

It is easy to get $r_1(\theta) = 1$, $r_2(\theta) = \int_0^\theta R_2(\theta)d\theta$ and

$$r_3(\theta) = \int_0^\theta (R_3(\theta) + 2R_2(\theta)r_2(\theta))d\theta = \int_0^\theta R_3(\theta)d\theta + r_2^2(\theta).$$

But computation of $r_i(\theta)$ becomes more and more involved by direct integration, as i grows. To overcome this difficulty, we present a new method to compute $r_i(\theta)$, which is closely related to the proof of the following theorem.

Theorem 7. *Suppose $r(\theta,\rho)$ is the solution of system (9) with $r(0,\rho) = \rho$, and let (12) hold. Then, for any $i \geq 1$, we have*

$$r_i(\theta) = \sum_{j=1}^{3i-3} (S_{i,j}(\theta)\sin^j(\theta) + C_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta)) + C_{i,0}(\theta), \tag{17}$$

where $S_{i,j}(\theta)$ and $C_{i,j}(\theta)$ are polynomials in θ.

Proof. We apply the method of mathematics induction to prove this lemma. It is easy to see that the conclusion is true for $i = 1$, since $r_1(\theta) = 1$. Then, suppose (17) holds for $i-1$ and we will show that (17) is also true for i.

Firstly, we need to prove $\deg(r_{j,i}(\theta), \{\sin(\theta), \cos(\theta)\}) = 3(i-j)$ for any $2 \leq j \leq i-1$. Note that

$$r^j(\theta,\rho) = \rho^j(1 + r_2(\theta)\rho + r_3(\theta)\rho^2 + \cdots)^j = \rho^j(1 + r_{j,j+1}(\theta)\rho + r_{j,j+2}(\theta)\rho^2 + \cdots).$$

Thus, $r_{j,i}(\theta)\rho^{i-j}$ should be a linear combination of the products of $r_k(\theta)\rho^{k-1}$, $2 \leq k \leq i-1$. Suppose that $r_{j,i}(\theta) = \sum r_{i_1}r_{i_2}\cdots r_{i_n}$, where $i_k \leq i-1$, $k = 1, \ldots, n$. Then $\sum_{k=1}^n (i_k - 1) = i - j$. Since $\deg(r_{i_k}(\theta), \{\sin(\theta), \cos(\theta)\}) = 3(i_k - 1)$, we have

$$\deg(r_{j,i}(\theta), \{\sin(\theta), \cos(\theta)\}) = \max\left(\sum_{k=1}^n 3(i_k - 1)\right) = 3(i-j).$$

From Lemma 6, we know $\deg(R_j(\theta), \{\sin(\theta), \cos(\theta)\}) = 3(j-1)$. Then, the right hand-side of Eq. (16) has degree $3(i-1)$ in $\sin(\theta)$ and $\cos(\theta)$. Applying $\sin^2(\theta)+\cos^2(\theta)=1$ to Eq. (16) and decreasing the degree in $\cos(\theta)$ gives

$$r_i'(\theta) = \sum_{j=1}^{3i-3} (T_{i,j}(\theta)\sin^j(\theta) + D_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta)) + D_{i,0}(\theta), \tag{18}$$

where $T_{i,j}(\theta)$ and $D_{i,j}(\theta)$ are polynomials in θ. Then,

$$r_i(\theta) = \sum_{j=1}^{3i-3} \Big(\int_0^\theta T_{i,j}(\theta)\sin^j(\theta)d\theta + \int_0^\theta D_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta)d\theta \Big) + \int_0^\theta D_{i,0}(\theta)d\theta.$$

On the other hand, for any polynomial $f(\theta)$ and number j we have

$$\int f(\theta)\sin^j(\theta)\cos(\theta)d\theta = \frac{1}{j+1} f(\theta)\sin^{j+1}(\theta) - \frac{1}{j+1}\int f'(\theta)\sin^{j+1}(\theta)d\theta, \tag{19}$$

and

$$\begin{aligned}
&\int f(\theta)\sin^{j+1}(\theta)d\theta \\
&= \int f(\theta)\sin^j(\theta)d(-\cos(\theta)) \\
&= -f(\theta)\sin^j(\theta)\cos(\theta) + \int f'(\theta)\sin^j(\theta)\cos(\theta)d\theta + j\int f(\theta)\sin^{j-1}(\theta)\cos^2(\theta)d\theta \\
&= -f(\theta)\sin^j(\theta)\cos(\theta) + \frac{1}{j+1} f'(\theta)\sin^{j+1}(\theta) - \frac{1}{j+1}\int f''(\theta)\sin^{j+1}(\theta)d\theta \\
&\quad + j\int f(\theta)\sin^{j-1}(\theta)d\theta - j\int f(\theta)\sin^{j+1}(\theta)d\theta.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int f(\theta)\sin^{j+1}(\theta)d\theta = &-\frac{1}{j+1} f(\theta)\sin^j(\theta)\cos(\theta) + \frac{1}{(j+1)^2} f'(\theta)\sin^{j+1}(\theta) \\
&- \frac{1}{(j+1)^2}\int f''(\theta)\sin^{j+1}(\theta)d\theta + \frac{j}{j+1}\int f(\theta)\sin^{j-1}(\theta)d\theta.
\end{aligned} \tag{20}$$

It follows from Eqs. (19) and (20) that the conclusion is true for i, and thus the proof is complete. □

From the above proof, we have seen that the procedure of computing $r_i(\theta)$ contains the following four steps:

(1) computing $r_{j,i}(\theta)$, $2 \le j \le i-1$;
(2) substituting $r_{j,i}(\theta)$ into (16), and applying $\cos^2(\theta) = 1 - \sin^2(\theta)$ to get (18);
(3) for any j in descending order, using (19) and (20) repeatedly to compute $\int_0^\theta T_{i,j}(\theta)\sin^j(\theta)d\theta$ and $\int_0^\theta D_{i,j}(\theta)\sin^{j-1}(\theta)\cos(\theta)d\theta$ by decreasing the degrees of polynomials $T_{i,j}(\theta)$ and $D_{i,j}(\theta)$; and finally,
(4) computing $\int_0^\theta D_{i,0}(\theta)d\theta$.

4. Proofs of Theorems 1 and 2

Now, we are ready to prove Theorems 1 and 2.

Proof of Theorem 1. If $a_6 b_6 \neq 0$, solving the multivariate polynomial equations based on the Lyapunov constants becomes extremely difficult, even if we could compute the Lyapunov constants up to an order we wish. If we assume $a_6 b_6 = 0$, then the third-order Lyapunov constant can be factorized and thus the computation is simplified. Now under the condition $a_6 b_6 = 0$, without loss of generality, we may let $b_6 = 0$. Denote by $C(\mathcal{E})$ the condition which is obtained from the condition C with the interchange of variables $\mathcal{E}$.

For system (4), as discussed in the previous section, we have $\delta = 0$ due to $V_1 = 0$. From the second Lyapunov constant $V_2 = \frac{2}{3}(a_5 - b_5)$, we solve $V_2 = 0$ to get $b_5 = a_5$. Then, we obtain $V_3 = -\frac{\pi}{8}(a_2 - a_5)a_6$.

First, we assume $a_6 \neq 0$. Then, $V_3 = 0$ yields $a_2 = a_5$. Further, by linearly solving $V_4 = 0$ for b_4, we have

$$b_4 = \frac{1}{a_5 b_3}\big[2a_5^3 - b_2 a_5^2 - (3a_3^2 - a_3 a_4 + 6a_3 a_6 - 2a_4 a_6 + b_2^2)a_5 + 3b_2 b_3^2\big], \quad a_5 b_3 \neq 0. \tag{21}$$

In the case $a_5 = 0$, we have $V_4 = \frac{2}{5} b_2 b_3^2$, which yields the center condition I by solving $V_4 = 0$. If $a_5 \neq 0$ and $b_3 = 0$, we obtain

$$a_4 = \frac{1}{a_3 + 2a_6}(3a_3^2 + 6a_3 a_6 - 2a_5^2 + a_5 b_2 + b_2^2), \tag{22}$$

by solving $V_4 = 0$ when $a_3 + 2a_6 \neq 0$. Under the condition (22), V_5 is given by

$$V_5 = \frac{\pi a_5 a_6}{48(a_3 + 2a_6)^2}(b_2 + 2a_5)(b_2 - a_5)(5a_3 a_6 + 2a_5^2 - a_5 b_2 + 10a_6^2 - b_2^2).$$

From $V_5 = 0$, we have condition II if $(b_2 + 2a_5)(b_2 - a_5) = 0$, or get another equation,

$$a_3 = -\frac{1}{5a_6}(2a_5^2 - a_5 b_2 + 10a_6^2 - b_2^2).$$

When the above equation holds, V_6 and V_7 are given by $V_6 = \frac{2a_5}{875a_6^2}F_{11}$ and $V_7 = \frac{\pi a_5 a_6}{64}F_{12}$, where

$$
\begin{aligned}
F_{11} = & -3b_2^6 - 9a_5b_2^5 + (9a_5^2 + 30a_6^2)b_2^4 + (33a_5^3 + 60a_5a_6^2)b_2^3 - (18a_5^4 + 90a_5^2a_6^2 \\
& - 50a_6^4)b_2^2 - (36a_5^5 + 120a_5^3a_6^2 - 50a_5a_6^4)b_2 + 24a_5^6 + 120a_5^4a_6^2 - 350a_5^2a_6^4, \\
F_{12} = & \, b_2^4 + 2a_5b_2^3 - (3a_5^2 + 5a_6^2)b_2^2 - (4a_5^3 + 5a_5a_6^2)b_2 + 4a_5^4 + 35a_5^2a_6^2.
\end{aligned}
$$

Then, $\mathrm{Res}(F_{11}, F_{12}, b_2) = 244\,140\,625a_6^{12}a_5^8(9a_5^2 + 40a_6^2)^2 \neq 0$ since $a_5a_6 \neq 0$, which means V_6 and V_7 do not have common solutions.

If $b_3 = a_3 + 2a_6 = 0$, we have

$$
\begin{aligned}
V_4 &= -\frac{2a_5}{15}(b_2 + 2a_5)(b_2 - a_5), \quad V_5 = -\frac{\pi a_5a_6}{48}(a_4 + 6a_6)(a_4 + a_6), \\
V_6 &= \frac{4a_5a_6}{315}(a_4 + 6a_6)\big[2(a_4 + a_6)(a_4 - 4a_6) + 9a_5^2\big].
\end{aligned}
$$

Thus, $V_4 = V_5 = V_6 = 0$ yields $(b_2 + 2a_5)(b_2 - a_5) = a_4 + 6a_6 = 0$, which are clearly included in the condition II.

When (21) holds, we obtain

$$
V_5 = -\frac{\pi a_5a_6}{48}(3a_3 - a_4)(3a_3 - a_4 + 5a_6). \tag{23}
$$

Taking $a_4 = 3a_3$ yields $V_5 = 0$ and $V_6 = -\frac{2b_3^2}{21}b_2(a_5 - b_2)(4a_5 + 3b_2)$. Setting $V_6 = 0$ yields $b_2(a_5 - b_2) = 0$, which gives the conditions III and IV, or $b_2 = -\frac{4}{3}a_5$ which results in $V_7 \equiv 0$ but $V_8 = \frac{448}{2187}a_5^5b_3^2 \neq 0$ since $a_5b_3 \neq 0$.

For (23), if $a_4 = 3a_3 + 5a_6$, we have $V_5 = 0$, and then obtain

$$
B_3 = \frac{3a_5a_6}{b_2D_{21}}(3a_3^3 + 12a_3^2a_6 + 10a_3a_6^2 + 2a_5^2a_6 - 4a_6^3), \quad b_2D_{21} \neq 0, \tag{24}
$$

by linearly solving $V_6 = 0$, where $B_3 = b_3^2$ and $D_{21} = 9a_3a_6 + 4a_5^2 - a_5b_2 + 18a_6^2 - 3b_2^2$. If $b_2 = 0$, we have $V_6 = \frac{2a_5a_6}{7}F_{21}$ and $V_7 = \frac{25\pi a_5a_6^3}{64}F_{22}$, where

$$
F_{21} = 3a_3^3 + 12a_3^2a_6 + 10a_3a_6^2 + 2a_5^2a_6 - 4a_6^3, \quad F_{22} = a_3^2 + 3a_3a_6 + a_5^2 + 2a_6^2.
$$

Then, $\mathrm{Res}(F_{21}, F_{22}, b_2) = a_5^4(9a_5^2 + 40a_6^2) \neq 0$, which means that there do not exist center conditions for this case. If $D_{21} = 0$, we have $a_3 = -\frac{1}{9a_6}(4a_5^2 - a_5b_2 + 18a_6^2 - 3b_2^2)$, and $V_6 = -\frac{2a_5}{1701a_6^2}F_{23}$, $V_7 = \frac{25\pi a_5a_6}{5184}F_{24}$, where F_{23} and F_{24} are polynomials in a_5, a_6 and b_2. Similarly, it can be easily shown that the two equations, $V_6 = V_7 = 0$, do not have solutions by verifying $\mathrm{Res}(F_{23}, F_{24}, b_2) \neq 0$.

Now suppose (24) holds. Then, we have $V_7 = \frac{25\pi a_5a_6^3}{64}F_{31}$, $V_8 = \frac{2a_5a_6}{27D_{21}}(F_{32}a_3 + D_{31})$, $V_9 \equiv 0$ and $V_{10} = \frac{2a_5a_6}{18\,711D_{21}^2}F_{33}$ with

$$
\begin{aligned}
F_{31} &= a_3^2 + 3a_3a_6 + a_5^2 + 2a_6^2, \\
F_{32} &= a_6[-a_5^6 + b_2a_5^5 - (17a_6^2 + 18b_2^2)a_5^4 - (13a_6^2b_2 - 9b_2^3)a_5^3 \\
&\quad - (30a_6^4 + 60a_6^2b_2^2 - 9b_2^4)a_5^2 + (-30a_6^4b_2 + 90a_6^2b_2^3)a_5 + 90a_6^2b_2^4],
\end{aligned}
$$

$$D_{31}=16a_5^6+14b_2a_5^5+(16a_6^2+24b_2^2)a_5^4+(14a_6^2b_2-27b_2^3)a_5^3 \\ +(15a_6^4+30a_6^2b_2^2-27b_2^4)a_5^2+(15a_6^4b_2-45a_6^2b_2^3)a_5-45a_6^2b_2^4.$$

If $D_{31}=0$, it follows from $V_8=0$ that $F_{32}=0$. Note that D_{31} and F_{32} are homogeneous polynomials in a_6, a_5 and b_2. Thus, by a variable scaling: $a_5\to a_5a_6$ and $b_2\to b_2a_6$, we can eliminate a_6. Without loss of generality, we take $a_6=1$, and then obtain $\mathrm{Res}(F_{32},D_{31},b_2)=-21\,870a_5^{24}(3\,862\,879a_5^6+35\,074\,080a_5^4+92\,750\,400a_5^2+50\,112\,000)\neq 0$ for nonzero a_5. This indicates that there are no solutions for the equations: $D_{31}=F_{32}=0$. If $D_{31}\neq 0$, we have $a_3=-\frac{F_{32}}{D_{31}}$, and $F_{31}=\frac{a_5^4}{D_{31}^2}\widetilde{F}_{31}$, $F_{33}=\frac{a_5^4a_6}{D_{31}^8}\widetilde{F}_{33}$, where $\widetilde{F}_{31}$ and $\widetilde{F}_{33}$ are homogeneous polynomials in a_6, a_5 and b_2. Similarly, by verifying $\mathrm{Res}(\widetilde{F}_{31},\widetilde{F}_{33},b_2)\neq 0$, we conclude that $V_7=V_{10}=0$ do not have common zeros when $a_5a_6\neq 0$.

Now we consider the case $a_6=0$, for which $V_3=0$, and get

$$b_4=\frac{1}{a_5b_3}[(a_2-b_2)a_5^2+(a_2^2+a_3a_4-b_2^2)a_5-3a_2a_3^2+3b_2b_3^2],\quad a_5b_3\neq 0, \tag{25}$$

by solving $V_4=0$. If $b_3=0$, $V_4=0$ yields $a_4=-\frac{1}{a_3a_5}[a_5a_2^2-(3a_3^2+a_5^2)a_2-b_2a_5^2-b_2^2a_5]$ provided $a_3a_5\neq 0$. Further, we have $V_5\equiv 0$, $V_6=-\frac{2a_3^2}{105}a_2F_{41}$, $V_7\equiv 0$, and $V_8=-\frac{2a_3^2}{2835a_5}a_2F_{42}$, where

$$F_{41}=15a_2^2+5a_2a_5-2a_5^2-9a_5b_2-9b_2^2,$$
$$F_{42}=315a_5a_2^4+(675a_3^2+315a_5^2)a_2^3+(225a_3^2a_5+1890a_5^3+225a_5^2b_2 \\ +225a_5b_2^2)a_2^2-(90a_3^2a_5^2+405a_3^2a_5b_2+405a_3^2b_2^2-602a_5^4-54a_5^3b_2 \\ -54a_5^2b_2^2)a_2-248a_5^5-1176a_5^4b_2-1446a_5^3b_2^2-540a_5^2b_2^3-270a_5b_2^4.$$

Obviously, $a_2=0$ is a solution of $V_6=V_8=0$, resulting in condition V. For $F_{41}=F_{42}=0$, we have

$$\mathrm{Res}(F_{41},F_{42},a_2)=2700a_5^2(a_5+3b_2)^2(2a_5+3b_2)^2(b_2-a_5)(b_2+2a_5) \\ \times(29a_5^2+108a_5b_2+108b_2^2).$$

Solving $F_{41}=F_{42}=\mathrm{Res}(F_{41},F_{42},a_2)=0$, we obtain condition VI, derived from $(a_5+3b_2)(2a_5+3b_2)=0$, and other center conditions derived from $(b_2-a_5)(b_2+2a_5)=0$ are already included in condition II. If $b_3=a_3=0$, $V_4=\frac{2}{15}a_5(a_2-b_2)(a_2+a_5+b_2)$. Solving $V_4=0$ we have the conditions VII and VIII. If $b_3=a_5=0$, center conditions obtained from $V_4=0$ are included in the condition I or the condition VII, where $V_4=-\frac{2}{5}a_2a_3^2$. If $b_3\neq 0$ and $a_5=0$, we obtain $b_2=\frac{1}{b_3^2}a_2a_3^2$ from $V_4=0$. Then $V_5\equiv 0$ and $V_6=\frac{2}{35b_3^4}a_2a_3^2(8a_2^2a_3^4-8a_2^2b_3^4-3a_3a_4b_3^4+3b_3^5b_4)$. When $a_2a_3=0$, we get subcases of I and I($\mathcal{E}$). Otherwise, we linearly solve $V_6=0$ using b_4, for which $V_7\equiv 0$, and further obtain

$$V_8=\frac{2a_2^3a_3^2}{945b_3^8}(b_3^2-a_3^2)\big[75a_3b_3^4(a_3^2+b_3^2)a_4-105a_3^4(a_3^2+b_3^2)a_2^2+95b_3^4(a_3^2+b_3^2)a_2^2-21a_3^2b_3^6\big].$$

When $b_3^2 - a_3^2 = 0$, we obtain subcases of X and XI. Otherwise, we linearly solve $V_8 = 0$ using a_4 to obtain

$$\begin{aligned}
V_{10} &= -\frac{4a_2^3 a_3^2 (a_3^2 - b_3^2)}{37\,125 b_3^{12}(a_3^2 + b_3^2)} F_{43}, \qquad V_{12} = -\frac{4a_2^3 a_3^2 (a_3^2 - b_3^2)}{4\,021\,875 b_3^{16}(a_3^2 + b_3^2)^2} F_{44}, \\
V_{14} &= -\frac{4a_2^3 a_3^2 (a_3^2 - b_3^2)}{4\,524\,609\,375 b_3^{20}(a_3^2 + b_3^2)^3} F_{45}, \qquad V_9 = V_{11} = V_{13} \equiv 0,
\end{aligned} \tag{26}$$

where F_{43}, F_{44} and F_{45} are homogeneous polynomials in a_3, a_2 and b_3. Thus, without loss of generality, taking $b_3 = 1$ yields

$$\begin{aligned}
F_{43} &= 150a_2^4 a_3^{12} + 300a_2^4 a_3^{10} - (50a_2^4 - 465a_2^2)a_3^8 - (400a_2^4 - 1240a_2^2)a_3^6 - (50a_2^4 \\
&\quad - 1240a_2^2 + 48)a_3^4 + (300a_2^4 + 465a_2^2)a_3^2 + 150a_2^4, \\
F_{44} &= 35\,250a_2^6 a_3^{18} + 105\,750a_2^6 a_3^{16} + (90\,250a_2^6 + 111\,675a_2^4)a_3^{14} - (11\,250a_2^6 \\
&\quad - 403\,975a_2^4)a_3^{12} - (62\,000a_2^6 - 675\,150a_2^4 + 4335a_2^2)a_3^{10} - (62\,000a_2^6 \\
&\quad - 765\,700a_2^4 - 4165a_2^2)a_3^8 - (11\,250a_2^6 - 675\,150a_2^4 - 4165a_2^2 + 2352)a_3^6 \\
&\quad + (90\,250a_2^6 + 403\,975a_2^4 - 4335a_2^2)a_3^4 + (105\,750a_2^6 + 111\,675a_2^4)a_3^2 \\
&\quad + 35\,250a_2^6, \\
F_{45} &= 64\,267\,500a_2^8 a_3^{24} + 257\,070\,000a_2^8 a_3^{22} + (374\,412\,500a_2^8 + 209\,282\,250a_2^6)a_3^{20} \\
&\quad + (212\,300\,000a_2^8 + 963\,353\,000a_2^6)a_3^{18} - (5\,995\,000a_2^8 + 2\,057\,488\,500a_2^6 \\
&\quad + 9\,179\,400a_2^4)a_3^{16} - (57\,200\,000a_2^8 - 2\,903\,448\,125a_2^6 - 82\,855\,050a_2^4)a_3^{14} \\
&\quad - (41\,030\,000a_2^8 - 3\,282\,832\,625a_2^6 - 209\,672\,700a_2^4 + 8\,036\,550a_2^2)a_3^{12} \\
&\quad - (57\,200\,000a_2^8 + 3\,282\,832\,625a_2^6 + 271\,994\,100a_2^4 - 11\,441\,925a_2^2)a_3^{10} \\
&\quad - (5\,995\,000a_2^8 + 2\,903\,448\,125a_2^6 + 209\,672\,700a_2^4 - 11\,441\,925a_2^2 \\
&\quad - 889\,056)a_3^8 + (212\,300\,000a_2^8 + 2\,057\,488\,500a_2^6 + 82\,855\,050a_2^4 \\
&\quad - 8\,036\,550a_2^2)a_3^6 + (374\,412\,500a_2^8 + 963\,353\,000a_2^6 + 9\,179\,400a_2^4)a_3^4 \\
&\quad + (257\,070\,000a_2^8 + 209\,282\,250a_2^6)a_3^2 + 64\,267\,500a_2^8,
\end{aligned}$$

from which we have

$$\begin{aligned}
\mathrm{Res}(F_{43}, F_{44}, a_3) &= 4.7937764808 \cdots \times 10^{56} a_2^{80} E_c^2 E_{41}^2, \\
\mathrm{Res}(F_{43}, F_{45}, a_3) &= 1.4812312887 \cdots \times 10^{77} a_2^{104} E_c^2 E_{42}^2,
\end{aligned}$$

where $E_c = (5a_2^2 + 1)^2 + 5a_2^2 \neq 0$, and E_{41} and E_{42} are polynomials in a_2 of degrees 16 and 24, respectively, satisfying $\mathrm{Res}(E_{41}, E_{42}, a_2) \neq 0$. Therefore, there are no solutions to satisfy the equations: $V_{10} = V_{12} = V_{14} = 0$, given in (26).

Next, with (25) holding, we get $V_5 \equiv 0$ and further solve $V_6 = 0$ to obtain

$$a_4 = \frac{1}{9a_5a_3E_0}\big[(-2a_2a_3^2 + 2b_2b_3^2)a_5^3 + (-4a_2^2a_3^2 + 9a_2b_2b_3^2 - 5b_2^2b_3^2)a_5^2 + (6a_2^3a_3^2 + 9a_2^2b_2b_3^2 - 15b_2^3b_3^2)a_5 + 27a_2^2a_3^4 - 27a_2a_3^2b_2b_3^2\big], \tag{27}$$

provided $a_5a_3E_0 \neq 0$, where $E_0 = a_2a_3^2 - b_2b_3^2$. The equation $a_3E_0 = 0$ yields the conditions IX–XI, V($\mathcal{E}$) and VI($\mathcal{E}$), as well as a subcase of II($\mathcal{E}$). Here, we omit the details of the discussion for the case $a_5a_3E_0 \neq 0$, since it is similar to the case $a_5b_3 = 0$ for $V_4 = 0$.

When (25) and (27) hold, we have $a_3a_5b_3E_0 \neq 0$, $V_7 = V_9 = V_{11} = V_{13} = V_{15} \equiv 0$ and

$$V_8 = \frac{2F_1}{1701a_5E_0}, \quad V_{10} = \frac{2F_2}{56\,133a_5^2E_0^2}, \quad V_{12} = \frac{2F_3}{19\,702\,683a_5^3E_0^3},$$
$$V_{14} = \frac{2F_4}{6\,206\,345\,145a_5^4E_0^4}, \quad V_{16} = \frac{2F_5}{949\,570\,807\,185a_5^5E_0^5},$$

where F_j, $1 \le j \le 5$, is a homogeneous polynomial in a_2, a_3, a_5, b_2, b_3, and also a polynomial in a_3^2 and b_3^2. Taking $a_5 = 1$, and letting $A_3 = a_3^2$ and $B_3 = b_3^2$, we obtain

$$\begin{aligned}
F_1 &= a_2^2[135b_2(3a_2 + 3b_2 + 1)(a_2 - b_2)B_3 + 2a_2(a_2 - 1)(3a_2 + 1)(6a_2 + 1)]A_3^2 \\
&\quad + a_2b_2[135b_2(3a_2 + 3b_2 + 1)(a_2 - b_2)B_3 - 189a_2^4 + 450a_2^2b_2^2 - 189b_2^4 - 189a_2^3 \\
&\quad + 171a_2^2b_2 + 171a_2b_2^2 - 189b_2^3 - 48a_2^2 + 64a_2b_2 - 48b_2^2 - 2a_2 - 2b_2)]B_3A_3 \\
&\quad + 2b_2^3(b_2 - 1)(3b_2 + 1)(6b_2 + 1)B_3^2, \\
F_2 &= 63[135b_2(3a_2 + 3b_2 + 1)(a_2 - b_2)B_3i + 2a_2(a_2 - 1)(3a_2 + 1)(6a_2 + 1)]a_2^4A_3^4 \\
&\quad - a_2^3[8505b_2^2(3a_2 + 3b_2 + 1)(a_2 - b_2)B_3^2 - 18b_2(2898a_2^4 - 3150a_2^2b_2^2 + 2421a_2^3 \\
&\quad - 1050a_2^2b_2 - 1245a_2b_2^2 + 5387a_2^2 - 415a_2b_2 - 4860b_2^2 + 1634a_2 - 1620b_2)B_3 \\
&\quad - 2a_2(a_2 - 1)(3a_2 + 1)(672a_2^3 + 416a_2^2 + 1349a_2 + 216)]A_3^3 \\
&\quad - a_2^2b_2B_3[8505b_2^2(3a_2 + 3b_2 + 1)(a_2 - b_2)B_3^2 + 18b_2(a_2 - b_2)(3024a_2^3 + 3024a_2^2b_2 \\
&\quad + 3024a_2b_2^2 + 3024b_2^3 + 2358a_2^2 + 2553a_2b_2 + 2358b_2^2 + 10\,191a_2 + 10\,191b_2 \\
&\quad + 3247)B_3 - 16\,821a_2^6 + 40\,635a_2^4b_2^2 - 6615a_2^2b_2^4 - 5103b_2^6 - 26\,397a_2^5 \\
&\quad + 15\,495a_2^4b_2 + 28\,218a_2^3b_2^2 - 7182a_2^2b_2^3 - 2205a_2b_2^4 - 8505b_2^5 - 47\,076a_2^4 \\
&\quad + 10\,706a_2^3b_2 + 99\,634a_2^2b_2^2 - 2394a_2b_2^3 - 45\,612b_2^4 - 47\,564a_2^3 + 38\,500a_2^2b_2 \\
&\quad + 36\,100a_2b_2^2 - 41\,832b_2^3 - 13\,986a_2^2 + 13\,724a_2b_2 - 10\,424b_2^2 - 864a_2 - 432b_2]A_3^2 \\
&\quad + a_2b_2^2B_3^2[8505b_2^2(3a_2 + 3b_2 + 1)(a_2 - b_2)B_3^2 + 18b_2(3150a_2^2b_2^2 - 2898b_2^4 \\
&\quad + 1245a_2^2b_2 + 1050a_2b_2^2 - 2421b_2^3 + 4860a_2^2 + 415a_2b_2 - 5387b_2^2 + 1620a_2 \\
&\quad - 1634b_2)B_3 - 5103a_2^6 - 6615a_2^4b_2^2 + 40\,635a_2^2b_2^4 - 16\,821b_2^6 - 8505a_2^5 \\
&\quad - 2205a_2^4b_2 - 7182a_2^3b_2^2 + 28\,218a_2^2b_2^3 + 15\,495a_2b_2^4 - 26\,397b_2^5 - 45\,612a_2^4
\end{aligned}$$

$$
\begin{aligned}
&- 2394a_2^3b_2 + 99\,634a_2^2b_2^2 + 10\,706a_2b_2^3 - 47\,076b_2^4 - 41\,832a_2^3 + 36\,100a_2^2b_2 \\
&+ 38\,500a_2b_2^2 - 47\,564b_2^3 - 10\,424a_2^2 + 13\,724a_2b_2 - 13\,986b_2^2 - 432a_2 - 864b_2]A_3 \\
&- 2b_2^4(3b_2+1)(b_2-1)(378B_3b_2^2 + 672b_2^3 + 63B_3b_2 + 416b_2^2 + 1349b_2 + 216)B_3^3.
\end{aligned}
$$

The other three lengthy polynomials F_3, F_4 and F_5 are omitted here for brevity. In order to solve $F_1 = F_2 = F_3 = F_4 = F_5 = 0$, we compute the following resultants:

$$
\begin{aligned}
&\mathrm{Res}(F_1, F_2, A_3) = -5292F_aE_1, && \mathrm{Res}(F_1, F_3, A_3) = -47\,628F_aF_bE_2, \\
&\mathrm{Res}(F_1, F_4, A_3) = -7\,001\,316F_aF_b^2E_3, && \mathrm{Res}(F_1, F_5, A_3) = -7\,001\,316F_aF_b^3E_4,
\end{aligned}
\tag{28}
$$

where

$$
\begin{aligned}
F_a &= B_3^6a_2^8b_2^7(3b_2+1)(b_2-1)(3a_2+3b_2+1)^2(a_2-b_2)^6(405B_3a_2^2b_2 - 405B_3b_2^3 \\
&\quad + 36a_2^4 + 135B_3a_2b_2 - 135B_3b_2^2 - 18a_2^3 - 16a_2^2 - 2a_2), \\
F_b &= B_3^2a_2^4b_2^2(3a_2+3b_2+1)(a_2-b_2)^2,
\end{aligned}
$$

and E_j, $1 \le j \le 4$, is a polynomial in B_3, a_2 and b_2. Note that all the resultants, $\mathrm{Res}(F_1, F_j, A_3)$, $j = 2, 3, 4, 5$, contain the common factor F_a. Thus, the conditions III($\mathcal{E}$), XII, XIII and XIII($\mathcal{E}$) can be easily obtained from the equation $a_2b_2(a_2 - b_2) = 0$. For example, taking $a_2 = 0$ we have $F_1 = 2b_2^3(b_2-1)(3b_2+1)(6b_2+1)B_3^2$, and then $b_2 \neq 0$ since $E_0 \neq 0$. We can get the condition III($\mathcal{E}$) if $b_2 = 1$, and the condition XII if $b_2 = -\frac{1}{3}$. If $b_2 = -\frac{1}{6}$, then $F_2 = -\frac{49}{139\,968}B_3^3 < 0$. Note that $E_0 \neq 0$, $A_3 > 0$ and $B_3 > 0$ due to (25) and (27). The rest factors in F_a can not lead to new center conditions. Here, we only present the details for the case $b_2 - 1 = 0$ with $a_2b_2(a_2 - b_2) \neq 0$. Similar procedures can be applied to discuss other cases.

Assume $a_2b_2(a_2 - b_2) \neq 0$. When $b_2 = 1$, and so $a_2 \neq 1$, we have

$$
\begin{aligned}
&F_1 = a_2A_3(a_2-1)G_1, \quad F_2 = a_2A_3(a_2-1)G_2, \\
&F_3 = a_2A_3(a_2-1)G_3, \quad F_4 = a_2A_3(a_2-1)G_4,
\end{aligned}
\tag{29}
$$

where G_j, $1 \le j \le 4$, is a polynomial in a_2, A_3 and B_3. Then,

$$
\begin{aligned}
&\mathrm{Res}(G_1, G_2, B_3) = -714\,420\,A_3^2a_2^3(1+3a_2)(3a_2+4)^3(a_2-1)^2G_5, \\
&\mathrm{Res}(G_1, G_3, B_3) = 173\,604\,060\,A_3^3a_2^4(1+3a_2)(3a_2+4)^4(a_2-1)^3G_6, \\
&\mathrm{Res}(G_1, G_4, B_3) = -2\,067\,103\,542\,420\,A_3^4a_2^5(1+3a_2)(3a_2+4)^5(a_2-1)^4G_7,
\end{aligned}
$$

where G_j, $5 \le j \le 7$, is a polynomial in a_2 and A_3. We first consider the condition from the common factor, $(1+3a_2)(3a_2+4) = 0$. If $a_2 = -\frac{1}{3}$, then $G_1 = -B_3I_{11}$ and $G_2 = \frac{1}{9}B_3I_{12}$, where $I_{11} = 135A_3 + 405B_3 + 328$ and

$$
\begin{aligned}
I_{12} &= 229\,635B_3^3 + (76\,545A_3 + 1\,499\,310)B_3^2 - (25\,515A_3^2 - 758\,700A_3 \\
&\quad - 1\,052\,136)B_3 - 8505A_3^3 + 86\,310A_3^2 + 256\,312A_3,
\end{aligned}
$$

satisfying $\mathrm{Res}(I_{11}, I_{12}, B_3) = 14\,696\,640(1080A_3 + 42\,107) > 0$. Thus, there are no solutions satisfying the equations: $G_1 = G_2 = G_3 = G_4 = 0$, if $a_2 = -\frac{1}{3}$. If $a_2 = -\frac{4}{3}$, then $G_1 = \frac{56}{3}(4A_3 + 3B_3) > 0$. Next we consider the equations, $G_5 = G_6 = G_7 = 0$, and have

$$\begin{aligned}
&\mathrm{Res}(G_5, G_6, A_3) = 2\,789\,427\,520\,800\, a_2^8 G_s G_a G_{81} G_{82},\\
&\mathrm{Res}(G_5, G_7, A_3) = -376\,572\,715\,308\,000\, a_2^{12}(3a_2 + 4) G_s G_a G_{91} G_{92},
\end{aligned}$$

where G_{j1} and G_{j2}, $j = 8, 9$ are polynomials in a_2 and

$$\begin{aligned}
G_s &= (a_2 - 8)(3a_2 + 5)(3a_2 - 8)(3a_2 + 4)^2,\\
G_a &= 486a_2^4 + 945a_2^3 - 1227a_2^2 - 2779a_2 - 428.
\end{aligned}$$

If $a_2 = 8$, we have $G_5 = 12\,544(9A_3 + 196)I_{21}$ and $G_6 = 39\,337\,984(9A_3 + 196)I_{22}$, where $I_{21} = 244\,111\,680A_3 - 240\,599\,605\,103$ and

$$\begin{aligned}
I_{22} = {}& 672\,679\,027\,641\,600A_3^3 + 95\,795\,633\,236\,828\,680A_3^2\\
&- 282\,894\,493\,179\,800\,916\,477A_3 - 4\,376\,089\,823\,211\,446\,777\,789,
\end{aligned}$$

satisfying $\mathrm{Res}(I_{21}, I_{22}, A_3) \neq 0$. Hence, there do not exist solutions to satisfy the equations: $G_5 = G_6 = G_7 = 0$, when $a_2 = 8$. In a similar way, it can be shown that no solutions exist for the equations: $G_5 = G_6 = G_7 = 0$, if $(3a_2 + 5)(3a_2 - 8)(3a_2 + 4) = 0$. If $G_a = 0$, we compute $I_{31} = \mathrm{Res}(G_a, G_5, a_2)$ and $I_{32} = \mathrm{Res}(G_a, G_6, a_2)$, with $\mathrm{Res}(I_{31}, I_{32}, A_3) \neq 0$. Moreover, we get $\mathrm{Res}(G_{8i}, G_{9j}, a_2) \neq 0$ for $i, j = 1, 2$, and thus there are no solutions to satisfy the equations: $G_{8i} = G_{9j} = 0$. Therefore, for $b_2 = 1$, no common zeros exist for the equations: $F_1 = F_2 = F_3 = F_4 = 0$.

For (28), we consider the equations: $E_1 = E_2 = E_3 = E_4 = 0$, under the condition $a_5 A_3 E_0 F_a \neq 0$, and get

$$\begin{aligned}
&\mathrm{Res}(E_1, E_2, B_3) = -3\,050\,238\,993\,994\,800\, F_c E_5,\\
&\mathrm{Res}(E_1, E_3, B_3) = -900\,567\,811\,781\,994\,726\,000\, F_c F_d E_6,\\
&\mathrm{Res}(E_1, E_4, B_3) = -387\,664\,913\,353\,600\,397\,935\,934\,460\,000\, F_c F_d^2 E_7,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
F_c &= a_2^3 b_2^{19}(3a_2 + 3b_2 + 1)^2(3a_2 + 1)^2(a_2 - 1)^2(3b_2 + 2 + 3a_2)\\
&\quad \times (a_2 + b_2 - 1) E_a E_b E_c E_d,\\
F_d &= a_2 b_2^9(3a_2 + 3b_2 + 1)(3a_2 + 1)(a_2 - 1),\\
E_a &= a_2^2 - (7b_2 + 1)a_2 + b_2^2 - b_2,\\
E_b &= 3a_2^2 - (6b_2 + 2)a_2 + 3b_2^2 - 2b_2 - 1,\\
E_c &= 486a_2^4 + (486b_2 + 459)a_2^3 - (1134b_2^2 + 207b_2 - 114)a_2^2\\
&\quad - (1134b_2^3 + 1323b_2^2 + 321b_2 + 1)a_2 - 189b_2^3 - 189b_2^2 - 48b_2 - 2,
\end{aligned}$$

$$E_d = (1134b_2 + 189)a_2^3 + (1134b_2^2 + 1323b_2 + 189)a_2^2 - (486b_2^3 - 207b_2^2 - 321b_2 - 48)a_2 - 486b_2^4 - 459b_2^3 - 114b_2^2 + b_2 + 2.$$

Note that all the resultants, $\mathrm{Res}(E_1, E_j, B_3)$, $j = 2, 3, 4$, contain the common factor F_c.

If $a_2 + b_2 - 1 = 0$, we have $a_2 = -b_2 + 1$ and

$$E_1 = 2b_2 I_a I_{41}, \quad E_2 = 4b_2^2 I_a I_{42}, \quad E_3 = 8b_2^3 I_a I_{43},$$

where $I_a = (3b_2 - 2)B_3 + (2b_2 - 1)^2$. Then, $I_a = 0$ yields $E_1 = E_2 = E_3 = 0$, i.e., $\mathrm{Res}(F_1, F_j, A_3) = 0$, $j = 2, 3, 4$, when $a_2 = -b_2 + 1$. We substitute $a_2 = -b_2 + 1$ into F_j to yield $\widetilde{F}_j$, $j = 1, 2, 3$. Next, we need to solve the equations: $I_a = \widetilde{F}_1 = \widetilde{F}_2 = \widetilde{F}_3 = 0$, and obtain

$$\mathrm{Res}(I_a, \widetilde{F}_1, B_3) = 2b_2(b_2 - 1)I_b I_{51}, \quad \mathrm{Res}(I_a, \widetilde{F}_2, B_3) = 2b_2(b_2 - 1)I_b I_{52},$$
$$\mathrm{Res}(I_a, \widetilde{F}_3, B_3) = 2b_2(b_2 - 1)I_b I_{53}, \quad \text{with } I_b = (3b_2 - 1)A_3 - (2b_2 - 1)^2.$$

Note that $b_2(b_2 - 1) \neq 0$ since $F_a \neq 0$. If $I_a = I_b = 0$, then $\widetilde{F}_1 = \widetilde{F}_2 = \widetilde{F}_3 = 0$, and we obtain the condition XIV. For $I_{51} = I_{52} = I_{53} = 0$, we have

$$\mathrm{Res}(I_{51}, I_{52}, A_3) = 7b_2^3(3b_2 + 1)(b_2 - 1)^3(3b_2 - 2)^5(2b_2 - 1)^6 J_a J_1,$$
$$\mathrm{Res}(I_{51}, I_{53}, A_3) = -21b_2^4(3b_2 + 1)(b_2 - 1)^5(3b_2 - 2)^8(2b_2 - 1)^9 J_a J_2,$$

where $(3b_2 + 1)(b_2 - 1) \neq 0$, and

$$\begin{aligned}
J_a &= 18b_2^3 + 651b_2^2 - 748b_2 + 214,\\
J_1 &= 605\,304b_2^4 - 2\,895\,060b_2^3 + 2\,555\,877b_2^2 - 373\,639b_2 - 148\,730,\\
J_2 &= 378\,882\,563\,472b_2^{10} - 29\,071\,087\,999\,056b_2^9 + 180\,968\,668\,598\,610b_2^8\\
&\quad - 499\,455\,418\,644\,927b_2^7 + 1\,319\,463\,134\,471\,394b_2^6 - 2\,296\,405\,188\,740\,916b_2^5\\
&\quad + 2\,157\,213\,472\,303\,974b_2^4 - 1\,020\,839\,133\,559\,269b_2^3 + 181\,189\,011\,015\,338b_2^2\\
&\quad + 20\,664\,548\,818\,076b_2 - 8\,460\,097\,956\,280.
\end{aligned}$$

If $3b_2 - 2 = 0$, then $I_a = \frac{1}{9} \neq 0$. If $2b_2 - 1 = 0$, then $I_a = -\frac{1}{2}B_3 \neq 0$. Moreover, we get $\mathrm{Res}(J_a, I_{51}, b_2) \neq 0$ and $\mathrm{Res}(J_1, J_2, b_2) \neq 0$. Thus, there are no solutions to satisfy the equations: $I_{51} = I_{52} = I_{53} = 0$. For $I_{41} = I_{42} = I_{43} = 0$, we have

$$\mathrm{Res}(I_{41}, I_{42}, B_3) = 18\,075\,490\,334\,784\,000\, b_2^8(b_2 - 1)^2(3b_2 - 4)^2 I_c I_d J_3,$$
$$\mathrm{Res}(I_{41}, I_{43}, B_3) = 1\,067\,339\,6287\,786\,604\,160\,000\, b_2^{12}(b_2 - 1)^3(3b_2 - 4)^3 I_c I_d J_4,$$

where $I_c = 186b_2^2 + 235b_2 - 528$, $I_d = 186b_2^2 - 607b_2 - 107$, and J_3 and J_4 are polynomials in b_2. Similarly, we can show that there are no solutions to satisfy the equations: for $I_{41} = I_{42} = I_{43} = 0$.

For the other factors contained in F_c, using similar procedures, we can show that no more center conditions exist, and thus the details are omitted. Since E_j's given in (30), $j = 5, 6, 7$, are polynomials in a_2 and b_2, it is straightforward to prove that the equations: $E_5 = E_6 = E_7 = 0$ can not result in more center conditions. It should be pointed out that although the computations are straightforward, it is very time-consuming and memory demanding.

Finally, we prove the sufficiency for the center conditions I–XIV by deriving their corresponding first integrals. We shall not discuss all the cases one by one. Actually, most of the cases belong to three special types of systems. We use the following notation in the remaining proof: for any $\mathrm{C} \in \{\mathrm{I}, \dots, \mathrm{XIV}\}$, C^+ denotes the upper system of system (4) under the condition C, C^- the lower system of (4) under the condition C.

First, it is well known that a quadratic Hamiltonian system is given by

$$\dot{x} = -y - Ax^2 + 2Bxy + (C + A)y^2, \quad \dot{y} = x + Bx^2 + 2Axy - By^2$$

with the Hamiltonian $H = \frac{1}{2}(x^2 + y^2) + \frac{1}{3}Bx^3 + Ax^2y - Bxy^2 - \frac{1}{3}(A + C)y^3$. Under the conditions I–XIV, the upper systems of II^+–IV^+ are Hamiltonian systems. The general form for I^+ and I^- ($b_2 = 0$) is given by

$$\dot{x} = -y + Ax^2 + By^2, \quad \dot{y} = x + Cxy,$$

with the first integral

$$H = (Cy + 1)^{-\frac{2A}{C}} \left[\frac{x^2}{2} + \frac{By^2}{2(A - C)} - \frac{(A - B - C)y}{(A - C)(2A - C)} - \frac{A - B - C}{2A(A - C)(2A - C)} \right],$$

if $C(A - C)(2A - C) \neq 0$; or

$$H = e^{-2Ay} \left(\frac{1}{2}x^2 + \frac{B}{2A}y^2 - \frac{A - B}{2A^2}y - \frac{A - B}{4A^3} \right), \quad \text{if } C = 0,\ A \neq 0,$$

$$\text{or} \quad H = -\frac{C^3x^2 + B + C}{2(Cy + 1)^2} + \frac{2B + C}{Cy + 1} + B\ln(Cy + 1), \quad \text{if } C \neq 0,\ A = C,$$

$$\text{or} \quad H = -\frac{4A^3x^2 + 2A + B}{8A^3(2Ay + 1)} - \frac{A + B}{4A^3}\ln(2Ay + 1) + \frac{By}{4A^2}, \quad \text{if } C \neq 0,\ C = 2A.$$

Systems VI^+, XII^+ ($b_3 \neq 0$) and $\mathrm{XIII}^\pm$ can be written in the form,

$$\dot{x} = -y - Ax^2 + 2Bxy + Ay^2, \quad \dot{y} = x - Bx^2 - 2Axy + By^2,$$

with the first integral,

$$H = \frac{4A^2x^2 + 4A^2y^2 - 2Ay - 2Bx + 1}{2Ay + 2Bx - 1}.$$

All the remaining upper systems and lower systems except $\mathrm{X}^\pm$, $\mathrm{XI}^\pm$ and $\mathrm{XIV}^\pm$ can be written in the form,

$$\dot{x} = -y + Axy, \quad \dot{y} = x + Bx^2 + Cxy - By^2, \tag{31}$$

with the first integral,

$$H = (-Ax+1)^{2B\omega}\left(Bx + \frac{C}{2}y - \frac{\omega}{2}y + 1\right)^{(\omega+C)A}\left(Bx + \frac{C}{2}y + \frac{\omega}{2}y + 1\right)^{(\omega-C)A},$$

if $AB(\omega^2 - C^2)\omega \neq 0$, where $\omega = \sqrt{4AB + 4B^2 + C^2}$. When $B = 0$, system (31) has the first integral,

$$\begin{aligned} & H = \frac{1}{2}y^2 - \frac{1}{A^2}(Ax + \ln(1 - Ax)), \quad \text{if } A \neq 0,\ C = 0, \\ \text{or} \quad & H = \frac{1}{C^2}(Cy - \ln(Cy+1)) - \frac{1}{A^2}(Ax + \ln(1 - Ax)), \quad \text{if } AC \neq 0, \\ \text{or} \quad & H = \frac{1}{C^2}(Cy - \ln(Cy+1)) + \frac{1}{2}x^2, \quad \text{if } A = 0,\ C \neq 0. \end{aligned}$$

When $B \neq 0$, we have the first integral given by

$$\begin{aligned} H = {} & 4\ln(2Bx + Cy + 2) - \frac{16B^2}{4B^2 + C^2}\ln(|(4B^2 + C^2)x + 4B|) \\ & + \frac{8(Bx+1)}{2Bx + Cy + 2}, \quad \text{if } \omega = 0. \end{aligned}$$

For $B\omega \neq 0$, we obtain the first integral,

$$H = \frac{C - \omega}{2B^2\omega}\ln(2Bx + Cy + \omega y + 2) - \frac{C + \omega}{2B^2\omega}\ln(2Bx + Cy - \omega y + 2) + \frac{x}{B},$$

if $A = 0$, or

$$H = -\frac{1}{C^2}\ln(Bx + Cy + 1) + \frac{B^2 + C^2}{B^2C^2}\ln(Bx + 1) + \frac{B^2 y + C}{B^2C(Bx+1)},$$

if $\omega^2 = C^2$.

For the center condition XIV, we have the following first integrals,

$$\begin{aligned} H^{\pm} = {} & \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{a_5}{3}x^3 \mp \frac{a_5(a_5 - 2b_2)}{\alpha_\pm}x^2y - (a_5 - b_2)xy^2 \mp \frac{\alpha_\pm}{3}y^3 \\ & + \frac{b_2(a_5 - b_2)}{4}x^4 \mp \frac{a_5b_2(a_5 - 2b_2)}{\alpha_\pm}x^3y - \frac{3a_5b_2(\alpha_\pm^2 - b_2^2)}{2\alpha_\pm^2}x^2y^2 \\ & \pm \frac{a_5b_2(a_5 - b_2)}{\alpha_\pm}xy^3 - \frac{a_5^2b_2(a_5 - 2b_2)}{4\alpha_\pm^2}y^4, \end{aligned}$$

where $\alpha_+ = \sqrt{-a_5^2 + 3a_5b_2} \neq 0$, $\alpha_- = \sqrt{2a_5^2 - 3a_5b_2} \neq 0$. XIV$^\pm(\alpha_+ = 0)$ and XIV$^\pm(\alpha_- = 0)$ are in the form of I$^\pm$. Under the center condition X, system (4) is smooth, and has a center at the origin. Under the center condition XI, system (4) is symmetric with respect to the x-axis.

Therefore, for the fourteen center conditions we have obtained the first integrals $H^+(x, y)$ and $H^-(x, y)$ for the upper system and the lower system in (4) near the origin. More specifically, for any center conditions I, ..., XIV, either both $H^+(x, 0)$ and $H^-(x, 0)$ are even functions, or $H^+(x, 0) \equiv H^-(x, 0)$, or $H^+(x, 0) = H^+(\rho, 0)$ and $H^-(x, 0) = H^-(\rho, 0)$ have common zeros $x(\rho)$ satisfying $x(\rho) \to 0^-$ as $\rho \to 0^+$.

This finishes the proof of Theorem 1.

Proof of Theorem 2. For system (3) with $a_1 = b_1 = 0$, $\delta = b_6 = 0$, $b_5 = a_5$, we add perturbations on a_k as $a_k \to a_k + \varepsilon p_k$ and $b_k \to b_k + \varepsilon q_k$, $k = 1, \dots, 6$, and $\delta = \varepsilon p_0$, where $0 < \varepsilon \ll 1$. Then, $V_{1,1} = e^{2p_0\pi\varepsilon} - 1$, which is the ε-order term in V_1. Taking $p_0 = 0$, we get $V_{1,1} = 0$, and then compute the Lyapunov constants, which are polynomials of ε. To prove the existence of 10 small-amplitude limit cycles, we need to solve the ε-order Lyapunov constants, i.e., the coefficient $V_{k,1}$ of ε in the kth-order Lyapunov constant V_k for all $k > 1$.

First, we get

$$V_{2,1} = \frac{2}{3}(2p_1 + p_5 - 2q_1 - q_5).$$

Setting $p_5 = -2p_1 + 2q_1 + q_5$ yields $V_{2,1} = 0$ and then we obtain

$$V_{3,1} = -\frac{\pi}{8}\big[(a_4 - 3)(p_1 + q_1) + (1 - a_5)(p_6 + q_6)\big].$$

Letting

$$p_6 = -q_6 - \frac{(a_4 - 3)(p_1 + q_1)}{1 - a_5},$$

results in $V_{3,1} = 0$. Similarly, we can linearly solve the polynomial equations one by one, for $V_{4,1} = 0$ using p_4, for $V_{5,1} = 0$ using q_1, for $V_{6,1} = 0$ using p_2, for $V_{8,1} = 0$ using p_3, for $V_{10,1} = 0$ using q_6 ($V_{7,1} = V_{9,1} \equiv 0$) and then obtain

$$V_{12,1} = -\frac{32p_1}{125E_0}F_aF_b, \quad V_{14,1} = -\frac{32p_1}{73\,125E_0}F_aF_c, \quad V_{11,1} = V_{13,1} \equiv 0,$$

where

$$\begin{aligned}
F_a &= -(a_4 - a_5 - 2)(a_4^2a_5 + a_4a_5^2 - 4a_4a_5 - 2a_5^2 - 3a_4 + a_5 + 10),\\
F_b &= 94\,623\,744a_4^{14}a_5^6 + 930\,466\,816a_4^{15}a_5^4 + 615\,054\,336a_4^{14}a_5^5 - 275\,404\,800a_4^{13}a_5^6\\
&\quad + 1\,342\,162\,944a_4^{12}a_5^7 + 2\,270\,969\,856a_4^{16}a_5^2 + 5\,488\,177\,152a_4^{15}a_5^3\\
&\quad - 2\,424\,275\,968a_4^{14}a_5^4 + 10\,977\,847\,296a_4^{13}a_5^5 + 9\,213\,454\,848a_4^{12}a_5^6\\
&\quad + 924\,797\,952a_4^{11}a_5^7 + 70\,958\,592a_4^{10}a_5^8 + \cdots,
\end{aligned}$$

$$
\begin{aligned}
F_c = {} & 3\,643\,883\,520 a_4^{16} a_5^6 + 703\,622\,160\,384 a_4^{14} a_5^8 + 35\,831\,521\,280 a_4^{17} a_5^4 \\
& + 23\,685\,242\,880 a_4^{16} a_5^5 + 7\,044\,986\,537\,984 a_4^{15} a_5^6 + 4\,776\,306\,345\,984 a_4^{14} a_5^7 \\
& - 2\,047\,910\,092\,800 a_4^{13} a_5^8 + 9\,980\,323\,651\,584 a_4^{12} a_5^9 + 87\,453\,204\,480 a_4^{18} a_5^2 \\
& + 211\,345\,244\,160 a_4^{17} a_5^3 + 18\,137\,210\,559\,488 a_4^{16} a_5^4 + 43\,606\,528\,505\,856 a_4^{15} a_5^5 \\
& - 16\,936\,192\,867\,328 a_4^{14} a_5^6 + 83\,213\,921\,538\,048 a_4^{13} a_5^7 + 70\,628\,108\,476\,416 a_4^{12} a_5^8 \\
& + 6\,876\,797\,571\,072 a_4^{11} a_5^9 + 527\,648\,090\,112 a_4^{10} a_5^{10} + \cdots, \\
E_0 = {} & 436\,926\,698\,208 a_4^7 a_5^7 + 4\,296\,445\,865\,712 a_4^8 a_5^5 + 318\,301\,099\,644\,528 a_4^7 a_5^6 \\
& + 436\,926\,698\,208 a_4^6 a_5^7 - 314\,587\,222\,709\,760 a_4^5 a_5^8 + 10\,486\,240\,756\,992 a_4^9 a_5^3 \\
& + 3\,127\,375\,663\,540\,128 a_4^8 a_5^4 + 2\,056\,104\,220\,650\,480 a_4^7 a_5^5 \\
& - 4\,828\,695\,405\,245\,712 a_4^6 a_5^6 - 1\,589\,648\,559\,755\,256 a_4^5 a_5^7 \\
& + 509\,238\,066\,761\,424 a_4^4 a_5^8 - 520\,052\,002\,542\,072 a_4^3 a_5^9 \cdots.
\end{aligned}
$$

By solving $F_b = F_c = 0$, we obtain a solution pair,

$$a_4 = 5.9943463371\cdots, \qquad a_5 = -8.1486126831\cdots, \tag{32}$$

which satisfies

$$\det\left[\frac{\partial(V_{12,1}, V_{14,1})}{\partial(a_4, a_5)}\right] = -49.555\cdots \neq 0.$$

Setting the non-used parameters $q_2 = q_3 = q_4 = q_5 = 0$, and $p_1 = 1$, we obtain the following critical parameter values:

$$
\begin{aligned}
& p_2 = 0.3000212842\cdots, \quad p_3 = 0.8220632161\cdots, \quad p_4 = 15.5929246779\cdots, \\
& p_6 = -4.6242893306\cdots, \quad q_6 = 4.6242893306\cdots, \quad p_5 = -4, \quad q_1 = -1,
\end{aligned}
\tag{33}
$$

under which the Lyapunov constants become $V_{j,1} = 0$, $j = 2, 3, \ldots, 15$, and $V_{16,1} = 13.3\cdots$. Thus, with (32) and (33) holding, we have $V_{j,1} = 0$, $j = 2, \ldots, 14$, but $V_{16,1} \neq 0$. Therefore, we can take perturbations in the backward order: on a_5 for $V_{14,1}$, on a_4 for $V_{12,1}$, on q_6 for $V_{10,1}$, on p_3 for $V_{8,1}$, on p_2 for $V_{6,1}$, on q_1 for $V_{5,1}$, on p_4 for $V_{4,1}$, on p_6 for $V_{3,1}$, on p_5 for $V_{2,1}$, on p_0 for V_1, to obtain 10 small-amplitude limit cycles bifurcating from the origin.

The proof of Theorem 2 is complete.

Remark 8. Theorem 2 guarantees the existence of 10 small-amplitude limit cycles in system (3) with the perturbations in a neighborhood of the critical point, defined by the center conditions given in Theorem 2, near the origin (i.e., near $\rho = 0$). In order to estimate the small-amplitude limit cycles, one needs to obtain the approximation of the 10 positive roots solved from the truncated polynomial equation of (6), $d(\rho) = V_{1,1}\,\rho + V_{2,1}\,\rho^2 + \cdots + V_{16,1}\,\rho^{16} = 0$. This requires to find a set of explicit perturbation values to have a true realization, which is not an easy task, in particular, for high multiple limit cycles bifurcations.

5. Conclusion

In this paper, we have studied planar switching systems, in particular, a switching Bautin system. We have developed a computationally efficient algorithm to compute the Lyapunov constants for planar switching systems. With the help of this algorithm and Maple built-in command 'resultant', we present, with rigorous proof, a complete classification on the center problem for the Bautin switching system under the condition $a_6 b_6 = 0$. Moreover, we have selected one of the center conditions to construct a special integrable system and then perturbed this system to obtain 10 small-amplitude limit cycles, which improves the existing result. The case $a_6 b_6 \neq 0$ causes extreme difficulty in solving multivariate polynomial equations based on the Lyapunov constants. We hope to develop more efficient methodology to find the solutions from these polynomial equations in order to classify the center problem and obtain more limit cycles. An even more challenging research project is to study the center problem of the generic planar switching system (3).

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References

[1] A.F. Filippov, Differential Equation with Discontinuous Right-Hand Sides, Kluwer Academic, Netherlands, 1988.
[2] M. Kunze, Non-Smooth Dynamical Systems, Springer-Verlag, Berlin, 2000.
[3] R.A. Ibrahim, Friction-induced vibration, chatter, squeal, and chaos – part II: dynamics and modeling, Appl. Mech. Rev. 47 (7) (1994) 227–253.
[4] J. Badertscher, K.A. Cunefare, A.A. Ferri, Braking impact of normal dither signals, J. Vib. Acoust. 129 (1) (2007) 17–23.
[5] A. Kaplan, N. Friedman, M. Andersen, N. Davidson, Observation of islands of stability in soft wall atom-optics billiards, Phys. Rev. Lett. 87 (27) (2001) 27410114.
[6] T. Kinoshita, T. Wenger, D.S. Weiss, A quantum Newton's cradle, Nature 440 (2006) 900–903.
[7] H.E. Nusse, J.A. Yorke, Border-collision bifurcations including "period two to period three" for piecewise smooth systems, Phys. D 57 (1992) 39–57.
[8] Y. Zou, T. Kupper, W.-J. Beyn, Generalized Hopf bifurcation for planar Filippov systems continuous at the origin, J. Nonlinear Sci. 16 (2) (2006) 159–177.
[9] D.J.W. Simpson, J.D. Meiss, Andronov–Hopf bifurcations in planar, piecewise-smooth, continuous flows, Phys. Lett. A 371 (3) (2007) 213–220.
[10] C.J. Budd, Non-smooth dynamical systems and the grazing bifurcation, in: Nonlinear Mathematics and Its Applications, Cambridge University Press, Cambridge, 1996, pp. 219–235.
[11] M.D. Bernardo, C.J. Budd, A.R. Champneys, Grazing, skipping and sliding: analysis of the nonsmooth dynamics of the DC/DC buck converter, Nonlinearity 11 (1998) 859–890.
[12] A. Gasull, J. Torregrosa, Center-focus problem for discontinuous planar differential equations, Internat. J. Bifur. Chaos 13 (7) (2003) 1755–1765.
[13] X. Liu, M. Han, Hopf bifurcation for non-smooth Liénard systems, Internat. J. Bifur. Chaos 19 (7) (2009) 2401–2415.
[14] Y. Tian, M. Han, Hopf bifurcation for two types of Liénard systems, J. Differential Equations 251 (4–5) (2011) 834–859.
[15] X. Chen, Z. Du, Limit cycles bifurcate from centers of discontinuous quadratic systems, Comput. Math. Appl. 59 (2010) 3836–3848.
[16] X. Chen, W. Zhang, Isochronicity of centers in a switching Bautin system, J. Differential Equations 252 (2012) 2877–2899.

[17] J. Llibre, A.C. Mereu, Limit cycles for discontinuous quadratic differential systems with two zones, J. Math. Anal. Appl. 413 (2) (2014) 763–775.
[18] B. Coll, R. Prohens, A. Gasull, The center problem for discontinuous Liénard differential equation, Internat. J. Bifur. Chaos 9 (1999) 1751–1761.
[19] B. Coll, A. Gasull, R. Prohens, Degenerate Hopf bifurcation in discontinuous planar systems, J. Math. Anal. Appl. 253 (2001) 671–690.
[20] M. Han, W. Zhang, On Hopf bifurcation in non-smooth planar systems, J. Differential Equations 248 (2010) 2399–2416.
[21] N. Bautin, On the number of limit cycles appearing from an equilibrium point of the focus or center type under varying coefficients, Mat. Sb. 30 (1952) 181–196.
[22] P. Yu, Y. Tian, Twelve limit cycles around a singular point in a planar cubic-degree polynomial system, Commun. Nonlinear Sci. Numer. Simul. 19 (2014) 2690–2705.
[23] M. Han, Bifurcation Theory of Limit Cycles, Science Press, Beijing, 2013.
[24] B. Mishra, Algorithmic Algebra, Springer-Verlag, New York, 1993.