# Bifurcation of small limit cycles in cubic integrable systems using higher-order analysis th 

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#### Abstract

In this paper, we present a method of higher-order analysis on bifurcation of small limit cycles around an elementary center of integrable systems under perturbations. This method is equivalent to higher-order Melinikov function approach used for studying bifurcation of limit cycles around a center but simpler. Attention is focused on planar cubic polynomial systems and particularly it is shown that the system studied by Żoła̧dek (1995) [24] can indeed have eleven limit cycles under perturbations at least up to 7th order. Moreover, the pattern of numbers of limit cycles produced near the center is discussed up to 39th-order perturbations, and no more than eleven limit cycles are found.


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MSC: 34C07; 34C23
Keywords: Bifurcation of limit cycles; Higher-order analysis; Darboux integral; Focus value

## 1. Introduction

Bifurcation theory of limit cycles is important for both theoretical development of qualitative analysis and applications in solving real problems. It is closely related to the well-known

[^0]Hilbert's 16th problem [2], whose second part asks for the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that the following system,

$$
\begin{equation*}
\frac{d x}{d t}=P_{n}(x, y), \quad \frac{d y}{d t}=Q_{n}(x, y) \tag{1}
\end{equation*}
$$

can have, where $P_{n}(x, y)$ and $Q_{n}(x, y)$ represent $n^{\text {th }}$-degree polynomials in $x$ and $y$. This problem has motivated many mathematicians and researchers in other disciplines to develop mathematical theories and methodologies in the areas of differential equations and dynamical systems. However, this problem has not been completely solved even for quadratic systems since Hilbert proposed the problem in the Second Congress of World Mathematicians in 1900. The maximal number of limit cycles obtained for some quadratic systems is 4 [3,4]. However, whether $H(2)=4$ is still open. For cubic polynomial systems, many results have been obtained on the lower bound of the number of limit cycles. So far, the best result for cubic systems is $H(3) \geq 13[5,6]$. Note that the 13 limit cycles obtained in [5,6] are distributed around several singular points.

When the problem is restricted to consider the maximum number of small-amplitude limit cycles, denoted by $M(n)$, bifurcating from a focus or a center in system (1), one of the best-known results is $M(2)=3$, which was obtained by Bautin in 1952 [10]. For $n=3$, a number of results in this research direction have been obtained. So far the best result for the number of small limit cycles around a focus is 9 [11-13], and that around a center is 12 [14].

One of powerful tools used for analyzing local bifurcation of limit cycles around a focus or a center is normal form theory (e.g., see [15-18]). Suppose system (1) has an elementary focus or an elementary center at the origin. With the computation methods using computer algebra systems (e.g., see [9,19-22]), we obtain the normal form expressed in polar coordinates as

$$
\begin{align*}
& \frac{d r}{d t}=r\left(v_{0}+v_{1} r^{2}+v_{2} r^{4}+\cdots+v_{k} r^{2 k}+\cdots\right) \\
& \frac{d \theta}{d t}=\omega_{c}+\tau_{0}+\tau_{1} r^{2}+\tau_{2} r^{4}+\cdots+\tau_{k} r^{2 k}+\cdots \tag{2}
\end{align*}
$$

where $r$ and $\theta$ represent the amplitude and phase of motion, respectively. $v_{k}(k=0,1,2, \cdots)$ is called the $k$ th-order focus value. $v_{0}$ and $\tau_{0}$ are obtained from linear analysis. The first equation of (2) can be used for studying bifurcation and stability of limit cycles, while the second equation can be used to determine the frequency of the bifurcating periodic motion. Moreover, the coefficients $\tau_{j}$ can be used to determine the order or critical periods of a center (when $v_{j}=0, j \geq 0$ ).

A particular attention has been paid to near-integrable polynomial systems, described in the form of $[7,8]$

$$
\begin{align*}
\frac{d x}{d t} & =M^{-1}(x, y, \mu) H_{y}(x, y, \mu)+\varepsilon p(x, y, \varepsilon, \delta) \\
\frac{d y}{d t} & =-M^{-1}(x, y, \mu) H_{x}(x, y, \mu)+\varepsilon q(x, y, \varepsilon, \delta) \tag{3}
\end{align*}
$$

where $0<\varepsilon \ll 1, \mu$ and $\delta$ are vector parameters; $H(x, y, \mu)$ is an analytic function in $x, y$ and $\mu ; p(x, y, \varepsilon, \delta)$ and $q(x, y, \varepsilon, \delta)$ are polynomials in $x$ and $y$, and analytic in $\delta$ and $\varepsilon . M(x, y, \mu)$ is an integrating factor of the unperturbed system (3) $\left.\right|_{\varepsilon=0}$.

Suppose the unperturbed system (3) $\left.\right|_{\varepsilon=0}$ has an elementary center. Then, considering limit cycles bifurcation in system (3) around the center, we may use the normal form theory to obtain the first equation of (2) as follows:

$$
\begin{equation*}
\frac{d r}{d t}=r\left[v_{0}(\varepsilon)+v_{1}(\varepsilon) r^{2}+v_{2}(\varepsilon) r^{4}+\cdots+v_{i}(\varepsilon) r^{2 i}+\cdots\right] \tag{4}
\end{equation*}
$$

where

$$
v_{i}(\varepsilon)=\sum_{k=1}^{\infty} \varepsilon^{k} V_{i k}, \quad i=0,1,2, \ldots
$$

in which $V_{i k}$ denotes the $i$ th $\varepsilon^{k}$-order focus value, and will be used throughout this paper. Note that $v_{i}(\varepsilon)=O(\varepsilon)$ since the unperturbed system (3) $\left.\right|_{\varepsilon=0}$ is an integrable system. Further, because system (3) is analytic in $\varepsilon$, we can rearrange the terms in (4), and obtain

$$
\begin{equation*}
\frac{d r}{d t}=V_{1}(r) \varepsilon+V_{2}(r) \varepsilon^{2}+\cdots+V_{k}(r) \varepsilon^{k}+\cdots \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
V_{k}(r)=\sum_{i=0}^{\infty} V_{i k} r^{2 i+1}, \quad k=1,2, \ldots \tag{6}
\end{equation*}
$$

Similarly, for the normal form of system (3) we have the second differential equation in (2), given by

$$
\frac{d \theta}{d t}=T_{0}(r)+O(\varepsilon)
$$

with $T_{0}(0) \neq 0$, and thus

$$
\begin{equation*}
\frac{d r}{d \theta}=\frac{V_{1}(r) \varepsilon+V_{2}(r) \varepsilon^{2}+\cdots+V_{k}(r) \varepsilon^{k}+\cdots}{T_{0}(r)+O(\varepsilon)} . \tag{7}
\end{equation*}
$$

Assume the solution $r(\theta, \rho, \varepsilon)$ of (7), satisfying the initial condition $r(0, \rho, \varepsilon)=\rho$, is given in the form of

$$
r(\theta, \rho, \varepsilon)=r_{0}(\theta, \rho)+r_{1}(\theta, \rho) \varepsilon+r_{2}(\theta, \rho) \varepsilon^{2}+\cdots+r_{k}(\theta, \rho) \varepsilon^{k}+\cdots
$$

with $0<\rho \ll 1$. Then, $r_{0}(0, \rho)=\rho$ and $r_{k}(0, \rho)=0$, for $k \geq 1$.
If there exists a positive integer $K$ such that $V_{k}(r) \equiv 0,1 \leq k<K$, and $V_{K}(r) \not \equiv 0$, then it follows from (7) that

$$
r_{0}(\theta, \rho)=\rho, \quad r_{k}(\theta, \rho)=0, \quad 1 \leq k<K, \quad \text { and } \quad r_{K}(\theta, \rho)=\frac{V_{K}(\rho)}{T_{0}(\rho)} \theta
$$

Thus, the displacement function $d(\rho)$ of system (7) can be written as

$$
\begin{equation*}
d(\rho)=r(2 \pi, \rho, \varepsilon)-\rho=2 \pi \frac{V_{K}(\rho)}{T_{0}(\rho)} \varepsilon^{K}+O\left(\varepsilon^{K+1}\right) \tag{8}
\end{equation*}
$$

Therefore, if we want to determine the number of small-amplitude limit cycles bifurcating from the center in system (3), we only need to study the number of isolated zeros of $V_{K}(\rho)$ for $0<\rho \ll 1$, and have to obtain the expression of the first non-zero coefficient $V_{K}(r)$ in (5) by computing $V_{i K}$, for $i \geq 0$.

The above discussions show that the basic idea of using focus values is actually the same as that of the Melnikov function method. Using $H(x, y)=h$ to parameterize the section (i.e. the Poincaré map), we obtain the displacement function of (3), given by

$$
\begin{equation*}
d(h)=M_{1}(h) \varepsilon+M_{2}(h) \varepsilon^{2}+\cdots+M_{k}(h) \varepsilon^{k}+\cdots, \tag{9}
\end{equation*}
$$

where

$$
\begin{equation*}
M_{1}(h)=\oint_{H(x, y, \mu)=h} M(x, y, \mu)[q(x, y, 0, \delta) d x-p(x, y, 0, \delta) d y], \tag{10}
\end{equation*}
$$

evaluated along closed orbits $H(x, y, \mu)=h$ for $h \in\left(h_{1}, h_{2}\right)$. Then, we can study the first nonzero Melnikov function $M_{k}(h)$ in (9) to determine the number of limit cycles in system (3). In the following, we remark on the comparison of the Melnikov function method and the method of normal forms (or focus values).

## Remark 1.

(1) Let $H=h, 0<h-h_{1} \ll 1$ define closed orbits around the center of system (5) $\left.\right|_{\varepsilon=0}$. It is easy to see that for any integer $K \geq 1$, equation (8) holds if and only if $M_{k}(h) \equiv 0,1 \leq k<K$ and $M_{K}(h) \not \equiv 0$ in (9). Moreover, $V_{K}(\rho)$ for $0<\rho \ll 1$ and $M_{K}(h)$ for $0<h-h_{1} \ll 1$ have the same maximum number of isolated zeros.
(2) As we can see, $V_{k}(r)$ can be obtained by the computation of normal forms or focus values.
(3) In particular, when the original system is not a Hamiltonian system but an integrable system, then even computing the coefficients of the first-order Melnikov function is much more involved than the computation of using the method of normal forms.
(4) However, the method of normal forms (or focus values) is restricted to Hopf and generalized Hopf bifurcations, while the Melnikov function method can also be applied to study bifurcation of limit cycles from homoclinic/heteroclinic loops or any closed orbits.
(5) Another method of using high-order perturbations of focus values can be found in [23] by Christopher. Like in our approach, linear terms of focus values are used firstly in [23] to estimate the cyclicity of centers. If the number of independent linear terms is less than the codimension, then higher order terms of focus values would be needed to obtain the cyclicity.

When we apply the method of normal form computation, some unnecessary perturbation parameters are involved in the computation of high-order focus values, which could be extremely computation demanding (in both time and memory), and makes it much more difficult to solve the problem. Meanwhile, before we use the first non-zero coefficient $V_{K}(r)$ in (5) to find limit
cycles, we need to prove $V_{k}(r) \equiv 0,1 \leq k<K$. The unnecessary parameters involved could greatly increase the difficulty of proving that.

In this paper, without loss of limit cycles, we introduce a linear transformation to eliminate unnecessary parameters from system (3). With less parameters in (3), we can use the approximation of first integrals to prove $V_{k}(r) \equiv 0$. The idea will be illuminated in Section 2.

We will apply our method to study the bifurcation of small-amplitude limit cycles in the system

$$
\begin{align*}
& \frac{d x}{d t}=a+\frac{5}{2} x+x y+x^{3}+\sum_{k=1}^{n} \varepsilon^{k} p_{k}(x, y) \\
& \frac{d y}{d t}=-2 a x+2 y-3 x^{2}+4 y^{2}-a x^{3}+6 x^{2} y+\sum_{k=1}^{n} \varepsilon^{k} q_{k}(x, y) \tag{11}
\end{align*}
$$

where

$$
\begin{equation*}
p_{k}(x, y)=a_{00 k}+\sum_{i+j=1}^{3} a_{i j k} x^{i} y^{j}, \quad q_{k}(x, y)=b_{00 k}+\sum_{i+j=1}^{3} b_{i j k} x^{i} y^{j} \tag{12}
\end{equation*}
$$

in which $a_{i j k}$ and $b_{i j k}$ are $\varepsilon^{k}$ th-order coefficients (parameters). The unperturbed system (11) $\left.\right|_{\varepsilon=0}$ has a rational Darboux integral [24],

$$
\begin{equation*}
H_{0}=\frac{f_{1}^{5}}{f_{2}^{4}}=\frac{\left(x^{4}+4 x^{2}+4 y\right)^{5}}{\left(x^{5}+5 x^{3}+5 x y+5 x / 2+a\right)^{4}} \tag{13}
\end{equation*}
$$

with the integrating factor $M=20 f_{1}^{4} f_{2}^{-5}$. It can be shown that for $a<-2^{5 / 4}$, system (11) $\left.\right|_{\varepsilon=0}$ has a center at $\mathrm{E}_{0}=\left(-\frac{a}{2},-\frac{a^{2}+2}{4}\right)$. The system (11) $\left.\right|_{\varepsilon=0}$ was proposed in [24], and it was claimed that this system could have 11 limit cycles around the center by studying the second-order Melnikov function. Later, Yu and Han applied the normal form computation method and got only 9 limit cycles around $\mathrm{E}_{0}$ [25] by analyzing the $\varepsilon$ - and $\varepsilon^{2}$-order focus values. Recently, it has been shown [26] that errors are made in [24] for choosing 12 integrals as the basis of the linear space of corresponding Melnikov functions of system (11) $\left.\right|_{\varepsilon=0}$. In fact, among the 12 chosen integrals, two of them can be expressed as linear combinations of the other ten integrals, and therefore only 9 limit cycles can exist, agreeing with that shown in [25].

It has been shown in $[23,27]$ that another two cubic systems can have 11 small limit cycles produced from a center. Recently, the existence of 12 small limit cycles around a center is proved in a cubic Darboux system with cubic perturbations [14].

The rest of the paper is organized as follows. In the next section, we consider system (3), and construct a transformation to reduce the number of perturbation parameters, which greatly simplifies the analysis in the following section. Section 3 is devoted to the computation of higher $\varepsilon^{k}$-order focus values and the existence of 11 limit cycles in system (11), which needs computing at least $\varepsilon^{7}$-order focus values. Finally, conclusion is drawn in Section 4.

## 2. Preliminaries

The method of focus values (or normal forms) is one of important and powerful tools for the study of small-amplitude limit cycles generated from Hopf bifurcation. In general, a sufficient number of focus values would be needed if one wants to find more small-amplitude limit cycles. One main challenge is that the computation of focus values becomes more and more difficult as the order of focus values goes up. That is why computer algebra systems such as Maple and Mathematica have been used for computing the focus values to improve the computational efficiency (e.g. see [21,22]). Another approach is to eliminate certain parameters from the system, which is the method we shall develop here for near-integrable systems.

In most studies of near-integrable systems, full perturbations like those polynomials $p(x, y, \varepsilon, \delta)$ and $q(x, y, \varepsilon, \delta)$ given in system (3) are considered. The parameter vector $\delta$ usually represents the coefficients in $p$ and $q$. When normal forms are used to study small limit cycles, it is easy to get and solve the focus values of $\varepsilon$ order (coefficients in $V_{1}(r)$ ), because they are linear functions of the system parameters, namely the coefficients in $p(x, y, 0, \delta)$ and $q(x, y, 0, \delta)$. For the $\varepsilon^{k}$-order focus values (coefficients in $V_{k}(r)$ ), more parameters would be involved in the computation. One can observe that some parameters are not necessary for obtaining the maximum number of limit cycles, and they only increase the difficulty in finding limit cycles.

When the first $n$ functions $V_{k}(r)$ in (5), $1 \leq k \leq n$ are applied to studying bifurcation of limit cycles, in order to remove unnecessary parameters without reducing the number of limit cycles, we may use the following transformation:

$$
\left\{\begin{array}{l}
x \rightarrow x+e_{1}(\varepsilon) x+e_{2}(\varepsilon) y+e_{3}(\varepsilon),  \tag{14}\\
y \rightarrow y+e_{4}(\varepsilon) x+e_{5}(\varepsilon) y+e_{6}(\varepsilon), \\
t \rightarrow t+e_{7}(\varepsilon) t, \\
\mu \rightarrow \mu+e_{8}(\varepsilon),
\end{array}\right.
$$

where

$$
e_{i}(\varepsilon)=e_{i 1} \varepsilon+e_{i 2} \varepsilon^{2}+\cdots+e_{i n} \varepsilon^{n}, \quad i=1, \cdots, 8
$$

Note that (14) $\left.\right|_{\varepsilon=0}$ is an identity map. Thus, (14) keeps the unperturbed system of (3) unchanged. Furthermore, the new system obtained by using (14) can be still written in the same form of (3). So we only need to find proper $e_{i}(\varepsilon)$ 's to get simpler perturbation functions without loss of generality.

To illustrate how to obtain $e_{i}(\varepsilon)$, we take system (11) as an example. The coefficients $a_{i j k}$ and $b_{i j k}$ in (11) are the parameters. Substituting the transformation (14) into system (11) yields

$$
\begin{align*}
& \frac{d x}{d t}=a+\frac{5}{2} x+x y+x^{3}+\sum_{k=1}^{n} \varepsilon^{k} \tilde{p}_{k}(x, y)+o\left(\varepsilon^{n}\right)  \tag{15}\\
& \frac{d y}{d t}=-2 a x+2 y-3 x^{2}+4 y^{2}-a x^{3}+6 x^{2} y+\sum_{k=1}^{n} \varepsilon^{k} \tilde{q}_{k}(x, y)+o\left(\varepsilon^{n}\right),
\end{align*}
$$

where

$$
\begin{equation*}
\tilde{p}_{k}(x, y)=\tilde{a}_{00 k}+\sum_{i+j=1}^{3} \tilde{a}_{i j k} x^{i} y^{j}, \quad \tilde{q}_{k}(x, y)=\tilde{b}_{00 k}+\sum_{i+j=1}^{3} \tilde{b}_{i j k} x^{i} y^{j} \tag{16}
\end{equation*}
$$

Obviously, the coefficients $\tilde{a}_{i j k}$ and $\tilde{b}_{i j k}$ in (16) are linear in $e_{m k}, m=1, \ldots, 8$. Let $E_{k}=$ $\left(e_{1 k}, e_{2 k}, \cdots, e_{8 k}\right)^{T}$. For any $1 \leq k \leq n, \tilde{a}_{i j k}$ and $\tilde{b}_{i j k}$ can be written in the form of

$$
\tilde{a}_{i j k}=A_{i j} E_{k}+\eta_{i j k}, \quad \tilde{b}_{i j k}=B_{i j} E_{k}+\zeta_{i j k},
$$

where $A_{i j}$ and $B_{i j}$ are $1 \times 8$ matrices, and $\eta_{i j k}$ and $\zeta_{i j k}$, given by

$$
\begin{align*}
\eta_{i j k} & =\eta_{i j k}\left(E_{1}, \cdots, E_{k-1}, a_{m l 1}, \cdots, a_{m l k}, b_{m l 1}, \cdots, b_{m l k}\right) \\
\zeta_{i j k} & =\zeta_{i j k}\left(E_{1}, \cdots, E_{k-1}, a_{m l 1}, \cdots, a_{m l k}, b_{m l 1}, \cdots, b_{m l k}\right) \tag{17}
\end{align*}
$$

are polynomials in $e_{m l}, l \leq l \leq k-1$, and the coefficients in the perturbation functions (12).
Note that $A_{i j}$ and $B_{i j}$ are not dependent on $k$. We hope that we can find some proper values for $e_{i k}$ to make some of the coefficients $\tilde{a}_{i j k}$ and $\tilde{b}_{i j k}$ vanish or satisfy some conditions, so that the computation of the focus values would become easier. For instance, we can choose for $1 \leq k \leq n$,

$$
\begin{align*}
& \tilde{a}_{10 k}=\tilde{a}_{01 k}=\tilde{a}_{20 k}=\tilde{a}_{11 k}=\tilde{a}_{02 k}=\tilde{a}_{30 k}=0, \quad \text { and } \\
& \tilde{a}_{p_{k}} \triangleq \tilde{p}_{k}\left(-\frac{a}{2},-\frac{a^{2}+4}{4}\right)=0, \quad \tilde{a}_{q_{k}} \triangleq \tilde{q}_{k}\left(-\frac{a}{2},-\frac{a^{2}+4}{4}\right)=0 . \tag{18}
\end{align*}
$$

The last two equations in (18) keep the equilibrium of system (11) in a neighborhood of $\mathrm{E}_{0}$ with radius $o\left(\varepsilon^{n}\right)$. A direct computation yields

$$
\begin{align*}
\tilde{a}_{10 k}= & 2 a e_{2 k}+e_{6 k}+\frac{5}{2} e_{7 k}+\eta_{10 k}, \quad \tilde{a}_{01 k}=\frac{1}{2} e_{2 k}+e_{3 k}+\eta_{01 k}, \\
\tilde{a}_{20 k}= & 3 e_{2 k}+3 e_{3 k}+e_{4 k}+\eta_{20 k}, \quad \tilde{a}_{11 k}=e_{5 k}+e_{7 k}+\eta_{11 k}, \\
\tilde{a}_{02 k}= & -3 e_{2 k}+\eta_{02 k}, \quad \tilde{a}_{30 k}=2 e_{1 k}+a e_{1 k}+e_{7 k}+\eta_{30 k}, \\
\tilde{a}_{p_{k}}= & -\frac{1}{4} a\left(4+a^{2}\right) e_{1 k}-\frac{1}{8}\left(4+a^{2}\right)\left(2+a^{2}\right) e_{2 k}+\frac{1}{4}\left(4+a^{2}\right) e_{3 k}  \tag{19}\\
& +\frac{1}{4} a^{2} e_{4 k}+\frac{1}{8} a\left(2+a^{2}\right) e_{5 k}-\frac{1}{2} a e_{6 k}+e_{8 k}+\tilde{\eta}_{k}, \\
\tilde{a}_{q_{k}}= & -\frac{1}{8} a^{2}\left(16+3 a^{2}\right) e_{1 k}-\frac{1}{16} a\left(16+3 a^{2}\right)\left(2+a^{2}\right) e_{2 k} \\
& +\frac{1}{4} a\left(16+3 a^{2}\right) e_{3 k}+\frac{1}{4} a\left(4+a^{2}\right) e_{4 k}+\frac{1}{8}\left(4+a^{2}\right)\left(2+a^{2}\right) e_{5 k} \\
& -\frac{1}{4}\left(4+a^{2}\right) e_{6 k}+\frac{1}{8} a\left(a^{2}+8\right) e_{8 k}+\tilde{\zeta}_{k},
\end{align*}
$$

where $\tilde{\eta}_{k}$ and $\tilde{\zeta}_{k}$ are also functions in $\eta_{i j l}$ and $\zeta_{i j l}$ with $1 \leq l \leq k-1$, respectively.
Because

$$
\operatorname{det}\left[\frac{\partial\left(\tilde{a}_{10 k}, \tilde{a}_{01 k}, \tilde{a}_{20 k}, \tilde{a}_{11 k}, \tilde{a}_{02 k}, \tilde{a}_{30 k}, \tilde{a}_{p_{k}}, \tilde{a}_{q_{k}}\right)}{\partial\left(e_{1 k}, e_{2 k}, e_{3 k}, e_{4 k}, e_{5 k}, e_{6 k}, e_{7 k}, e_{8 k}\right)}\right]=\frac{3}{4}\left(32-a^{4}\right)<0
$$

for $a<-2^{-5 / 4}$, we can solve (19) for $e_{m k}$ to obtain

$$
e_{m k}=e_{m k}\left(\eta_{10 k}, \eta_{01 k}, \eta_{20 k}, \eta_{11 k}, \eta_{02 k}, \eta_{30 k}, \tilde{\eta}_{k}, \tilde{\zeta}_{k}\right), \quad 1 \leq m \leq 8
$$

which can be rewritten by using (17) as

$$
e_{m k}=\tilde{e}_{m k}\left(E_{1}, \cdots, E_{k-1}, a_{i j 1}, \cdots, a_{i j k}, b_{i j 1}, \cdots, b_{i j k}\right)
$$

Note that $e_{m 1}$ only depends on $a_{i j 1}$ and $b_{i j 1}$. Therefore, for all $1 \leq m \leq 8,1 \leq k \leq n, e_{m k}$ can be expressed as a polynomial in $a_{i j l}$ and $b_{i j l}, 1 \leq l \leq k$. In other words, (18) has solutions for all $1 \leq k \leq n$.

Thus, without loss of generality, we assume that (12) takes the following form,

$$
\begin{align*}
p_{k}(x, y)= & a_{00 k}+a_{21 k} x^{2} y+a_{12 k} x y^{2}+a_{03 k} y^{3}, \\
q_{k}(x, y)= & b_{00 k}+b_{10 k} x+b_{01 k} y+b_{20 k} x^{2}+b_{11 k} x y+b_{02 k} y^{2}  \tag{20}\\
& +b_{30 k} x^{3}+b_{21 k} x^{2} y+b_{12 k} x y^{2}+b_{03 k} y^{3},
\end{align*}
$$

with

$$
\begin{align*}
a_{00 k}= & \frac{1}{64}\left(a^{2}+2\right)\left[\left(a^{2}+2\right)^{2} a_{03 k}+2 a\left(a^{2}+2\right) a_{12 k}+4 a^{2} a_{21 k}\right], \\
b_{00 k}= & \frac{1}{64}\left\{8 a^{3} b_{30 k}+16 a\left(2 b_{10 k}-a b_{20 k}\right)+4\left(a^{2}+2\right)\left(4 b_{01 k}-2 a b_{11 k}+a^{2} b_{21 k}\right)\right.  \tag{21}\\
& \left.-\left(a^{2}+2\right)^{2}\left[4 b_{02 k}-2 a b_{12 k}-\left(a^{2}+2\right) b_{03 k}\right]\right\} .
\end{align*}
$$

As mentioned in Section 1, to find limit cycles around $\mathrm{E}_{0}$ in system (11), we apply normal form theory to compute the focus values and then solve the multivariate polynomial equations based on the focus values. Particularly, we have

$$
\begin{align*}
b_{01 k}= & \frac{1}{16}\left[4 a\left(2 b_{11 k}-a b_{21 k}\right)-\left(a^{2}+2\right)^{2}\left(a_{12 k}+3 b_{03 k}\right)\right. \\
& \left.+4\left(a^{2}+2\right)\left(2 b_{02 k}-a a_{21 k}-a b_{12 k}\right)\right], \tag{22}
\end{align*}
$$

solved from the zeroth-order focus value $V_{0 k}=0$, where

$$
\begin{align*}
V_{0 k}= & \frac{1}{32}\left\{16 b_{01 k}-4 a\left(2 b_{11 k}-a b_{21 k}\right)+\left(a^{2}+2\right)^{2}\left(a_{12 k}+3 b_{03 k}\right)\right.  \tag{23}\\
& \left.-4\left(a^{2}+2\right)\left(2 b_{02 k}-a a_{21 k}-a b_{12 k}\right)\right] .
\end{align*}
$$

Higher-order focus values are relatively complex, and we shall study them in Section 3.
When we want to use focus values $V_{i K}$ in $V_{K}(r), i=0,1,2, \cdots$, to study limit cycles, we first need to show $V_{k}(r) \equiv 0,1 \leq k<K$, or $\frac{d r}{d t}=O\left(\varepsilon^{K}\right)$ in (5). In order to prove this, we use the approximation of first integrals, and claim that if there exists an analytic function $H_{K}(x, y, \varepsilon)$ such that

$$
\begin{equation*}
\left(M^{-1} H_{y}+\varepsilon p\right) \frac{\partial H_{K}}{\partial x}+\left(-M^{-1} H_{x}+\varepsilon q\right) \frac{\partial H_{K}}{\partial y}=O\left(\varepsilon^{K}\right) \tag{24}
\end{equation*}
$$

then $\frac{d r}{d t}=O\left(\varepsilon^{K}\right)$. This claim can be easily proved by using the closed contour $H_{K}=h$ as the parameter to express the displacement function.

Usually, like that considered in [14,25] the method of focus values is used only to prove how many limit cycles around the equilibrium point that system (3) can have. Combining it with the approximation of first integrals, we can obtain the maximal number of small limit cycles for parameters in a neighborhood of critical conditions. Furthermore, if the focus values are linear functions in parameters, we have a global result as follows.

Theorem 1. Consider system (5) and assume $V_{k}(r) \equiv 0,1 \leq k<K$. Suppose that for an integer $m \geq 1$, each $V_{i K}, 0 \leq i<m$ is linear in $\delta$, and further the following two conditions hold:
(i) $\operatorname{rank}\left[\frac{\partial\left(V_{0 K}, \cdots, V_{m-1, K}\right)}{\partial\left(\delta_{1}, \cdots, \delta_{m}\right)}\right]=m$,
(ii) $V_{K}(r) \equiv 0$, if $V_{i K}=0, i=0,1, \cdots, m-1$.

Then, for any given $N>0$, there exist $\varepsilon_{0}>0$ and a neighborhood $V$ of the origin such that system (3) has at most $m-1$ limit cycles in $V$ for $0<|\varepsilon|<\varepsilon_{0},|\delta| \leq N$. Moreover, $m-1$ limit cycles can appear in an arbitrary neighborhood of the origin for some values of $(\varepsilon, \delta)$.

The above theorem can be proved following the proof of Theorem 2.4.3 given in [1] with a minor modification. So the proof is omitted here.

## 3. Higher-order analysis leading to 11 limit cycles in system (11)

In this section, we focus on system (11) and show that it can have 11 limit cycles by using perturbations at least up to 7th order. In the following, we will use the transformed system (11) with the simplified perturbations given in (19) for the analysis.

In order to compute the focus values of this system, we first shift the equilibrium of system (11), $(x, y)=\left(-\frac{a}{2}+o\left(\varepsilon^{n}\right),-\frac{a^{2}+2}{4}+o\left(\varepsilon^{n}\right)\right)$ to the origin and then use a computer algebra system and software package (e.g., the Maple program in [19]) to obtain the focus values in terms of the parameters $a, a_{i j k}$ and $b_{i j k}$. We shall give detailed analysis for the first few lower-order focus values, and then summarize the results obtained from higher-order analysis.

For convenience, define the vectors:

$$
\begin{align*}
W_{k}^{8} & =\left(V_{1 k}, V_{2 k}, \cdots, V_{8 k}\right) \\
W_{k}^{9} & =\left(V_{1 k}, V_{2 k}, \cdots, V_{9 k}\right) \\
W_{k}^{10} & =\left(V_{1 k}, V_{2 k}, \cdots, V_{10 k}\right) \\
S_{k}^{8} & =\left(b_{10 k}, b_{20 k}, b_{11 k}, b_{02 k}, b_{30 k}, b_{21 k}, b_{12 k}, b_{03 k}\right)  \tag{25}\\
S_{k}^{9} & =\left(b_{10 k}, b_{20 k}, b_{11 k}, b_{02 k}, b_{30 k}, b_{21 k}, b_{12 k}, b_{03 k}, a_{03 k}\right) \\
S_{k}^{10} & =\left(b_{10 k}, b_{20 k}, b_{11 k}, b_{02 k}, b_{30 k}, b_{21 k}, b_{12 k}, b_{03 k}, a_{03 k}, a_{12(3 m)}\right)
\end{align*}
$$

where in $S_{k}^{10}, k=7 m$ for Case (A) and $k=13 m$ for Case (B) ( $m \geq 1$, integer) to be considered in Sections 3.4 and 3.5; and the determinants:

$$
\begin{equation*}
\operatorname{det}_{k}^{8}=\operatorname{det}\left[\frac{\partial W_{k}^{8}}{\partial S_{k}^{8}}\right], \quad \operatorname{det}_{k}^{9}=\operatorname{det}\left[\frac{\partial W_{k}^{9}}{\partial S_{k}^{9}}\right], \quad \operatorname{det}_{k}^{10}=\operatorname{det}\left[\frac{\partial W_{k}^{10}}{\partial S_{k}^{10}}\right] ; \tag{26}
\end{equation*}
$$

and the functions:

$$
\begin{align*}
& F_{1}=-\frac{373423834799904305184768}{5 a^{36}\left(a^{4}-32\right)^{8}}, \\
& F_{2}=\frac{3013505105717894236809449177088}{5 a^{45}\left(a^{4}-32\right)^{9}}, \\
& F_{3}=-\frac{57397219210893210316046010501071634432}{a^{55}\left(a^{4}-32\right)^{10}}, \\
& F_{4}=-\frac{27963847691641519344238256641414767487418}{a^{66}\left(a^{4}-32\right)^{11}}, \\
& G_{1}=-\frac{258237837}{32 a^{9}\left(a^{4}-32\right)}, \\
& G_{2}=\frac{2347167}{64 a^{11}\left(a^{4}-32\right)^{2}}\left(57697 a^{4}-35728 a^{2}-88704\right), \\
& G_{3}=-\frac{223476167}{1024 a^{13}\left(a^{4}-32\right)^{3}}\left(2304313595 a^{8}-1702233920 a^{6}-11829269248 a^{4}\right. \\
& \left.\quad-39211065344 a^{2}+8642101248\right), \\
& G_{4}=-\frac{75246080}{a^{10}\left(a^{4}-32\right)},  \tag{27}\\
& G_{5}=\frac{9405760}{3 a^{12}\left(a^{4}-32\right)^{2}}\left(75767 a^{4}-46944 a^{2}-96768\right), \\
& G_{6}=-\frac{180880}{3 a^{14}\left(a^{4}-32\right)^{3}}\left(11681524055 a^{8}-8555309984 a^{6}-56944147200 a^{4}\right. \\
& \left.\quad-204210659328 a^{2}+30640177152\right), \\
& G_{7}=\frac{2006968901247765}{2883584 a^{11}\left(a^{4}-32\right)}, \\
& G_{8}=-\frac{15438223172905}{2883584 a^{13}\left(a^{4}-32\right)^{2}}\left(48667 a^{4}-30160 a^{2}-52416\right), \\
& G_{9}=\frac{6616380993155}{46137344 a^{15}\left(a^{4}-32\right)^{3}}\left(6314158847 a^{8}-4591849024 a^{6}\right. \\
& \left.\quad-29599122432 a^{4}-112639700992 a^{2}+11915624448\right) .
\end{align*}
$$

Note that $F_{i} \neq 0, i=1,2,3$, and $G_{i} \neq 0, i=1,2, \ldots, 9$, since $a^{4}-32>0$ for $a<-2^{-5 / 4}$.

## 3.1. $\varepsilon$ - and $\varepsilon^{2}$-order analysis

The $\varepsilon$-order focus values $V_{11}, V_{21}, \cdots, V_{111}$ are obtained by using the algorithm and Maple program developed in [19]. Their expressions are lengthy, and here we only present the first one for brevity,

$$
\begin{align*}
V_{11}= & -\frac{1}{64 a\left(a^{4}-32\right)}\left\{6912 b_{101}-5760 a b_{201}\right. \\
& +16\left(a^{4}-36 a^{2}+40\right) b_{111}+48 a\left(a^{4}+36 a^{2}+160\right) b_{021} \\
& +3456 a^{2} b_{301}-24 a\left(a^{4}-12 a^{2}+40\right) b_{211} \\
& -16\left(3 a^{6}+68 a^{4}+300 a^{2}+20\right) b_{121}  \tag{28}\\
& -24 a\left(3 a^{6}+65 a^{4}+300 a^{2}+224\right) b_{031} \\
& +27\left(a^{2}+2\right)\left(7 a^{6}+82 a^{4}+320 a^{2}+128\right) a_{031}
\end{align*}
$$

$$
\begin{aligned}
& +8 a\left(24 a^{6}+223 a^{4}+1140 a^{2}+1180\right) a_{121} \\
& \left.+8\left(21 a^{6}-73 a^{4}+480 a^{2}+320\right) a_{211}\right\}
\end{aligned}
$$

It is noted that all $V_{i 1}$ 's are linear polynomials in $a_{i j 1}$ and $b_{i j 1}$. It can be shown that

$$
\begin{equation*}
\operatorname{det}_{1}^{8}=F_{1} \neq 0, \quad \operatorname{det}_{1}^{9}=F_{2} \neq 0, \quad \operatorname{det}_{1}^{10}=0 . \tag{29}
\end{equation*}
$$

In fact, with the solution of $S_{1}^{8}$ solved from $W_{1}^{8}=0$, we obtain

$$
\begin{equation*}
V_{91}=G_{1} a_{031}, \quad V_{101}=G_{2} a_{031}, \quad V_{111}=G_{3} a_{031}, \tag{30}
\end{equation*}
$$

where $G_{i}$ 's are given in (27). Noticing $G_{1} \neq 0$ for $a<-2^{5 / 4}$, we have $V_{91} \neq 0$ if $a_{031} \neq 0$. Moreover, $\operatorname{det}_{1}^{8} \neq 0$ and (23) indicate that perturbing $W_{1}^{8}$ and $V_{01}$ around the solutions $S_{1}^{8}$ and $b_{011}$ (see (22)) does yield 9 small limit cycles around the equilibrium $\mathrm{E}_{0}$.

It is seen from (30) that $V_{91}=V_{101}=V_{111}=0$ for $a_{031}=0$. For convenience, define the critical condition $S_{1 c}^{8}$, satisfying (22), $W_{1}^{8}=0$ and $a_{031}=0$, as

$$
\mathrm{S}_{1 \mathrm{c}}^{8}:\left\{\begin{array}{l}
b_{011}=C_{1} a_{121}-\frac{9}{8} a^{3} a_{211},  \tag{31}\\
a_{031}=b_{211}=0, \quad b_{121}=\frac{7}{2} a_{211}, \quad b_{021}=-6 a_{121}, \quad b_{031}=\frac{8}{3} a_{121}, \\
b_{111}=9 a a_{121}+\frac{9}{2} a_{211}, \quad b_{101}=C_{2} a_{121}+C_{3} a_{211}, \\
b_{201}=C_{4} a_{121}+C_{5} a_{211}, \quad b_{301}=C_{6} a_{121}+C_{7} a_{211},
\end{array}\right.
$$

where $C_{i}$ 's are given in Appendix A.
We have the following result.
Theorem 2. The equilibrium $\mathrm{E}_{0}$ of system (11) is a center up to $\varepsilon$-order, i.e. all $\varepsilon$-order focus values vanish if and only if the condition $\mathrm{S}_{1 \mathrm{c}}^{8}$ holds. Furthermore, there exist at most 9 small limit cycles around $\mathrm{E}_{0}$ for all parameters $a_{i j 1}$ and $b_{i j 1}$, and 9 small limit cycles can be obtained for some parameter values near $\mathrm{S}_{1 \mathrm{c}}^{8}$.

Proof. The existence of 9 small limit cycles has been shown under the solution $S_{1}^{8}$ with $a_{031} \neq 0$ and $\operatorname{det}_{1}^{8} \neq 0$. It is obvious that the critical condition $S_{1 c}^{8}$ is necessary for all $\varepsilon$-order focus values to vanish since $V_{i 1}=0,0 \leq i \leq 11$ under this condition. To prove sufficiency, under the critical condition $\mathrm{S}_{1 \mathrm{c}}^{8}$, we use (24) to obtain the following $\varepsilon$-order approximation of the first integral,

$$
\begin{equation*}
H_{1}(x, y, \varepsilon)=\frac{f_{1}+\varepsilon f_{11}}{f_{2}+\varepsilon f_{21}} \tag{32}
\end{equation*}
$$

where $f_{1}$ and $f_{2}$ are given in (13), $f_{11}=a_{121} r_{1}+a_{211} r_{2}$ and $f_{21}=a_{121} r_{3}+a_{211} r_{4}$ with

$$
\begin{align*}
r_{1}= & -\frac{1}{48}\left[a^{2}\left(3 a^{2}+4\right)\left(5+2 y+2 x^{2}+x^{4}\right)+220-192 a x+280 y\right. \\
& \left.+120 x^{2}-64 y^{2}+128 a x^{3}+64 x^{2} y+76 x^{4}\right], \\
r_{2}= & -\frac{1}{8} a\left(5 a^{2}-4\right)+5 x-\frac{1}{8} a\left(a^{2}-4\right)\left(2 y+2 x^{2}+x^{4}\right)+2 x y-2 x^{3}, \tag{33}
\end{align*}
$$

$$
\begin{aligned}
r_{3}= & \frac{1}{192}\left[a^{2}\left(3 a^{2}+4\right)\left(4 a-15 x+10 x y+10 x^{3}\right)+304 a+16 a^{3}-180 x\right. \\
& \left.-40\left(16 a y-8 a x^{2}+5 x y-23 x^{3}-8 x y^{2}+16 a x^{4}+8 x^{3} y\right)\right], \\
r_{4}= & \frac{a^{2}}{8}\left(a^{2}-1\right)-a\left(\frac{15}{32} a^{2}+\frac{5}{8}\right) x+\frac{5}{2} x^{2}\left(\frac{3}{2}+y-x^{2}\right)+\frac{5}{16} a\left(a^{2}-4\right) x\left(y+x^{2}\right) .
\end{aligned}
$$

This implies that setting the first 10 focus values $V_{i 1}=0, i=0, \cdots, 9$ yields $V_{i 1}=0$ for all $i \geq 10$. Moreover, due to that all $V_{i 1}$ are linear in all parameters $a_{i j 1}$ and $b_{i j 1}$, by Theorem 1 at most 9 small limit cycles can be obtained for this case. The proof is complete.

Now suppose the condition $\mathrm{S}_{1 \mathrm{c}}^{8}$ holds and so all $\varepsilon$-order focus values vanish, we then need to use the $\varepsilon^{2}$-order focus values to study bifurcation of limit cycles. With an almost exact same procedure as that used in the $\varepsilon$-order analysis, we can find a solution $S_{2}^{8}$ such that $W_{2}^{8}=0$, and then

$$
\begin{equation*}
V_{92}=G_{1} a_{032}, \quad V_{102}=G_{2} a_{032}, \quad V_{112}=G_{3} a_{032}, \quad \operatorname{det}_{2}^{8}=F_{1} \neq 0 \tag{34}
\end{equation*}
$$

where $F_{1}$ and $G_{i}$ 's are given in (27). Note that the above equations are exactly the same as those given in (29) and (30), if we replace $k=1$ by $k=2$ in (29) and (30). This clearly shows that there can exist 9 limit cycles around the equilibrium $\mathrm{E}_{0}$ when all $\varepsilon$-order focus values vanish. It is also noted that all $V_{i 2}$ are linear polynomials in $a_{i j 2}$ and $b_{i j 2}$.

Similarly, we see that setting $a_{032}=0$ in (34) yields $V_{92}=V_{102}=V_{112}=0$, implying that the solution $S_{2}^{8}$ with $a_{032}=0$ and $b_{012}$ given in (22) defines a necessary condition for all $\varepsilon^{2}$-order focus values to vanish. This critical condition is given below:

$$
\mathrm{S}_{2 \mathrm{c}}^{8}:\left\{\begin{array}{l}
b_{012}=\frac{9}{64} a^{4} a_{211}^{2}-\frac{9}{8} a^{3} a_{212}+C_{1} a_{122}+C_{8} a_{121} a_{211}+C_{9} a_{121}^{2},  \tag{35}\\
a_{032}=0, b_{032}=\frac{8}{3} a_{122}+5 a_{121}^{2}, \quad b_{212}=\frac{a}{2} a_{121}\left(5 a a_{121}-9 a_{211}\right), \\
b_{122}=\frac{7}{2} a_{212}-\frac{1}{4} a_{121}\left(31 a a_{121}-45 a_{211}\right), \\
b_{102}=C_{2} a_{122}+C_{3} a_{212}+C_{10} a_{121}^{2}+C_{11} a_{211}^{2}+C_{12} a_{121} a_{211}, \\
b_{202}=C_{4} a_{122}+C_{5} a_{212}+C_{13} a_{121}^{2}+C_{14} a_{211}^{2}+C_{15} a_{121} a_{211}, \\
b_{112}=9 a a_{122}+\frac{9}{2} a_{212}+\frac{9 a^{3}}{32} a_{211}^{2}+C_{16} a_{121}^{2}+C_{17} a_{121} a_{211}, \\
b_{022}=-6 a_{122}+C_{18} a_{121}^{2}+C_{19} a_{121} a_{211}, \\
b_{302}=C_{6} a_{122}+C_{7} a_{212}+C_{20} a_{121}^{2}+C_{21} a_{211}^{2}+C_{22} a_{211} a_{121},
\end{array}\right.
$$

where $C_{i}$ 's are given in Appendix A.
We have the following theorem.
Theorem 3. Assume $\mathrm{S}_{1 \mathrm{c}}^{8}$ holds. The equilibrium $\mathrm{E}_{0}$ of system (11) is a center up to $\varepsilon^{2}$-order, if and only if $\mathrm{S}_{2 \mathrm{c}}^{8}$ holds. Furthermore, there exist at most 9 small limit cycles around $\mathrm{E}_{0}$ for all parameters $a_{i j 2}$ and $b_{i j 2}$, and 9 small limit cycles exist for some parameter values near $\mathrm{S}_{2 \mathrm{c}}^{8}$.

Proof. Similarly, we only need to prove sufficiency. With $S_{1 c}^{8}$ and $S_{2 c}^{8}$ holding, we can use (24) to find the following $\varepsilon^{2}$-order approximation of the first integral,

$$
\begin{equation*}
H_{2}(x, y, \varepsilon)=\frac{f_{1}+\varepsilon f_{11}+\varepsilon^{2} f_{12}}{f_{2}+\varepsilon f_{21}+\varepsilon^{2} f_{22}} \tag{36}
\end{equation*}
$$

where $f_{11}$ and $f_{21}$ are given in $H_{1}(x, y, \varepsilon)$ (see Eq. (32)), and

$$
\begin{aligned}
& f_{21}=a_{122} r_{1}+a_{212} r_{2}+a_{121}^{2} s_{1}+a_{211}^{2} s_{2}+a_{121} a_{211} s_{3}, \\
& f_{22}=a_{122} r_{3}+a_{212} r_{4}+a_{121}^{2} s_{4}+a_{211}^{2} s_{5}+a_{121} a_{211} s_{6},
\end{aligned}
$$

in which $r_{i}, i=1,2,3,4$ are given in (33), and $s_{i}, i=1,2, \ldots, 8$, are listed in Appendix B.
The existence of 9 small limit cycles is easily seen from $V_{92} \neq 0$ and $\operatorname{det}_{2}^{8} \neq 0$ when $a_{032} \neq 0$ under the critical condition $\mathrm{S}_{2 \mathrm{c}}^{8}$. On the other hand, the above results show that setting $V_{i 2}, 0 \leq$ $i \leq 9$ results in $V_{i 2}=0$ for all $i \geq 10$. Further, all $V_{i 2}$ 's are linear in $a_{i j 2}$ and $b_{i j 2}$, and $\mathrm{S}_{2 \mathrm{c}}^{8}$ is the unique solution of $V_{i 2}=0,0 \leq i \leq 9$. Then by Theorem 1, at most 9 small limit cycles can be obtained around $\mathrm{E}_{0}$ for all parameters $a_{i j 2}$ and $b_{i j 2}$.

## 3.2. $\varepsilon^{3}$-order analysis

In this section, we assume the critical condition $\left\{S_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}\right\}$, which stands for that both the critical conditions $S_{1 \mathrm{c}}^{8}$ and $\mathrm{S}_{2 \mathrm{c}}^{8}$ hold, under which all $\varepsilon$ - and $\varepsilon^{2}$-order focus values vanish. Thus, we use $\varepsilon^{3}$-order focus values $V_{i 3}$ to study bifurcation of limit cycles around the equilibrium $\mathrm{E}_{0}$. With a similar procedure, but for this order, we solve 9 equations $W_{3}^{9}=0$ to obtain the solution $S_{3}^{9}$ for which

$$
\begin{equation*}
V_{103}=G_{4} a_{121}^{3}, \quad V_{113}=G_{5} a_{121}^{3}, \quad V_{123}=G_{6} a_{121}^{3}, \quad \operatorname{det}_{3}^{9}=F_{2} \neq 0, \tag{37}
\end{equation*}
$$

where $F_{2}$ and $G_{i}$ 's are given in (27). Note that for this order, there is one more independent coefficient $a_{033}$ in $S_{3}^{9}$ for solving $W_{3}^{9}=0$, compared to the solutions $S_{1}^{8}$ and $S_{2}^{8}$ which have only 8 independent coefficients to be used for solving the first 8 focus value equations. The equations in (37) show that when all $\varepsilon$ - and $\varepsilon^{2}$-order focus values vanish, the $\varepsilon^{3}$-order focus values can have solutions such that $V_{i 3}=0, i=0,1, \cdots, 9$ but $V_{103} \neq 0$, as well as $\operatorname{det}_{3}^{9} \neq 0$, implying that 10 small limit cycles can bifurcate from the equilibrium $\mathrm{E}_{0}$.

Setting $a_{121}=0$ in (37), we have $V_{103}=V_{113}=V_{123}=0$, implying that under the solution $S_{3}^{9}$ with $a_{121}=0$ and $b_{013}$ given in (22), the equilibrium $\mathrm{E}_{0}$ might be a center up to $\varepsilon^{3}$ order. This critical condition is given by

$$
\mathrm{S}_{3 \mathrm{c}}^{9}:\left\{\begin{array}{l}
b_{013}=\frac{9}{32} a^{4} a_{211} a_{212}-\frac{9}{8} a^{3} a_{213}+C_{1} a_{123}+C_{8} a_{122} a_{211}+C_{23} a_{211}^{3},  \tag{38}\\
a_{121}=a_{033}=0, \quad b_{033}=\frac{8}{3} a_{123}, \quad b_{023}=-6 a_{123}+C_{19} a_{122} a_{211} \\
b_{213}=-\frac{9 a}{16} a_{211}\left(8 a_{122}+a_{211}^{2}\right), \quad b_{123}=\frac{7}{2} a_{213}+\frac{45}{32} a_{211}\left(8 a_{122}+a_{211}^{2}\right), \\
b_{113}=9 a a_{123}+\frac{9}{2} a_{213}+a_{211}\left[\frac{9}{16} a^{3} a_{212}+C_{17} a_{122}+C_{24} a_{211}^{2}\right] \\
b_{103}=C_{2} a_{123}+C_{3} a_{213}+a_{211}\left[2 C_{11} a_{212}+C_{12} a_{122}+C_{25} a_{211}^{2}\right] \\
b_{203}=C_{4} a_{123}+C_{5} a_{213}+a_{211}\left[2 C_{14} a_{212}+C_{15} a_{122}+C_{26} a_{211}^{2}\right] \\
b_{303}=C_{6} a_{123}+C_{7} a_{213}+a_{211}\left[2 C_{21} a_{212}+C_{22} a_{122}+C_{27} a_{211}^{2}\right],
\end{array}\right.
$$

under which the critical conditions $S_{1 c}^{8}$ and $S_{2 c}^{8}$ are simplified. Here, $C_{i}$ 's are given in Appendix A.

We have the following theorem.
Theorem 4. Let $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}\right\}$ hold. The equilibrium $\mathrm{E}_{0}$ of system (11) is a center up to $\varepsilon^{3}$-order, if and only if the condition $\mathrm{S}_{3 \mathrm{c}}^{9}$ holds. Furthermore, there exist 10 small limit cycles around $\mathrm{E}_{0}$ for some parameter values of $a_{i j 3}$ and $b_{i j 3}$ near the critical value defined by $S_{3 c}^{9}$ when $V_{103} \neq 0$.

Proof. Similarly again, we only need to prove sufficiency. Under the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}\right\}$, we obtain the following $\varepsilon^{3}$-order approximation of first integral,

$$
\begin{equation*}
H_{3}(x, y, \varepsilon)=\frac{f_{1}+\varepsilon a_{211} r_{1}+\varepsilon^{2}\left(a_{122} r_{1}+a_{212} r_{2}+a_{211}^{2} s_{2}\right)+\varepsilon^{3} f_{31}}{f_{2}+\varepsilon a_{211} r_{4}+\varepsilon^{2}\left(a_{122} r_{3}+a_{212} r_{4}+a_{211}^{2} s_{5}\right)+\varepsilon^{3} f_{32}} \tag{39}
\end{equation*}
$$

where

$$
\begin{aligned}
& f_{31}=a_{123} r_{1}+a_{213} r_{2}+a_{211}\left(a_{122} t_{1}+a_{212} t_{2}+a_{211}^{2} t_{3}\right) \\
& f_{32}=a_{123} r_{3}+a_{213} r_{4}+a_{211}\left(a_{122} t_{4}+a_{212} t_{5}+a_{211}^{2} t_{6}\right)
\end{aligned}
$$

in which $r_{i}, i=1,2,3,4$ are given in (33), and $s_{2}, s_{5}$ and $t_{i}, i=1,2, \ldots, 6$ are listed in Appendix B. This implies that setting $V_{i 3}=0,0 \leq i \leq 10$ yields $V_{i 3}=0$ for all $i \geq 11$. Then, there exist at most 10 small limit cycles for this case. On the other hand, 10 small limit cycles exist since when $a_{121} \neq 0, V_{101} \neq 0$ and $\operatorname{det}_{3}^{9} \neq 0$.

## 3.3. $\varepsilon^{4}-\varepsilon^{6}$-order analysis

The analyses for $\varepsilon^{4}-, \varepsilon^{5}$ - and $\varepsilon^{6}$-order are similar to that of $\varepsilon^{1}-, \varepsilon^{2}$ - and $\varepsilon^{3}$-order, respectively.
Let $\left\{S_{1 c}^{8}, S_{2 c}^{8}, S_{3 c}^{9}\right\}$ hold. Following the same procedure used in the previous sections, we can solve the equations $W_{4}^{8}=0$ to obtain a solution $S_{4}^{8}$ such that

$$
\begin{equation*}
V_{94}=G_{1} a_{034}, \quad V_{104}=G_{2} a_{034}, \quad V_{114}=G_{3} a_{034}, \quad \operatorname{det}_{4}^{8}=F_{1} \neq 0 \tag{40}
\end{equation*}
$$

which has the exactly same form of the equations as those given in (30) and (34), implying that perturbing the $\varepsilon^{4}$-order focus values from the solution $S_{4}^{8}$ and $b_{014}$ (see (22)) can yield 9 limit cycles around the equilibrium $\mathrm{E}_{0}$. Similarly, the solution $S_{4}^{8}$ and $b_{014}$ with $a_{034}=0$ yields a critical condition $\mathrm{S}_{4 \mathrm{c}}^{8}$, under which the equilibrium $\mathrm{E}_{0}$ is a center up to $\varepsilon^{4}$ order.

Then let $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}\right\}$ hold. In the same line, we can solve the equations $W_{5}^{8}=0$ to obtain a solution $S_{5}^{8}$ such that

$$
\begin{equation*}
V_{95}=G_{1} A_{035}, \quad V_{105}=G_{2} A_{035}, \quad V_{115}=G_{3} A_{035}, \quad \operatorname{det}_{5}^{8}=F_{1} \neq 0 \tag{41}
\end{equation*}
$$

where

$$
\begin{equation*}
A_{035}=a_{035}+\frac{1}{48} a_{122} a_{211}\left(140 a_{122}+35 a_{211}^{2}\right) \tag{42}
\end{equation*}
$$

This shows that perturbing the $\varepsilon^{5}$-order focus values near the solution $S_{5}^{8}$ and $b_{015}$ given in (22) can also yield 9 limit cycles around the equilibrium $\mathrm{E}_{0}$. It is easy to see that the solution of $A_{035}=0$,

$$
\begin{equation*}
a_{035}=-\frac{35}{48} a_{122} a_{211}\left(4 a_{122}+a_{211}^{2}\right) \tag{43}
\end{equation*}
$$

yields $V_{95}=V_{105}=V_{115}=0$. Now, we combine the solution $S_{5}^{8}, b_{015}$ and $a_{035}$ to obtain the critical condition $\mathrm{S}_{5 \mathrm{c}}^{8}$, under which the equilibrium $\mathrm{E}_{0}$ becomes a center up to $\varepsilon^{5}$ order.

The lengthy critical conditions $S_{4 c}^{8}$ and $S_{5 c}^{8}$ are omitted here for brevity. Summarizing the above results leads to the following theorem.

Theorem 5. System (11) can have maximal 9 limit cycles around the equilibrium $\mathrm{E}_{0}$ under the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}\right\}$ by perturbing the $\varepsilon^{4}$-order focus values around the critical value $\mathrm{S}_{4 \mathrm{c}}^{8}$; and under the critical condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}\right\}$ by perturbing the $\varepsilon^{5}$-order focus values near the critical point $\mathrm{S}_{5 \mathrm{c}}^{8}$. The equilibrium $\mathrm{E}_{0}$ becomes a center up to $\varepsilon^{4}$ order under the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}\right\}$, and a center up to $\varepsilon^{5}$ order under the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}, \mathrm{~S}_{5 \mathrm{c}}^{8}\right\}$.

Remark 2. The proof for the center conditions in Theorem 5 is similar to that in proving Theorems 2,3 and 4 by finding the $\varepsilon^{4}$-order and $\varepsilon^{5}$-order approximations of the first integrals. This is the major and tedious part. For higher-order analysis, the proofs are similar. We omit the detailed proofs in the following analysis for brevity.

Next, suppose the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}, \mathrm{~S}_{5 \mathrm{c}}^{8}\right\}$ is satisfied, then all $\varepsilon^{k}, k=1,2, \ldots, 5$, order focus values vanish. Following a similar analysis as that for $\varepsilon^{3}$ order, we solve the equations $W_{6}^{9}=0$ to obtain a solution $S_{6}^{9}$ such that

$$
\begin{align*}
& V_{106}=G_{4} a_{122}^{2}\left(a_{122}+\frac{9}{8} a_{211}^{2}\right), \\
& V_{116}=G_{5} a_{122}^{2}\left(a_{122}+\frac{9}{8} a_{211}^{2}\right),  \tag{44}\\
& V_{126}=G_{6} a_{122}^{2}\left(a_{122}+\frac{9}{8} a_{211}^{2}\right), \quad \operatorname{det}_{6}^{9}=F_{2} \neq 0,
\end{align*}
$$

which indeed shows the existence of 10 limit cycles around the equilibrium $\mathrm{E}_{0}$, generated from perturbing the $\varepsilon^{6}$-order focus values near the solution $S_{6}^{9}$ under the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}, \mathrm{~S}_{5 \mathrm{c}}^{8}\right\}$. Moreover, when $a_{122}=-\frac{9}{8} a_{211}^{2}$ or $a_{122}=0$, we have $V_{106}=V_{116}=$ $V_{126}=0$, indicating that the solution $S_{6}^{9}$ with either $a_{122}=-\frac{9}{8} a_{211}^{2}$ or $a_{122}=0$, plus $b_{016}$ given by (22) $\left.\right|_{k=6}$, yields a critical condition $\mathrm{S}_{6 \mathrm{c}}^{9 \mathrm{a}}$ (corresponding to the former) or $\mathrm{S}_{6 \mathrm{c}}^{9 \mathrm{~b}}$ (corresponding to the latter) under which all $\varepsilon^{6}$-order focus values vanish. Thus, under the critical condition $\left\{S_{1 c}^{8}, S_{2 c}^{8}, S_{3 c}^{9}, S_{4 c}^{8}, S_{5 c}^{8}, S_{6 c}^{9}\right\}\left(S_{6 c}^{9}\right.$ equals either $S_{6 c}^{9 a}$ or $\left.S_{6 c}^{9 b}\right)$, the equilibrium $E_{0}$ becomes a center up to $\varepsilon^{6}$ order.

We have the following theorem for this order.
Theorem 6. System (11) can have maximal 10 limit cycles bifurcating from the equilibrium $\mathrm{E}_{0}$ under the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}, \mathrm{~S}_{5 \mathrm{c}}^{8}\right\}$ by perturbing the $\varepsilon^{6}$-order focus values near the critical point $\mathrm{S}_{6 \mathrm{c}}^{9 \mathrm{a}}$ or $\mathrm{S}_{6 \mathrm{c}}^{9 \mathrm{~b}}$. Further, the equilibrium $\mathrm{E}_{0}$ becomes a center up to $\varepsilon^{6}$ order under the condition $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}, \mathrm{~S}_{5 \mathrm{c}}^{8}, \mathrm{~S}_{6 \mathrm{c}}^{9}\right\}$, for which all $\varepsilon^{k}$-order $(k=1,2, \ldots, 6)$ focus values vanish.

Suppose the condition $\left\{S_{1 c}^{8}, S_{2 c}^{8}, S_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}, \mathrm{~S}_{5 \mathrm{c}}^{8}, \mathrm{~S}_{6 \mathrm{c}}^{9}\right\}$ holds. Then, all the $\varepsilon^{k}$-order $(k=$ $1,2, \ldots, 6$ ) focus values vanish. We have two cases for higher-order analysis, defined as

$$
\begin{align*}
& \text { Case (A) }\left\{S_{1 c}^{8}, S_{2 c}^{8}, S_{3 c}^{9}, S_{4 c}^{8}, S_{5 c}^{8}, S_{6 c}^{9 a}\right\}, \\
& \text { Case (B) }\left\{S_{1 c}^{8}, S_{2 c}^{8}, S_{3 c}^{9}, S_{4 c}^{8}, S_{5 c}^{8}, S_{6 c}^{9 b}\right\} . \tag{45}
\end{align*}
$$

### 3.4. Higher-order analysis for Case (A)

First we consider Case (A), under which we will show that 11 limit cycles can bifurcate from the equilibrium $\mathrm{E}_{0}$ based on the $\varepsilon^{7}$-order focus values.

### 3.4.1. $\varepsilon^{7}$-order analysis

Under the condition (A) defined in (45) with $a_{122}=-\frac{9}{8} a_{211}^{2}$, we obtain

$$
\begin{equation*}
\operatorname{det}_{7}^{10}=F_{3} a_{211}^{4}, \quad \operatorname{det}_{7}^{11}=F_{4} a_{211}^{10} \tag{46}
\end{equation*}
$$

which shows that $\operatorname{det}_{7}^{10} \neq 0$ and $\operatorname{det}_{7}^{11} \neq 0$ when $a_{211} \neq 0$, implying that we may have solutions such that the first ten focus values vanish but $V_{117} \neq 0$ and so 11 small limit cycles may be obtained. Indeed, we can solve the first ten focus values equations: $W_{k}^{10}=0$ to obtain a solution $S_{7}^{10}$ such that

$$
\begin{equation*}
V_{117}=G_{7} a_{211}^{7}, \quad V_{127}=G_{8} a_{211}^{7}, \quad V_{137}=G_{9} a_{211}^{7} \tag{47}
\end{equation*}
$$

which clearly shows that $V_{117} \neq 0$ if $a_{211} \neq 0$. In addition, due to $\operatorname{det}_{7}^{10} \neq 0$ when $a_{211} \neq 0$, implying that 11 small limit cycles exist.

Letting $a_{211}=0$, we have $V_{117}=V_{127}=V_{137}=0$, leading to a critical condition $\mathrm{S}_{7 \mathrm{c}}^{10}$, defined by

$$
\mathrm{S}_{7 \mathrm{c}}^{10}:\left\{\begin{array}{l}
b_{017}=C_{1} a_{127}-\frac{9}{8} a^{3} a_{217}+C_{8} C_{28}+\frac{9}{32} a^{4} C_{29}+C_{23} C_{30},  \tag{48}\\
a_{211}=a_{123}=a_{037}=0, b_{037}=\frac{8}{3} a_{127}, \\
b_{027}=-6 a_{127}+C_{19} C_{28}, \quad b_{217}=-\frac{9 a}{16}\left(8 C_{28}+C_{30}\right), \\
b_{127}=\frac{7}{2} a_{217}+\frac{45}{32}\left(8 C_{28}+C_{30}\right), \\
b_{107}=C_{2} a_{127}+C_{3} a_{217}+2 C_{11} C_{29}+C_{12} C_{28}+C_{25} C_{30}, \\
b_{207}=C_{4} a_{127}+C_{5} a_{217}+2 C_{14} C_{29}+C_{15} C_{28}+C_{26} C_{30}, \\
b_{117}=9 a a_{127}+\frac{9}{2} a_{217}+\frac{9 a^{3}}{16} C_{29}+C_{17} C_{28}+C_{24} C_{30}, \\
b_{307}=C_{6} a_{127}+C_{7} a_{217}+2 C_{21} C_{29}+C_{22} C_{28}+C_{27} C_{30},
\end{array}\right.
$$

where $C_{i}$ 's are given in Appendix A.
We have the following result.
Theorem 7. Let $\left\{\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}, \mathrm{~S}_{3 \mathrm{c}}^{9}, \mathrm{~S}_{4 \mathrm{c}}^{8}, \mathrm{~S}_{5 \mathrm{c}}^{8}, \mathrm{~S}_{6 \mathrm{c}}^{9 \mathrm{a}}\right\}$ hold. The equilibrium $\mathrm{E}_{0}$ of (11) becomes a center up to $\varepsilon^{7}$ order under $\mathrm{S}_{7 \mathrm{c}}^{10}$ for which all $\varepsilon^{7}$-order focus values vanish. Furthermore, there exist 11 small limit cycles around $\mathrm{E}_{0}$ for parameter values of $a_{i j 7}$ and $b_{i j 7}$ near the critical point $\mathrm{S}_{7 \mathrm{c}}^{10}$.

### 3.4.2. Higher-order analysis

For higher-order analysis ( $k \geq 8$ ), we first briefly list the results for a few orders to see the patterns and then summarize the results in a table for higher orders.

The analysis on $\varepsilon^{k}(k=8,9,10,11)$ orders show the same pattern, giving 9 limit cycles for each order, as follows:

Order $k$ :

$$
(k=8,9,10,11)
$$

$$
\left\{S_{k}^{8}, W_{k}^{8}\right\},\left\{\begin{align*}
V_{9 k} & =G_{1} A_{03 k}, & & V_{10 k}=G_{2} A_{03 k},  \tag{49}\\
V_{11 k} & =G_{3} A_{03 k}, & & \operatorname{det}_{k}^{8}=F_{1},
\end{align*}\right.
$$

where $\left\{S_{k}^{m}, W_{k}^{m}\right\}$ denotes the solution $S_{k}^{m}$ solved from $W_{k}^{m}=0$, and

$$
\begin{aligned}
A_{038} & =a_{038}, \quad A_{039}=a_{039} \\
A_{0310} & =a_{0310}+\frac{35}{48} a_{124} a_{212}\left(4 a_{124}+a_{212}^{2}\right), \\
A_{0311} & =a_{0311}+\frac{35}{48}\left[a_{125} a_{212}\left(8 a_{124}+a_{212}^{2}\right)+a_{124} a_{213}\left(4 a_{124}+3 a_{212}^{2}\right)\right]
\end{aligned}
$$

This clearly shows that for each order of $k=8,9,10,11$, one can solve $A_{03 k}=0$ to get a unique solution for $a_{03 k}$ under which (together with the solutions $S_{k}^{m}$ and $b_{01 k}$ obtained in the previous orders and the current order) the equilibrium $\mathrm{E}_{0}$ becomes a center up to that order.

When the equilibrium $\mathrm{E}_{0}$ is a center up to 11th order, as given in (49) we obtain the following result for order 12:

$$
\text { Order 12: }\left\{S_{12}^{9}, W_{12}^{9}\right\},\left\{\begin{array}{l}
V_{1012}=G_{4} a_{124}^{2}\left(a_{124}+\frac{9}{8} a_{212}^{2}\right),  \tag{50}\\
V_{1112}=G_{5} a_{124}^{2}\left(a_{124}+\frac{9}{8} a_{212}^{2}\right), \\
V_{1212}=G_{6} a_{124}^{2}\left(a_{124}+\frac{9}{8} a_{212}^{2}\right), \operatorname{det}_{12}^{9}=F_{2}
\end{array}\right.
$$

which has the exactly the same pattern as order 6 , shown in (44), indicating that 10 limit cycles can be obtained from this order, and there are two solutions from the equations $V_{1012}=V_{1112}=$ $V_{1212}=0: a_{124}=-\frac{9}{8} a_{212}^{2}$ and $a_{124}=0$, which are again similar to that as in order 6 . When $a_{124}=0$, it will be shown in Section 3.6 that it yields the same pattern as that for Case (B) in higher orders. So in this section, we choose $a_{124}=-\frac{9}{8} a_{212}^{2}$, like we chose $a_{122}=-\frac{9}{8} a_{212}^{2}$ in order 6 to obtain the center condition.

Let $a_{124}=-\frac{9}{8} a_{212}^{2}$, under which (together with the solutions obtained from previous orders and this order) the equilibrium $\mathrm{E}_{0}$ becomes a center up to $\varepsilon^{12}$ order. Then, we have the result for $\varepsilon^{13}$ order:

$$
\text { Order 13: }\left\{S_{13}^{9}, W_{13}^{9}\right\},\left\{\begin{array}{l}
V_{1013}=\frac{81}{64} G_{4} a_{212}^{4}\left(a_{125}+\frac{9}{4} a_{212} a_{213}\right),  \tag{51}\\
V_{1113}=\frac{81}{64} G_{5} a_{212}^{4}\left(a_{125}+\frac{9}{4} a_{212} a_{213}\right), \\
V_{1213}=\frac{81}{64} G_{6} a_{212}^{4}\left(a_{125}+\frac{9}{4} a_{212} a_{213}\right), \operatorname{det}_{13}^{9}=F_{2},
\end{array}\right.
$$

which shows that perturbing $\varepsilon^{13}$-order focus values can also yield 10 small limit cycles around the equilibrium $\mathrm{E}_{0}$. It can be seen from (51) that either $a_{212}=0$ or $a_{125}=-\frac{9}{4} a_{212} a_{213}$ leads to the equilibrium $\mathrm{E}_{0}$ being a center. However, it can be shown that setting $a_{212}=0$ at this order will not yield 11 small limit cycles at the next order though it will resume the same pattern at higher orders.

So let $a_{125}=-\frac{9}{4} a_{212} a_{213}$. Then, we obtain the following result for $\varepsilon^{14}$ order:

$$
\text { Order 14: }\left\{S_{14}^{10}, W_{14}^{10}\right\}, \begin{cases}V_{1114}=G_{7} a_{212}^{7}, & V_{1214}=G_{8} a_{212}^{7}  \tag{52}\\ V_{1314}=G_{9} a_{212}^{7}, & \operatorname{det}_{14}^{10}=F_{3} \neq 0,\end{cases}
$$

which shows that perturbing $\varepsilon^{14}$-order focus values can yield 11 limit cycles around the equilibrium $\mathrm{E}_{0}$, and setting $a_{212}=0$ leads to $\mathrm{E}_{0}$ being a center up to $\varepsilon^{14}$ order. It has been noted that choosing $a_{212}=0$ at order 13 or 14 makes differences. More precisely, as shown in Table 1, if taking $a_{125}=-\frac{9}{4} a_{212} a_{213}$ at order 13 , we have small limit cycles $11,9,9,9,9$ for the orders 14-18; while if taking $a_{212}=0$ at order 13, then the limit cycles obtained for the orders 14-18 are $9,10,9,9,10$, and then the two different choices merge into the same pattern from order 19. Note that the choice $a_{212}=0$ at order 13 does not yield 11 small limit cycles at order 14, but gives two more 10 small limit cycles at orders 15 and 18 . However, it returns to the general pattern at order 19. So we treat $a_{212}=0$ as a special case of the case $a_{125}=-\frac{9}{4} a_{212} a_{213}$.

Summarizing the above results we have the following pattern: 11 limit cycles are obtained from $\varepsilon^{7}$ order, then 9 limit cycles from four consecutive $\varepsilon^{k}$ orders ( $k=8,9,10,11$ ), and then 10 limit cycles from two consecutive $\varepsilon^{k}$ orders ( $k=12,13$ ), and finally return to 11 limit cycles at $\varepsilon^{14}$ order. This pattern, starting from order 8 , four 9 limit cycles, followed by two 10 limit cycles, and then 11 limit cycle, has been verified up to $\varepsilon^{35}$ order. We call this as $9^{4}-10^{2}-11^{1}$ generic pattern, and the corresponding solution (or center condition) is called generic solution (or generic center condition). By generic we mean that one should always choose a non-zero solution (if it exists) when one solves the center conditions at each order. Other types of solutions are called non-generic. For example, as discussed above, if choosing the non-generic solution $a_{212}=0$ at order 13, then 11 limit cycles will be missed at order 14 but the solution procedure will return to the generic $9^{4}-10^{2}-11^{1}$ pattern at order 19 . However, it should be noted that a non-generic solution in Case (A) does not always lead to the generic $9^{4}-10^{2}-11^{1}$ pattern. For instance, choosing the non-generic solution $a_{124}=0$ at order 12 will generate solutions in the form of generic patter of Case (B) at a higher order, as shown in the next section.

Remark 3. It has been observed from the above analysis, the values of the parameter $a$ in the Hamiltonian function does not affect the number of limit cycles. In other words, $a$ can not be used to increase the number of bifurcating limit cycles. Thus, to simply the computations in higher order analysis, we set $a=-3$ in higher-order $(k \geq 15)$ computations, which greatly simplify the computations.

We summarize the results of Case (A) in Table 1, where $k$ is the order of $\varepsilon^{k}$ focus values, $\left(S_{k}^{m}, W_{k}^{m}\right)$ represents the solution $S_{k}^{m}$ solved from $W_{k}^{m}=0$, and LC denotes limit cycles around the equilibrium $\mathrm{E}_{0}$ obtained by perturbing the $\varepsilon^{k}$-order focus values. The "Condition for Center" in each row only lists the condition for the current row, which assumes that the conditions in the previous rows hold. For example, when $k=4, \mathrm{~S}_{4 \mathrm{c}}^{8}$ only gives the center condition for $k=4$, which should be combined with the conditions given in the previous rows: $\mathrm{S}_{1 \mathrm{c}}^{8}, \mathrm{~S}_{2 \mathrm{c}}^{8}$ and $S_{3 c}^{9}$ to get a complete center condition $\left\{S_{1 c}^{8}, S_{2 c}^{8}, S_{3 c}^{9}, S_{4 c}^{8}\right\}$. Note that the critical condition $\mathrm{S}_{\mathrm{kc}}^{8}$ contains the solutions $S_{k}^{8}$, the $b_{01 k}$ given in (22) and a particular coefficient. For example, $S_{2 \mathrm{c}}^{8}=\left\{S_{2}^{8}, b_{012}, a_{032}\right\}, S_{3 \mathrm{c}}^{9}=\left\{S_{3}^{9}, b_{013}, a_{121}\right\}$, and $\mathrm{S}_{7 \mathrm{c}}^{10}=\left\{S_{7}^{10}, b_{017}, a_{211}\right\}$, etc. The solutions of these key coefficients are given below.

$$
\begin{aligned}
& 9 \mathrm{LC}: \quad k=1,2,4,8,9 \quad a_{03 k}=0 \\
& k=5 \quad a_{035}=-\frac{35}{48} a_{122} a_{211}\left(4 a_{122}+a_{211}^{2}\right) \\
& k=10 \quad a_{0310}=-\frac{35}{48} a_{124} a_{212}\left(4 a_{124}+a_{212}^{2}\right) \\
& k=11 \quad a_{0311}=-\frac{35}{48}\left[a_{212} a_{125}\left(8 a_{124}+a_{212}^{2}\right)+a_{213} a_{124}\left(4 a_{124}+3 a_{212}^{2}\right)\right] \\
& k=15 \quad a_{0315}=-\frac{735}{256} a_{213}^{5} \\
& k=16 \quad a_{0316}=\frac{35}{768} a_{213}^{3}\left(128 a_{127}-27 a_{214} a_{213}\right) \\
& k=17 \quad a_{0317}=-\frac{35}{768} a_{213}\left[32 a_{127}\left(2 a_{127}-3 a_{214} a_{213}\right)\right. \\
& \left.-a_{213}^{2}\left(128 a_{128}+54 a_{214}^{2}-27 a_{215} a_{213}\right)\right] \\
& k=18 \quad a_{0318}=\frac{35}{6} a_{213}^{3} a_{129}-\frac{35}{24} a_{213} a_{128}\left(4 a_{127}-3 a_{214} a_{213}\right) \\
& -\frac{35}{48} a_{127}\left[a_{214}\left(4 a_{127}+3 a_{214} a_{213}\right)-6 a_{213}^{2} a_{215}\right] \\
& +\frac{3}{4096} a_{213}^{2}\left[1120 a_{214}\left(a_{214}^{2}+6 a_{215} a_{213}\right)\right. \\
& \left.+3 a_{213}^{2}\left(2269 a_{213}^{2}-560 a_{216}\right)\right] \\
& k=22 \quad a_{0322}=\frac{35}{768} a_{214}^{3}\left(128 a_{1210}-486 a_{215}^{2}-27 a_{216} a_{214}\right) \\
& k=23 \quad a_{0323}=\frac{35}{768} a_{214}^{2}\left[a_{214}\left(128 a_{1211}-27 a_{217} a_{214}\right)\right. \\
& \left.+a_{215}\left(384 a_{1210}-108 a_{214} a_{216}-198 a_{215}^{2}\right)\right] \\
& k=24,25 \quad a_{03 k}=\cdots \\
& k=29-32 \quad a_{03 k}=\cdots \\
& 10 \mathrm{LC}: k=3 \quad a_{121}=0 \\
& k=6 \quad a_{122}=-\frac{9}{8} a_{211}^{2} \\
& k=12 \quad a_{124}=-\frac{9}{8} a_{212}^{2} \\
& k=13 \quad a_{125}=-\frac{9}{4} a_{213} a_{212} \\
& k=19 \quad a_{127}=-\frac{9}{4} a_{213} a_{214} \\
& k=20 \quad a_{128}=-\frac{9}{8}\left(a_{214}^{2}+2 a_{215} a_{213}\right) \\
& k=26 \quad a_{1210}=-\frac{9}{8}\left(a_{215}^{2}+2 a_{216} a_{214}\right) \\
& k=27 \quad a_{1211}=-\frac{9}{4}\left(a_{217} a_{214}+a_{216} a_{215}\right) \\
& k=33 \quad a_{1213}=-\frac{9}{4}\left(a_{218} a_{215}+a_{217} a_{216}\right) \\
& k=34 \quad a_{1214}=-\frac{9}{8}\left(a_{217}^{2}+2 a_{219} a_{215}+2 a_{218} a_{216}\right) \\
& 11 \mathrm{LC}: \quad k=7 m \\
& m=1-5 \quad a_{21 m}=0
\end{aligned}
$$

where '.. ' represents the omitted lengthy expressions for brevity. In addition, in Table 1 , the green and red colors ${ }^{1}$ denote the solutions and center conditions corresponding to the 10 and 11 small limit cycle, respectively.

[^1]Table 1
Bifurcation of limit cycles for generic Case (A).

| $k$ | $\left(S_{k}^{m}, W_{k}^{m}\right)$ | No. of LC | Condition for Center |
| :--- | :--- | :--- | :--- |
| 1,2 | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| 3 | $\left(S_{3}^{9}, W_{3}^{9}\right)$ | 10 | $\mathrm{~S}_{3 \mathrm{c}}^{9}$ |
| 4,5 | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| 6 | $\left(S_{6}^{9}, W_{6}^{9}\right)$ | 10 | $\mathrm{~S}_{6 \mathrm{c}}^{9}$ |
| 7 | $\left(S_{7}^{10}, W_{7}^{10}\right)$ | 11 | $\mathrm{~S}_{7 \mathrm{c}}^{10}$ |
| $8-11$ | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| 12,13 | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{~S}_{\mathrm{kc}}^{9}$ |
| 14 | $\left(S_{14}^{10}, W_{14}^{10}\right)$ | 11 | $\mathrm{~S}_{14 \mathrm{c}}^{10}$ |
| $15-18$ | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| 19,20 | $\left(S_{k}^{8}, W_{k}^{9}\right)$ | 10 | $\mathrm{~S}_{\mathrm{kc}}^{9}$ |
| 21 | $\left(S_{21}^{10}, W_{21}^{10}\right)$ | 11 | $\mathrm{~S}_{21 \mathrm{c}}^{10}$ |
| $22-25$ | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| 26,27 | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{~S}_{\mathrm{kc}}^{9}$ |
| 28 | $\left(S_{28}^{10}, W_{28}^{10}\right)$ | 11 | $\mathrm{~S}_{28 \mathrm{c}}^{10}$ |
| $29-32$ | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| 33,34 | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{~S}_{\mathrm{kc}}^{9}$ |
| 35 | $\left(S_{35}^{10}, W_{35}^{10}\right)$ | 11 | $\mathrm{~S}_{35 \mathrm{c}}^{10}$ |

### 3.5. Higher-order analysis for Case (B)

We now turn to Case (B) for which we choose $a_{122}=0$ at $\varepsilon^{6}$ order. Thus, the results starting from $\varepsilon^{6}$ order are different from those given in Table 1. Now under the condition $a_{122}=0$, together with the conditions obtained in previous orders, the equilibrium $\mathrm{E}_{0}$ becomes a center up to $\varepsilon^{6}$ order. Then for $\varepsilon^{7}$-order focus values we solve $W_{7}^{8}=0$ to obtain $S_{7}^{8}$ and then

$$
V_{97}=G_{1} A_{037}, \quad V_{107}=G_{2} A_{037}, \quad V_{117}=G_{3} A_{037}, \quad \operatorname{det}_{7}^{8}=F_{1},
$$

where

$$
A_{037}=a_{037}+\frac{35}{768} a_{211}\left[a\left(a^{2}-4\right) a_{123} a_{211}^{3}+16 a_{124} a_{211}^{2}+16 a_{123}\left(4 a_{123}+3 a_{212} a_{211}\right)\right]
$$

which shows that for Case (B) only 9 small limit cycles can be obtained from $\varepsilon^{7}$-order. Then, solving $A_{037}=0$ gives a unique solution for $a_{037}$, under which, together with the conditions obtained in the previous orders as well as $S_{7}^{8}$ and $b_{017}$, the equilibrium $\mathrm{E}_{0}$ becomes a center up to $\varepsilon^{7}$ order.

Next, the $\varepsilon^{8}$-order analysis shows that 10 limit cycles can be obtained by solving $W_{8}^{9}=0$ to have the solution $S_{8}^{9}$, under which higher-order focus values become

$$
\begin{equation*}
V_{108}=\frac{9}{8} G_{4} a_{211}^{2} a_{123}^{2}, \quad V_{118}=\frac{9}{8} G_{5} a_{211}^{2} a_{123}^{2}, \quad V_{128}=\frac{9}{8} G_{6} a_{211}^{2} a_{123}^{2}, \quad \operatorname{det}_{8}^{9}=F_{2} . \tag{53}
\end{equation*}
$$

Table 2
Bifurcation of limit cycles for generic Case (B).

| $k$ | $\left(S_{k}^{m}, W_{k}^{m}\right)$ | LC | Condition for Center |
| :--- | :--- | :--- | :--- |
| 7 | $\left(S_{7}^{8}, W_{7}^{8}\right)$ | 9 | $\mathrm{~S}_{7 \mathrm{c}}^{8}$ |
| 8 | $\left(S_{8}^{9}, W_{8}^{9}\right)$ | 10 | $\mathrm{~S}_{8 \mathrm{c}}^{9}$ |
| 9 | $\left(S_{9}^{8}, W_{9}^{8}\right)$ | 9 | $\mathrm{~S}_{9 \mathrm{c}}^{8}$ |
| $10-12$ | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{~S}_{\mathrm{kc}}^{9}$ |
| 13 | $\left(S_{13}^{10}, W_{13}^{10}\right)$ | 11 | $\mathrm{~S}_{13 \mathrm{c}}^{10}$ |
| $14-19$ | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| $20-25$ | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{~S}_{\mathrm{kc}}^{9}$ |
| 26 | $\left(S_{26}^{10}, W_{26}^{10}\right)$ | 11 | $\mathrm{~S}_{26 \mathrm{c}}^{10}$ |
| $27-32$ | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{~S}_{\mathrm{kc}}^{8}$ |
| $33-38$ | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{~S}_{\mathrm{kc}}^{9}$ |
| 39 | $\left(S_{39}^{10}, W_{39}^{10}\right)$ | 11 | $\mathrm{~S}_{39 \mathrm{c}}^{10}$ |

This clearly indicates that either $a_{211}=0$ or $a_{123}=0$, together with $b_{018}$, leads to the equilibrium $\mathrm{E}_{0}$ being a center up to $\varepsilon^{8}$ order. If taking $a_{211}=0$, then we again obtain 10 small limit cycles from $\varepsilon^{9}$ order by solving $W_{9}^{9}=0$ to obtain the solution $S_{9}^{9}$ and

$$
V_{109}=G_{4} a_{123}^{3}, \quad V_{119}=G_{5} a_{123}^{3}, \quad V_{129}=G_{6} a_{123}^{3}, \quad \operatorname{det}_{9}^{9}=F_{2} .
$$

Thus, for the equilibrium $\mathrm{E}_{0}$ being a center up to $\varepsilon^{9}$ order, $a_{123}$ must be taken zero (with $b_{019}$ ), yielding the same result as that generated from Case (A) at order 9 (and so the result at order 8 also becomes same). In other words, choosing the non-generic solution $a_{211}=0$ at order 8 makes the higher-order solutions $(k \geq 9)$ follow the generic pattern of Case (A).

Now we consider the choice $a_{123}=0$ at $\varepsilon^{8}$ order. It can be shown that under this condition only 9 limit cycles exist for $\varepsilon^{9}$ order. Then for the $\varepsilon^{10}$ order, we solve $W_{10}^{9}=0$ to obtain the solution $S_{10}^{9}$ and then get

$$
\begin{align*}
& V_{1010}=\frac{9}{8} G_{4} a_{211}^{2}\left(a_{124}^{2}-\frac{5}{16} a_{211}^{4} a_{124}+\frac{429}{40960} a_{211}^{8}\right), \\
& V_{1110}=\frac{9}{8} G_{5} a_{211}^{2}\left(a_{124}^{2}-\frac{5}{16} a_{211}^{4} a_{124}+\frac{429}{40960} a_{211}^{8}\right),  \tag{54}\\
& V_{1210}=\frac{9}{8} G_{6} a_{211}^{2}\left(a_{124}^{2}-\frac{5}{16} a_{211}^{4} a_{124}+\frac{429}{40960} a_{211}^{8}\right), \quad \operatorname{det}_{10}^{9}=F_{2},
\end{align*}
$$

which gives two solutions leading to a center at $\mathrm{E}_{0}$, one of them is $a_{211}=0$, which yields the same solution as that obtained in Case (A) at order 10. Thus, choosing the non-generic solution $a_{211}=0$ at this order leads to the generic pattern of Case (A) starting from $\varepsilon^{11}$ order (i.e., for $k \geq 11$ ). The second solution, given by

$$
\begin{equation*}
a_{124}=\frac{1}{64}\left(10 \pm \frac{1}{10} \sqrt{5710}\right) a_{211}^{4}, \tag{55}
\end{equation*}
$$

is a generic solution for Case (B), different from the generic pattern of Case (A). Then, following a similar computation procedure as that used in Case (A), we obtain the generic solutions up to $\varepsilon^{39}$ order. The results are given in Table 2, showing a $9^{6}-10^{6}-11^{1}$ generic pattern, starting from
order 14. The notations used in this table are the same as that used in Table 1. For each $k$, the key coefficient used to obtain the center condition is give below.

$$
\begin{aligned}
& 9 \mathrm{LC}: \quad k=7 \quad a_{037}=-\frac{35}{768} a_{211}\left[64 a_{123}^{2}-a_{211}\left(15 a_{123} a_{211}^{2}\right.\right. \\
& \left.\left.-16 a_{124} a_{211}-48 a_{123} a_{212}\right)\right] \\
& k=9 \quad a_{039}=\cdots \\
& k=14 \quad a_{0314}=-\frac{35}{48} a_{212}^{3} a_{128} \\
& k=15 \quad a_{0315}=-\frac{35}{48} a_{212}^{2}\left(a_{129} a_{212}+3 a_{128} a_{213}\right) \\
& k=16 \quad a_{0316}=-\frac{35}{768} a_{212}\left[48 a_{213}^{2} a_{128}+a_{212}\left(16 a_{1210} a_{212}\right.\right. \\
& \left.\left.+48 a_{129} a_{213}+48 a_{128} a_{214}-15 a_{128} a_{212}^{2}\right)\right] \\
& k=17-19 \quad a_{03 k}=\cdots \\
& k=27-32 \quad a_{03 k}=\cdots \\
& 10 \text { LC: } k=8 \quad a_{123}=0 \\
& k=10 \quad a_{124}=\frac{100 \pm \sqrt{5710}}{640} a_{211}^{4} \\
& k=11 \quad a_{125}=\frac{100 \pm \sqrt{5710}}{10240} a_{211}^{3}\left[64 a_{212}+a\left(a^{2}-4\right) a_{211}^{2}\right] \\
& k=12 \quad a_{126}=\cdots \\
& k=20 \quad a_{128}=\frac{100 \pm \sqrt{5710}}{640} a_{212}^{4} \\
& k=21 \quad a_{129}=\frac{100 \pm \sqrt{5710}}{160} a_{212}^{3} a_{213} \\
& k=22-25 \quad a_{12(k-12)}=\cdots \\
& k=33 \quad a_{1215}=-\frac{100 \pm \sqrt{5710}}{10240} a_{213}\left[a_{213}^{2}\left(15 a_{213}^{2}-64 a_{216}\right)\right. \\
& \left.-64 a_{214}\left(a_{214}^{2}+3 a_{213} a_{215}\right)\right] \\
& k=34-38 \quad a_{12(k-18)}=\cdots \\
& 11 \mathrm{LC}: \quad k=13 m \\
& m=1,2,3 \quad a_{21 m}=0
\end{aligned}
$$

### 3.6. Non-generic solutions

Couple of non-generic solutions have been discussed in Case (B) (see Section 3.5), showing that setting $a_{211}=0$ at order 8 or 10 (see Eqns. (53) and (54)) leads to the $9^{4}-10^{2}-11^{1}$ generic pattern of Case (A) for orders greater than 10 or 11. These two examples give a route from Case (B) to Case (A). In this section, we present several more non-generic solutions to show other possibilities that they eventually return to either the $9^{4}-10^{2}-11^{1}$ generic pattern of Case (A) or $9^{6}-10^{6}-11^{1}$ generic pattern of Case (B). Other cases can be similarly discussed. Since the discussions for different cases are similar, we will not give the details but list the cases below and summarize the results in Table 3.
(A1) In Case (A), at order 13: $a_{212}=0$, leading to Case (A).
(A2) In Case (A), at order 12: $a_{124}=0$, leading to Case (B).
(B1) In Case (B), at order 11: $a_{211}=0$, leading to Case (B).

Table 3
Non-generic solutions.

| Case | $k$ | $\left(S_{k}^{m}, W_{k}^{m}\right)$ | LC | Condition for Center |
| :---: | :---: | :---: | :---: | :---: |
| (A1) | 13, 15 | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{S}_{\mathrm{kc}}^{9}$ |
|  | 14, 16, 17 | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{S}_{\mathrm{kc}}^{8}$ |
|  | 18 | $\left(S_{18}^{9}, W_{18}^{9}\right)$ | 10 | $\mathrm{S}_{18 \mathrm{c}}^{9} \Longrightarrow$ generic Case (A) |
| (B1) | 11,14, 16 | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{S}_{\mathrm{kc}}^{9}$ |
|  | 12, 13, 15, 17 | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{S}_{\mathrm{kc}}^{8}$ |
|  | 18 | $\left(S_{18}^{9}, W_{18}^{9}\right)$ | 10 | $\mathrm{S}_{18 \mathrm{c}}^{9} \Longrightarrow$ generic Case (B) |
| (A2) | 12, 14, 16 | $\left(S_{k}^{9}, W_{k}^{9}\right)$ | 10 | $\mathrm{S}_{\mathrm{kc}}^{9}$ |
|  | 13, 15, 17 | $\left(S_{k}^{8}, W_{k}^{8}\right)$ | 9 | $\mathrm{S}_{\mathrm{kc}}^{8}$ |
|  | 18 | $\left(S_{18}^{9}, W_{18}^{9}\right)$ | 10 | $\mathrm{S}_{18 \mathrm{c}}^{9} \Longrightarrow$ generic Case (B) |

For each $k$, the key coefficient used to obtain the center condition is given below.

$$
\begin{aligned}
& \text { Case (A1) } k=13 \quad a_{212}=0 \\
& k=14 \quad a_{0314}=-\frac{35 a_{125}}{48}\left[4 a_{125} a_{214}+a_{213}\left(8 a_{126}+a_{213}^{2}\right)\right] \\
& k=16 \quad a_{0316}=-\frac{35}{48}\left[a_{127} a_{213}\left(8 a_{126}+a_{213}^{2}\right)\right. \\
& \left.+a_{126} a_{214}\left(4 a_{126}+3 a_{213}^{2}\right)\right] \\
& k=17 \quad a_{0317}=-\frac{35}{48} a_{128} a_{213}\left(a_{213}^{2}+8 a_{126}\right) \\
& -\frac{35}{48} a_{127}\left(4 a_{213} a_{127}+8 a_{214} a_{126}+3 a_{214} a_{213}^{2}\right) \\
& -\frac{35}{48} a_{126}\left(3 a_{214}^{2} a_{213}+4 a_{126} a_{215}+3 a_{215} a_{213}^{2}\right) \\
& k=15 \quad a_{125}=0 \\
& k=18 \quad a_{126}=-\frac{9}{8} a_{213}^{2} \\
& \text { Case (B1) } k=11 \quad a_{211}=0 \\
& k=12 \\
& k=13,15,17 \\
& a_{0312}=-\frac{35}{48} a_{212}\left(a_{126} a_{212}^{2}+3 a_{125} a_{212} a_{213}+4 a_{125}^{2}\right) \\
& a_{03 k}=\cdots \\
& k=14,16,18 \\
& a_{12(k / 2-2)}=0 \\
& \text { Case (A2) } k=13 \\
& \left.\begin{array}{c}
a_{0313}=-\frac{35}{48} a_{212}\left[a_{127} a_{212}^{2}+a_{126}\left(3 a_{213} a_{212}+8 a_{125}\right)\right] \\
+\frac{35}{768} a_{125}\left(15 a_{212}^{4}-48 a_{212}^{2} a_{214}\right. \\
-
\end{array}\right) \\
& k=15,17 \quad a_{03 k}=\cdots \\
& k=12,14,16,18 \quad a_{12(k / 2-2)}=0
\end{aligned}
$$

Therefore, there are four possible routes for the non-generic solutions: from Case (A) to Case (A) or Case (B); and from Case (B) to Case (A) or Case (B).

### 3.7. Summary of this section

Summarizing the results obtained in sections 3.4, 3.5 and 3.6, we have the following theorem.
Theorem 8. For system (11), based on the higher-order focus values, there exist two generic patterns: One is $9^{4}-10^{2}-11^{1}$ pattern starting from order 8 with four consecutive 9 limit cycles, followed by two consecutive 10 limit cycles, and then one 11 limit cycles up to $\varepsilon^{35}$ order; and the other is $9^{6}-10^{6}-11^{1}$ pattern, starting from order 14 with six consecutive 9 limit cycles, followed by six consecutive 10 limit cycles, and then one 11 limit cycles up to $\varepsilon^{39}$ order. Other non-generic solutions deviate from the current pattern for certain orders and eventually return to either the $9^{4}-10^{2}-11^{1}$ pattern or the $9^{6}-10^{6}-11^{1}$ pattern.

Finally, we propose a conjecture on the number of limit cycles around $\mathrm{E}_{0}$ for system (11).
Conjecture. For the perturbed system (11), the maximal number of small limit cycles which can bifurcate from the equilibrium $\mathrm{E}_{0}$ is 11 .

## 4. Conclusion

In this paper, we have applied high-order focus value computation to prove that system (11) can have 11 limit cycles around the equilibrium of (11), obtained by perturbing at least $\varepsilon^{7}$-order focus values. Moreover, no more than 11 limit cycles can be found up to $\varepsilon^{39}$-order analysis. It is believed that system (11) can have maximal 11 small limit cycles around the equilibrium.

## Acknowledgments

This work was supported by the National Natural Science Foundation of China (NSFC No. 11501370), and the Natural Sciences and Engineering Research Council of Canada (NSERC No. R2686A02).

## Appendix A

The coefficients $C_{i}$ 's in (31), (35) and (38) are given below.

$$
\begin{array}{ll}
C_{1}=-\frac{3}{16}\left(3 a^{4}+4 a^{2}+44\right) & C_{2}=-\frac{a}{48}\left(3 a^{4}+12 a^{2}+116\right) \\
C_{3}=-\frac{1}{8}\left(a^{4}+2 a^{2}+5\right) & C_{4}=-\frac{1}{16}\left(9 a^{4}-20 a^{2}+172\right) \\
C_{5}=-\frac{3 a}{8}\left(3 a^{2}-8\right) & C_{6}=-\frac{a}{11}\left(a^{2}-120\right) \\
C_{7}=-\frac{1}{8}\left(3 a^{2}-80\right) & C_{8}=\frac{3}{128} a\left(7 a^{4}+68 a^{2}-900\right) \\
C_{9}=\frac{1}{64}\left(3 a^{6}+40 a^{4}-860 a^{2}-1600\right) & C_{10}=-\frac{a}{576}\left(303 a^{4}+1596 a^{2}+8096\right) \\
C_{11}=-\frac{a}{64}\left(55 a^{2}-256\right) & C_{12}=-\frac{1}{384}\left(569 a^{4}-1660 a^{2}+1420\right) \\
C_{13}=\frac{1}{192}\left(21 a^{6}+80 a^{4}-2996 a^{2}-9040\right) & C_{14}=\frac{3}{64}\left(7 a^{4}+8 a^{2}+160\right) \\
C_{15}=\frac{a}{128}\left(49 a^{4}+220 a^{2}-380\right) & C_{16}=\frac{a}{32}\left(9 a^{4}+36 a^{2}-172\right) \\
C_{17}=\frac{3}{64}\left(15 a^{4}+28 a^{2}-916\right) & C_{18}=-\frac{1}{16}\left(3 a^{4}+4 a^{2}+340\right)
\end{array}
$$

$$
\begin{array}{ll}
C_{19}=-\frac{3 a}{8}\left(a^{2}-16\right) & C_{20}=\frac{a}{96}\left(45 a^{4}+266 a^{2}-644\right) \\
C_{21}=\frac{a}{32}\left(41 a^{2}+72\right) & C_{22}=\frac{1}{192}\left(303 a^{4}+1728 a^{2}-3620\right) \\
C_{23}=-\frac{9}{512} a\left(a^{4}+12 a^{2}+200\right) & C_{24}=-\frac{9}{256}\left(a^{4}+160\right) \\
C_{25}=-\frac{1}{512}\left(70 a^{6}-471 a^{4}+128 a^{2}-300\right) & C_{26}=-\frac{9}{512} a\left(a^{4}-108 a^{2}+696\right) \\
C_{27}=\frac{3}{256}\left(21 a^{4}+58 a^{2}-1880\right) & C_{28}=a_{125} a_{212}+a_{124} a_{213} \\
C_{29}=a_{215} a_{212}+a_{214} a_{213} & C_{30}=3 a_{213} a_{212}^{2}
\end{array}
$$

## Appendix B

The coefficients $s_{i}$ 's involved in $H_{2}$ (see Eq. (36)) and $t_{i}$ 's involved in $H_{3}$ (see Eq. (39)) are given as follows.

$$
\left.\begin{array}{rl}
s_{1}= & \frac{1}{3072} a^{6}\left(3 a^{2}+16\right)\left(10+4 y+4 x^{2}+3 x^{4}\right)-\frac{1}{192}\left(2850+1824 a^{2}-85 a^{4}\right) \\
& +\frac{a}{18}\left(6 a^{2}-319\right) x-\frac{1}{288}\left(5902+1568 a^{2}+45 a^{4}\right) y-\frac{a}{9}\left(53+10 x^{2}\right) x y+\frac{13}{6} y^{2} \\
& -\frac{1}{72} a^{2}\left(3 a^{2}+4\right) y\left(y-x^{2}\right)-\frac{1}{96}\left(1074+200 a^{2}+45 a^{4}\right) x^{2}-\frac{29}{18} x^{2} y-\frac{2}{9} y^{3} \\
& +\frac{a}{36}\left(12-4 a^{2}+3 a^{4}\right) x^{3}-\frac{1}{1152}\left(6746+3140 a^{2}+91 a^{4}+3 a^{6}\right) x^{4}+\frac{2}{3} x^{2} y^{2} \\
s_{2}= & \frac{1}{128} a^{2}\left(24-10 a^{2}+5 a^{4}\right)-\frac{1}{64}\left(1120-24 a^{2}+2 a^{4}-a^{6}\right)\left(y+x^{2}\right)-\frac{1}{2} x^{2} y \\
& +\frac{a}{16}\left(4+5 a^{2}\right) x-\frac{25}{4} x^{2}-\frac{a}{256}\left(a^{2}-4\right) x\left[32\left(y-x^{2}\right)-3 a^{3} x^{3}\right]-\frac{1}{8}\left(73-2 a^{2}\right) x^{4} \\
s_{3}= & \frac{1}{384} a\left(-4716+376 a^{2}+25 a^{4}+15 a^{6}\right)-\frac{1}{96}\left(2260-164 a^{2}-15 a^{4}\right) x \\
& +\frac{1}{192} a^{5}\left(5+3 a^{2}\right)\left(y+x^{2}\right)-\frac{1}{12} a\left(a^{2}-4\right) y\left(y-x^{2}\right)-\frac{2}{3} x y^{2}+2 x^{3} y \\
& -\frac{1}{48}\left[a\left(1151+10 a^{2}\right) y-a\left(1511+40 a^{2}\right) x^{2}+\left(100-4 a^{2}-3 a^{4}\right) x y\right. \\
& \left.+\left(140-76 a^{2}+11 a^{4}\right) x^{3}\right]-\frac{1}{768} a\left(10216+100 a^{2}-2 a^{4}-9 a^{6}\right) x^{4} \\
s_{4}= & \frac{1}{9216} a\left(96656+17952 a^{2}+3640 a^{4}+24 a^{6}+9 a^{8}\right) \\
& -\frac{5}{4608}\left(2256+7272 a^{2}-56 a^{4}+6 a^{6}+9 a^{8}\right) x-\frac{5}{72} a\left(224+8 a^{2}+3 a^{4}\right) y \\
& -\frac{5}{144} a\left(642-8 a^{2}-3 a^{4}\right) x^{2}-\frac{5}{1152}\left(368-1628 a^{2}-92 a^{4}+3 a^{6}\right) x y \\
& -\frac{5}{2} a y^{2}+\frac{5}{1152}\left(5272+86 a^{4}+2964 a^{2}-3 a^{6}\right) x^{3}-\frac{275}{36} a x^{2} y \\
& +\frac{5}{288}\left(372+4 a^{2}+3 a^{4}\right) x y^{2}-\frac{5}{144} a\left(140+12 a^{2}+3 a^{4}\right) x^{4} \\
& -\frac{5}{288}\left(268+3 a^{4}+4 a^{2}\right) x^{3} y-\frac{5}{18} x y^{3}-\frac{25}{18} a x^{4} y+\frac{5}{6} x^{3} y^{2} \\
s_{5}= & \frac{1}{256} a\left(-16 a^{2}-8 a^{4}+a^{6}+1120\right)+\frac{5}{256}\left(560+9 a^{4}-2 a^{6}-4 a^{2}\right) x \\
& -\frac{5}{64} a\left(20-11 a^{2}\right) x^{2}-\frac{5}{128} a^{2}\left(3 a^{2}+4\right) x\left(y+x^{2}\right)+\frac{175}{8} x y+\frac{265}{16} x^{3} \\
& +\frac{5}{32} a\left(a^{2}-4\right) x^{2}\left(y-x^{2}\right)-\frac{5}{8} x^{3} y \\
s_{6}= & \frac{1}{768} a^{2}\left(540 a^{2}+3904-8 a^{4}+3 a^{6}\right)-\frac{5}{1536} a\left(484-100 a^{2}-23 a^{4}+12 a^{6}\right) x \\
& -\frac{5}{12} a^{2}\left(a^{2}-1\right) y-\frac{35}{384}\left(244-4 a^{2}-7 a^{4}\right) x^{2}+\frac{5}{768} a\left(5132+84 a^{2}-13 a^{4}\right) x y \\
& -\frac{5}{768} a\left(-3756-76 a^{2}+13 a^{4}\right) x^{3}+\frac{5}{192}\left(380+3 a^{4}+4 a^{2}\right) x^{2} y \\
& (30
\end{array}\right)
$$

$$
\left.\begin{array}{rl} 
& +\frac{5}{48} a\left(a^{3}-4\right) x y\left(y-x^{2}\right)-\frac{5}{192}\left(11 a^{4}+20 a^{2}+12\right) x^{4}-\frac{5}{6} x^{2} y^{2}+\frac{5}{2} x^{4} y . \\
t_{1}= & \frac{1}{384} a\left(-4716+376 a^{2}+25 a^{4}+15 a^{6}\right)-\frac{1}{96}\left(2260-164 a^{2}-15 a^{4}\right) x \\
& +\frac{1}{192} a^{5}\left(5+3 a^{2}\right)\left(y+x^{2}\right)-\frac{1}{48} a\left(1151+10 a^{2}\right) y-\frac{2}{3} x y^{2}+2 x^{3} y \\
& -\frac{1}{48} a\left(1511+40 a^{2}\right) x^{2}+\frac{1}{48}\left(100-4 a^{2}-3 a^{4}\right) x y-\frac{1}{12} a\left(a^{2}-4\right) y\left(y-x^{2}\right) \\
& +\frac{1}{48}\left(140-76 a^{2}+11 a^{4}\right) x^{3}+\frac{1}{768} a\left(-100 a^{2}-10216+9 a^{6}+2 a^{4}\right) x^{4} \\
t_{2}= & \frac{1}{64} a^{2}\left(24-10 a^{2}+5 a^{4}\right)-\frac{1}{32}\left(1120-24 a^{2}+2 a^{4}-a^{6}\right)\left(y+x^{2}\right) \\
& +\frac{1}{8} a\left(5 a^{2}+4\right) x-\frac{25}{2} x^{2}-\frac{1}{4} a\left(a^{2}-4\right) x\left(y-x^{2}\right)-x^{2} y \\
& -\frac{1}{128}\left(2336-64 a^{2}+12 a^{4}-3 a^{6}\right) x^{4} \\
t_{3}= & -\frac{1}{2048} a^{5}\left(20+a^{4}\right)\left(5+2 y+2 x^{2}+2 x^{4}\right)-\frac{1}{128} a\left(550-343 a^{2}\right) \\
& -\frac{1}{256}\left(6800-24 a^{2}-10 a^{4}+5 a^{6}\right) x-\frac{3}{64} a\left(25 a^{2}+34\right) y-\frac{1}{32} a\left(45 a^{2}+61\right) x^{2} \\
& -\frac{1}{1024} a^{4}\left(2-a^{2}\right) x\left(8 y-8 x^{2}+3 a x^{3}\right)-\frac{3}{16}\left(50-a^{2}\right) x y+\frac{3}{16}\left(80-a^{2}\right) x^{3} \\
& +\frac{1}{32} a\left(a^{2}-4\right) x^{2} y-\frac{1}{64} a\left(197-2 a^{2}\right) x^{4}+\frac{1}{4} x^{3} y \\
t_{4}= & \frac{1}{768} a^{2}\left(3904+540 a^{2}-8 a^{4}+3 a^{6}\right)-\frac{5}{1536} a\left(484-100 a^{2}-23 a^{4}+12 a^{6}\right) x \\
& -\frac{5}{12} a^{2}\left(a^{2}-1\right) y-\frac{35}{384}\left(244-7 a^{4}-4 a^{2}\right) x^{2} \\
& -\frac{5}{768} a\left(-5132-84 a^{2}+13 a^{4}\right) x y-\frac{5}{768} a\left(-3756-76 a^{2}+13 a^{4}\right) x^{3} \\
& +\frac{5}{192}\left(380+4 a^{2}+3 a^{4}\right) x^{2} y+\frac{5}{48} a\left(a^{2}-4\right) x y\left(y-x^{2}\right) \\
& -\frac{5}{192}\left(12+20 a^{2}+11 a^{4}\right) x^{4}-\frac{5}{6} x^{2} y^{2}+\frac{5}{2} x^{4} y \\
t_{5}= & \frac{1}{128} a\left(1120-16 a^{2}-8 a^{4}+a^{6}\right)+\frac{5}{128}\left(560-4 a^{2}+9 a^{4}-2 a^{6}\right) x \\
& +\frac{5}{32} a\left(11 a^{2}-20\right) x^{2}-\frac{5}{64} a^{2}\left(4+3 a^{2}\right) x\left(y+x^{2}\right)+\frac{175}{4} x y \\
& +\frac{265}{8} x^{3}+\frac{5}{16} a\left(a^{2}-4\right) x^{2}\left(y-x^{2}\right)-\frac{5}{4} x^{3} y \\
t_{6}= & -\frac{1}{2048} a^{2}\left(2544-2336 a^{2}+8 a^{4}+3 a^{6}\right)-\frac{5}{2048} a\left(2632+276 a^{2}+3 a^{4}-8 a^{6}\right) x \\
& +\frac{5}{512}\left(1240-4 a^{2}-41 a^{4}+4 a^{6}\right) x^{2}+\frac{5}{1024} a\left(-1832+892 a^{2}+a^{4}\right) x y \\
& +\frac{5}{1024} a\left(-1368+696 a^{2}+a^{4}\right) x^{3}-\frac{5}{256} a^{2}\left(3 a^{2}+4\right) x^{2}\left(y-x^{2}\right) \\
& +\frac{375}{32} x^{2} y-\frac{65}{16} x^{4}-\frac{5}{128} a\left(a^{2}-4\right) x^{3} y+\frac{5}{16} x^{4} y \\
& 2 \\
\end{array}\right)
$$

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[^0]:    4y The first draft of this paper has been posted on arXiv.org since August 25, 2017, No. 1708.07864v1.

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[^1]:    ${ }^{1}$ For interpretation of the colors in this table, the reader is referred to the web version of this article.

