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Isolated periodic wave solutions arising from Hopf and Poincaré bifurcations in a class of single species model

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Abstract

In this paper, we consider the bifurcations of local and global isolated periodic traveling waves in a single species population model described by a reaction-diffusion equation. Based on the singular point quantity algorithm of conjugate symmetric complex systems, we investigate Hopf bifurcation from all equilibrium points for the corresponding planar traveling wave system. We obtain all center conditions and construct one perturbed Hamiltonian system to study Poincaré bifurcation. Further, using the Chebyshev criterion, we develop a utilized approach to prove the existence of at most two limit cycles in a piecewise continuous parameter interval. Finally, the existence of double isolated periodic traveling waves for the model is established, and the results are illustrated by numerical simulation. It is shown that in a population model with density-dependent migrations and Allee effect, two large amplitude oscillations (isolated periodic traveling waves) can exist simultaneously.

Keywords: Single species population model; Multiple isolated periodic wave; Hopf bifurcation; Poincaré bifurcation; Limit cycle

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1. Introduction

In recent years, increasing attention has been paid to practical problems described by reaction-diffusion equations arising from engineering, physical and biochemical fields. To understand the nonlinear phenomenon arising from population invasion, many researchers have utilized reaction-diffusion models to study the population dynamics, such as survival range, patterns of spread, oscillating motion and traveling front propagation [6,12,26].

One of the earliest single species models was introduced by Malthus in 1798, with the consideration of limited resources and environmental constraints in modifying the simple growth model $\frac{\mathrm{d}U(T)}{\mathrm{d}T} = \alpha U$, $\alpha > 0$, leading to the following model,

$$\frac{\mathrm{d}U(T)}{\mathrm{d}T} = f(U)U, \tag{1.1}$$

where the per capita growth rate f should be a decreasing function in U when the carrying capacity $K < U$, yielding the simplest form $f = \alpha(K - U)$, called logistic population model. However, in the reality the per capita growth rate is not always monotonically decreasing in U. In fact, when the Allee effect is introduced, the function f becomes

$$f = \alpha(U - U_0)(K - U), \tag{1.2}$$

which establishes a relation between the per capita growth rate and the population density. The Allee effect is strong when $0 < U_0 < K$, and weak when $-K < U_0 < 0$, while no Allee effect exists when $U_0 < -K$ [17]. Obviously, the single species model with the strong Allee effect has rich and complex dynamics compared with the logistic model. In particular, the population experiences extinction when the population density U falls below the Allee threshold U_0 [3], but the Allee effect guarantees the growth of the biological invasion when the per capita growth rate is larger than the threshold U_0, i.e. if $U \in (U_0, K)$, see [5,16,17,25] and reference therein.

Other mechanisms related to U with time are associated with the redistribution of the population in space due to the motion of its individuals. Usually, the population flux through the area boundary is described by Fick's law [2]. In general case, the motion of the individuals can be regarded as random, with the flux described by the equation $\mathrm{J} = -D\nabla U(\mathrm{R}, T)$, where R is the position in space, D is the diffusivity [2,13,17]. However, except random motion, another widely observed dynamics is advection or migration when the individuals exhibit a correlated motion toward a certain direction. Later, in order to study biological invasion problem, particularly the interplay between these two types of motions regarding species invasion, Petrovskii and Li [24] introduced the correlated motion into the equation by combining the random motion, resulting in

$$\mathrm{J} = A(U)\,U(\mathrm{R}, T) - D\,\nabla U(\mathrm{R}, T), \tag{1.3}$$

where $A(U)$ is the average speed of all the individuals at a given position, implying that the migration is density dependent. Considering the migrations and diffusion restricted to the one-dimensional case, the spatiotemporal dynamics of a given population is then described by the following equation:

$$\begin{aligned} \frac{\partial U(X,T)}{\partial T} &= -\operatorname{div}(\mathrm{J}) + f(U)U \\ &= -\frac{\partial[A(U)\,U]}{\partial U}\frac{\partial U}{\partial X} + D\frac{\partial^2 U}{\partial X^2} + \alpha U(U - U_0)(K - U). \end{aligned} \tag{1.4}$$

In [24], the $A(U)$ was assumed as $A_0 + A_1U$, in which A_0 is the speed of advection due to the impact of wind or water current, and A_1U is the speed of the migration due to biological mechanisms. However, in reality the speed of the migration is not necessarily proportional to the density. It is often observed that higher density can damp the directional movement in the population migration, such as crowding and trampling. This indicates that the per capita migration speed A_1 should be modified as a decreasing function in U when the population density U is greater than certain critical value, similar to deriving the logistic population model. Therefore, we add a nonlinear term A_2U^2 to the migration speed, yielding the following more realistic model, which will be studied in this paper:

$$\frac{\partial U(X,T)}{\partial T} + (A_0 + 2A_1U + 3A_2U^2)\frac{\partial U}{\partial X} = D\frac{\partial^2 U}{\partial X^2} + \alpha U(U - U_0)(K - U). \tag{1.5}$$

To simplify the analysis on the model (1.5), we introduce the transformation and time rescaling: $X = x\left(\frac{D}{\alpha K^2}\right)^{1/2}$, $U = Ku$, $T = \frac{t}{\alpha K^2}$, to obtain the following dimensionless model:

$$u_t + (a_0 + a_1u + a_2u^2)u_x = u_{xx} - \beta u + (1+\beta)u^2 - u^3, \tag{1.6}$$

where the new parameters are defined as

$$\beta = U_0K^{-1},\ a_0 = A_0K^{-1}(\alpha D)^{-1/2},\ a_1 = 2A_1(\alpha D)^{-1/2},\ a_2 = 3A_2K(\alpha D)^{-1/2}.$$

When $A_2 = 0$, model (1.5) has been studied in [24] using an appropriate ansatz, namely a reasonable substitution of variables to obtain an exact solution, and then to investigate the properties of the solution with the method of parameter variation, revealing the impacts from environmental and biological mechanisms such as wind or water current, population density, and the Allee effect. Thus, the interplay between the diffusion and different factors is shown thoroughly, and the direction of described propagation of traveling population fronts, e.g. either species invasion or species retreat is determined. Numerical simulation was given in [1] to verify the theoretical result for the case $A_2 = 0$.

The aim of this paper is to investigate traveling wave solutions in the model (1.6). Thus, assume the solution is given in the form of

$$u(x,t) = v(\xi), \quad \xi = x - ct, \tag{1.7}$$

where c is the propagation speed of a wave. Then substituting (1.7) into (1.6) yields

$$v''(\xi) = (a_0 - c + a_1v(\xi) + a_2v^2(\xi))v'(\xi) + \beta v(\xi) - (1+\beta)v^2(\xi) + v^3(\xi), \tag{1.8}$$

which is a second order ordinary differential equation. Hence, by applying bifurcation theory of planar dynamical systems, we can examine periodic oscillations on the density of invasion population to reveal certain patterns of the species invasion propagation. Further, letting $y = v'(\xi)$ in (1.8) yields the following planar dynamic system,

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}\xi} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}\xi} &= (a_0 - c + a_1v + a_2v^2)y + \beta v - (1+\beta)v^2 + v^3. \end{aligned} \tag{1.9}$$

If system (1.9) is integrable, then some exact traveling wave solutions of the original equation (1.5), such as periodic wave solution, solitary wave solution and monotone kink solitary wave solution can be easily obtained [18,19].

If system (1.9) is not integrable, then in general studying the traveling wave solutions of the model is not easy. One may apply Hopf bifurcation and Poincaré bifurcation theory to determine the existence of isolated periodic wave solutions for the original system (1.5). In particular, through Hopf bifurcation, small-amplitude periodic traveling waves of reaction-diffusion equation have been extensively studied, see [14,15,22,28]. In [30], the authors considered the case $A_2 = 0$, (that is, $a_2 = 0$ in system (1.9)), and applied the method of computing singular point quantify to study Hopf bifurcation, and proved the existence of one stable limit cycle. This implies that an isolated periodic traveling wave solution in the original model (1.5) exists, revealing a particular pattern of population fronts in real biological world.

As for the bifurcation of isolated periodic traveling wave solutions via Poincaré bifurcation, some results can be found in [29,31,32]. Especially, Sun *et al.* [29] studied model (1.5) for the case $A_2 = 0$, by examining the monotonicity of the ratio of the related Abelian integrals, and applying global bifurcation theory to prove that there exists maximal one periodic solution which can be reached in a large feasible parameter regime, implying the existence of one global isolated periodic wave solution.

In this paper, we will investigate isolated periodic wave solutions for model (1.5). Firstly, applying the same algorithm as used in [30], we determine center conditions and Hopf cyclicity of the auxiliary planar system (1.9) and then find the local isolated periodic traveling wave solutions of (1.5). Then based on one center condition, we construct a near-Hamiltonian system of (1.9), and choose the center of the Hamiltonian system to investigate the zeros of corresponding Abelian integral inside the saddle loop, namely perturbing an elliptic Hamiltonian of degree four. It is worth mentioning that for this case the detailed discussion has been given in [7], and shown that the maximum number of zeros is two by proving that a planar convex curve has a non-zero curvature everywhere. However, in this paper we use the method of Chebyshev criterion [9,23], with pure symbolic computation to determine the interval of the parameter β for the existence of at most two zeros, leading to the existence of double global isolated periodic traveling waves for model (1.5). Usually, in application of Chebyshev criterion, one can obtain a continuous parameter interval for the existence of bifurcating limit cycles. However, for the model studied in this paper, such a continuous interval is difficult to obtain. To overcome this difficulty, we develop a utilized approach to obtain a piecewise continuous interval.

The paper is organized as follows. In Section 2, we first compute the singular point quantities for three equilibria, and then determine center conditions and the cyclicity of Hopf bifurcation for the corresponding planar dynamical system. In Section 3, we introduce some preliminary results which are needed to determine the number of zeros of Abelian integrals, and give the main results associated with the equilibrium point $(1, 0)$ of system (1.9). In Section 4, we prove our main theorem for certain parameter regime. In Section 5, the existence of two global limit cycles via Poincaré bifurcation is proved, and a concrete example with simulation is given to illustrate the existence of two isolated traveling periodic waves, which implies the co-existence of two large-amplitude oscillations (isolated periodic traveling waves). This is a new interesting phenomenon discovered in single species models.

2. Center conditions and Hopf bifurcation for system (1.9)

In this section, we first briefly introduce the method of computing singular point quantities, which is needed to investigate the integrability and Hopf bifurcation of real dynamical systems (more details can be found in [4,20,21]). Consider the following real polynomial system,

$$\begin{aligned} \frac{dx}{dt} &= -y + \sum_{k+j=2}^{\infty} A_{kj} x^k y^j = X(x,y), \\ \frac{dy}{dt} &= x + \sum_{k+j=2}^{\infty} B_{kj} x^k y^j = Y(x,y), \end{aligned} \tag{2.1}$$

where $x,\ y,\ t,\ A_{kj},\ B_{kj} \in \mathbb{R};\ k,\ j \in \mathbb{N}$. By introducing the transformation,

$$z = x + \mathrm{i}\, y, \quad w = x - \mathrm{i}\, y, \quad T = \mathrm{i}\, t, \quad \mathrm{i} = \sqrt{-1},$$

we obtain the conjugate complex system with symmetry as follows:

$$\begin{aligned} \frac{dz}{dT} &= z + \sum_{k+j=2}^{\infty} a_{kj} z^k w^j = Z(z,w), \\ \frac{dw}{dT} &= -w - \sum_{k+j=2}^{\infty} b_{kj} w^k z^j = -W(z,w), \end{aligned} \tag{2.2}$$

where $z,\ w,\ T,\ a_{kj},\ b_{kj} \in \mathbb{C};\ k,\ j \in \mathbb{N}$.

Lemma 2.1 *([4,21]). For system (2.2), with $c_{11} = 1,\ c_{20} = c_{02} = 0,\ c_{kk} = 0,\ k = 2, 3, \cdots$, the following formal series,*

$$F(z,w) = zw + \sum_{p+q=2}^{\infty} c_{pq} z^p w^q,$$

can be obtained, satisfying

$$\frac{dF}{dT} = \sum_{m=1}^{\infty} \mu_m (zw)^{m+1}.$$

When $p \neq q$, c_{pq} is determined from the following recursive formula:

$$c_{pq} = \frac{1}{q-p} \sum_{k+j=3}^{\infty} \big[(p-k+1) a_{k,j-1} - (q-j+1) b_{j,k-1}\big] c_{p-k+1,q-j+1},$$

and for any positive integer m, μ_m is determined from the following recursive formula:

$$\mu_m = \sum_{k+j=3}^{\infty} \big[(m-k+2) a_{k,j-1} - (m-j+2) b_{j,k-1}\big] c_{m-k+2,m-j+2}.$$

When $p = q > 0$ *or* $p < 0$ *or* $q < 0$, $c_{pq} = 0$.

Remark 2.1. The coefficient μ_m, $m = 1, 2, \cdots$, given in Lemma 2.1 is called the mth singular point quantity at the origin of the system. Note that for the mth focal value v_{2m+1} and μ_m, the following relation holds [20]:

$$v_3(2\pi) = \mathrm{i}\pi\mu_1,$$

and if $v_{2k+1} = \mu_k = 0$ for $k = 1, 2, \cdots, m-1$, then

$$v_{2m+1} = \mathrm{i}\pi\mu_m, \quad m = 2, 3, \cdots.$$

Thus, the stability and integrability for the origin of system (2.1) can be directly determined by computing the singular point quantities of the origin of system (2.2).

Clearly, system (1.9) has three singular points located at $(v, y) = (0, 0)$, $(1, 0)$ and $(\beta, 0)$, denoted by O, O_1 and O_2 respectively. Let $DX(v^*, 0)$ represent the differential matrix at the singular point $(v^*, 0)$ and

$$\mathrm{Spec}(DX(v^*, 0)) = \frac{\gamma_{v^*} \pm \sqrt{\Delta}_{v^*}}{2}$$

denote the characteristic roots of $DX(v^*, 0)$, where $\gamma_{v^*} = a_0 - c + a_1 v^* + a_2 v^{*2}$ and

$$\Delta_{v^*} = \gamma_{v^*}^2 + 4(\beta - 2v^* - 2\beta v^* + 3v^{*2}).$$

By the qualitative theory of ordinary differential equations, under the condition: $\gamma_{v^*} = 0$ and $\Delta_{v^*} < 0$, i.e. $4(\beta - 2v^* - 2\beta v^* + 3v^{*2}) < 0$, it is easy to show that the singular point $(v^*, 0)$ is a center-focus. Next, we consider Hopf bifurcation from the three singular points $O(0, 0)$, $O_1(1, 0)$ and $O_2(\beta, 0)$.

Case (i): for $O(0, 0)$, namely $v^* = 0$. When $c = a_0$ and $\beta < 0$, it is a center-focus. We introduce the transformation: $z = y + \mathrm{i}\sqrt{-\beta}\, v$, $w = y - \mathrm{i}\sqrt{-\beta}\, v$, $T = \mathrm{i}\sqrt{-\beta}\,\xi$, $\mathrm{i} = \sqrt{-1}$ into system (1.9) to obtain the following system in the form of (2.2),

$$\frac{\mathrm{d}z}{\mathrm{d}T} = z + Z_2 + Z_3, \quad \frac{\mathrm{d}w}{\mathrm{d}T} = -w - W_2 - W_3, \tag{2.3}$$

in which

$$\begin{aligned}
Z_2 &= \tfrac{1}{4(-\beta)^{3/2}}(w - z)\big[z\big(a_1\sqrt{-\beta} + \mathrm{i}(1+\beta)\big) + w\big(a_1\sqrt{-\beta} - \mathrm{i}(1+\beta)\big)\big],\\
Z_3 &= \tfrac{1}{8\beta^2}(w - z)^2\big[z\big(1 + \mathrm{i}a_2\sqrt{-\beta}\big) - w\big(1 - \mathrm{i}a_2\sqrt{-\beta}\big)\big],\\
W_2 &= \bar{Z}_2, \quad W_3 = \bar{Z}_3,
\end{aligned}$$

where $\bar{Z}_k$ is the complex conjugate of Z_k, $k = 2, 3$.

Case (ii): for $O_1(1, 0)$, namely $v^* = 1$. When $c = a_0 + a_1 + a_2$ and $1 - \beta < 0$, it is a center-focus. We use the transformation: $z = y + \mathrm{i}\sqrt{\beta - 1}\,(v + 1)$, $w = y - \mathrm{i}\sqrt{\beta - 1}\,(v + 1)$, $T = \mathrm{i}\sqrt{\beta - 1}\,\xi$, $\mathrm{i} = \sqrt{-1}$ to make system (1.9) become

$$\frac{dz}{dT} = z + Z_2 + Z_3, \quad \frac{dw}{dT} = -w - W_2 - W_3, \tag{2.4}$$

where

$$\begin{aligned} Z_2 &= \frac{1}{4(\beta-1)^{3/2}}(w-z)\big[z\big((a_1+2a_2)\sqrt{\beta-1}+\mathrm{i}\,(\beta-2)\big) \\ &\qquad +w\big((a_1+2a_2)\sqrt{\beta-1}-\mathrm{i}\,(\beta-2)\big)\big], \\ Z_3 &= \frac{1}{8(\beta-1)^2}(w-z)^2\big[z\big(1+\mathrm{i}\,a_2\sqrt{\beta-1}\big)-w\big(1-\mathrm{i}\,a_2\sqrt{\beta-1}\big)\big], \\ W_2 &= \bar{Z}_2, \quad W_3 = \bar{Z}_3. \end{aligned}$$

Case (iii): for $O_2(\beta, 0)$, namely $v^* = \beta$. When $c = a_0 + a_1\beta + a_2\beta^2$ and $\beta^2 - \beta < 0$, it is a center-focus. Similarly, we apply the transformation: $z = y + \mathrm{i}\,\rho(v+\beta)$, $w = y - \mathrm{i}\,\rho(v+\beta)$, $T = \mathrm{i}\,\rho\xi$, $\mathrm{i} = \sqrt{-1}$ with $\rho = \sqrt{\beta-\beta^2}$, to transform system (1.9) to obtain

$$\frac{dz}{dT} = z + Z_2 + Z_3, \quad \frac{dw}{dT} = -w - W_2 - W_3, \tag{2.5}$$

where

$$\begin{aligned} Z_2 &= \frac{1}{4(\beta-\beta^2)^{3/2}}(w-z)(a_{11}z+b_{11}w), \\ Z_3 &= \frac{1}{8(\beta-\beta^2)^2}(w-z)^2\big[z\big(1+\mathrm{i}\,a_2\sqrt{\beta-\beta^2}\big)-w\big(1-\mathrm{i}\,a_2\sqrt{\beta-\beta^2}\big)\big], \\ a_{11} &= (a_1+a_2+a_2\,|2\beta-1|)\sqrt{\beta-\beta^2}-\mathrm{i}\,|2\beta-1|, \quad b_{11} = \bar{a}_{11}, \\ W_2 &= \bar{Z}_2, \quad W_3 = \bar{Z}_3. \end{aligned}$$

Obviously, it is impossible to have all the three singular points being center-focus simultaneously due to the restriction on the parameter β. Applying the recursive formulas in Lemma 2.1 with the help of Mathematica, we obtain the first 10 singular point quantities for system (2.3)) as follows:

$$\begin{aligned} \mu_1 &= \frac{\mathrm{i}(a_1(\beta+1)+a_2\beta)}{4(-\beta)^{5/2}}, \\ \mu_2 &= \frac{5\mathrm{i}a_1(\beta+1)}{24(-\beta)^{9/2}}, \\ \mu_3 &= \mu_4 = \cdots = \mu_{10} = 0, \end{aligned} \tag{2.6}$$

in which $\mu_{k-1} = 0$ has been used in computing μ_k, $k = 2, \cdots, 10$. Similarly, for system (2.4), we obtain that

$$\begin{aligned} \mu_1 &= \frac{\mathrm{i}(a_1(\beta-2)+a_2(\beta-3))}{4(\beta-1)^{5/2}}, \\ \mu_2 &= -\frac{5\mathrm{i}a_1(\beta-2)}{24(\beta-3)(\beta-1)^{7/2}}, \\ \mu_3 &= \mu_4 = \cdots = \mu_{10} = 0. \end{aligned} \tag{2.7}$$

For system (2.5), we get that

$$\begin{aligned} \mu_1 &= -\frac{\mathrm{i}(a_1(1-2\beta)+a_2(\beta-1)(3\beta-2))}{4(\beta-\beta^2)^{5/2}}, \\ \mu_2 &= -\frac{5\mathrm{i}a_1\beta(2\beta-1)}{24(3\beta-2)(\beta-\beta^2)^{9/2}}, \\ \mu_3 &= \mu_4 = \cdots = \mu_{10} = 0. \end{aligned} \tag{2.8}$$

For the above three cases, it is easy to find the conditions such that $\mu_1 = \mu_2 = 0$, yielding the center conditions. We have the following result.

Theorem 2.1. (i) *With $c = a_0$, the origin of system (2.3) or system (1.9) is a center if and only if $a_2 = a_1 = 0$, $\beta < 0$ or $a_2 = 0$, $\beta = -1$.*

(ii) *With $c = a_0 + a_1 + a_2$, the origin of system (2.4) or the point $O_1(1, 0)$ of system (1.9) is a center if and only if $a_2 = a_1 = 0$, $\beta > 1$ or $a_2 = 0$, $\beta = 2$.*

(iii) *With $c = a_0 + a_1\beta + a_2\beta^2$, the origin of system (2.5) or the point $O_2(\beta, 0)$ of system (1.9) is a center if and only if $a_2 = a_1 = 0$, $\beta(\beta - 1) < 0$ or $a_2 = 0$, $\beta = \frac{1}{2}$.*

Proof. The necessity directly follows the expressions of the first 10 singular point quantities given in (2.6), (2.7) and (2.8). To prove sufficiency, we consider two cases.

(I) when $a_1 = a_2 = 0$, $c = a_0$, for all the three cases. System (1.9) is reduced to

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = v(v-1)(v-\beta), \tag{2.9}$$

which is obviously an integrable system, with the first integral,

$$H_0(v, y) = \frac{y^2}{2} - \frac{1}{12}v^2(3v^2 - 4v - 4\beta v + 6\beta). \tag{2.10}$$

(II) when $a_2 = 0$, $a_1 \neq 0$, under the conditions respectively for the three cases: $c = a_0$, $\beta = -1$; $c = a_0 + a_1$, $\beta = 2$ and $c = a_0 + \frac{a_1}{2}$, $\beta = \frac{1}{2}$, system (1.9) respectively becomes

$$\begin{aligned} &\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = v(v^2 + a_1 y - 1) = Y_1, \\ &\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = (v-1)(v^2 - 2v + a_1 y) = Y_2, \\ &\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = (v - \tfrac{1}{2})(v^2 - v + a_1 y) = Y_3 \end{aligned} \tag{2.11}$$

which are integrable with the integrating factors:

$$M(v, y) = h_i^{\frac{2}{\kappa^2}-1}, \quad i = 1, 2, 3, \tag{2.12}$$

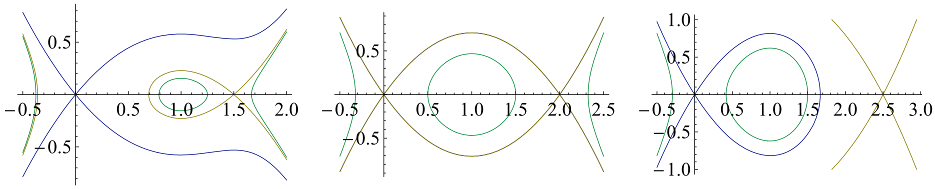
where $h_1 = v^2 + \kappa y - 1$, $h_2 = v^2 - 2v + \kappa y$, $h_3 = v^2 - v + \kappa y$ and $\kappa = \frac{1}{2}(a_1 \pm \sqrt{a_1^2 + 8})$. □

Further, according to Theorem 2.1, it is straightforward to obtain the conditions that $\mu_1 = 0$, $\mu_2 \neq 0$, yielding 2nd-order weak focus. We have the following theorem.

Theorem 2.2. *The three equilibria of system (1.9): $(0, 0)$, $(1, 0)$ and $(\beta, 0)$ can be respectively at most a 2nd-order weak focus, and there exist at most two limit cycles bifurcating from the three equilibria via Hopf bifurcation.*

3. The zeros of the Abelian integral around (1, 0) of system (1.9)

For practical problems, population density u in the model (1.5) is often nonnegative. Thus we only need to study the dynamics of system (1.9) in the right-half of the v-y plane. That is, we only need to disscuss Poincaré bifurcation from the period annulus around one center inside



(i) (ii) (iii)

Fig. 1. The phase portraits of system (1.9) for three cases with $c = a_0,\ a_1 = a_2 = 0$ for (i) $1 < \beta < 2$; (ii) $\beta = 2$; and (iii) $\beta > 2$.

either the two heteroclinic orbits or one homoclinic orbit. According to Theorem 2.1, we only need to consider the center $O_1(1,0)$ for $\beta > 1$ or $O_2(\beta,0)$ for $0 < \beta < 1$. In this paper, we consider the center $(1,0)$ for $\beta > 1$ and $c = a_0 + a_1 + a_2$, with concentration on the general case, $a_1 = a_2 = 0$, while leaving the case $\beta = 2$, $a_2 = 0$ (which has a very complex Abelian integral) for future study.

Now, under the conditions: $\beta > 1$, $a_1 = a_2 = 0$, $c = a_0$, we know from the first integral (2.10) that $O_1(1,0)$ is a center, $O(0,0)$ and $O_1(\beta,0)$ are two saddles, which yields three types of phase portraits for different values of β, as shown in Fig. 1.

In order to consider Poincaré bifurcation around the equilibrium point $(1,0)$ of system (1.9), we take some appropriate perturbations on the coefficients, by assuming $0 < |a_0 - c| \ll 1$ and $0 < |a_1|, |a_2| \ll 1$, and thus introduce the rescaling: $c = a_0 - \varepsilon b_0$, $a_1 = \varepsilon b_1$, $a_2 = \varepsilon b_2$ into system (1.9) to obtain the following perturbed Hamiltonian system,

$$\begin{aligned} \frac{dv}{d\xi} &= y, \\ \frac{dy}{d\xi} &= \beta v - (1+\beta)v^2 + v^3 + \varepsilon(b_0 + b_1 v + b_2 v^2)y. \end{aligned} \tag{3.1}$$

To apply the Chebyshev criterion in determining the number of zeros of the Abelian integral (see [9,23]), we need to perform a translation: $v \mapsto v + 1$ to bring the center $(1,0)$ to the origin, for which system (3.1) becomes

$$\begin{aligned} \frac{dv}{d\xi} &= y, \\ \frac{dy}{d\xi} &= (v+1)v(v+1-\beta) + \varepsilon f(v)y, \end{aligned} \tag{3.2}$$

where

$$f(v) = b_0 + b_1(v+1) + b_2(v+1)^2 = d_0 + d_1 v + d_2 v^2, \tag{3.3}$$

with $d_0 = b_0 + b_1 + b_2$, $d_1 = b_1 + 2b_2$, $d_2 = b_2$. Obviously, the unperturbed system $(3.2)_{\varepsilon=0}$ has the Hamiltonian function,

$$H(v, y) = \frac{y^2}{2} - \frac{1}{12}v^2(6 - 6\beta + 8v - 4\beta v + 3v^2). \tag{3.4}$$

It can be seen from the phase portraits shown in Fig. 1 that the cases (iii) and (i) are symmetric. In fact, under the transformation $v \mapsto \beta - (\beta-1)v$, $y \mapsto (\beta-1)^2 y$ and time rescaling $\xi \mapsto \frac{\xi}{(\beta-1)^2}$, system (3.1) with $\varepsilon = 0$ is changed to

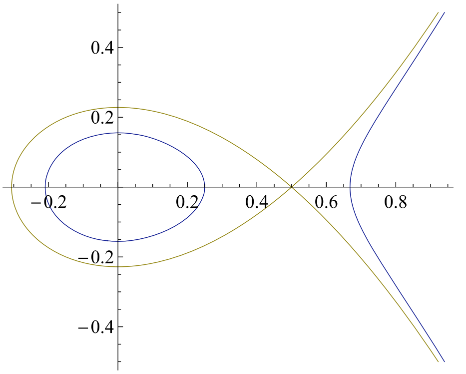


Fig. 2. The phase portrait of system $(3.2)_{\varepsilon=0}$ for the case (i) shown in Fig. 1 via the translation $v \mapsto v+1$.

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = v(v-1)\big(v - \tfrac{\beta}{\beta-1}\big).$$

If $\beta > 2$, then $1 < \frac{\beta}{\beta-1} < 2$, namely case (iii) becomes (i). So we only consider the case (i). The translated phase portrait of system $(3.2)_{\varepsilon=0}$ for case (i) is shown in Fig. 2.

The closed orbits depicted in Fig. 2 are defined by the following function:

$$\Gamma_h : \quad H(v, y) = h, \quad h \in (0, h_0), \tag{3.5}$$

where $h_0 = \frac{1}{12}(\beta-1)^3(\beta+1)$ with $1 < \beta < 2$. The period annulus Γ_h around the origin (the center) is inside the homoclinic loop, passing through the hyperbolic saddle $(\beta-1, 0)$, intersecting the v-axis at (v_0^-, v_0^+), where

$$v_0^- = -\tfrac{1}{3}(1+\beta+\sqrt{4+2\beta-2\beta^2}), \quad v_0^+ = \beta - 1. \tag{3.6}$$

Moreover, we rewrite the Hamiltonian function (3.4) of the unperturbed system $(3.2)_{\varepsilon=0}$ as

$$H(v, y) = A(v) + B(v)y^2, \tag{3.7}$$

where $A(v) = -\frac{1}{12}v^2(6 - 6\beta + 8v - 4\beta v + 3v^2)$ and $B(v) = \frac{1}{2}$. Thus, the Melnikov function (Abelian integral) corresponding to system (3.2) can be written as

$$\begin{aligned} I(h) = \oint_{\Gamma_h} f(v) y dv &= \oint_{\Gamma_h} (d_0 + d_1 v + d_2 v^2) y dv \\ &= d_0 \tilde{I}_0(h) + d_1 \tilde{I}_1(h) + d_2 \tilde{I}_2(h), \end{aligned} \tag{3.8}$$

where $\tilde{I}_i(h) = \oint_{\Gamma_h} v^i y dv,\ i = 0, 1, 2$. The number of zeros of $I(h)$ corresponds to the number of limit cycles bifurcating in system (3.2). Before giving a detailed analysis on the problem, in the following we present some relevant definitions and lemmas, more details can be found in [9,23].

Definition 3.1. Assume that $\{f_0(x), f_1(x), \cdots, f_{n-1}(x)\}$ is an ordered set of analytic functions on an open interval $\mathbb{L}$ in $\mathbb{R}$. This set is called an extended complete Chebyshev system *(ECT-system)* if, for all $i = 1, 2, \cdots, n$, any nontrivial linear combination,

$$k_0 f_0(x) + k_1 f_1(x) + \cdots + k_{i-1} f_{i-1}(x),$$

has at most $i-1$ isolated zeros on $\mathbb{L}$ counted multiplicities.

Lemma 3.1. [9] *$\{f_0(x), f_1(x), \cdots, f_{n-1}(x)\}$ is an* ECT*-system on $\mathbb{L}$ if and only if, for each $i = 1, 2, \cdots, n$, the Wronskian $W\big[f_0(x), f_1(x), \cdots, f_{i-1}(x)\big] \neq 0$ for all $x \in \mathbb{L}$.*

Thus, if we can prove that $\tilde{I}_0$, $\tilde{I}_1$ and $\tilde{I}_2$ in (3.8) form an ECT-system, we could conclude that at most two isolated zeros with respect to h exist in $\mathbb{L} = (0, h_0)$, where $h_0 = \frac{1}{12}(\beta-1)^3(\beta+1)$. To prove this conclusion, we need the following lemmas.

Lemma 3.2. [9] *Let Γ_h be an oval inside the level curve $\{A(v) + \frac{1}{2}y^2 = h\}$ and a function F such that F/A' is analytic at $v = 0$. Then, for any $k \in \mathbb{N}$,*

$$\oint_{\Gamma_h} F(v) y^{k-2} \mathrm{d}v = \oint_{\Gamma_h} G(v) y^k \mathrm{d}v,$$

where $G(v) = \frac{1}{k}\big(\frac{F}{A'}\big)'(v)$.

Lemma 3.3. [9] *Consider the Abelian integrals,*

$$I_i(h) = \oint_{\Gamma_h} f_i(v) y^{2s-1} \mathrm{d}v, \quad i = 0, 1, \cdots, n-1,$$

where, for each $h \in \mathbb{L}$, Γ_h is the oval surrounding the origin inside the level curve $\{A(v) + \frac{1}{2}y^2 = h\}$. Let σ be the involution associated to $A(v)$ and define

$$\ell_i(v) := \frac{f_i(v)}{A'(v)} - \frac{f_i(\sigma(v))}{A'(\sigma(v))}.$$

Then $\{I_0, I_1, I_2, \cdots, I_{n-1}\}$ is an ECT*-system on $\mathbb{L}$ if $s > n-2$ and $\{\ell_0, \ell_1, \cdots, \ell_{n-1}\}$ is a* CT*-system on $(0, v_0^+)$.*

The following lemma will be used to prove our main result.

Lemma 3.4. [8] *Set $\Omega = \mathbb{R}$ and let*

$$G_\beta(x) = g_n(\beta)x^n + g_{n-1}(\beta)x^{n-1} + \cdots + g_1(\beta)x + g_0(\beta)$$

be a family of real polynomials depending also polynomially on a real parameter β. Assume that there exists an open interval $L \subset \mathbb{R}$ such that: (i) *There exists some $\beta_0 \in L$ such that $G_{\beta_0}(x) > 0$ on Ω.* (ii) *For all $\beta \in L$, the discriminant of G_β with respect to x is not equal to zero.* (iii) *For all $\beta \in L$, $g_n(\beta) \neq 0$. Then, for all $\beta \in L$, $G_\beta(x) > 0$ on Ω.*

Our main result is given in the following theorem. Its lengthy proof will be given in the next two sections.

Theorem 3.1. *For the perturbed system (3.2), there exists an interval $L \subset (1, 2)$ such that for all $\beta \in L$, the number of zeros of the Abelian integral (3.8) is at most two and can be reached, accounting multiplicity.*

4. Proof of Theorem 3.1

In this section, we divide five steps to prove Theorem 3.1. Firstly, it is easy to get

$$\begin{aligned}\tilde{I}_i(h)=\oint_{\Gamma_h} v^i y\mathrm{d}v &= \tfrac{1}{h}\oint_{\Gamma_h}(A(v)+\tfrac{1}{2}y^2)v^i y\mathrm{d}v \\ &= \tfrac{1}{h}\oint_{\Gamma_h} A(v)v^i y\mathrm{d}v+\tfrac{1}{2h}\oint_{\Gamma_h} v^i y^3\mathrm{d}v, \quad i=0,\ 1,\ 2.\end{aligned} \tag{4.1}$$

Then we apply Lemma 3.2 to obtain

$$\oint_{\Gamma_h} A(v)v^i y\mathrm{d}v=\tfrac{1}{3}\oint_{\Gamma_h}\big(\tfrac{Av^i}{A'}\big)' y^3\mathrm{d}v.$$

Rewriting $\tilde{I}_0(h)=\frac{1}{h}I_0(h)$, $\tilde{I}_1(h)=\frac{1}{h}I_1(h)$, $\tilde{I}_2(h)=\frac{1}{h}I_2(h)$ with

$$I_0(h)=\oint_{\Gamma_h} f_0(v)y^3\mathrm{d}v, \quad I_1(h)=\oint_{\Gamma_h} f_1(v)y^3\mathrm{d}v, \quad I_2(h)=\oint_{\Gamma_h} f_2(v)y^3\mathrm{d}v,$$

we obtain $f_0(v)$, $f_1(v)$ and $f_2(v)$, expressed in the following forms:

$$f_i(v)=-\frac{v^{i-1}(2vAA''-2iAA'-5vA'^2)}{6A'^2}, \quad i=0,\ 1,\ 2. \tag{4.2}$$

Further, applying Lemma 3.3 yields

$$\ell_i(v)=\frac{f_i(v)}{A'(v)}-\frac{f_i(\sigma(v))}{A'(\sigma(v))}, \quad i=0,\ 1,\ 2,$$

where $\sigma(v)$ is the involution associated with $A(v)$. Actually $\sigma(v)$ and v together serve as the abscissae of the two intersection points of the period annulus Γ_h with the v-axis. Moreover, letting $\sigma(v)=\varphi$ yields $H(v,y)=H(\varphi,y)$ or $A(v)-A(\varphi)=0$, namely

$$A(v)-A(\varphi)=-\tfrac{1}{12}(v-\varphi)g(v,\varphi)=0, \tag{4.3}$$

where

$$g(v,\varphi)=3(v^2+\varphi^2)(v+\varphi)-4(\beta-2)(v^2+v\varphi+\varphi^2)-6(\beta-1)(v+\varphi). \tag{4.4}$$

Due to $v\varphi<0$, we have $v-\varphi\neq 0$, and thus $\varphi=\sigma(v)$ is an implicit function defined by $g(v,\varphi)=0$.

Next, a direct calculation yields the following Wronskians:

$$\begin{aligned} &W[\ell_1(v)]=\ell_1(v)=\Phi_1(v,\varphi,\beta),\\ &W[\ell_1(v),\ell_2(v)]=\Phi_2(v,\varphi,\beta),\\ &W[\ell_1(v),\ell_2(v),\ell_0(v)]=\Phi_3(v,\varphi,\beta),\end{aligned} \tag{4.5}$$

where Φ_1, Φ_2 and Φ_3 are rational functions in v,φ and β. Thus, by investigating the nonexistence of zeros for Φ_1, Φ_2 and Φ_3 with respect to $v\in(0,\beta-1)$ in certain intervals of β, we determine that $\{I_0,I_1,I_2\}$ is an ECT-system on $(0,\beta-1)$. In the following, we present the detailed proof in four steps.

Step 1. We first compute the resultants of Φ_1, Φ_2 and Φ_3 respectively with $g(v,\varphi)$ to obtain

$$\begin{aligned}\text{Resultant}[\Phi_1, g, \varphi] &= R_1(v,\beta),\\ \text{Resultant}[\Phi_2, g, \varphi] &= R_2(v,\beta),\\ \text{Resultant}[\Phi_3, g, \varphi] &= R_3(v,\beta),\end{aligned}$$

where R_1, R_2 and R_3 are rational functions in v and β. It is easy to verify that there exist some specific values of β such that $R_1R_2R_3 \neq 0$ for $v \in (0, \beta-1)$ under the condition $g(v,\varphi)=0$. We have the following result.

Lemma 4.1. *There exist some specific values of* $\beta = \beta_0 = \frac{11}{10}, \frac{3}{2}$, *such that* $\{I_1, I_2, I_0\}$ *is an* ECT-*system on* $(0, \beta-1)$.

However, we cannot claim the existence of certain interval of β for which $\{I_1, I_2, I_0\}$ is an ECT-system. Therefore, further analysis is needed.

Step 2. Substituting $v = \frac{t^2}{t^2+1}(\beta-1) \in (0, \beta-1)$ where $t \in (-\infty, +\infty)$, into R_1, R_2 and R_3, we obtain

$$\begin{aligned}R_1(v,\beta) &= \tfrac{(\beta-2)t^2(1+t^2)^7 h_{11}}{3888(\beta-1)^3(1+\beta t^2)^9(d_{11})^3(d_{12})^3} := r_1(t,\beta),\\ R_2(v,\beta) &= \tfrac{(\beta-2)t^6(1+t^2)^{13} h_{21}}{1889568(\beta-1)^4(1+\beta t^2)^{15}(d_{21})(d_{11})^6(d_{12})^6} := r_2(t,\beta),\\ R_3(v,\beta) &= \tfrac{(\beta-2)(\beta+1)(2\beta-1)(1+t^2)^{30} h_{31}}{4251528(\beta-1)^{16}t^{12}(1+\beta t^2)^{20}(d_{21})^6(d_{11})^{10}(d_{12})^{10}} := r_3(t,\beta),\end{aligned}$$

where h_{11}, h_{21} and h_{31} are polynomials in t and β, with degrees 44, 80 and 136 in t, respectively, and

$$\begin{aligned}d_{11} &= 1+\beta+4t^2+4\beta t^2+6\beta t^4,\\ d_{12} &= 1-2\beta+4t^2-10\beta t^2+4\beta^2t^2-2\beta t^4+\beta^2t^4,\\ d_{21} &= 6+4t^2+4\beta t^2+t^4+\beta t^4.\end{aligned}$$

It is not difficult to verify that $d_{11} > 0$, $d_{12} < 0$ and $d_{21} < 0$ for $\beta \in (1,2)$. Thus, we only need to find certain intervals of β such that the h_{i1} in $r_i(t,\beta)$ is always positive or negative with respect to $t \in \mathbb{R}$, $i = 1, 2, 3$, yielding an interval such that all R_i's (or Φ_i's) do not vanish.

Step 3. Compute the discriminants of h_{11}, h_{21} and h_{31} respectively with respect to t. For convenience, we use "Discriminant[P, t]" to denote the discriminant of a polynomial $P(t) = a_nt^n + \cdots + a_1t + a_0$, that is,

$$\text{Discriminant}[P,t] = (-1)^{\frac{n(n-1)}{2}}\frac{1}{a_n}\text{Resultant}[P(t), P'(t), t],$$

where Resultant[P, P', t] is the resultant of P and P' with respect to t. Then we have

$$\begin{aligned}\text{Discriminant}[h_{11}, t] &= \delta_1(\beta),\\ \text{Discriminant}[h_{21}, t] &= \delta_2(\beta),\\ \text{Discriminant}[h_{31}, t] &= \delta_3(\beta)\end{aligned}$$

where $\delta_1(\beta)$, $\delta_2(\beta)$ and $\delta_3(\beta)$ are polynomials in β.

Next, we try to obtain certain intervals of β such that $\delta_i(\beta) \neq 0$ by determining the roots of $\delta_i(\beta)$, $i = 1, 2, 3$. With the help of Mathematica, for $\beta \in (1, 2)$, we identify one real root, β_{11}, for $\delta_1(\beta)$; 5 real roots β_{2i}, $i = 1, 2, \cdots, 5$, for $\delta_2(\beta)$, satisfying $\beta_{21} < \beta_{22} < \cdots < \beta_{25}$; and 7 real roots β_{3i}, $i = 1, 2, \cdots, 7$, for $\delta_3(\beta)$, satisfying $\beta_{31} < \beta_{32} < \cdots < \beta_{37}$. By applying Sturm's theorem, we can also verify the existence of these roots with a determined existence interval. For example,

$$\beta_{11} \in (\underbrace{1.108\cdots441,\ 1.108\cdots442}_{10^{-50}}),$$
$$\beta_{31} \in (\underbrace{1.496\cdots720,\ 1.496\cdots721}_{10^{-50}}),$$
$$\beta_{32} \in (\underbrace{1.747\cdots957,\ 1.747\cdots572}_{10^{-50}}).$$

Further, we use Lemma 3.4 to find some values of β such that the condition (i) holds. Firstly, for h_{11}, we find some values, for example $\beta_0 = \frac{11}{10}, \frac{3}{2}$, which are not equal to β_{11} and satisfying $h_{11}(t, \beta_0) > 0, \ \forall t \in \mathbb{R}$, i.e.

$$\begin{aligned} h_{11}(t, \tfrac{11}{10}) &= \frac{13265280000000000+269110195200000000t^2+\cdots+29115938088468t^{44}}{1000000000000002187} > 0, \\ h_{11}(t, \tfrac{3}{2}) &= \frac{43130880000+903536640000t^2+\cdots+88573500t^{44}}{16384} > 0. \end{aligned}$$

Then, according to Lemma 3.4, we conclude that when $\beta \in L_1 = (1, \beta_{11}) \cup (\beta_{11}, 2)$, $h_{11}(t, \beta) > 0$ for all $t \in \mathbb{R}$.

Next, for h_{31}, we can also verify that the values $\beta_0 = \frac{11}{10}, \frac{3}{2}$ are not equal to β_{3i}, $i = 1, 2, \cdots, 7$, satisfying $h_{31}(t, \beta_0) > 0, \ \forall t \in \mathbb{R}$, i.e.

$$\begin{aligned} h_{31}(t, \tfrac{11}{10}) &= \frac{68630377364883(24759631762948096000000000000000000000000000000+\cdots+s_0t^{136})}{62500000000000000000000000000000000000000} > 0, \\ h_{31}(t, \tfrac{3}{2}) &= \frac{179504811717705282355200000000000+\cdots+576440890438474757812500000t^{136}}{274877906944} > 0, \end{aligned}$$

where $s_0 = 21624980641055812753537766898291269412960$. Then by Lemma 3.4, we can conclude that when $\beta \in L_3 = (1, \beta_{31}) \cup (\beta_{31}, \beta_{32})$, $h_{31}(t, \beta) > 0$ for all $t \in \mathbb{R}$. Therefore, we can choose $L_{13} = L_1 \cap L_3 = (1, \beta_{11}) \cup (\beta_{11}, \beta_{31}) \cup (\beta_{31}, \beta_{32})$ to have $h_{11}(t, \beta) \neq 0$ and $h_{31}(t, \beta) \neq 0$ for all $t \in \mathbb{R}$ as long as $\beta \in L_{13}$. The above results give the following lemma.

Lemma 4.2. *There exists an interval* $L_{13} = (1, \beta_{11}) \cup (\beta_{11}, \beta_{31}) \cup (\beta_{31}, \beta_{32})$ *such that* $\Phi_1 \neq 0$ *and* $\Phi_3 \neq 0$ *for* $\beta \in L_{13}$ *and all* $v \in (0, \beta - 1)$.

Note that the interval L_{13} given in Lemma (4.2) is a pretty large subinterval in $(1, 2)$. However, unfortunately we cannot use the above approach to find a value of $\beta \in (1, 2)$ such that h_{21} is always positive or negative, that is, we cannot determine an interval for β in which $h_{21} \neq 0$. Hence, we need to develop an alternative for the remaining proof.

Step 4. Compute the discriminant of f_{21} with respect to t, which is generated in (4.9) and represents the derivative of function ϕ_2 with respect to β, with ϕ_2 determined from (4.5) such that

$$\Phi_2 = \frac{(v-\varphi)^3 \phi_2}{d_{20}}, \tag{4.6}$$

where ϕ_2 is a polynomial in v, φ and β, and

$$\begin{aligned} d_{20} = \; & 1296(1+v)^5(1-\beta+v)^5(\beta-1-\varphi)^5(1+\varphi)^5 \\ & \times(6-6\beta+8\varphi-4\beta\varphi+3\varphi^2+16\varphi-8b\varphi+6v\varphi+9\varphi^2). \end{aligned}$$

Then, we have

$$\begin{aligned} \frac{\mathrm{d}\phi_2(v,\varphi,\beta)}{\mathrm{d}\beta} := \; & d_\beta(v,\varphi,\beta), \\ \text{Resultant}[d_\beta, g, \varphi] = \; & R_{21}(v,\beta), \\ \text{Resultant}[d_{20}, g, \varphi] = \; & R_{22}(v,\beta), \end{aligned} \tag{4.7}$$

where R_{21} and R_{22} are polynomials in v and β.

Similarly, substituting $v = \frac{t^2}{t^2+1}(\beta-1)$ with $t \in (-\infty, +\infty)$, into R_{21} and R_{22} gives

$$R_{22}(v,\beta) = \frac{313456656384}{(1+t^2)^{66}}(\beta-1)^{27}(1+\beta t^2)^{20}(d_{11})^6(d_{12})^6 d_{21}, \tag{4.8}$$

and

$$\begin{aligned} & R_{21}(v,\beta) = \frac{72}{(1+t^2)^{42}}(\beta-1)^{14}(1+\beta t^2)^2 f_{21} := r_{21}(t,\beta), \\ & \text{Discriminant}[f_{21}, t] = \delta_4(\beta), \end{aligned} \tag{4.9}$$

where f_{21} is a polynomial in t and β with degree 76 of t, and $\delta_4(\beta)$ is a polynomial in β.

Then, we again try to find certain intervals of β such that $\delta_4 \neq 0$. To achieve this, letting $\delta_4 = 0$ yields 15 real roots $\beta_{4i} \in (1,2)$ with $\beta_{41} < \beta_{42} < \cdots < \beta_{415}$. For example,

$$\beta_{41} \in \underbrace{(1.501\cdots050,\ 1.501\cdots051)}_{10^{-50}}.$$

Moreover, we can verify that the two values $\beta_0 = \frac{11}{10}, \frac{3}{2}$ are not equal to β_{4i} satisfying $f_{21}(t,\beta_0) > 0, \forall t \in \mathbb{R}$, i.e.

$$\begin{aligned} f_{21}(t, \tfrac{11}{10}) \; &= \frac{1594323(3555860412501196800000000000000000000+\cdots+s_1 t^{76})}{50000000000000000000000000} > 0, \\ f_{21}(t, \tfrac{3}{2}) \; &= \frac{7535726068003504128000+\cdots+926572879928542500t^{76}}{33554432} > 0, \end{aligned}$$

where $s_1 = 174741133897900845319596327468$. Then by Lemma 3.4, we obtain that when $\beta \in L_4 = (1, \beta_{41})$, $f_{21}(t,\beta) > 0$ for all $t \in \mathbb{R}$, which gives the following lemma.

Lemma 4.3. *There exists an interval $L_{40} = [\beta_0, \beta_{41})$ such that when $\beta \in L_{40}$, $\phi_2 > 0$ for all $v \in (0, \beta_0 - 1)$, where $\beta_0 \in L_4$, yielding $\phi_2(v,\varphi,\beta_0) > 0$ for all $v \in (0, \beta_0 - 1)$, with $\beta_0 = \frac{11}{10}$ or $\frac{3}{2}$.*

Proof. It has been shown in the above that when $\beta \in L_4 = 1, \beta_{41})$, $f_{21}(t, \beta) > 0$ for all $t \in \mathbb{R}$. Thus, the Resultant R_{21} in (4.9) between d_β and $g(v, \varphi)$ with respect to φ, does not vanish, implying that $d_\beta \neq 0$ in (4.7) always holds. Moreover, it can be verified that $d_\beta > 0$ for $v \in (0, \beta - 1)$, meaning that $\phi_2(v, \varphi, \beta)$ in (4.6) is monotonically increasing with respect to β.

Since $\varphi = \sigma(v)$ is a continuous function in v, $\phi_2(v, \varphi, \beta)$ is also a continuous function with respect to v and β in the domain:

$$D = \{(v, \beta) | 0 \leq v \leq \beta - 1,\ 1 \leq \beta \leq 2\}.$$

Thus, when $\beta > \beta_0$, $\phi_2(v, \varphi, \beta) > \phi_2(v, \varphi, \beta_0)$ for any $v \in (0, \beta_0 - 1)$. On the other hand, we know that $\phi_2(v, \varphi, \beta_0) > 0$, and so $\phi_2(v, \varphi, \beta) > 0$ for any $v \in (0, \beta_0 - 1)$. In fact, according to Lemma 4.1, there exist $\beta_0 = \frac{11}{10}, \frac{3}{2}$ such that $\Phi_2(v, \varphi, \beta_0) \neq 0$ always holds, and so it can be verified that $\phi_2(v, \varphi, \beta_0) > 0$ when $v \in (0, \beta_0 - 1)$. □

Step 5. Compute the discriminant of g_{21} with respect to t, which is generated in (4.11) and represents the derivative of function ϕ_2 defined in (4.6) with respect to v. We have

$$\begin{aligned} \frac{\mathrm{d}\phi_2(v,\varphi,\beta)}{\mathrm{d}v} :=&\ d_v(v, \varphi, \beta), \\ \mathrm{Resultant}[d_v, g, \varphi] =&\ R_{23}(v, \beta), \end{aligned} \tag{4.10}$$

where R_{23} is a polynomial in v and β. Similarly, substituting

$$v = \frac{t^2}{t^2+1}(\beta - \beta_0) + \beta_0 - 1, \quad \beta_0 = \frac{3}{2}, \quad t \in (-\infty, +\infty)$$

into R_{23} we have

$$R_{23}(v, \beta) = \frac{3(\beta-2)(2\beta-3)}{2147483648(t^2+1)^{41} d_{23}}(2\beta t^2 + 3)(2\beta t^2 - 2t^2 + 1 g_{21} := r_{23}(t, \beta)$$

and

$$\begin{aligned} \mathrm{Discriminant}[g_{21}, t] = \delta_5(\beta), \\ \mathrm{Discriminant}[d_{23}, t] = \delta_6(\beta) \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} d_{23} =&\ (4\beta^2 t^4 + 16\beta^2 t^2 - 4\beta t^2 + 32\beta - 4t^4 - 20t^2 - 43) \\ &\times (4\beta^2 t^4 + 16\beta^2 t^2 - 8\beta t^4 - 44\beta t^2 + 24t^2 - 3) \\ &\times (24\beta^2 t^4 + 16\beta^2 t^2 + 4\beta^2 - 24\beta t^4 + 16\beta t^2 + 4\beta - 24t^2 + 3), \\ \delta_6(\beta) =&\ -113755476875026872166594815941782079061589407540707328 \\ &\times (\beta-2)^5(\beta-1)^{18}\beta^{18}(\beta+1)^5(2\beta-3)^{60}(2\beta-1)^4(32\beta-43)(4\beta^2+4\beta+3), \end{aligned}$$

and g_{21} is a polynomial in t and β with degree 88 of t, and $\delta_6(\beta)$ is a polynomial in β.

Then, it can be verified that when $\beta \in (\beta_0, 2) = (\frac{3}{2}, 2)$, $\delta_6 \neq 0$. Thus we only need try to find certain intervals of β such that $\delta_5 \neq 0$. To do this, setting $\delta_5 = 0$ yields 6 real roots $\beta_{5i} \in (\frac{3}{2}, 2)$ with $\beta_{51} < \beta_{52} < \cdots < \beta_{56}$. For example,

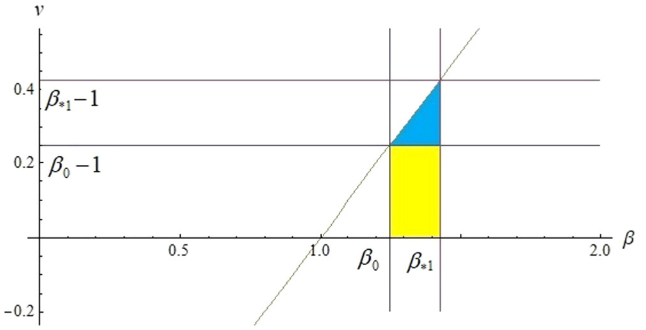


Fig. 3. The interval $[\beta_0, \beta_{*1}) = [\frac{11}{10}, \beta_{61})$ or $[\frac{3}{2}, \beta_{41})$ satisfying $\phi_2 > 0$ in (4.6) for $\beta \in [\beta_0, \beta_{*1})$.

$$\beta_{51} \in (\underbrace{1.515\cdots596,\ 1.515\cdots597}_{10^{-50}}), \quad \beta_{52} \in (\underbrace{1.562\cdots032,\ 1.562\cdots033}_{10^{-50}}).$$

Moreover, we can verify that the two values $\beta_* = \frac{151}{100}, \frac{155}{100}$ are not equal to β_{6i} satisfying $g_{21}(t, \beta_*) < 0,\ \forall t \in \mathbb{R}$, i.e.

$$g_{21}(t, \tfrac{151}{100}) = \frac{908166180608\cdots00000 + \cdots + 83655\cdots9882t^{88}}{4547473508864\cdots0000} < 0,$$

$$g_{21}(t, \tfrac{155}{100}) = -\frac{3486784401(95582750\cdots00000 + \cdots + 14611\cdots5242t^{88})}{8000\cdots0000} < 0.$$

At the same time, we can get $d_{23}(t, \beta_*) < 0, \forall t \in \mathbb{R}$. Then by Lemma 3.4, we obtain that when $\beta \in L_5 = (\frac{3}{2}, \beta_{51}) \cup (\beta_{51}, \beta_{52})$, $g_{21}(t, \beta) < 0$ and $d_{23}(t, \beta) < 0$ for all $t \in \mathbb{R}$, yielding $d_v(v, \varphi, \beta) > 0$ in (4.10) for all $v \in (\beta_0 - 1, \beta - 1) = (\frac{1}{2}, \beta - 1)$.

Furthermore, when $\beta_0 = \frac{11}{10}$, using the above process, we get an interval $L_6 = (\frac{11}{10}, \beta_{61})$ with $\beta_{61} \approx 1.11316496347$, such that when $\beta \in L_6$, $d_v(v, \varphi, \beta) > 0$ in (4.10) for all $v \in (\beta_0 - 1, \beta - 1) = (\frac{1}{10}, \beta - 1)$.

Summarizing the above discussions in Step 5 yields the following lemma.

Lemma 4.4. *When $\beta \in L_{50} = [\frac{11}{10}, \beta_{61}) \cup [\frac{3}{2}, \beta_{41})$, $\Phi_2 \neq 0$ for all $v \in (0, \beta - 1)$.*

Proof. First, one can see that the factor $(v - \varphi)$ in (4.6) does not vanish for $v \in (0, \beta - 1)$. Moreover, it follows from $d_{11}d_{12}d_{21} \neq 0$ that the Resultant of φ and R_{22} in (4.8) cannot equal zero, which implies that $d_{20} \neq 0$ in (4.6) for $v \in (0, \beta - 1)$.

Next, we prove that $\phi_2 > 0$ in (4.6) for $v \in (0, \beta - 1)$. According to Lemma 4.3, we have obtained that when $\beta \in L_{40} = [\beta_0, \beta_{41})$, $\phi_2 > 0$ for all $v \in (0, \beta_0 - 1)$ with $\beta_0 = \frac{11}{10}$ or $\frac{3}{2}$, as shown in Fig. 3. Moreover, in the Step 5, it has been verified that $d_v(v, \varphi, \beta) > 0$ in (4.10) for all $v \in (\beta_0 - 1, \beta - 1)$, meaning that $\phi_2(v, \varphi, \beta)$ in (4.6) is monotonically increasing with respect to v. Thus, when $\beta - 1 > v > v_0 > \beta_0 - 1$, $\phi_2(v, \varphi, \beta) > \phi_2(v_0, \varphi(v_0), \beta)$ for any $\beta \in L_5 \cup L_{5*}$.

Since $\phi_2(v, \varphi, \beta)$ is a continuous function with respect to v and β in the domain: $D = \{(v, \beta) | 0 \le v \le \beta - 1,\ 1 \le \beta \le 2\}$, moreover when $\beta = \beta_0$ and $v_0 = \beta_0 - 1$, $\phi_2(v_0, \varphi(v_0), \beta_0) = 0$, we have $d_\beta(v, \varphi, \beta) > 0$ in (4.7) which implies that $\phi_2(v_0, \varphi(v_0), \beta) > 0$ for any $\beta \in (\beta_0, \beta_{41})$. Applying the local invariance of signs for the continuous functions in two variables, we have that when $v \in [\beta_0 - 1, \beta - 1)$, $\phi_2(v, \varphi, \beta) > \phi_2(v_0, \varphi(v_0), \beta) > 0$ for any $\beta \in (L_5 \cup L_{5*}) \cap (\beta_0, \beta_{41})$, as shown in Fig. 3.

Thus, for $\beta \in L_{50} = [\frac{11}{10}, \beta_{61}) \cup [\frac{3}{2}, \beta_{41})$, we have $\phi_2(v, \varphi, \beta) > 0$ for any $v \in (0, \beta - 1)$, and so Φ_2 in (4.5) does not vanish. □

Remark 4.1. As a matter of fact, for the β_0 considered in the above process, except for $\frac{11}{10}$ and $\frac{3}{2}$, there exist many other such values $\beta_0 \in L_4$ with the corresponding intervals $L_* = [\beta_0, \beta_{*1})$ such that when $\beta \in L_*$, $\Phi_2 \neq 0$ in (4.5) for any $v \in (0, \beta - 1)$. However, it is not easy to identify all such values.

Combining all the results obtained above, we let $L = L_{50} \cap L_{13} = [\frac{11}{10}, \beta_{11}) \cup [\frac{3}{2}, \beta_{41})$. Then according to Lemma 3.3, we combine Lemmas 4.2 and 4.4, to obtain that for $\beta \in L$, Φ_1, Φ_2 and Φ_3 in (4.5) do not vanish for all $v \in (0, \beta - 1)$. This proves that $\{I_1, I_2, I_3\}$ is an ECT-system on $(0, h_0)$, implying that the maximum number of zeros of the Abelian integral (3.8) is at most two for $\beta \in L$. The second part of Theorem 3.1 that the number two can be reached will be proved in the next section.

5. Double isolated periodic traveling waves and simulations

An isolated closed orbit in system (1.9) corresponds to an isolated periodic traveling wave in the original model (1.5). According to Theorem 2.2, there exist at most two small-amplitude limit cycles in the neighborhood of the three equilibria of system (1.9) arising from Hopf bifurcation. In this section, we show that these two limit cycles can be obtained by appropriate parameter perturbations. Further, according to Theorem 3.1, we also know that at most two global limit cycles exist which surround the equilibrium $(1, 0)$ of system (3.1) via Poincaré bifurcation. In particular, we give a concrete example to achieve the two global limit cycles. We have the following result.

Proposition 5.1. *For the perturbed system* (3.1)*, under the condition: $\beta = \beta_0 \in L$, with $\beta_0 = \frac{11}{10}, \frac{3}{2}$, there exist two and at most two limit cycles bifurcating from the equilibrium point* $(1, 0)$ *via Poincaré bifurcation.*

Proof. By applying the method of asymptotic expansion of the Abelian integral $I(h)$ (or the first Melnikov function, see [10,11,27]), we can search the zeros, via the asymptotic expansion at $h = 0$ to match the Hopf bifurcation values, and the asymptotic expansion at $h = h_0 = \frac{1}{12}(\beta_0 - 1)^3(\beta_0 + 1)$ for the homoclinic bifurcation.

Based on (3.8) and (4.2), an alternative is to consider

$$I(h) = d_0\tilde{I}_0(h) + d_1\tilde{I}_1(h) + d_2\tilde{I}_2(h) = \frac{1}{h}(d_0 I_0 + d_1 I_1 + d_2 I_2), \tag{5.1}$$

where $I_i = I_i(h) = \oint_{\Gamma_h} f_i(v) y \mathrm{d}v$, $i = 0, 1, 2$. Without loss of generality, setting $d_0 = 1$ and choosing two arbitrarily different values $h_1^*, h_2^* \in (0, h_0)$, yield two equations:

$$\begin{aligned} d_1 I_1(h_1^*) + d_2 I_2(h_1^*) &= -I_0(h_1^*), \\ d_1 I_1(h_2^*) + d_2 I_2(h_2^*) &= -I_0(h_2^*). \end{aligned} \tag{5.2}$$

According to Lemmas 4.2 and 4.3, $\{I_1, I_2\}$ is an ECT-system on $(0, h_0)$. Thus, it follows from Theorem B and Lemma 2.3 in [9] that all discrete Wronskians of $\{I_1, I_2\}$ are not equal to zero.

On the other hand, it is seen that the determinant of the coefficient matrix on the left-hand side of (5.2) is exactly the Wronskian of $\{I_1, I_2\}$, which is thus not equal to zero on (h_1^*, h_2^*), implying that (5.2) has a unique solution $\{d_1^*, d_2^*\}$. Let $\mathbf{d} = (d_1^*, d_2^*, 1)$. Then, h_1^* and h_2^* are naturally two different zeros of the Abelian integral $I(h, \mathbf{d})$ in (3.8). Moreover, it is noted that the number of zeros is at most two, and so no more zeros exist. Correspondingly, there exist some parameter values such that system (3.2) or (3.1) has two large-amplitude limit cycles which bifurcate from an period annulus surrounding the origin.

The proof is complete. □

By Proposition 5.1, the second part of Theorem 3.1 is proved, that is, for $\beta \in L$, the maximum number of zeros of the Abelian integral (3.8) can reach two.

To demonstrate the theoretical result, we give a concrete example to simulate the two large-amplitude limit cycles. We choose $\beta = \frac{3}{2}$, $h_1 = \frac{1}{100}$ and $h_2 = \frac{1}{40}$ to obtain

$$\begin{aligned} &I_0(h_1) \approx 0.045632, \quad &&I_1(h_1) \approx 0.000531, \quad &&I_2(h_1) \approx 0.000493, \\ &I_0(h_2) \approx 0.122854, \quad &&I_1(h_2) \approx 0.005536, \quad &&I_2(h_2) \approx 0.004248. \end{aligned}$$

Then, it follows from (5.2) that $d_0\varepsilon = a_0 + a_1 + a_2 - c$, $d_1\varepsilon = a_1 + 2a_2$, $d_2\varepsilon = a_2$. Further, setting $\varepsilon = 0.002$ yields

$$a_0 \approx c - 1.35695, \quad a_1 \approx 2.15294, \quad a_2 \approx -0.79399. \tag{5.3}$$

The oscillating time histories for the above parameter values with three different initial points are depicted in Fig. 4(a), (b) and (c), respectively. In addition, the phase portrait given in Fig. 4(d) clearly shows two limit cycles with the inner one stable and the outer one unstable. Moreover, for the above example, we provide the simulation in Fig. 5 to illustrate the existence of the two global isolated periodic traveling waves.

Based on Theorem 2.2 and Proposition 5.1, we have the following result.

Proposition 5.2. *In the single species model* (1.5)*, at most two local isolated periodic traveling waves can bifurcate from $u(x,t) = 0$ or $u(x,t) = 1$ or $u(x,t) = \beta$ via Hopf bifurcation; and double global isolated periodic traveling waves can bifurcate from $u(x,t) = 1$ via Pioncaré bifurcation when the parameter β is chosen from some appropriate subinterval L of* $(1, 2)$.

Remark 5.1. It should be noted that a_0 corresponds to the speed of advection A_0, while a_1 and a_2 correspond to the speed coefficients of migration A_1 and A_2, the latter is due to the biological mechanisms while the former is responsible for the impacts of wind and water current, etc. The parameter $\beta = U_0 K^{-1}$ depends on the Allee effect and species carrying capacity. Therefore, the above influential factors can appear in reality, implying that double isolated periodic traveling waves can coexist, revealing special pattern of spread in population dynamics.

6. Conclusion

In this paper, isolated periodic traveling waves have been studied for a class of modified single species population model and particular attention is focused on the real pattern of spread for population dynamics. Based on the Chebyshev criterion, we have developed a utilized approach

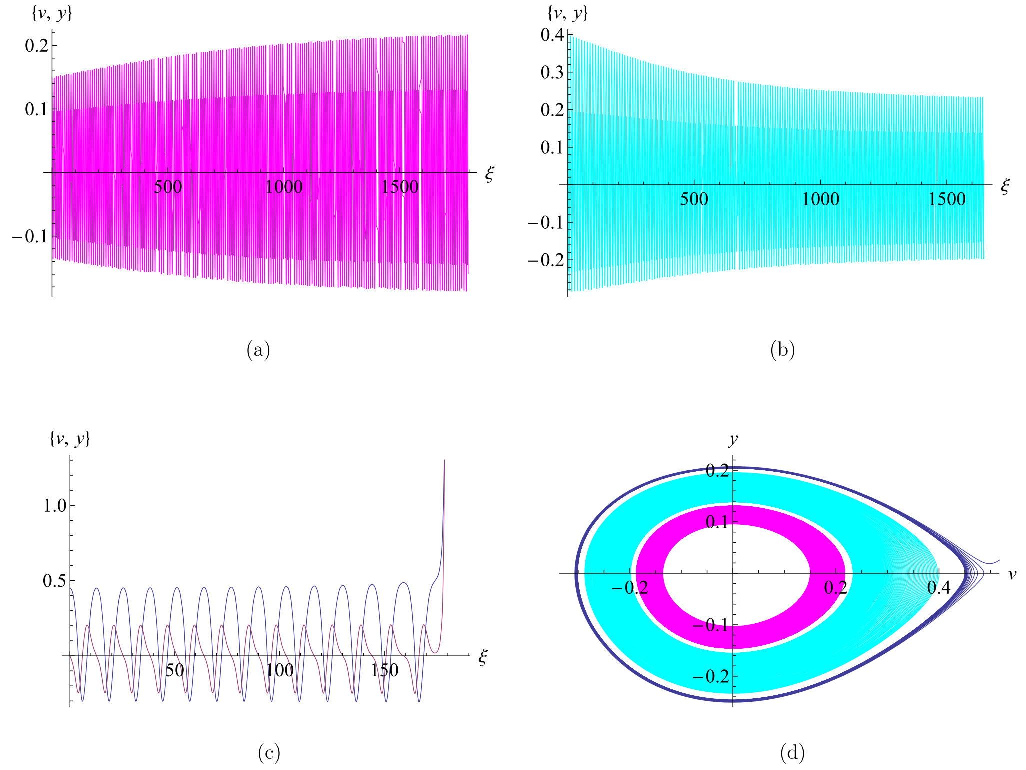


Fig. 4. Time histories of system (3.2) under the condition (5.3) and $\beta = 3/2$ for $v(\xi)$ and $y(\xi)$ with the initial points: (a) $(v, y) = (0.15, 0)$; (b) $(v, y) = (0.4, 0)$; and (c) $(v, y) = (0.45, 0)$; and (d) the phase portrait showing two limit cycles with the inner one stable and the outer one unstable.

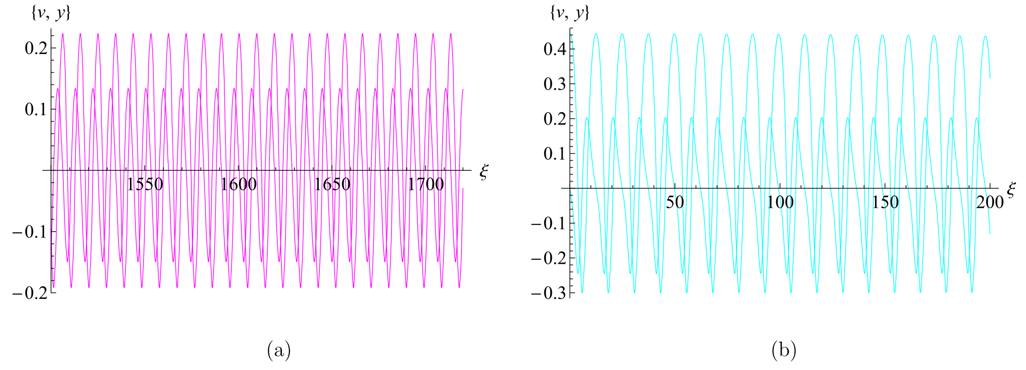


Fig. 5. Two global isolated periodic traveling waves of system (3.2) with the condition (5.3) and $\beta = \frac{3}{2}$, bifurcating from $u(x,t) = 0$ for (a) $h_1 = \frac{1}{100}$, and (b) $h_2 = \frac{1}{40}$.

to determine a piecewise continuous interval of the parameter β for the existence of the corresponding two zeros of the Abelian integral and thus to prove the existence of limit cycles. It is shown that double global isolated periodic traveling waves, with one stable and one unstable, can bifurcate from $u(x,t) = 1$ via Pioncaré bifurcation. It has also shown that when the species carrying capacity K, together with the measure of the Allee effect U_0 (yielding $\beta = U_0 K^{-1}$) satisfies appropriate conditions, with the population density and the rate of its change taking the corresponding initial values, the density of population keeps oscillating, or approaches a periodic oscillation, or diverges from an equilibrium to infinity. This may explain the complex behavior in real population models.

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References

[1] R.C. Almeida, A.D. Simone, I.S.C. Michel, A numerical model to solve single-species invasion problems with Allee effects, Ecol. Model. 192 (3–4) (2006) 601–617.
[2] H.C. Berg, Random Walks in Biology, Princeton University Press, Princeton, 1983.
[3] D.S. Boukal, L. Berec, Single-species models of the Allee effect: extinction boundaries, sex ratios and mate encounters, J. Theor. Biol. 218 (3) (2002) 375–394.
[4] H. Chen, Y. Liu, Linear recursion formulas of quantities of singular point and applications, Appl. Math. Comput. 148 (1) (2004) 163–171.
[5] F. Courchamp, L. Berek, J. Gascoigne, Allee Effects in Ecology and Conservation, Oxford University Press, Oxford, 2008.
[6] M.A. Davis, Invasion Biology, Oxford University Press, Oxford, 2009.
[7] F. Dumortier, C. Li, Perturbations from an elliptic Hamiltonian of degree four: I. Saddle loop and two saddle cycle, J. Differ. Equ. 176 (1) (2001) 114–157.
[8] J.D. García-Saldaña, A. Gasull, H. Giacomini, Bifurcation values for a family of planar vector fields of degree five, Discrete Contin. Dyn. Syst. 35 (2) (2012) 669–701.
[9] M. Grau, F. Mañosas, J. Villadelprat, A Chebyshev criterion for Abelian integrals, Trans. Am. Math. Soc. 363 (2011) 109–129.
[10] M. Han, J. Yang, A.A. Tarta, G. Yang, Limit cycles near homoclinic and heteroclinic loops, J. Dyn. Differ. Equ. 20 (4) (2008) 923–944.
[11] M. Han, J. Yang, P. Yu, Hopf bifurcations for near-Hamiltonian systems, Int. J. Bifurc. Chaos 19 (12) (2009) 4117–4130.
[12] R. Hengeveld, Dynamics of Biological Invasions, Chapman and Hall, London, 1989.
[13] E.E. Holmes, M.A. Lewis, J.E. Banks, R.R. Veit, Partial differential equations in ecology: spatial interactions and population dynamics, Ecology 75 (1) (1994) 17–29.
[14] W. Huang, T. Chen, J. Li, Isolated periodic wave train and local critical wave lengths for a nonlinear reaction-diffusion equation, Commun. Nonlinear Sci. Numer. Simul. 74 (2019) 84–96.
[15] N. Kopell, L.N. Howard, Plane wave solutions to reaction-diffusion equations, Stud. Appl. Math. 52 (4) (1973) 291–328.
[16] M.A. Lewis, P. Kareiva, Allee dynamics and the spread of invading organisms, Theor. Popul. Biol. 43 (2) (1993) 141–158.
[17] M.A. Lewis, S.V. Petrovskii, J.R. Potts, The Mathematics Behind Biological Invasions, Springer International Publishing, 2016.
[18] J. Li, H. Dai, On the Study of Singular Nonlinear Traveling Wave Equations: Dynamical System Approach, Science Press, 2007.
[19] J. Li, Z. Liu, Smooth and non-smooth traveling waves in a nonlinearly dispersive equation, Appl. Math. Model. 25 (1) (2000) 41–56.
[20] Y. Liu, W. Huang, A cubic system with twelve small amplitude limit cycles, Bull. Sci. Math. 129 (2) (2005) 83–98.
[21] Y. Liu, J. Li, Theory of values of singular point in complex autonomous differential system, Sci. China Math. 1 (1990) 10–23.

[22] V. Mañosa, Periodic travelling waves in nonlinear reaction-diffusion equations via multiple Hopf bifurcation, Chaos Solitons Fractals 18 (2003) 241–257.

[23] F. Mañosas, J. Villadelprat, Bounding the number of zeros of certain Abelian integrals, J. Differ. Equ. 251 (2011) 1656–1669.

[24] S.V. Petrovskii, B.L. Li, An exactly solvable model of population dynamics with density-dependent migrations and the Allee effect, Math. Biosci. 186 (2003) 79–91.

[25] S.V. Petrovskii, A.Y. Morozov, E. Venturino, Allee effect makes possible patchy invasion in a predator-prey system, Ecol. Lett. 5 (3) (2002) 345–352.

[26] S.V. Petrovskii, E. Venturino, Spatiotemporal Patterns in Ecology and Epidemiology: Theory, Models, and Simulation, Chapman and Hall/CRC Press, NY, 2008.

[27] R. Roussarie, On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields, Bull. Braz. Math. Soc. 17 (2) (1986) 67–101.

[28] J.A. Sherratt, M.J. Smith, Periodic traveling waves in cyclic populations: field studies and reaction-diffusion models, J. R. Soc. Interface 5 (22) (2008) 483–505.

[29] X. Sun, P. Yu, B. Qin, Global existence and uniqueness of periodic waves in a population model with density-dependent migrations and Allee effect, Int. J. Bifurc. Chaos 27 (12) (2017) 1750192.

[30] Q. Wang, W. Huang, Limit periodic travelling wave solution of a model for biological invasions, Appl. Math. Lett. 34 (2014) 13–16.

[31] Y. Zeng, X. Sun, P. Yu, Dynamical analysis on traveling wave of a reaction-diffusion model, Appl. Math. Lett. 109 (2020) 106550.

[32] Y. Zhou, Q. Liu, W. Zhang, Bounded traveling waves of the Burgers-Huxley equation, Nonlinear Anal. 74 (4) (2011) 1047–1060.