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Study of hidden attractors, multiple limit cycles from Hopf bifurcation and boundedness of motion in the generalized hyperchaotic Rabinovich system

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Abstract Based on Rabinovich system, a 4D Rabinovich system is generalized to study hidden attractors, multiple limit cycles and boundedness of motion. In the sense of coexisting attractors, the remarkable finding is that the proposed system has hidden hyperchaotic attractors around a unique stable equilibrium. To understand the complex dynamics of the system, some basic properties, such as Lyapunov exponents, and the way of producing hidden hyperchaos are analyzed with numerical simulation. Moreover, it is proved that there exist four small-amplitude limit cycles bifurcating from the unique equilibrium via Hopf bifurcation. Finally, boundedness of motion of the hyperchaotic attractors is rigorously proved.

Keywords Rabinovich system · Hidden attractor · Hopf bifurcation · Boundedness of motion

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1 Introduction

It is well known that nonlinear dynamical systems with at least three (autonomous) or two (driven) degrees of freedom can exhibit irregular, noise-like behavior [1]. A great number of nonlinear systems exhibit complicated types of attractors for certain parameter values. It turns out that some real dynamical systems are basically nonlinear and have multiple attractors, depending on the choice of the initial conditions or system parameter values [2]. Although a chaotic attractor is often a global attracting set, the coexistence of structurally different chaotic attractors is not totally impossible, especially if some complex nonlinearity is present [3–6].

Moreover, due to recent theoretical development and practical applications in relevant fields, such as in secure communications, lasers, nonlinear circuits, neural networks, generation, control and synchronization [7–11], hyperchaos has also become a hot topic. Therefore, it is necessary to determine the nature or the type of chaos observed in experiments or in simulation, or even proved analytically. As we all know, horseshoe-type or Shilnikov chaos is one of the analytic criteria for proving chaos in autonomous systems with homoclinic orbit or heteroclinic loop [12]. We may roughly classify four kinds of chaos: homoclinic chaos, heteroclinic chaos, a combination of homoclinic and heteroclinic chaos, and chaos without homoclinic orbits and heteroclinic orbits [13]. From computational point of view, this allows one to use numerical method toidentify the trajectories that start from the unstable manifold near the equilibrium and finally reach an attractor.

Recently, there has discovered another type of attractors called hidden attractor by Leonov et al. [14–16], whose attracting basin does not contain neighborhoods of equilibria and so it can not be simulated with the help of classical methods. Therefore, increasing attention has led to develop some unusual examples, such as 3D autonomous quadratic systems which may have no equilibria [17,18], or stable equilibria [19–22] or coexisting attractors [23], as well as 4D autonomous quadratic system with no equilibria [24–27].

The investigation of hyperchaos is still in its early stage, and the dynamics of hyperchaotic systems are not well understood by researchers. During resent years, reports about hyperchaos are mostly focused on the generation of hyperchaos [28–31]. As far as we know, there is very little known on the coexistence of hyperchaotic attractor in 4D autonomous systems with only stable equilibria. This opens an interesting field to study the existence of chaotic (hyperchaotic) attractors, in particular hidden attractors in systems with only stable equilibria. As a matter of fact, important questions have been raised such as how to identify integrable dynamical systems and how to determine the basin of chaos in phase space. Therefore, it is interesting to ask whether similar 4D autonomous systems with only stable equilibria can have non-Shilnikov hyperchaos. The present paper will report a modified hyperchaotic Rabinovich system, which shows the seemingly impossible phenomenon.

In addition, interesting bifurcations, such as Hopf, double zero, Hopf zero and double Hopf, may also occur in chaotic systems or hyperchaotic systems [32,33]. Detailed study for some types of these bifurcations can be found in [34–39] by applying normal form theory, and in [40,41] by averaging theory. In this paper, we will investigate the maximum number of limit cycles which can bifurcate from the equilibrium in modified hyperchaotic Rabinovich system, due to Hopf bifurcation.

The rest of this paper is organized as follows. In Sect. 2, the new hyperchaotic system is introduced and the influence of the initial condition on the dynamics of system is analyzed. The way of producing hidden hyperchaos is present in Sect. 3. In Sect. 4, by using the normal form theory and symbolic computation, we obtain four small-amplitude limit cycles bifurcating from the unique equilibrium via Hopf bifurcation. In Sect. 5, boundedness of motion for the hyperchaotic system is discussed. Finally, conclusions are drawn in Sect. 6.

2 The proposed system and hidden hyperchaos

2.1 Formulation of the system

In this section, we first present the Rabinovich system given by Pikovsky et al. [42]

$$\begin{cases} \dot{x} = hy - ax + yz, \\ \dot{y} = hx - by - xz, \\ \dot{z} = -cz + xy, \end{cases} \tag{1}$$

where $(x, y, z) \in R^3$ is the state vector and $(h, a, b, c) \in R^4$ is the parameter vector. One can easily use simulation to show that system (1) is chaotic when $h = 0.04, a = 1.5, b = -0.3, c = 1.67$ [42,43].

In contrast to the study of 3D dynamical systems, dynamical behaviors (such as bifurcation) of 4D hyperchaotic systems have not yet been well studied. Recently, a 4D hyperchaotic Rabinovich system is generated from system (1) as follows:

$$\begin{cases} \dot{x} = hy - ax + yz, \\ \dot{y} = hx - by - xz + w, \\ \dot{z} = -cz + xy, \\ \dot{w} = -ky, \end{cases} \tag{2}$$

where k is a positive parameter. When $(a, b, c, h, k) = (4, 1, 1, 6.75, 2)$, system (2) has a unique unstable equilibrium and exhibits a hyperchaotic attractor [44–47].

It is worth noting that the proposed new hyperchaotic system should satisfy two of the criteria introduced by Sprott [48]: The system should be a realistic model for some important unsolved problem in nature and shed insight on that problem, and the system should exhibit some behavior previously unobserved. Therefore, we propose a new system which can have hidden hyperchaos:

$$\begin{cases} \dot{x} = my - ax + syz, \\ \dot{y} = -by - xz + w, \\ \dot{z} = -cz + xy - n, \\ \dot{w} = -ky, \end{cases} \tag{3}$$

where $(m, a, b, c, n, k, s) \in R^7$ is the parameter vector. Under the coordinate transform,

$$(x, y, z, w) \to \left(x, y, z - \frac{n}{c}, w\right),$$

system (3) becomes

$$\begin{cases} \dot{x} = \left(m - \frac{n}{c}s\right)y - ax + syz, \\ \dot{y} = \frac{n}{c}x - xz - by + w, \\ \dot{z} = -cz + xy, \\ \dot{w} = -ky, \end{cases} \tag{4}$$

which contains system (2) as a special case when setting $s = 1, n = hc$ and $m = 2h$.

2.2 Hidden hyperchaotic attractors in system (3) with a unique stable equilibrium

In the following, we focus on some behavior previously unobserved in 4D systems: hyperchaotic hidden attractors associated with a unique stable equilibrium: The basin of the attractors does not intersect with small neighborhood of the equilibrium.

Under the parameter values, $m = 35, a = 35, s = 35, b = -17, c = 0.8, n = 4$ and $k = 12$, and the choice of the initial condition: (0.2, 0, 1, 1), system (3) has a unique equilibrium $(x, y, z, w) = (0, 0, -5, 0)$, whose characteristic eigenvalues are $\lambda_1 = -0.8$, $\lambda_2 = -10.7510$, $\lambda_{3,4} = -3.6245 \pm 5.0921i$. In this case, system (3) has no homoclinic or heteroclinic orbits, but a hyperchaotic attractor (see Fig. 1). The corresponding Lyapunov exponents [49,50] are given by $L_1 = 1.3012$, $L_2 = 0.2825$, $L_3 = 0.0$ and $L_4 = -20.3892$. The Kaplan–Yorke dimension [51] is defined by

$$D_L = j + \frac{1}{|L_{j+1}|}\sum_{i=1}^{j} L_i,$$

where j is the largest integer satisfying $\sum_{i=1}^{j} L_i \geq 0$ and $\sum_{i=1}^{j+1} L_i < 0$. For the hyperchaotic attractor shown in Fig. 1, the Kaplan–Yorke dimension is $D_L = 3.0777$, indicating that the Lyapunov dimension of the hyperchaotic attractor is fractional. Existence of the hyperchaotic attractor can be seen via plotting Poincaré maps on different sections. In Fig. 2, the Poincare image on the plane $y = w$ has no regular limbs, further indicating that the system has extremely rich dynamics, different from that of normal hyperchaotic systems with one or more unstable equilibria. Such a system belongs to a newly discovered category of chaotic systems with hidden attractors.

To further study the properties of hyperchaos of system (3), we introduce the following definition.

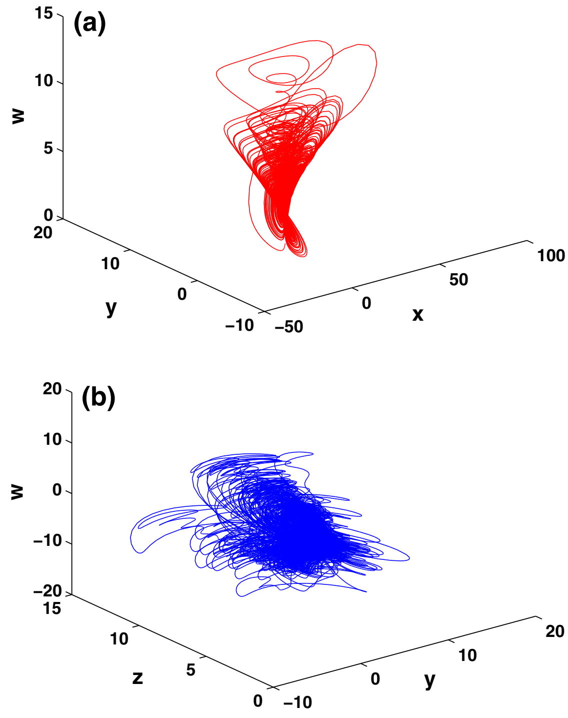

Fig. 1 Hidden hyperchaotic attractor for the 4D chaoticsystem (3) with the initial condition (0.2, 0, 1, 1) for the parameter values $m = 35, a = 35, s = 35, b = -17, c = 0.8, n = 4$ and $k = 12$

Definition 2.1 [52]. Let $f(x)$ and $g(x)$ be vector fields on R^n, and

$$\dot{x} = f(x), x \in R^n \tag{5}$$

$$\dot{y} = f(y), y \in R^n, \tag{6}$$

are two systems of differential equations on R^n. If there exists a diffeomorphism h on R^n such that

$$f(x) = M^{-1}(x)g(h(x)),$$

where $M(x)$ is the Jacobian of h at the point x , then (5) and (6) are said to be *smoothly equivalent (or diffeomorphic)*.

Remark 2.2 If systems (5) and (6) are smoothly equivalent, and suppose that x_0 and $y_0 = h(x_0)$ are their corresponding equilibria, $A(x_0)$ and $B(y_0)$ are the Jacobians of $f(x)$ and $g(x)$, respectively; then, $A(x_0)$ and $B(y_0)$ are similar, i.e., their characteristic polynomials and eigenvalues are identical.

Remark 2.3 Based on the concept and techniques of the equilibrium and resultant elimination, it is easy to verify that the existing horseshoe chaos is not smoothly equivalent to hidden hyperchaos for system (3) with

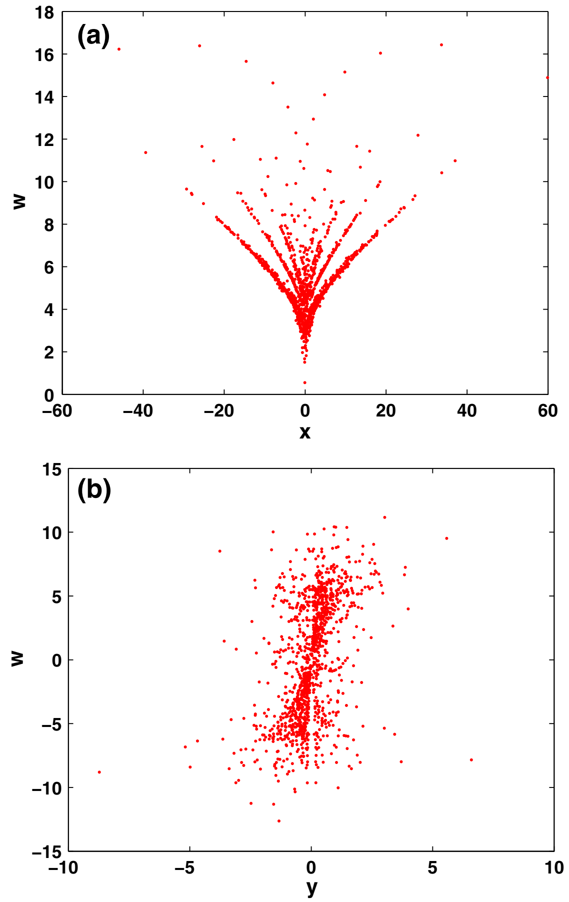

Fig. 2 Poincaré maps of the hyperchaotic attractor of system (3) with parameter values $m = 35, a = 35, s = 35, b = -17, c = 0.8, n = 4$ and $k = 12$. (a) projected on the section $y = 0$, (b) projected on the section $x = 0$

only one stable equilibrium. Therefore, system (3) and the previously reported hyperchaotic systems are different and, in fact, nonequivalent in topological structures.

In addition, system (3) cannot have chaotic solutions for certain parameter values. More precisely, we have the following result.

Theorem 2.4 *Assume that one of the following conditions holds:*

(1) $m = 0, s = 0$ *and* $a \neq 0$*;*
(2) $n = 0, c = 0$ *and* $a \neq 0$*;*
(3) $n = 0, m = 0$ *and* $asc < 0$*;*
(4) $s = 0, c = 0$ *and* $mna < 0$,

and then, system (3) does not have bounded chaotic solutions or hyperchaotic solutions.

Proof It is obvious that system (3) does not have bounded chaotic solutions or hyperchaotic solutions if case (1) holds. Furthermore, using the first and third equations of system (3) with integration yields

$$-\frac{x^2}{2} + \frac{sz^2}{2} + mz = \int_0^t \left[ax^2 - scz^2 - (mc + sn)z - mn\right] dt. \quad (7)$$

Under assumption (2–4), the expression of the right-hand side of (7) is monotonic. Thus, the polynomial $-\frac{x^2}{2} + \frac{sz^2}{2} + mz$, as a function of time, has a limit $L \in R$ as t tends to infinity. If L is finite, then any attractor for system (3) lies on the surface $-\frac{x^2}{2} + \frac{sz^2}{2} + mz$ and is not chaotic. If $L = \pm\infty$, then at least one of the two variables is not bounded and cannot be chaotic. The proof is complete. □

2.3 Initial conditions and coexisting attractors in system (3)

Now, we investigate the influence of initial conditions on the dynamics of system (3). A small change in the initial condition of the system can cause wide difference in trajectories. Here, we give the two groups of parameters for coexisting attractors:

Case I. $(m, a, s, b, c, n, k) = (35, 35, 35, -17, 0.8, 4, 20)$

(IA) For the initial condition $(0, 1, -0.5, 0)$, the Lyapunov exponents of system (3) are found to be $L_1 = 1.2908$, $L_2 = 0.2086$, $L_3 = 0.0 and L_4 = -20.2994$. A hidden hyperchaotic attractor exists with the unique equilibrium.

(IB) For the initial condition (0,1,-1,0), trajectories converge to the stable equilibrium, with the Lyapunov exponents $L_1 = -0.7997$, $L_2 = -2.2035$, $L_3 = -2.2479 and L_4 = -13.5489$. There do not exist chaotic attractors.

Case II. $(m, a, s, b, c, n, k) = (35, 35, 32, -17, 0.8, 4, 20)$

(IIA) For the initial condition $(1, 1, 5, 0)$, the Lyapunov exponents of the system are $L_1 = 1.3115$, $L_2 = 0.1402$, $L_3 = 0.0 and L_4 = -20.2517$. A hyperchaotic attractor is obtained.

(IIB) For the initial condition $(0.1, 0, 0, 1)$, the Lyapunov exponents of the system are $L_1 = 1.2438$, $L_2 = 0.0003$, $L_3 = -0.0433 and L_4 = -20.0008$. A chaotic attractor is found.

Slightly changing the initial values can cause large variations in dynamical behavior of system (3). For different initial conditions, trajectories may go to different attractors. When attractors occur in a system, researchers are usually interested in obtaining the basins of attractors, defined as the set of initial points whose trajectories finally reach the attractor. It is noted that the attraction not only depends on the parameter values but also the initial conditions.

3 Generation of hidden attractors

Hartman–Grobman theorem tells us that the dynamical behavior of an autonomous system in the neighborhood of a hyperbolic equilibrium is qualitatively the same as (or topologically equivalent to) that of its linearized system near this equilibrium. Therefore, we can find the hidden attractors by amplitude control.

It is worth noting that coexisting attractors and thus the fractal basin may not be observed in a controlled experiment where system parameters are smoothly varied. In such instances, the initial condition and coordinate transformation for each parameter value are the final conditions (or state) for the previous parameter and the trajectories are thus locked on only one of the attracting sets.

Under the following linear transformation

$$x_2 = x,\ y_2 = y,\ z_2 = \frac{z}{\mu},\ w_2 = w,$$

system (3) becomes

$$\begin{cases} \dot{x}_2 = my_2 - ax_2 + s\mu y_2 z_2, \\ \dot{y}_2 = -by_2 - \mu x_2 z_2 + w_2, \\ \dot{z}_2 = -cz_2 + \frac{2}{\mu}x_2 y_2 - \frac{n}{\mu}, \\ \dot{w}_2 = -ky_2, \end{cases} \tag{8}$$

where μ is a real parameter ($\mu \neq 0$).

No matter how μ changes, the characteristic equations of system (8) and system (3) are identical at the corresponding equilibrium, meaning that system (8) is topologically equivalent to system (3). Although system (3) has only one hyperbolic stable equilibrium, the transformed system is still chaotic for certain values of the parameter μ.

Considering $m = 35, s = 35, a = 35, b = -17, c = 0.8, n = 4, k = 12$ and the initial condition $(0.2, 0, -5, 1)$, when the parameter μ is arbitrarily changing ($\mu \neq 0$), the equilibrium of system (8) is stable. We can still obtain the coexistence of hidden hyperchaotic attractors and the unique equilibrium for $\mu \in [-5, -0.01]$, confirmed by computing the Lyapunov exponents as shown in Fig. 3.

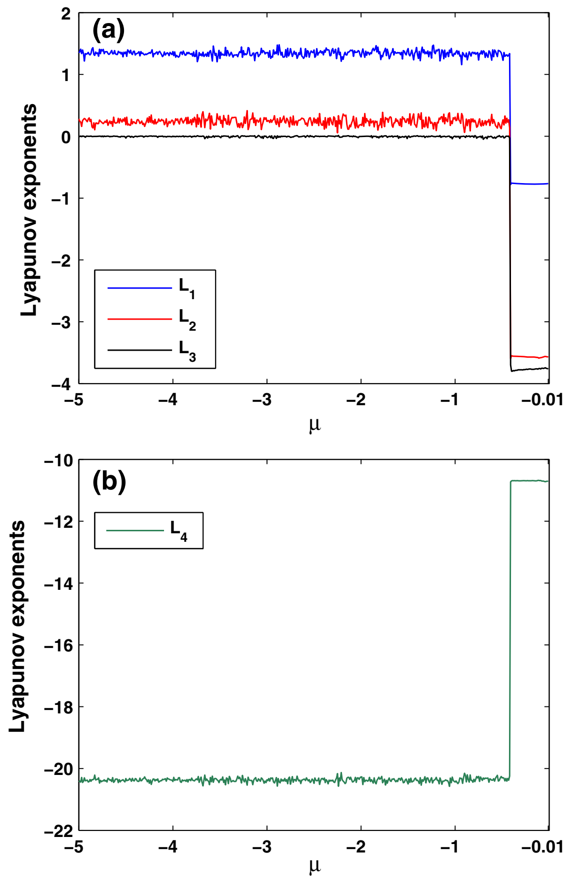

Fig. 3 Lyapunov exponents of system (9) with only one stable equilibrium for the parameter values $m = 35, a = 35, s = 35, b = -17, c = 0.8, n = 4, k = 12$, and $\mu \in$ in $[-5, -0.01]$

4 Local bifurcation in the generalized hyperchaotic Rabinovich system

4.1 Equilibrium and stability

Firstly, it is easy to see that under the coordinate transformation,

$$S : (x, y, z, w) \to (-x, -y, z, -w),$$

system (3) is invariant.

When $c = 0$, there is no equilibrium. When $c \neq 0$, system (3) has a unique equilibrium E_0 $(0, 0, -\frac{n}{c}, 0)$.

By liberalization around E_0, the Jacobian matrix of system (3) is given by

$$J(E_0) = \begin{pmatrix} -a & m+sz & sy & 0 \\ -z & -b & -x & 1 \\ y & x & -c & 0 \\ 0 & -k & 0 & 0 \end{pmatrix}. \tag{9}$$

The characteristic equation evaluated at E_0 is given by

$$\det(\lambda I - J(E_0)) = (\lambda + c) \left[ak + \left(ab + k - \frac{mn}{c} + \frac{n^2 s}{c^2}\right) \lambda + (a+b)\lambda^2 + \lambda^3\right] = 0. \tag{10}$$

According to the Routh–Hurwitz criterion [38], the real parts of all the roots of the polynomial are negative if and only if

$$c > 0,\ \Delta_1 = a + b > 0,$$
$$\Delta_2 = \begin{vmatrix} a+b & 1 \\ ab + k - \frac{mn}{c} + \frac{n^2 s}{c^2} & ak \end{vmatrix} > 0,$$
$$\Delta_3 = ak\Delta_2 > 0.$$

Therefore, E_0 is asymptotically stable if and only if the following conditions

$$a + b > 0,\ c > 0,\ ak > 0 \tag{11}$$

and

$$s > -\frac{c\left(a^2 bc + ab^2 c + bck - amn - bmn\right)}{(a+b)n^2} \tag{12}$$

hold.

4.2 Hopf bifurcation

It has been shown in [32] that for general n-dimensional dynamical systems, the necessary condition for the system to have a Hopf bifurcation is $\Delta_{n-1} = 0$. For our system, this condition leads to

$$s = -\frac{c\left(a^2 bc + ab^2 c + bck - amn - bmn\right)}{(a+b)n^2}.$$

In this section, we want to investigate generalized Hopf bifurcations which may occur from the unique equilibrium, and in particular, we are interested in studying how many limit cycles which can bifurcate from this critical point.

The method for studying Hopf and generalized Hopf bifurcations is mainly based on center manifold theory and normal form theory, which can be found in many textbooks (e.g., see [34]), and the computation of finding the normal forms associated with Hopf bifurcation can be found, for example, in [35]. Roughly speaking, for a general system

$$\dot{x} = f(x, \mu), \quad x \in R^n, \quad \xi \in R, \tag{13}$$

where x is n-dimensional vector and ξ is a parameter. Suppose $x = 0$ is an equilibrium solution of the system for any real values of ξ. (When the equilibrium is not at the origin, one can apply a simple shift to make it zero.) Further, assuming the Jacobian matrix of system (13), $J(\xi) = D_x f(0, \xi)$ has a purely pair of imaginary, $\lambda_{1,2} = \pm i\,\omega_c$ ($\omega_c > 0$) at a critical point $\xi = \xi_c$, and other eigenvalues have nonzero real part. Thus, we consider perturbation to the critical point and assume $\xi = \xi_c + \mu$. Then, the purely imaginary pair becomes a complex conjugate $\lambda_{1,2} = \alpha(\xi) \pm i\,\omega(\xi)$, satisfying $\alpha(\xi_c) = 0$ and $\omega(\xi_c) = \omega_c$. Further, if $\frac{\mathrm{d}\alpha(\xi)}{\mathrm{d}\xi}|_{\xi=\xi_c} \neq 0$, then Hopf bifurcation appears, giving rise to bifurcating of a family of limit cycles. In order to study the stability of bifurcating limit cycles, we may apply center manifold theory and normal form theory to obtain the normal form associated with Hopf bifurcation as

$$\frac{\mathrm{d}r}{\mathrm{d}t} = r\left(v_0\mu + v_1 r^2 + v_2 r^4 + \cdots\right),$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega_c + t_0\mu + t_1 r^2 + t_2 r^4 + \cdots, \tag{14}$$

where r and θ represent the amplitude and phase of periodic solutions, respectively; v_0 and t_0 can be obtained from a linear analysis, while computing v_i and t_i needs a nonlinear analysis such as normal form computation (e.g., see [36]); v_i is called the ith-order focus value. When $v_1 \neq 0$, it is called Hopf bifurcation; when $v_i = 0,\ i = 1, 2, \ldots, k$ ($k \geq 1$), it is called a generalized Hopf bifurcation. $v_1 \neq 0$ will usually give one isolated limit cycle. To obtain more limit cycles, we use system parameters to set $v_i = 0,\ i = 0, 1, \ldots, k-1$, but $v_k \neq 0$, and obtain at most k limit cycles near the equilibrium around the critical point. Next, proper perturbations on these critical values will yield k limit cycles.

For our system (3), in order to simplify the computation of normal forms, we first use scaling to simplify system. To achieve this, introduce the time scaling $\tau = ct$ and the following state and parameter scaling:

$$x = cX, \quad y = \frac{n}{c}Y, \quad x = \frac{n}{c}Z, \quad w = nW,$$

and

$$m = \frac{c^3}{n} M, \quad a = cA, \quad s = \frac{c^4}{n^2} S, \quad b = cB, \quad k = c^2 K,$$

where M, A, S, B and K are scaled parameters. Without loss of generality, we assume $c > 0$ and $n > 0$, which keeps the sign of all new state variables and new parameters unchanged. Under the above scaling, we obtain the following new system,

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}\tau} = MY - AX + SYZ, \\ \frac{\mathrm{d}Y}{\mathrm{d}\tau} = -BY - XZ + W, \\ \frac{\mathrm{d}Z}{\mathrm{d}\tau} = -Z + XY - 1, \\ \frac{\mathrm{d}W}{\mathrm{d}\tau} = -KY, \end{cases} \tag{15}$$

and so the new equilibrium is given by $X = Y = W = 0$, $Z = -1$, and the Hopf critical point becomes

$$S = S_c = M - AB - \frac{BK}{A + B}. \tag{16}$$

Note that the number of parameters has been reduced from seven to five. In order to apply the Maple programs [36] to compute the normal form coefficients (or the focus values), we first need to transform system (15) such that the linear part of the resulting system is in Jordan canonical form. Therefore, introducing the following transformation

$$\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} + T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{17}$$

where the linear transformation T is given by

$$T = \begin{bmatrix} -\frac{B}{A+B} & -\frac{B}{K}\sqrt{\frac{AK}{A+B}} & 0 & -\frac{1}{K}[A(A+B)+K] \\ 0 & -\frac{1}{K}\sqrt{\frac{AK}{A+B}} & 0 & \frac{A+B}{K} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

into the scaled system (16), we obtain

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}\tau} &= \sqrt{\frac{AK}{A+B}}\, x_2 + f_1(x, \beta), \\ \frac{\mathrm{d}x_2}{\mathrm{d}\tau} &= -\sqrt{\frac{AK}{A+B}}\, x_1 + f_2(x, \beta), \\ \frac{\mathrm{d}x_3}{\mathrm{d}\tau} &= -x_3 + f_3(x, \beta), \\ \frac{\mathrm{d}x_4}{\mathrm{d}\tau} &= -(A + B)\, x_3 + f_4(x, \beta), \end{aligned} \tag{19}$$

where x is a 4D state vector, $x = (x_1, x_2, x_3, x_4)^T$, and β is a parameter vector, $\beta = (M, A, B, K)^T$. It is now seen that the linear part of the above system is in Jordan canonical form, consisting of a purely pair of imaginary $\pm i\sqrt{\frac{AK}{A+B}}$ and two real eigenvalues -1 and $-(A + B)$. To ensure that the equilibrium is stable before the Hopf bifurcation, we assume $A + B > 0$.

Now we apply the Maple programs [36] to obtain the normal form (14) with

$$\begin{aligned} v_1 = &- \frac{BK^2(A+B)^2}{2(A+B+4AK)[(A+B)^3+AK]^3} \\ &\Big\{4AB(4A+4B-1)\,K^2 + (A+B) \\ &\Big[4A^2B(4A+4B-1) + 6B(A+B) \\ &- \Big(4(A+B)(2A-1) + 2B - 1\Big)\,M\Big]\,K \\ &+ 3(A+B)^3(2AB - M)\Big\}, \\ v_2 = &\ \cdots \\ v_3 = &\ \cdots \\ v_4 = &\ \cdots \end{aligned} \tag{20}$$

where the lengthy expressions of v_2, v_3 and v_4 are omitted for brevity. The focus values v_i are functions of the four free parameters: M, A, B, K. Note that we have restrictions on these parameters: $(A+B) > 0$ and $AK > 0$. Without these restrictions, the best result we can obtain is five small-amplitude limit cycles bifurcating from the Hopf critical point, that is, we may find the critical values of the four parameters such that $v_1 = v_2 = v_3 = v_4 = 0$, but $v_5 \neq 0$. (Note that the parameter S has been used to set $v_0 = 0$.)

Since v_1 is linear with respect to M, we can solve M from $v_1 = 0$, and then, v_i, $i = 2, 3, 4$ becomes functions of three parameters A, B, K. But they are nonlinear polynomial functions. Applying the approaches used in, we can show that there do not exist solutions such that $v_2 = v_3 = v_4 = 0$ satisfying $A + B > 0$ and $AK > 0$. Therefore, the best choice we can have is $v_2 = v_3 = 0$ but $v_4 \neq 0$, leaving one parameter free. This way, we can have infinitely many solutions for which there may exist four small-amplitude limit cycles. One solution is given below:

$$\begin{aligned} &A = 3, \quad B = -2.5087446615\cdots, \\ &K = 2.1605267634\cdots, \quad M = -54.8827664176\cdots \end{aligned} \tag{21}$$

and the Hopf critical point is defined by

$S_c = -36.3231460051\cdots.$

For these critical values, the focus values become $v_0 = v_1 = v_2 = v_3 = 0$ and $v_4 = 0.0000387891\cdots$. Further, we can show that

$$\frac{\partial(v_2, v_3)}{\partial(A, K)} = -0.7404608381\cdots \times 10^{31} \neq 0, \qquad (22)$$

implying that there exist four small-amplitude limit cycles which can bifurcate from the equilibrium from the Hopf critical point.

Remark 4.1 Based on precise symbolic computation of singular point quantities, we discuss the center focus problem for the system (3) restricted to the center manifold, which closely relates to the maximum number of limit cycles bifurcating from the equilibrium. The work can be applied to general differential systems associated with Hopf bifurcation. It is expected that more new detailed theory and results about characters of limit cycles for the system (3).

5 Boundedness of motion for the hyperchaotic system

In this section, we will discuss the boundedness of motion for the hyperchaotic systems (3).

Theorem 5.1 *Denote* $\Omega_\phi = \left\{(x, y, z, w)|x^2 + (s+\phi)y^2 + \phi(z - \frac{m}{\phi})^2 + \frac{s+\phi}{k}w^2 \leq R^2\right\}$, *where*

$$R^2 = \begin{cases} \dfrac{(cm+n\phi)^2}{c^2\phi}, c < min\{2a, 2b\} \\ \dfrac{(cm+n\phi)^2}{4a\phi(c-a)}, c \geq 2a \\ \dfrac{(cm+n\phi)^2}{4b\phi(c-b)}, c \geq 2b. \end{cases} \qquad (23)$$

If $m > 0, a > 0, b > 0, c > 0, s > 0, k > 0$ *and* $\phi > 0$, *then all orbits of system (3), including hidden hyperchaotic attractors, are trapped into a bounded region, and so the hyperellipsoid* Ω_ϕ *is an ultimate bound and positively invariant set for system (3).*

Proof First, we define the following generalized positive definite and radially unbounded Lyapunov function

$$V_p(x, y, z, w) = x^2 + (s+\phi)y^2 + \phi\left(z - \frac{m}{\phi}\right)^2 + \frac{s+\phi}{k}w^2,$$

where $\phi > 0, s > 0, k > 0$. Computing the derivative of $V_p(x, y, z, w)$ with respect to time t along a trajectory of (3), we have

$$\frac{1}{2}\frac{V_p(x, y, z, w)}{dt}\bigg|_{(3)} = -ax^2 - b(s+\phi)y^2 - c\phi z^2 + (cm - n\phi)z + mn.$$

That is to say, for $a > 0, b > 0, c > 0, \phi > 0$, the surface, defined by

$$\Gamma = \left\{(x, y, z, w)\middle| ax^2 + b(s+\phi)y^2 + c\phi\left(z - \frac{cm - n\phi}{2c\phi}\right)^2 = mn + \frac{(cm-n\phi)^2}{4c\phi}\right\},$$

is an ellipsoid in 4D space for certain values of a, b, c, m, n, ϕ and k. Outside Γ, $\frac{V_p(x, y, z, w)}{\mathrm{d}t}\Big|_{(3)} < 0$, attracting the trajectories outside Γ to move toward Γ. The ultimate bound for system (3) can only be reached on Γ.

Next, we further use the Lagrange multiplier method to obtain the optimal value of V on Γ. Define

$$F = x^2 + (s+\phi)y^2 + \phi\left(z - \frac{m}{\phi}\right)^2 + \frac{s+\phi}{k}w^2 + \tau\left\{ax^2 + b(s+\phi)y^2 + c\phi\left(z - \frac{cm-n\phi}{2c\phi}\right)^2 - mn - \frac{(cm-n\phi)^2}{4c\phi}\right\},$$

and let

$$\frac{1}{2}F_x^{'} = x + ax\tau = 0,$$

$$\frac{1}{2}F_y^{'} = y(1+b\tau)(s+\phi) = 0,$$

$$\frac{1}{2}F_z^{'} = -\left(1 + \frac{c\tau}{2}\right)\left(m + \frac{n\phi}{c}\right) + \left(z + \frac{n}{c}\right)(1 + c\tau) = 0,$$

$$\frac{1}{2}F_w^{'} = \frac{w(s+\phi)}{k} = 0,$$

$$F_\tau^{'} = ax^2 + b(s+\phi)y^2 + c\phi\left(z - \frac{cm-n\phi}{2c\phi}\right)^2 - mn - \frac{(cm-n\phi)^2}{4c\phi} = 0.$$

Thus,

(1) when $\tau = -\frac{2}{c}, \tau \neq -\frac{1}{a}, \tau \neq -\frac{1}{b}$, we have

$$(x, y, z, w) = \left(0, 0, -\frac{n}{c}, 0\right),$$

and $V_p(x, y, z, w)_{\max} = \dfrac{(cm + n\phi)^2}{c^2\phi}$.

(2) when $\tau = -\frac{1}{a}, \tau \neq -\frac{2}{c}, \tau \neq -\frac{1}{b}$ and $c \geq 2a$, we have

$$(x, y, z, w) = \left(\pm\frac{\sqrt{c-2a}(cm+nq)}{2(a-c)\sqrt{a\phi}}, 0, \frac{2am - cm + n\phi}{2\phi(a-c)}, 0\right),$$

and $V_p(x, y, z, w)_{\max} = \dfrac{(cm + n\phi)^2}{4a\phi(c-a)}$.

(3) when $\tau = -\frac{1}{b}, \tau \neq -\frac{2}{c}, \tau \neq -\frac{1}{a}$ and $c \geq 2b$, we have

$$(x, y, z, w) = \left(0, \pm\frac{\sqrt{c-2b}(cm+n\phi)}{2(b-c)\sqrt{b\phi(s+\phi)}}, \frac{2bm - cm + n\phi}{2\phi(b-c)}, 0\right),$$

and $V_p(x, y, z, w)_{\max} = \dfrac{(cm + n\phi)^2}{4b\phi(c-b)}$.

Figure 4 shows the hyperchaos and the corresponding ultimate bound and positively invariant set for $a = 4, m = 10, s = 1, b = 0.5, n = 10, k = 2, c = 1$ and $\phi = 1$. According to (11) and (12), the unique equilibrium E_0 is stable. However, for the initial condition (0.2, 0, 4, 10), the Lyapunov exponents of system (3) are found to be $L_1 = 0.3156$, $L_2 = 0.0831$, $L_3 = 0.0 and L_4 = -5.8988$. Therefore, the hidden hyperchaotic attractor can be obtained and is in the domain bounded by

$$\Omega_\phi = \left\{(x, y, z, w) | x^2 + 2y^2 + (z-10)^2 + w^2 \leq 400\right\},$$

and thus, the estimation of the bounds given in Theorem 5.1 is feasible. □

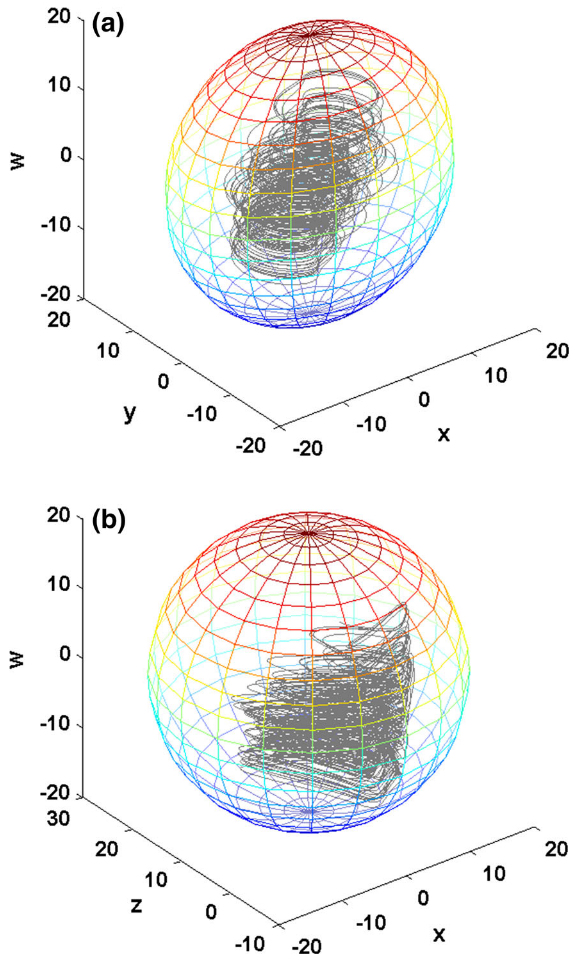

Fig. 4 The bound estimated for the hyperchaotic attractor of system (3) for $m = 10, a = 4, s = 1, b = 0.5, n = 10, k = 2, c = 1$ and $\phi = 1$. (a) projected in the (x, y, w) space on the section $z = 10$, (b) projected in the (x, z, w) space on the section $y = 0$

6 Conclusion

In this paper, a generalized hyperchaotic Rabinovich system has been investigated. The hyperchaotic attractors can coexist with only one stable equilibrium, which is different from that of the existing 4D chaotic systems. An amplitude control provides a tool for identifying hidden attractors with fixed initial conditions and parameter values. The existence of hidden attractors that may render the system's behavior unpredictable is not only due to variation of the system parameters but also due to change of the initial conditions. Some basic properties of the new system have been investigated in terms of chaotic motions, Lyapunov exponent spectrum, bifurcation diagram and associated Poincaré maps. Four limit cycles bifurcating from the unique equilibrium are obtained via Hopf bifurcation. The analysis of finding the ultimate bound of the chaotic attractors in the 4D dynamical system is given.

The results are not only identical with and complementary to the previous work on Hopf bifurcation in 4D Rabinovich system, but also helpful to compare the related hyperchaotic systems. There are still abundant and complex dynamical behaviors, and the topological structure of the new system should be completely and thoroughly investigated and exploited.

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