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Complex Dynamics Due to Multiple Limit Cycle Bifurcations in a Tritrophic Food Chain Model

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In this paper, we consider a tritrophic food chain model with Holling functional response types III and IV for the predator and superpredator, respectively. The main attention is focused on the stability and bifurcation of equilibria when the prey has a linear growth. Coexistence of different species is shown in the food chain, demonstrating bistable phenomenon. Hopf bifurcation is studied to show complex dynamics due to multiple limit cycles bifurcation. In particular, normal form theory is applied to prove that three limit cycles can bifurcate from an equilibrium in the vicinity of a Hopf critical point, yielding a new bistable phenomenon which involves two stable limit cycles.

Keywords: Food chain model; stability; Hopf bifurcation; limit cycle; bistable.

1. Introduction

After the pioneering work of Lotka and Volterra [Lotka, 1925; Volterra, 1926], the study of ditrophic food chains has received and been continuously receiving much attention in mathematical ecology. The classical ecological models of interacting populations often have focused on two species. Continuous time models of two interacting species, usually called prey–predator models, have been analyzed extensively, and some notable achievements have been made, for example, see [Bazykin, 1998; Kuang & Beretta, 1998]. Mathematically, these models can exhibit only two basic patterns: Trajectories approach a steady state or a limit cycle. However, the ecological communities in nature have been observed to exhibit much more complex dynamics. Price *et al.* [1980] argued that community behavior must be based on three or more trophic levels. Continuous time models with three species have been reported to have more complicated patterns. The existence of limit cycles, multiplicity of attractors, and catastrophic bifurcations are the characteristics of the models which have been used to explain complex behaviors observed in the field. The research in the past two decades has demonstrated that the complex dynamical behavior, including quasi-periodic motion or even chaos, can arise in continuous time ecological models with three or more species. Much attention has been paid to the study of the transition from periodic oscillation to chaotic motion. Understanding the mechanism of generating such behaviors constitutes an exercise of paramount importance. In the late 1970s, some interest in the mathematics of tritrophic food chain models emerged. One important model is called tritrophic food chain because each population except the lowest eats only the one on the immediate lower trophic level [Freedman & So, 1985]. Our attention in this article is to investigate whether or not three species can exist simultaneously. The typical

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ecological interaction has been studied from a mathematical point of view through a differential equation system, showing the coexistence of species translating to the existence of a stable limit cycle. Some authors dealt with the problem of persistence [Freedman & So, 1985; Gard, 1980], but did not provide information on the number and the geometry of the attractors. Hogeweg and Hesper [1978] showed through simulation that a particular food chain model can behave chaotically, however, this paper did not receive much attention. Later, Hastings and Powell showed in [Hastings & Powell, 1991] that food chains behave chaotically on a "tea-cup" strange attractor, and the three populations have diversified time responses increasing from bottom to top. Around the same time, Muratori and Rinaldi [1991] performed a singular perturbation analysis to confirm that the tea-cup geometry is the result of the interaction between high frequency (prey–predator) oscillation and low frequency (predator–top-predator) oscillation. Since then, particular effort has been devoted to the study of the complex dynamics of food chain systems, and bifurcation analysis has been the main tool of investigation.

There have been many studies on the subject of Hopf bifurcation. Among them, the main research is focused on the analysis of limit cycles and stability. Yu *et al.* [2017] studied a bacteriophage model that includes prophage, focusing on asymptotic behavior of the solutions, and proved the existence of a supercritical Hopf bifurcation with stable limit cycles. Zhang *et al.* [2014] studied a newly autoimmune four-dimensional disease model, and proved the existence of a supercritical Hopf bifurcation, leading to a family of stable limit cycles. The study of Hopf bifurcation has also been used to investigate the behaviors in simple disease models arising from epidemiology, in-host disease and autoimmunity (e.g. see [Zhang *et al.*, 2016]). Recently, there are also some studies on the tritrophic food chain models, in which the coexistence of multiple species is shown by proving the existence of a stable limit cycle. For example, Francois and Llibre [2011] used averaging method to prove the existence of a stable periodic orbit contained in the region where all variables are positive. Castellanos *et al.* [2013] studied linear growth of the prey and functional response of Holling type III [Dawes & Souza, 2013] for the middle and top species, and showed a double zero-Hopf bifurcation in the positive octant of $\mathbb{R}^3$ using the averaging method. In [Blé *et al.*, 2016], the authors studied the case when the prey has linear growth, while the middle and top species have functional responses of Holling types II and III [Dawes & Souza, 2013], respectively, and proved the existence of a stable limit cycle in the region of interest. In [Castellanos & Chan-Lòpez, 2017], the authors considered the case when the prey has logistic growth, while the predator and superpredator have the functional responses of Lotka–Volterra type and Holling type II [Dawes & Souza, 2013], respectively, forming a differential system based on the Leslie-scheme. In that paper, the authors also computed the first Lyapunov coefficient explicitly and showed the existence of a stable limit cycle. Moreover, they demonstrated by simulation a strange attractor which provides evidence that the model exhibits chaotic dynamics. In this paper, our main interest is in the tritrophic food chain model characterized by the fact that the prey growth rate is linear in the absence of the predators, while the functional responses for the middle and top species in the chain are Holling type III and Holling type IV, respectively.

The general tritrophic food chain model with three species is described by the following three ordinary differential equations [Castellanos *et al.*, 2018]:

$$\begin{aligned} \dot{x} &= h(x) - f(x)y, \\ \dot{y} &= c_1 y f(x) - g(y)z - \mu y, \\ \dot{z} &= c_3 g(y)z - d_2 z, \end{aligned} \tag{1}$$

where the dot denotes differentiation with respect to time t; x, y and z represent respectively the densities of the bottom, the middle and the top species in the chain. The function $h(x)$ represents the growth rate of prey in the absence of the other species, and $h(x)$ is assumed linear in this study. The functions $f(x)$ and $g(y)$ are the functional responses of the predator y and superpredator z, respectively. All the parameters are positive. The parameters c_1 and c_3 represent the benefits from the consumption of food, and the parameters μ and d_2 represent the mortality rate of the corresponding predators. For ecological study, the region of interest in $\mathbb{R}^3$ is the positive octant $\Omega = \{(x, y, z) \in \mathbb{R}^3 \,|\, x > 0, y > 0, z > 0\}$.

In this paper, we consider the case when f is Holling type III and g is Holling type IV, which are given explicitly in the form of

$$f(x) = \frac{a_1 x^2}{x^2 + b_1}, \quad g(y) = \frac{a_2 y}{y^2 + b_2}, \tag{2}$$

where a_1, b_1, a_2, b_2 are positive parameters.

The rest of the paper is organized as follows. In the next section, we present some methodologies and preliminary results, which are needed for the analysis in the following sections. In Sec. 3, we provide a linear analysis for system (1), in particular on the stability and bifurcation of the positive equilibrium. In Sec. 4, we prove the existence of three limit cycles around the positive equilibrium, arising from Hopf bifurcation. In Sec. 5, numerical simulation is given to show a good agreement with our analytical prediction. Finally, the conclusion and discussion are drawn in Sec. 6.

2. Methodology and Preliminary Results

In this section, we present some basic methods and preliminary results which will be used in the following sections.

2.1. *Linear theory*

Firstly, we present some results and formulas for general dynamical systems to study the stability of equilibrium solutions. Consider the general nonlinear differential system:

$$\dot{x} = f(x, \mu), \quad x \in R^n, \ \mu \in R^m, \ f : R^{n+m} \mapsto R^n, \tag{3}$$

where x and μ denote n-dimensional state variable and m-dimensional parameter variable, respectively. Assume that the nonlinear function $f(x, \mu)$ is analytic with respect to x and μ. Suppose that an equilibrium solution of (3) is given in the form of $x_e = x_e(\mu)$, which is determined from $f(x, \mu) = 0$. In order to analyze the stability of x_e, evaluating the Jacobian of system (3) at $x = x_e(\mu)$ yields $J(\mu) = D_x f|_{x=x_e(\mu)}$. If all eigenvalues of $J(\mu)$ have nonzero real parts, then the system is said to be hyperbolic, that means no complex dynamics exists in the vicinity of the equilibrium. Otherwise, at least one of the eigenvalues of $J(\mu)$ has zero real part at a critical point, defined by $\mu = \mu_c$, and bifurcations may occur from $x_e(\mu)$. To determine the stability of the equilibrium, we firstly find the eigenvalues of the Jacobian $J(\mu)$, which are the roots of the characteristic polynomial equation:

$$\begin{aligned} P_n(\lambda) &= \det[\lambda I - J(\mu)] \\ &= \lambda^n + a_1(\mu)\lambda^{n-1} + a_2(\mu)\lambda^{n-2} \\ &\quad + \cdots + a_{n-1}(\mu)\lambda + a_n(\mu) = 0. \end{aligned} \tag{4}$$

For a fixed value of μ, if all the roots of the polynomial $P_n(\lambda)$ have negative real part, then the equilibrium is asymptotically stable for this value of μ. If at least one of the eigenvalues has zero real part as μ is varied to cross a critical point μ_c, then the equilibrium becomes unstable at μ_c and bifurcation occurs from this critical point. When all the roots of $P_n(\lambda)$ have negative real part, we call $P_n(\lambda)$ a stable polynomial, otherwise an unstable polynomial.

In general, for $n \geq 3$, it is hard to find the roots of $P_n(\lambda)$. Thus we use the Routh–Hurwitz Criterion [Hinrichsen & Pritchard, 2005] to analyze the local stability of the equilibrium solution $x = x_e(\mu)$. The criterion gives sufficient conditions under which the equilibrium is locally asymptotically stable, i.e. all the roots of the characteristic polynomial $P_n(\lambda)$ have negative real part. The conditions are given by

$$\Delta_i(\mu) > 0, \quad i = 1, 2, \ldots, n, \tag{5}$$

where $\Delta_i(\mu)$ is called the ith-principal minor of the Hurwitz arrangements of order n, defined as follows (here, order n means that there are n coefficients a_i $(i = 1, 2, \ldots, n)$ in Eq. (4), which construct the Hurwitz principal minors):

$$\begin{aligned} &\Delta_1 = a_1, \quad \Delta_2 = \det\begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix}, \\ &\Delta_3 = \det\begin{bmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{bmatrix}, \quad \ldots, \quad \Delta_n = a_n \Delta_{n-1}. \end{aligned} \tag{6}$$

Assume that as μ is varied to reach a critical point $\mu = \mu_c$, at least one of the Δ_is becomes zero. Then the fixed point $x_e(\mu_c)$ becomes unstable, and μ_c is called critical point. It can be seen from Eq. (5) that if $a_n(\mu) = 0$, but other Hurwitz arrangements are still positive (i.e. $\Delta_n = 0$, $\Delta_i(\mu) > 0$, $i = 1, 2, \ldots, (n-1)$), then $P_n(\lambda) = 0$ has one zero root. In this case, system (3) has a simple zero singularity and a static bifurcation occurs from x_e. In other cases, for example, Hopf bifurcation occurs at a critical point when $P_n(\lambda) = 0$ has a pair of purely imaginary eigenvalues $\pm i\omega$ $(\omega > 0)$ at this point. However, the pair of purely imaginary eigenvalues are often difficult to be determined explicitly for high dimensional systems. Here, we present the following theorem without computing the eigenvalues of the Jacobian of a general system. The theorem gives the necessary and sufficient conditions for determining a Hopf critical point based on the Hurwitz criterion. Its proof can be found in [Yu, 2005].

Theorem 1 [Yu, 2005]. *The necessary and sufficient conditions for system (3) to have a Hopf bifurcation at an equilibrium solution $x = x_e$ is $\Delta_{n-1} = 0$, with other Hurwitz conditions being still held, i.e. $a_n > 0$ and $\Delta_i > 0$, for $i = 1, \ldots, n-2$.*

Next, we present a method for computing focus values. There are many methods developed for computing the focus values of planar vector fields, such as Poincaré–Takens method [Guckenheimer & Holmes, 1983], the perturbation method [Yu, 1998], the singular point value method [Liu & Li, 1990], etc. But, for higher dimensional dynamical systems, the computation is much involved. In the next two subsections, we briefly introduce the method of normal forms for computing the focus values of general n-dimensional dynamical systems. The general normal form theory can be found in [Guckenheimer & Holmes, 1983; Chow *et al.*, 1994] and computations using computer algebra systems can be found in [Han & Yu, 2012; Tian & Yu, 2013].

2.2. *Normal form theory*

Consider the following general n-dimensional differential system:

$$\dot{z} = Az + f(z), \quad z \in R^n, \quad f : R^n \to R^n, \tag{7}$$

where Az and $f(z)$ represent the linear and nonlinear parts of the system, respectively. We assume that $z = 0$ is a fixed point of the system, which implies that $f(0) = Df(0) = 0$. It is also assumed that $f(z)$ is analytic and can be expanded in Taylor series about z, and system (7) only contains stable and center manifolds. In the computation of normal forms, the first step is usually to introduce a linear transformation into (7) such that the linear part of (7) can be put in the Jordan canonical form. Without loss of generality, suppose that under the linear transformation $z = T(x, y)$, system (7) becomes

$$\begin{aligned}\dot{x} &= J_1 x + f_1(x, y), \quad x \in R^k, \quad f_1 : R^n \to R^k,\\ \dot{y} &= J_2 y + f_2(x, y), \quad y \in R^{n-k}, \quad f_2 : R^n \to R^{n-k},\end{aligned} \tag{8}$$

where $J_1 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$, and $J_2 = \mathrm{diag}(\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n)$, with $\mathrm{Re}(\lambda_j) = 0$, $j = 1, 2, \ldots, k$ and $\mathrm{Re}(\lambda_j) < 0$, $j = k+1, \ldots, n$.

Next, we apply center manifold theory [Carr, 2012] to system (8), leading to that y can be expressed as $y = H(x)$, satisfying $H(0) = DH(0) = 0$. Therefore, the first equation of (8) can be rewritten as

$$\begin{aligned}\dot{x} &= J_1 x + f_1(x, H(x))\\ &= J_1 x + f_1^2(x) + f_1^3(x) + \cdots + f_1^s(x) + \cdots,\end{aligned} \tag{9}$$

where $f_1^j \in M_j, j = 2, 3, \ldots,$ M_j defining a linear space of vector fields whose elements are homogeneous polynomials of degree j. Equation (9) describes the dynamics on the center manifold of system (8), and $H(x)$ can be determined from the following equation:

$$\begin{aligned}&DH(x)[J_1 x + f_1(x, H(x))]\\ &\quad - J_2(H(x)) - f_2(x, H(x)) = 0.\end{aligned} \tag{10}$$

Next, by using normal form theory, we introduce the near-identity transformation:

$$\begin{aligned}x &= u + Q(u)\\ &= u + q^2(u) + q^3(u) + \cdots + q^s(u) + \cdots,\end{aligned} \tag{11}$$

where $q^j \in M_j$, $j = 2, 3, \ldots,$ into (9) to obtain the normal form,

$$\begin{aligned}\dot{u} &= J_1 u + C(u)\\ &= J_1 u + c^2(u) + c^3(u) + \cdots + c^s(u) + \cdots,\end{aligned} \tag{12}$$

where $c^j \in M_j$, $j = 2, 3, \ldots$.

In the view point of computation, computing center manifold and normal form seem to be straightforward. However, to design an efficient algorithm is not an easy task. Recently, an explicit recursive formula has been developed for computing the normal form together with center manifold for general n-dimensional differential systems associated with semisimple singularities. We omit the detailed formulas and algorithms, as well as the Maple program here, which can be found in [Tian & Yu, 2013].

2.3. *Bifurcation of multiple limit cycles*

Now we turn to discuss how to determine the maximal number of limit cycles which may bifurcate from a Hopf critical point. Suppose that we have obtained the normal form of system (7), given in the polar coordinates up to the $(2k+1)$th-order term:

$$\begin{aligned}\dot{r} &= r(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k}),\\ \dot{\theta} &= \omega_c + t_1 r^2 + t_2 r^4 + \cdots + t_k r^{2k},\end{aligned} \tag{13}$$

where r and θ denote the amplitude and phase of motion, respectively. v_k and t_k are expressed in terms of the original system's coefficients. v_k is called the kth-order focus value of the origin. The zero-order focus value v_0 is obtained from a linear analysis.

To find k small-amplitude limit cycles of system (7) around the origin, we first find the conditions based on the original system's coefficients such that $v_0 = v_1 = v_2 = \cdots = v_{k-1} = 0$ (note that $v_0 = 0$ is automatically satisfied at the critical point), but $v_k \neq 0$. Then appropriate small perturbations are performed to prove the existence of k limit cycles. In the following theorem, we give sufficient conditions for the existence of k small-amplitude limit cycles. (The proof can be found in [Han & Yu, 2012].)

Theorem 2 [Han & Yu, 2012]. *Suppose that the focus values depend on k parameters, expressed as*

$$v_j = v_j(\epsilon_1, \epsilon_2, \ldots, \epsilon_k), \quad j = 0, 1, \ldots, k,$$

satisfying

$$v_j(0, \ldots, 0) = 0, \quad j = 0, 1, \ldots, k-1,$$

$$v_k(0, \ldots, 0) \neq 0, \quad \textit{and}$$

$$\det\left[\frac{\partial(v_0, v_1, \ldots, v_{k-1})}{\partial(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)}(0, \ldots, 0)\right] \neq 0.$$

Then, for any given $\epsilon_0 > 0$, there exist $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ and $\delta > 0$ with $|\epsilon_j| < \epsilon_0$, $j = 1, 2, \ldots, k$ such that the equation $\dot{r} = 0$ has exactly k real positive roots [*i.e. the system has exactly k limit cycles*] *in a δ-ball with its center at the origin.*

3. Bifurcation Analysis of System (1)

In this section, we consider the differential system (1) where the functional responses are given in (2) and the growth rate of prey is linear. So, the function h is written as $h(x) = \rho x$ and the differential system that we will analyze has the form,

$$\begin{aligned}\dot{x} &= \left(\rho - \frac{a_1 x y}{b_1 + x^2}\right) x,\\ \dot{y} &= \left(-\mu + \frac{a_1 c_1 x^2}{b_1 + x^2} - \frac{a_2 z}{b_2 + y^2}\right) y,\\ \dot{z} &= \left(-d_2 + \frac{a_2 c_3 y}{b_2 + y^2}\right) z.\end{aligned} \tag{14}$$

It is obvious that $p_b = (0, 0, 0)$ is a boundary equilibrium solution of system (14). The Jacobian matrix of system (14) evaluated at the equilibrium p_b is given by

$$\begin{bmatrix}\rho & 0 & 0\\ 0 & -\mu & 0\\ 0 & 0 & -d_2\end{bmatrix},$$

which clearly shows that p_b is a saddle point.

Because it is hard to obtain the explicit expression of the positive equilibrium solution of system (14), we derive the conditions based on the system parameters for the existence of the positive equilibrium in the region of interest. Moreover, we study the stability and find the conditions under which Hopf bifurcation occurs from the positive equilibrium. In [Castellanos *et al.*, 2018], the authors have given the conditions for the existence of the positive equilibrium, and a single limit cycle bifurcating from this positive equilibrium. Here, our main result is to prove the existence of three limit cycles. First, we cite some results from [Castellanos *et al.*, 2018].

It should be noted that for practical systems, proving the existence of a single limit cycle is usually not difficult, but proving the existence of two limit cycles is quite a challenge. It is extremely difficult to prove the existence of three limit cycles. Very few articles have been published to discuss the existence of three limit cycles, for example, see [Gonzàlez-Olivares *et al.*, 2011; Yu & Lin, 2016].

Lemma 1 [Castellanos *et al.*, 2018]. *For system* (*17*) *with positive parameters, the point $p_0 = (x_0, y_0, z_0) \in \Omega$ is an equilibrium if and only if the parameters a_1, b_2 and the third coordinate of p_0 satisfy*

$$a_1 = \frac{\rho(b_1 + x_0^2)}{x_0 y_0}, \quad b_2 = \frac{y_0}{d_2}(a_2 c_3 - d_2 y_0),$$

$$z_0 = \frac{c_3}{d_2}(-\mu y_0 + c_1 x_0 \rho),$$
$$y_0 < \frac{a_2 c_3}{d_2}, \quad -\mu y_0 + c_1 x_0 \rho > 0. \tag{15}$$

Corollary 3.1 [Castellanos *et al.*, 2018]. *With the conditions given as in Lemma 1, if*

$$c_1 = \frac{k_1 + \mu c_3 y_0}{c_3 x_0 \rho}, \quad y_0 < \frac{a_2 c_3}{d_2}, \tag{16}$$

with $k_1 > 0$, *then* $z_0 > 0$ *and the equilibrium is inside* Ω, *given in the form of* $p_0 = (x_0, y_0, \frac{k_1}{d_2})$.

With the parameter values given in Lemma 1 and Corollary 3.1, system (14) becomes

$$\begin{aligned}
\dot{x} &= \rho x - \frac{\rho x^2 y (b_1 + x_0^2)}{x_0 y_0 (b_1 + x^2)}, \\
\dot{y} &= -\frac{a_2 d_2 y z}{y_0(a_2 c_3 - d_2 y_0) + d_2 y^2} + \frac{x^2 y (b_1 + x_0^2)(k_1 + \mu c_3 y_0)}{c_3 x_0^2 y_0 (b_1 + x^2)} - \mu y, \\
\dot{z} &= d_2 z \left(\frac{a_2 d_2 c_3 y}{y_0(a_2 c_3 - d_2 y_0) + d_2 y^2} - 1 \right).
\end{aligned} \tag{17}$$

The Jacobian matrix of system (17) evaluated at $p_0 = (x_0, y_0, \frac{k_1}{d_2})$ is given as follows:

$$\mathbf{J}_{p_0} = \begin{bmatrix} \frac{(x_0^2 - b_1)\rho}{x_0^2 + b_1} & -\frac{x_0 \rho}{y_0} & 0 \\ \frac{2b_1(k_1 + \mu c_3 y_0)}{c_3 x_0^2 (b_1 + x_0^2)} & \frac{2 d_2 k_1}{a_2 c_3^2} & -\frac{d_2}{c_3} \\ 0 & \frac{k_1(a_2 c_3 - 2 d_2 y_0)}{a_2 c_3 y_0} & 0 \end{bmatrix},$$

which in turn yields a cubic characteristic polynomial, given by

$$P_1(\lambda) = \lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0, \tag{18}$$

where the coefficients A_1, A_2, A_3 are expressed in terms of the parameters in system (17) as

$$\begin{aligned}
A_1 &= \frac{c_3^2 \rho (b_1 - x_0^2) a_2 - 2 d_2 k_1 (b_1 + x_0^2)}{(b_1 + x_0^2) c_3^2 a_2}, \\
A_2 &= \frac{1}{(b_1 + x_0^2) c_3^2 a_2 y_0} \{[(2 a_2 c_3 \rho + (a_2 c_3 - 2 \rho y_0) d_2 \\
&\quad - 2 y_0 d_2^2) b_1 + d_2 x_0^2 (a_2 c_3 - 2 d_2 y_0 + 2 \rho y_0)] k_1 \\
&\quad + 2 a_2 b_1 c_3^2 \mu \rho y_0\}, \\
A_3 &= \frac{d_2 k_1 \rho (b_1 - x_0^2)(a_2 c_3 - 2 d_2 y_0)}{(b_1 + x_0^2) c_3^2 a_2 y_0}.
\end{aligned} \tag{19}$$

Based on the characteristic polynomial (18), we consider possible bifurcations from the equilibrium p_0, including both static and dynamic (Hopf) bifurcations. The static bifurcation occurs when $P_1(\lambda) = 0$ has zero roots (zero eigenvalues). Thus we let $A_3 = 0$ to get $b_1 = x_0^2$, which yields $A_1 < 0$. Thus, the static bifurcation occurs in the unstable region, which is not interesting physically.

Theorem 3. *For system* (*17*), *if one of the following conditions is satisfied*,

$$\begin{aligned}
&\text{(I)} \quad 0 < x_0 < \sqrt{b_1}, \quad 0 < y_0 < \frac{a_2 c_3}{2 d_2}, \quad k_1 \le k_{\mathrm{L}}, \quad \mu > 0; \\
&\text{(II)} \quad 0 < x_0 < \sqrt{b_1}, \quad 0 < y_0 < \frac{a_2 c_3}{2 d_2}, \quad k_{\mathrm{L}} < k_1 < k_{\mathrm{U}}, \quad \mu > \mu_{\mathrm{H}},
\end{aligned} \tag{20}$$

then the equilibrium $p_0 = (x_0, y_0, \frac{k_1}{d_2})$ *is locally asymptotically stable. Here,*

$$\begin{aligned}
\mu_{\rm H} &= \frac{k_1[d_2(b_1+x_0^2)(a_2c_3-2y_0d_2+2\rho y_0)+2b_1\rho(a_2c_3-2d_2y_0)](k_1-k_{\rm L})}{2b_1a_2c_3^2\rho y_0(k_{\rm U}-k_1)},\\
k_{\rm L} &= \frac{a_2c_3^2\rho^2(b_1-x_0^2)[b_1(a_2c_3-d_2y_0)+d_2y_0x_0^2]}{d_2(b_1+x_0^2)[d_2(b_1+x_0^2)(a_2c_3-2d_2y_0)+2d_2\rho y_0x_0^2+2b_1\rho(a_2c_3-d_2y_0)]},\\
k_{\rm U} &= \frac{c_3^2\rho(b_1-x_0^2)a_2}{2d_2(b_1+x_0^2)}.
\end{aligned} \tag{21}$$

Proof. According to the Hurwitz criterion, first we have

$$\begin{aligned}
&A_1>0 \;\Rightarrow\; c_3^2\rho(b_1-x_0^2)a_2>0 \;\Leftrightarrow\; b_1-x_0^2>0 \;\Rightarrow\; 0<x_0<\sqrt{b_1};\\
&A_1>0 \;\Rightarrow\; c_3^2\rho(b_1-x_0^2)a_2-2d_2k_1(b_1+x_0^2)>0 \;\Rightarrow\; k_1<\frac{c_3^2\rho(b_1-x_0^2)a_2}{2d_2(b_1+x_0^2)} \;\Leftrightarrow\; k_1<k_{\rm U};\\
&A_3>0 \;\Rightarrow\; (b_1-x_0^2)(a_2c_3-2y_0d_2)>0 \;\Rightarrow\; 0<y_0<\frac{a_2c_3}{2d_2}.
\end{aligned} \tag{22}$$

Then, we compute $\Delta_2 = A_1A_2 - A_3$ to obtain

$$\begin{aligned}
\Delta_2 &= \frac{2}{(b_1+x_0^2)^2c_3^4a_2^2y_0}\Delta_{2a},\\
\Delta_{2a} &= (b_1+x_0^2)y_0\rho c_3^4b_1a_2^2A_1\mu - k_1\{-2y_0(x_0^2+b_1)d_2^2+[(a_2c_3-2\rho y_0)b_1\\
&\quad + x_0^2(a_2c_3+2\rho y_0)]d_2+2a_2b_1c_3\rho\}d_2(x_0^2+b_1)(k_1-k_{\rm L}).
\end{aligned}$$

Solving Δ_{2a} for μ, we obtain the critical value $\mu_{\rm H}$. Further, it can be verified that $k_{\rm L} < k_{\rm U}$:

$$\begin{aligned}
k_{\rm L}<k_{\rm U} &\Leftrightarrow \frac{a_2c_3^2\rho^2(b_1-x_0^2)[b_1(a_2c_3-d_2y_0)+d_2y_0x_0^2]}{d_2(b_1+x_0^2)[d_2(b_1+x_0^2)(a_2c_3-2y_0d_2)+2d_2\rho y_0x_0^2+2b_1\rho(a_2c_3-d_2y_0)]}<\frac{c_3^2\rho(b_1-x_0^2)a_2}{2d_2(b_1+x_0^2)}\\
&\Leftrightarrow \frac{\rho[b_1(a_2c_3-d_2y_0)+d_2y_0x_0^2]}{[d_2(b_1+x_0^2)(a_2c_3-2y_0d_2)+2d_2\rho y_0x_0^2+2b_1\rho(a_2c_3-d_2y_0)]}<\frac{1}{2}\\
&\Leftrightarrow 2\rho[b_1(a_2c_3-d_2y_0)+d_2y_0x_0^2]<d_2(b_1+x_0^2)(a_2c_3-2y_0d_2)+2d_2\rho y_0x_0^2+2b_1\rho(a_2c_3-d_2y_0)\\
&\Leftrightarrow d_2(b_1+x_0^2)(a_2c_3-2y_0d_2)>0.
\end{aligned}$$

Thus, besides $A_1 > 0$, $A_2 > 0$, $A_3 > 0$ under the conditions given in (22), the above discussion shows that $\Delta_2 > 0$ for $k_1 \le k_{\rm L}$ ($\mu > 0$) or $k_{\rm L} < k_1 < k_{\rm U}$ ($\mu > \mu_{\rm H}$). Hence,

$$\Delta_2>0 \quad \begin{cases} \text{if } k_1\le k_{\rm L},\ \ \mu>0, \quad \text{or}\\ \text{if } k_{\rm L}<k_1<k_{\rm U},\ \ \mu>\mu_H. \end{cases} \tag{23}$$

Therefore, the equilibrium p_0 is asymptotically stable under the conditions given in (22) and (23), that is, under the condition (I) or (II). ■

Next, we derive the conditions for Hopf bifurcation. According to Theorem 1, Hopf bifurcation occurs from the equilibrium p_0 at $\mu = \mu_{\rm H}$ when the following conditions hold,

$$0<x_0<\sqrt{b_1}, \quad k_{\rm L}<k_1<k_{\rm U}, \quad 0<y_0<\frac{a_2c_3}{2d_2}. \tag{24}$$

Under the conditions in (24), we rewrite $P_1(\lambda)$ as

$$P_1(\lambda) = \frac{F_1F_2}{y_0a_2c_3^2[a_2c_3^2\rho(b_1-x_0^2)-2k_1d_2(b_1+x_0^2)](b_1+x_0^2)}, \tag{25}$$

where

$$F_1 = a_2c_3^2(b_1 + x_0^2)\left[\lambda + \frac{2d_2(k_U - k_1)}{a_2c_3^2}\right],$$

$$F_2 = 2y_0d_2(b_1 + x_0^2)(k_U - k_1) \times \left[\lambda^2 + \frac{\rho k_1(b_1 - x_0^2)(a_2c_3 - 2d_2y_0)}{2y_0(b_1 + x_0^2)(k_U - k_1)}\right].$$

Thus, we can get the three roots (three eigenvalues) of $P_1(\lambda)$ as α and $\pm\omega i$, where

$$\alpha = -\frac{2d_2(k_U - k_1)}{a_2c_3^2},$$

$$\omega_H = \sqrt{\frac{\rho k_1(b_1 - x_0^2)(a_2c_3 - 2d_2y_0)}{2y_0(b_1 + x_0^2)(k_U - k_1)}},$$

satisfying $\alpha < 0$ and $\omega_H > 0$ for $k_L < k_1 < k_U$.

In order to make the normal form computation feasible for the Hopf bifurcation, we further set

$$x_0 = \sqrt{X_0}, \quad X_0 > 0, \quad k_1 = k_L + \frac{1}{2}(k_U - k_L),$$

$$y_0 = \frac{a_2c_3}{4d_2}, \quad b_1 = x_0^2 + B_1, \quad B_1 > 0.$$

Then, we can rewrite k_L, k_U, A_1, A_2, A_3, μ_H, α and ω as

$$k_L = \frac{B_1\rho^2c_3^2a_2(3B_1 + 4X_0)}{2d_2[(d_2 + 3\rho)B_1 + 2X_0(d_2 + 2\rho)](2X_0 + B_1)},$$

$$k_U = \frac{c_3^2\rho B_1a_2}{2d_2(2X_0 + B_1)},$$

$$A_1 = \frac{B_1\rho d_2}{2[(d_2 + 3\rho)B_1 + 2X_0(d_2 + \rho)]},$$

$$A_2 = \frac{(d_2 + 6\rho)B_1 + 2X_0(d_2 + 4\rho)B_1\rho}{(2X_0 + B_1)^2},$$

$$A_3 = \frac{d_2\rho^2B_1^2[(d_2 + 6\rho)B_1 + 2X_0(d_2 + 4\rho)]}{2[(d_2 + 3\rho)B_1 + 2X_0(d_2 + 2\rho)](2X_0 + B_1)^2},$$

$$\alpha = -\frac{B_1\rho d_2}{2[(d_2 + 3\rho)B_1 + 2X_0(d_2 + 2\rho)]},$$

$$\mu_H = \frac{[(d_2 + 6\rho)B_1 + 2X_0(d_2 + 4\rho)]B_1}{4(X_0 + B_1)(2X_0 + B_1)},$$

$$\omega_H = \frac{\sqrt{[(d_2 + 6\rho)B_1 + 2X_0(d_2 + 4\rho)]B_1\rho}}{2X_0 + B_1}, \tag{26}$$

satisfying $A_1 > 0$, $A_2 > 0$, $A_3 > 0$, $\Delta_2 = 0$, $\alpha < 0$, $\mu_H > 0$, $\omega_H > 0$, as expected.

4. Existence of Three Limit Cycles Around p_0

In this section, we will present our main result. We will apply normal form theory to show that at least three small-amplitude limit cycles can bifurcate from the equilibrium p_0.

Theorem 4. *For system* (*17*), *at least three small-amplitude limit cycles can bifurcate from the equilibrium* p_0.

Proof. In order to study the limit cycle bifurcation around the equilibrium p_0 near the critical point $\mu = \mu_H$, we need to compute the focus values. To achieve this, we introduce the following transformation,

$$\begin{aligned} x &= x_0 + T_{11}u + T_{12}v + T_{13}w, \\ y &= y_0 + T_{21}u + T_{22}v + T_{23}w, \\ z &= z_0 + T_{31}u + T_{32}v + T_{33}w, \end{aligned} \tag{27}$$

where

$$T_{11} = -\frac{8d_2(2X_0 + B_1)\sqrt{X_0}[(d_2 + 3\rho)B_1 + 2X_0(d_2 + 2\rho)]}{c_3^2a_2[(d_2 + 7\rho)B_1 + 2X_0(d_2 + 4\rho)]B_1},$$

$$T_{12} = -\frac{8d_2(2X_0 + B_1)^2\sqrt{X_0}[(d_2 + 3\rho)B_1 + 2X_0(d_2 + 2\rho)]\omega_H}{c_3^2[(d_2 + 6\rho)B_1 + 2X_0(d_2 + 4\rho)]a_2[(d_2 + 7\rho)B_1 + 2X_0(d_2 + 4\rho)]B_1},$$

$$T_{13} = \frac{x_0\rho(-a_2c_3^2\rho x_0^2 + a_2b_1c_3^2\rho - 2d_2k_1x_0^2 - 2b_1d_2k_1)c_3a_2}{2(a_2c_3x_0^2 - 2d_2x_0^2y_0 + a_2b_1c_3 - 2b_1d_2y_0)k_1^2d_2},$$

$$T_{21} = 0,$$

$$
\begin{aligned}
T_{22} &= \frac{2[(d_2+3\rho)B_1 + 2X_0(d_2+2\rho)](2X_0+B_1)\omega_{\mathrm{H}}}{\rho c_3[(d_2+6\rho)B_1 + 2X_0(d_2+4\rho)]B_1}, \\
T_{23} &= -\frac{y_0(-a_2c_3^2\rho x_0^2 + a_2b_1c_3^2\rho - 2d_2k_1x_0^2 - 2b_1d_2k_1)}{(a_2c_3x_0^2 - 2d_2x_0^2y_0 + a_2b_1c_3 - 2b_1d_2y_0)c_3k_1}, \\
T_{31} &= 1, \quad T_{32} = 0, \quad T_{33} = 1,
\end{aligned} \tag{28}
$$

into system (17) to obtain a new system, expanded around the origin as

$$
\begin{aligned}
\dot{u} &= v + \sum_{i+j+k=2}^{5} a_{ijk}u^iv^jw^k, \\
\dot{v} &= -u + \sum_{i+j+k=2}^{5} b_{ijk}u^iv^jw^k, \\
\dot{w} &= c_{001}w_{\mathrm{H}} + \sum_{i+j+k=2}^{5} c_{ijk}u^iv^jw^k,
\end{aligned} \tag{29}
$$

where the coefficients a_{ijk}, b_{ijk} and c_{ijk} are expressed in terms of the parameters in system (17).

Then, using the Maple program developed by Yu [1998] or Tian and Yu [2013], we get the first three focus values V_1, V_2 and V_3:

$$
V_1 = \tilde{V}_1V_{1a}, \quad V_2 = \tilde{V}_2V_{2a}, \quad V_3 = \tilde{V}_3V_{3a}, \tag{30}
$$

where $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ are rational functions, and V_{1a}, V_{2a} and V_{3a} are polynomials in X_0, d_2, ρ and B_1. $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ are listed in the Appendix while the lengthy polynomials V_{1a}, V_{2a} and V_{3a} are given in supplementary material.

The computation shows that we may choose X_0 as a free parameter and express other parameters in the form of

$$
d_2 = D_2X_0, \quad \rho = RD_2X_0, \quad B_1 = BX_0.
$$

Then, we obtain

$$
V_{1a} = D_2^7X_0^{16}V_{1b}, \quad V_{2a} = D_2^{26}X_0^{56}V_{2b},
$$

$$
V_{3a} = D_2^{48}X_0^{102}V_{3b},
$$

where V_{1b}, V_{2b} and V_{3b} are expressed in terms of R and B.

Using the Maple built-in command, we eliminate R from $V_{1b} = V_{2b} = 0$ to obtain

$$
\begin{aligned}
V_{12b} &= \mathrm{resultant}(V_{1b}, V_{2b}, R) \\
&= -2093055571282759930109250749399841783152 64B^4 6(B+1)^2(B+2)^{182} \\
&\quad \times (3B+4)^{50}(8+9B)^3(15B+16)^2(23B+24)^2(3B^2+8B+8)F_1F_2^2F_3^2F_4^2F_5,
\end{aligned} \tag{31}
$$

where

$$
\begin{aligned}
F_1 &= 147B^5 + 874B^4 + 1668B^3 + 1104B^2 + 24B - 128, \\
F_2 &= 9279B^6 + 32016B^5 + 38156B^4 + 10960B^3 - 10700B^2 - 4864B + 2048, \\
F_3 &= 59193B^7 + 227328B^6 - 393560B^5 - 3227552B^4 - 5745904B^3 - 4141440B^2 - 1167360B - 98304, \\
F_4 &= 265186629B^{14} + 3162608064B^{13} + 18146078748B^{12} + 65973784152B^{11} + 166528741076B^{10} \\
&\quad + 297577658208B^9 + 366510497088B^8 + 283043369216B^7 + 89715855360B^6 - 57877204992B^5 \\
&\quad - 76036567040B^4 - 27416985600B^3 + 3643146240B^2 + 5555355648B + 1233125376
\end{aligned}
$$

and F_5 can be found in supplementary material.

Then, we find seven real positive roots from the polynomial equation $V_{12b} = 0$, however, only one satisfies $V_1 = V_2 = 0$. This solution is given by

$$
B = 0.1268297707\cdots, \quad R = 0.235165980\cdots,
$$

under which

$$V_1 = V_2 = 0,$$

$$V_3 = -2.677759596\cdots \times 10^{10}\frac{D_2^5 X_0^6}{c_3^{10} a_2^4} < 0.$$

Further, a simple computation shows that

$$\det\left[\frac{\partial(V_1, V_2)}{\partial(B, R)}\right] = 5.650407050\cdots \times 10^{11}\frac{D_2^4 X_0^6}{a_2^2 c_3^8} \neq 0.$$

Thus, according to Theorem 2, two small-amplitude limit cycles can bifurcate from the equilibrium point p_0 in the system (17). Finally, a small perturbation on μ from $\mu_{\rm H}$ is applied to obtain an additional small-amplitude limit cycle, giving a total of three limit cycles around the equilibrium p_0. ■

5. Simulation of Three Limit Cycles

In this section, we simulate the three limit cycles bifurcating from the equilibrium p_0. Set $D_2 = 1$, $X_0 = 1$, $a_2 = 2$, $c_3 = 10$. The equilibrium point becomes $p_0 = (1, 5, \frac{50RB(6BR+B+8R+2)}{(2+B)(3BR+B+4R+2)})$. Now, we perturb the parameters B and R as

$$\begin{aligned} B &= B_c - 0.000036455\cdots, \\ R &= R_c - 0.000124549\cdots, \end{aligned} \tag{32}$$

under which system (17) becomes

$$\begin{aligned} \dot{x} &= (19.86323342\cdots)x + (17.62810726\cdots)x^3 \\ &\quad - (7.498268139\cdots)x^2y + (0.264843112\cdots)xy^2 \\ &\quad + (0.235041430\cdots)x^3y^2 \\ &\quad - (0.099976908\cdots)x^2y^3, \\ \dot{y} &= -(4.679126724\cdots)y - (2.253586630\cdots)yz \\ &\quad + (7.643326115\cdots)x^2y - (0.062388356\cdots)y^3 \\ &\quad + (0.101911014\cdots)x^2y^3 - 2x^2yz, \\ \dot{z} &= -(84.50949863\cdots)z + (22.53586630\cdots)yz \\ &\quad - 75x^2z - (1.126793315\cdots)y^2z \\ &\quad + 20x^2yz - x^2y^2z. \end{aligned} \tag{33}$$

Based on system (33), the simulation result obtained using Matlab is shown in Fig. 1. Note that since $V_3 < 0$, the outer-most and inner-most limit cycles are stable and the middle one is unstable, and the equilibrium p_0 must be unstable. It is seen from the figure that all the three limit cycles are located on a two-dimensional invariant manifold near the equilibrium p_0.

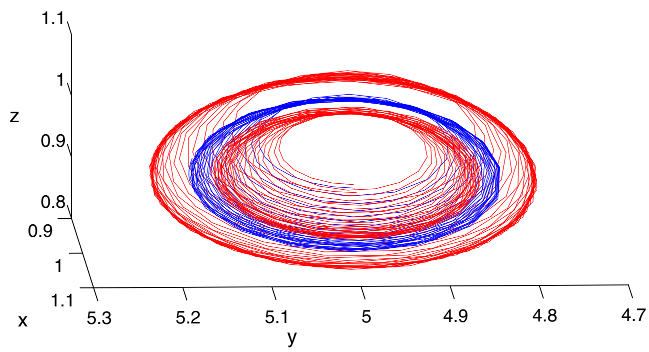

Fig. 1. Simulated three limit cycles of system (33), which enclose the unstable focus p_1, with inner-most and outer-most ones stable (in red) and middle one unstable (in blue).

6. Conclusion and Discussion

In this paper, we have considered a tritrophic food chain model with Holling functional response types III and IV for the predator and superpredator, respectively. We have studied the stability and bifurcation of the positive equilibrium when the prey has linear growth. We have applied center manifold theory and normal form theory to give a detailed analysis on the Hopf bifurcation. Moreover, we have investigated the bifurcation of multiple limit cycles, which can cause complex dynamics in biological systems.

Previous study on the tritrophic food chain model [Castellanos *et al.*, 2018] has shown that one stable limit cycle can bifurcate from a positive equilibrium, implying the coexistence of three different species in the food chain. In this paper, we further investigate this food chain model and show that it can exhibit at least three limit cycles due to Hopf bifurcation. Thus, depending upon initial conditions, the solution trajectories of the system may converge to different limit cycles (attractors), yielding a new bistable phenomenon consisting of two stable limit cycles (the inner-most and outer-most ones), restricted to an invariant manifold, and a separatory (unstable limit cycle) is between the two limit cycles. This may explain how some complex dynamical behaviors occur in such food chain systems.

Since periodic and quasi-periodic oscillations are often observed in real biological systems, it is anticipated that the multiple limit cycle bifurcation studied in this paper may lead to establishing a good methodology for investigating such complex dynamical behaviors. We hope that the method presented in this paper can be used to study other nonlinear dynamical systems, and promote further research in this field.

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Appendix

The rational functions $\tilde{V}_1$, $\tilde{V}_2$ and $\tilde{V}_3$ (see Sec. 4) are given below.

$$\begin{aligned}
\tilde{V}_1 = {} & -(B_1d_2+3B_1\rho+2X_0d_2+4X_0\rho)^2/[2B_1^3\rho(B_1d_2+7B_1\rho+2X_0d_2+8X_0\rho)(4B_1^3d_2^3+49B_1^3d_2^2\rho \\
& +180B_1^3d_2\rho^2+216B_1^3\rho^3+24B_1^2X_0d_2^3+260B_1^2X_0d_2^2\rho+840B_1^2X_0d_2\rho^2+864B_1^2X_0\rho^3+48B_1X_0^2d_2^3 \\
& +452B_1X_0^2d_2^2\rho+1280B_1X_0^2d_2\rho^2+1152B_1X_0^2\rho^3+32X_0^3d_2^3+256X_0^3d_2^2\rho+640X_0^3d_2\rho^2+512X_0^3\rho^3) \\
& \times(16B_1^3d_2^3+193B_1^3d_2^2\rho+720B_1^3d_2\rho^2+864B_1^3\rho^3+96B_1^2X_0d_2^3+1028B_1^2X_0d_2^2\rho+3360B_1^2X_0d_2\rho^2 \\
& +3456B_1^2X_0\rho^3+192B_1X_0^2d_2^3+1796B_1X_0^2d_2^2\rho+5120B_1X_0^2d_2\rho^2+4608B_1X_0^2\rho^3+128X_0^3d_2^3 \\
& +1024X_0^3d_2^2\rho+2560X_0^3d_2\rho^2+2048X_0^3\rho^3)c_3^2],
\end{aligned}$$

$$\begin{aligned}
\tilde{V}_2 = {} & -2(B_1d_2+3B_1\rho+2X_0d_2+4X_0\rho)^4d_2/[9c_3^6a_2^2(36B_1^3d_2^3+433B_1^3d_2^2\rho+1620B_1^3d_2\rho^2+1944B_1^3\rho^3 \\
& +216B_1^2X_0d_2^3+2308B_1^2X_0d_2^2\rho+7560B_1^2X_0d_2\rho^2+7776B_1^2X_0\rho^3+432B_1X_0^2d_2^3+4036B_1X_0^2d_2^2\rho \\
& +11520B_1X_0^2d_2\rho^2+10368B_1X_0^2\rho^3+288X_0^3d_2^3+2304X_0^3d_2^2\rho+5760X_0^3d_2\rho^2+4608X_0^3\rho^3) \\
& \times(2X_0+B_1)(16B_1^3d_2^3+193B_1^3d_2^2\rho+720B_1^3d_2\rho^2+864B_1^3\rho^3+96B_1^2X_0d_2^3+1028B_1^2X_0d_2^2\rho \\
& +3360B_1^2X_0d_2\rho^2+3456B_1^2X_0\rho^3+192B_1X_0^2d_2^3+1796B_1X_0^2d_2^2\rho+5120B_1X_0^2d_2\rho^2+4608B_1X_0^2\rho^3 \\
& +128X_0^3d_2^3+1024X_0^3d_2^2\rho+2560X_0^3d_2\rho^2+2048X_0^3\rho^3)^3\rho^3B_1^7(B_1d_2+7B_1\rho+2X_0d_2+8X_0\rho)^2 \\
& \times(4B_1^3d_2^3+49B_1^3d_2^2\rho+180B_1^3d_2\rho^2+216B_1^3\rho^3+24B_1^2X_0d_2^3+260B_1^2X_0d_2^2\rho+840B_1^2X_0d_2\rho^2 \\
& +864B_1^2X_0\rho^3+48B_1X_0^2d_2^3+452B_1X_0^2d_2^2\rho+1280B_1X_0^2d_2\rho^2+1152B_1X_0^2\rho^3+32X_0^3d_2^3 \\
& +256X_0^3d_2^2\rho+640X_0^3d_2\rho^2+512X_0^3\rho^3)^3(B_1d_2+6B_1\rho+2X_0d_2+8X_0\rho)^2],
\end{aligned}$$

$$\begin{aligned}
\tilde{V}_3 = {} & -d_2^2(B_1d_2+3B_1\rho+2X_0d_2+4X_0\rho)^6/[648(B_1d_2+6B_1\rho+2X_0d_2+8X_0\rho)^4B_1^{11}\rho^5(B_1d_2+7B_1\rho \\
& +2X_0d_2+8X_0\rho)^3c_3^{10}a_2^4(64B_1^3d_2^3+769B_1^3d_2^2\rho+2880B_1^3d_2\rho^2+3456B_1^3\rho^3+384B_1^2X_0d_2^3 \\
& +4100B_1^2X_0d_2^2\rho+13440B_1^2X_0d_2\rho^2+13824B_1^2X_0\rho^3+768B_1X_0^2d_2^3+7172B_1X_0^2d_2^2\rho \\
& +20480B_1X_0^2d_2\rho^2+18432B_1X_0^2\rho^3+512X_0^3d_2^3+4096X_0^3d_2^2\rho+10240X_0^3d_2\rho^2+8192X_0^3\rho^3) \\
& \times(36B_1^3d_2^3+433B_1^3d_2^2\rho+1620B_1^3d_2\rho^2+1944B_1^3\rho^3+216B_1^2X_0d_2^3+2308B_1^2X_0d_2^2\rho+7560B_1^2X_0d_2\rho^2 \\
& +7776B_1^2X_0\rho^3+432B_1X_0^2d_2^3+4036B_1X_0^2d_2^2\rho+11520B_1X_0^2d_2\rho^2+10368B_1X_0^2\rho^3+288X_0^3d_2^3 \\
& +2304X_0^3d_2^2\rho+5760X_0^3d_2\rho^2+4608X_0^3\rho^3)^2(2X_0+B_1)^2(4B_1^3d_2^3+49B_1^3d_2^2\rho+180B_1^3d_2\rho^2+216B_1^3\rho^3 \\
& +24B_1^2X_0d_2^3+260B_1^2X_0d_2^2\rho+840B_1^2X_0d_2\rho^2+864B_1^2X_0\rho^3+48B_1X_0^2d_2^3+452B_1X_0^2d_2^2\rho \\
& +1280B_1X_0^2d_2\rho^2+1152B_1X_0^2\rho^3+32X_0^3d_2^3+256X_0^3d_2^2\rho+640X_0^3d_2\rho^2+512X_0^3\rho^3)^5 \\
& \times(16B_1^3d_2^3+193B_1^3d_2^2\rho+720B_1^3d_2\rho^2+864B_1^3\rho^3+96B_1^2X_0d_2^3+1028B_1^2X_0d_2^2\rho+3360B_1^2X_0d_2\rho^2 \\
& +3456B_1^2X_0\rho^3+192B_1X_0^2d_2^3+1796B_1X_0^2d_2^2\rho+5120B_1X_0^2d_2\rho^2+4608B_1X_0^2\rho^3+128X_0^3d_2^3 \\
& +1024X_0^3d_2^2\rho+2560X_0^3d_2\rho^2+2048X_0^3\rho^3)^5].
\end{aligned}$$