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GLOBAL ANALYSIS ON n-SCROLL CHAOTIC ATTRACTORS OF MODIFIED CHUA'S CIRCUIT

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In this paper, we present a further mathematical study on the report of existence of n-scroll chaotic attractors in a modified Chua's circuit. A series of results based on mathematical theory are given. First, we show that the chaotic attractors of the modified Chua's circuit are globally attractive, with estimations given for the globally attractive set and positive invariant set. Then, we study the positions, number and local stability of the equilibrium points. We also design simple feedback control laws to globally exponentially stabilize any given equilibrium point. Finally, we use the theory and methodology of absolute stability of Luré nonlinear control systems and nonlinear feedback control to exponentially synchronize two modified Chua's circuits with the same structure. The design of constructive feedback control laws for synchronization is also discussed.

Keywords: n-scroll chaotic attractor; modified Chua's circuit; chaos control; chaos synchronization.

1. Introduction

For a long time, people thought that it was impossible to control chaos or to synchronize two chaotic systems. It was changed in the 1990's of the last century when the OGY method [Ott *et al.*, 1990] was proposed, and Pecora and Carroll [1990] used electrical circuits to synchronize two chaotic systems. Recently, great progress has been achieved, particularly for chaos control and chaos synchronization [Chua, 1993; Chen, 1998]. More and more new chaotic attractors (systems) have been developed.

Chua's circuit is the first chaotic system which was physically realized by using an electrical circuit. Not only does it have complex and interesting dynamical behavior but also it is widely used in many different areas [Tang, 2001; Yalcin, 2000]. Chua's circuit has been thoroughly studied by many scientists, especially from the field of chaos control and chaos synchronization, including some

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contribution of recent work [Liao & Yu, 2005; Liao & Chen, 2005]. However, there still exist fundamental problems that have not been resolved. For example, up to now, only the Lorenz chaotic system has been proved to be globally attractive and its globally attractive set and positive invariant set have been estimated. Whether the Chua's circuit has the same property is still unknown.

In 2001, Tang *et al.* [2001] proposed a modified Chua's circuit model, in which the piecewise linear function was replaced by a sinusoidal function. This modified system, which can generate n-scroll chaotic attractors, has broadened understanding of chaotic attractors. Nevertheless, this system still needs further study to explore its interesting dynamical property.

In this paper, we study qualitative behavior of the modified Chua's circuit mathematically. The rest of this paper is organized as follows. In Sec. 2, we present the mathematical model and give some definitions and lemmas, which will be used in the following sections to study the properties of the chaotic attractors. Section 3 is devoted to prove that the chaotic attractors in the modified Chua's circuit are globally attractive. The estimations of the globally attractive set and positive invariant set are derived. Therefore, there do not exist equilibrium points, periodic solutions, almost periodic solutions or other chaotic attractors outside the attractive set. In Sec. 4, we present the equilibrium solutions, and discuss the number of equilibrium solutions and their stability. In Sec. 5, we design simple linear feedback control laws to globally, exponentially stabilize the system for any given equilibrium point. Thus, chaotic motion and remaining equilibrium points disappear. Constructive algebra conditions for globally exponential stabilization are also given in this section. Section 6 is to study chaos synchronization. We apply the absolute stability theory and method of nonlinear control system [Liao, 1993], and linear feedback control principle to exponentially synchronize two n-scroll chaotic attractors with same structure. Numerical simulations are presented to show that the theories and conclusions obtained in this paper are applicable.

2. Modified Chua's Circuit with n-Scroll Chaotic Attractors

The modified Chua's circuit which exhibits n-scroll chaotic attractors, proposed by Tang *et al.* [2001], is described by the following equations:

$$\begin{aligned} \dot{x} &= \alpha[y - f(x)], \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned} \tag{1}$$

where

$$f(x) = \begin{cases} \dfrac{b\pi}{2a}(x - 2ac), & \text{when } x \geq 2ac, \\ -b\sin\left(\dfrac{\pi x}{2a} + d\right), & \text{when } -2ac < x < 2ac, \\ \dfrac{b\pi}{2a}(x + 2ac), & \text{when } x \leq -2ac, \end{cases}$$

and

$$d = \begin{cases} \pi, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}$$

Here, α, β, a, b and c are constants. The values of these coefficients used in this paper are taken as:

$$\alpha = 10.814, \quad \beta = 14,$$
$$a = 1.3, \quad b = 0.726 \quad \text{and} \quad c = n - 1,$$

for which this system can produce n-scroll chaotic attractors. See an example shown in Fig. 1.

From the mean value theorem, we have

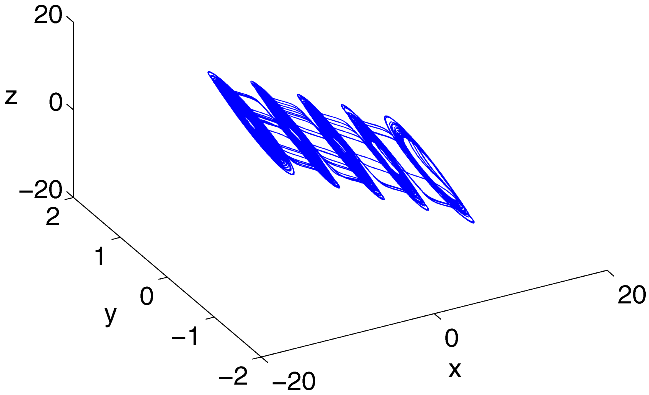
$$-\frac{b\pi}{2a} \leq \frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq \frac{b\pi}{2a}.$$

To compare with the classical Chua's circuit, we rewrite (1) as:

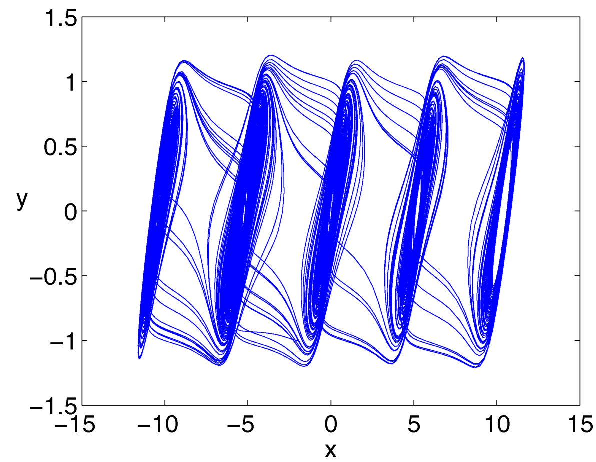
$$\begin{aligned} \dot{x} &= \alpha[y - x - g(x)], \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned} \tag{2}$$

where $f(x) = x + g(x)$. Thus

$$g(x) = -x + f(x) = \begin{cases} \left(\dfrac{b\pi}{2a} - 1\right)x - b\pi c, & \text{when } x \geq 2ac, \\ -x - b\sin\left(\dfrac{\pi x}{2a} + d\right), & \text{when } -2ac < x < 2ac, \\ \left(\dfrac{b\pi}{2a} - 1\right)x + b\pi c, & \text{when } x \leq -2ac. \end{cases}$$



(a)



(b)

Fig. 1. Simulated phase portrait of a five-scroll chaotic attractor for system (1) when $c = 4$: (a) in the x–y–z space; (b) projected on the x–y plane.

Obviously,

$$-1 - \frac{b\pi}{2a} \le \frac{g(x_1) - g(x_2)}{x_1 - x_2} \le -1 + \frac{b\pi}{2a}.$$

Let

$$g_{(x_1,x_2)} = \frac{g(x_1) - g(x_2)}{x_1 - x_2} \quad \text{and}$$

$$f_{(x_1,x_2)} = \frac{f(x_1) - f(x_2)}{x_1 - x_2}.$$

Denote the solution of (1) as

$$\begin{aligned} X(t) &= (x(t), y(t), z(t)) \\ &= (x(t, t_0, X_0), y(t, t_0, X_0), z(t, t_0, X_0)), \end{aligned}$$

or simply,

$$X = (x, y, z) \in R^3.$$

Definition 1. An equilibrium solution, (x^*, y^*, z^*), is called stable (or unstable) if it is locally stable (or unstable) in the sense of Lyapunov.

Definition 2. Let $\rho(X, \Omega) = \inf_{\hat{X}\in\Omega} \|X - \hat{X}\|$, denoting the distance between the point X and the set Ω. If there exists a compact set $\Omega \in R^3$ such that $\forall X_0 \in R^3$, $\rho(X(t, t_0, X_0), \Omega) \to 0$, when $t \to +\infty$, then Ω is said to be globally attractive. If $\forall X_0 \in R^3$, $X(t, t_0, X_0) \in \Omega$ for all $t \ge t_0$, then Ω is said to be a positive invariant set. A system with global attractive set is called an ultimately bounded dissipative system.

Obviously, the statement that $\rho(X(t, t_0, X_0), \Omega) \to 0$ when $t \to +\infty$ is equivalent to that $\forall\ \varepsilon > 0$, $\exists\ T > 0$, when $t \ge t_0 + T$, $X(t, t_0, X_0) \in \Omega_\varepsilon$, where Ω_ε is an ε-neighborhood of Ω, $\Omega_\varepsilon \supset \Omega$.

Definition 3. If all the zeros of a polynomial with real coefficients have negative real part, the polynomial is called a Hurwitz polynomial. If the corresponding characteristic polynomial of a real matrix is a Hurwitz polynomial, the matrix is called a Hurwitz matrix.

Now, assume that system (1) with feedback controls becomes

$$\begin{aligned} \dot{u} &= \alpha[v - u - \tilde{g}(u)], \\ \dot{v} &= u - v + w, \\ \dot{w} &= -\beta v, \end{aligned} \tag{3}$$

where $\tilde{g} \in G := \{\tilde{g}(u) | -l_1 \le \tilde{g}(u)/u \le l_2$, $\tilde{g}(u)$ is continuous, and $\tilde{g}(0) = 0\}$. $l_1 > 0$ and $l_2 > 0$ are constants.

Lemma 1 [Liao, 2001]. *The necessary and sufficient conditions for a real cubic polynomial,* $\lambda^3 + p\lambda^2 + q\lambda + r$, *to be a Hurwitz polynomial are* $q > 0$ *and* $pq > r > 0$.

Lemma 2. *The necessary and sufficient condition for the zero solution of system* (3) *to be exponentially stable is that the zero solution is exponentially stable with respect to the partial variable* u.

Proof. The necessity is obvious, since $u^2(t) + v^2(t) + w^2(t) \le Me^{-\alpha(t-t_0)}$. In particular, we have $u^2(t) \le Me^{-\alpha(t-t_0)}$. We only need to prove sufficiency. To achieve this, we first prove that the coefficient matrix of the linear part of system (3), given by

$$A = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix},$$

is a Hurwitz matrix. In fact,

$$\begin{aligned}\det(\lambda I_3 - A) &= \det\begin{bmatrix}\lambda+\alpha & -\alpha & 0\\ -1 & \lambda+1 & -1\\ 0 & \beta & \lambda\end{bmatrix}\\ &= (\lambda+\alpha)(\lambda+1)\lambda + (\lambda+\alpha)\beta - \alpha\lambda\\ &= \lambda^3 + (\alpha+1)\lambda^2 + \alpha\lambda + \beta\lambda\\ &\quad + \alpha\beta - \alpha\lambda\\ &= \lambda^3 + (\alpha+1)\lambda^2 + \beta\lambda + \alpha\beta,\end{aligned}$$

where $pq = (\alpha+1)\beta > \alpha\beta = r > 0$. From Lemma 1, we know that A is a Hurwitz matrix. Thus, there exist constants $M \ge 1$ and $\mu > 0$, such that

$$\|e^{A(t-t_0)}\| \le Me^{-\mu(t-t_0)}. \tag{4}$$

From the hypothesis of sufficiency, we know that there exist constants $h > 0$ and $\tilde{\mu} > 0$, such that

$$|u(t)| \le he^{-\tilde{\mu}(t-t_0)}. \tag{5}$$

Without loss of generality, assume $\mu \ne \tilde{\mu}$, otherwise one may decrease $\tilde{\mu}$ in (5) to reach $\mu \ne \tilde{\mu}$. Then, with variation of constants, we write the solution of (3) as

$$\begin{aligned}\begin{pmatrix}u(t)\\ v(t)\\ w(t)\end{pmatrix} &= \begin{pmatrix}u(t_0)\\ v(t_0)\\ w(t_0)\end{pmatrix} e^{A(t-t_0)}\\ &\quad + \int_{t_0}^{t} e^{A(t-\tau)} \begin{pmatrix}-\alpha\\ 0\\ 0\end{pmatrix} \tilde{g}(u(\tau))d\tau. \end{aligned}\tag{6}$$

Hence,

$$\begin{aligned}\left\|\begin{pmatrix}u(t)\\ v(t)\\ w(t)\end{pmatrix}\right\| &\le \left\|\begin{pmatrix}u(t_0)\\ v(t_0)\\ w(t_0)\end{pmatrix}\right\| Me^{-\mu(t-t_0)}\\ &\quad + \int_{t_0}^{t} Me^{-\mu(t-\tau)} \max\{l_1, l_2\}\\ &\quad \times \alpha h e^{-\tilde{\mu}(\tau-t_0)} d\tau. \end{aligned}\tag{7}$$

Apparently, the first term in (7) has negative exponential form. So we only need to prove that the second term also has negative exponential form. To show this, let

$$M\alpha \max(l_1, l_2)h := C.$$

Then,

$$\begin{aligned}&\int_{t_0}^{t} Ce^{-\mu(t-\tau)}e^{-\tilde{\mu}(\tau-t_0)}d\tau\\ &\quad = Ce^{-\mu t}\int_{t_0}^{t} e^{(\mu-\tilde{\mu})\tau}e^{\tilde{\mu}t_0}d\tau\\ &\quad = Ce^{-\mu t+\tilde{\mu}t_0}\frac{e^{(\mu-\tilde{\mu})t} - e^{(\mu-\tilde{\mu})t_0}}{\mu-\tilde{\mu}}\\ &\quad \le \begin{cases}\dfrac{C}{\mu-\tilde{\mu}}e^{-\tilde{\mu}(t-t_0)} & \text{when } \mu > \tilde{\mu},\\ \dfrac{C}{\tilde{\mu}-\mu}e^{-\mu(t-t_0)} & \text{when } \mu < \tilde{\mu},\end{cases}\end{aligned}\tag{8}$$

which shows that the second term of (7) has negative exponential estimation. Thus, the zero solution of system (3) is exponentially stable. ■

3. Estimates of Global Attractive Set and Positive Invariant Set

Study of global attraction (or dissipativeness) of a chaotic system is very important, since it is fundamental for the applications in chaos control, chaos tracking and chaos synchronization. However, proving a chaotic system to have global attractive set is a very challenging task. Up to now, only the Lorenz chaotic attractor has been proved to have global attractive set [Leonov, 2001; Yu & Liao, 2006], in which analytical expressions of global attractive set are also given. In this section, we first prove that the modified Chua's circuit (1) has a global attractive set and derive an estimate for the attractive set.

Theorem 1. *The modified Chua's circuit* (*1*) *is globally attractive, with the global attractive and positive invariant set*, Ω, *given by*

$$\Omega = \left\{ x, y, z \;\middle|\; \begin{array}{l} -L_1 \le x \le L_2,\\ -\lambda_{\max}(Q)\left(|y| + \dfrac{L}{\lambda_{\max}(Q)}\right)^2 - \lambda_{\max}(Q)\left(|z| + \dfrac{\varepsilon L}{2\lambda_{\max}(Q)}\right)^2\\ \le -\dfrac{L^2}{\lambda_{\max}(Q)} - \dfrac{\varepsilon^2 L^2}{4\lambda_{\max}(Q)}, \end{array}\right\}, \tag{9}$$

where the parameters are defined in the proof.

Proof. First we show that when $|x| \geq 2ac$, the coefficient matrix of the corresponding homogeneous system is

$$B := \begin{bmatrix} -\frac{\alpha b\pi}{2a} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix},$$

which is a Hurwitz matrix. In fact,

$$\begin{aligned}
&\det(\lambda I_3 - B) \\
&= \det \begin{bmatrix} \lambda + \frac{\alpha b\pi}{2a} & -\alpha & 0 \\ -1 & \lambda+1 & -1 \\ 0 & \beta & \lambda \end{bmatrix} \\
&= \left(\lambda + \frac{\alpha b\pi}{2a}\right)(\lambda+1)\lambda - \alpha\lambda + \beta\left(\lambda + \frac{\alpha b\pi}{2a}\right) \\
&= \lambda^3 + \left(\frac{\alpha b\pi}{2a} + 1\right)\lambda^2 + \left(\frac{\alpha b\pi}{2a} - \alpha + \beta\right)\lambda \\
&\quad + \frac{\alpha\beta b\pi}{2a} \\
&:= \lambda^3 + p\lambda^2 + q\lambda + r,
\end{aligned}$$

where I_3 is a 3×3 identity matrix. When $\alpha = 10.814$, $\beta = 14$, $a = 1.3$ and $b = 0.726$, we obtain

$$p = \left(\frac{\alpha b\pi}{2a} + 1\right) > 0, \quad q = \left(\frac{\alpha b\pi}{2a} - \alpha + \beta\right) > 0$$

$$\text{and} \quad r = \frac{\alpha\beta b\pi}{2a} > 0,$$

showing that

$$pq = \left(\frac{\alpha b\pi}{2a} + 1\right)\left(\frac{\alpha b\pi}{2a} - \alpha + \beta\right) > \frac{\alpha\beta b\pi}{2a} = r.$$

Thus, B is a Hurwitz matrix.

For $x \leq -2ac$, system (1) can be written as

$$\begin{aligned}
\dot{x} &= -\frac{\alpha b\pi}{2a}x + \alpha y + 0 - \alpha bc\pi, \\
\dot{y} &= x - y + z, \qquad\qquad (x \leq -2ac). \\
\dot{z} &= 0 - \beta y + 0,
\end{aligned}$$

The vector form of the above system is

$$\begin{aligned}
\dot{X} &= \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \\
&= \begin{bmatrix} -\frac{\alpha b\pi}{2a} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -\alpha bc\pi \\ 0 \\ 0 \end{pmatrix} \\
&:= BX - h, \qquad (10)
\end{aligned}$$

when $x \leq -2ac$, where

$$h = \begin{pmatrix} \alpha bc\pi \\ 0 \\ 0 \end{pmatrix}.$$

Since

$$B = \begin{bmatrix} -\frac{\alpha b\pi}{2a} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$$

is a Hurwitz matrix, there exists positive definite matrix P for the following matrix equation:

$$PB + B^T P = -I_3.$$

Construct positive definitive quadratic function

$$V = X^T P X.$$

Thus, differentiating V with respect to t along the trajectory of (10) yields

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(10)} &= X^T P\dot{X} + \dot{X}^T P X \\
&= X^T PBX - X^T Ph + (BX - h)^T PX \\
&= -X^T X - X^T Ph - h^T PX \\
&= -x^2 - y^2 - z^2 - 2xp_{11}\alpha bc\pi \\
&\quad - 2yp_{21}\alpha bc\pi - 2zp_{31}\alpha bc\pi \\
&= -(x + p_{11}\alpha bc\pi)^2 - (y + p_{21}\alpha bc\pi)^2 \\
&\quad - (z + p_{31}\alpha bc\pi)^2 + (p_{11}\alpha bc\pi)^2 \\
&\quad + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2 \\
&< 0,
\end{aligned}$$

when

$$\begin{aligned}
&(x + p_{11}\alpha bc\pi)^2 + (y + p_{21}\alpha bc\pi)^2 + (z + p_{31}\alpha bc\pi)^2 \\
&\quad > (p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2,
\end{aligned}$$

and particularly for

$$\begin{aligned}
x &< -\sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} \\
&\quad - p_{11}\alpha bc\pi.
\end{aligned}$$

Thus, when

$$x < \min\{-\sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} - p_{11}\alpha bc\pi, -2ac\},$$

$V = X^T PX$ strictly decreases along the solution curve $(x(t), y(t), z(t))$, implying that

$$x \geq \min\{-\sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} - p_{11}\alpha bc\pi, -2ac\}$$

defines the attractive set.

Similarly, for $x \geq 2ac$, system (1) can be written as

$$\begin{aligned} \dot{x} &= -\frac{\alpha b\pi}{2a}x + \alpha y + 0 + \alpha bc\pi, \\ \dot{y} &= x - y + z, \qquad (x \geq 2ac). \\ \dot{z} &= 0 - \beta y + 0, \end{aligned}$$

Its vector form is

$$\begin{aligned} \dot{X} &= \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \\ &= \begin{bmatrix} -\frac{\alpha b\pi}{2a} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \alpha bc\pi \\ 0 \\ 0 \end{pmatrix} \\ &:= BX + h, \end{aligned} \tag{11}$$

when $x \geq 2ac$, where

$$h = \begin{pmatrix} \alpha bc\pi \\ 0 \\ 0 \end{pmatrix}.$$

Since

$$B = \begin{bmatrix} -\frac{\alpha b\pi}{2a} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}$$

is a Hurwitz matrix, there exists positive definite matrix P satisfying the matrix equation:

$$PB + B^T P = -I_3.$$

Similarly, we construct positive definitive quadratic function: $V = X^T PX$, and then obtain

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(11)} &= X^T P\dot{X} + \dot{X}^T PX \\ &= X^T PBX + X^T Ph + (BX + h)^T PX \\ &= -X^T X + X^T Ph + h^T PX \\ &= -x^2 - y^2 - z^2 + 2xp_{11}\alpha bc\pi \\ &\quad + 2yp_{21}\alpha bc\pi + 2zp_{31}\alpha bc\pi \\ &= -(x - p_{11}\alpha bc\pi)^2 - (y - p_{21}\alpha bc\pi)^2 \\ &\quad - (z - p_{31}\alpha bc\pi)^2 + (p_{11}\alpha bc\pi)^2 \\ &\quad + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2 \\ &< 0, \end{aligned}$$

when

$$\begin{aligned} &(x - p_{11}\alpha bc\pi)^2 + (y - p_{21}\alpha bc\pi)^2 + (z - p_{31}\alpha bc\pi)^2 \\ &\quad > (p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2, \end{aligned}$$

in particular, for

$$x > \sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} + p_{11}\alpha bc\pi.$$

Thus, when

$$x > \max\{\sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} + p_{11}\alpha bc\pi, 2ac\},$$

$V = X^T PX$ strictly decreases along the solution curve $(x(t), y(t), z(t))$, indicating that

$$x \leq \max\{\sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} + p_{11}\alpha bc\pi, 2ac\}$$

is the attractive set.

Let

$$-L_1 = \min\{-\sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} - p_{11}\alpha bc\pi, -2ac\},$$

and

$$L_2 = \max\{\sqrt{(p_{11}\alpha bc\pi)^2 + (p_{21}\alpha bc\pi)^2 + (p_{31}\alpha bc\pi)^2} + p_{11}\alpha bc\pi, 2ac\}.$$

Then the attractive region for the variable x is

$$-L_1 \le x \le L_2. \tag{12}$$

Next, constructing positive definite and radially unbounded Lyapunov function for the second and third equations of system (1) yields

$$V = y^2 + \frac{z^2}{\beta} - \varepsilon yz = \begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} 1 & -\frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \frac{1}{\beta} \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix}. \tag{13}$$

We choose $\varepsilon = 1/10$. Then differentiating V for the second and third equations of system (1), where x is treated as part of the nonhomogeneous terms about y and z, we have

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(1)} &= 2xy - 2y^2 + 2yz - 2yz - \varepsilon xz \\ &\quad + \varepsilon yz - \varepsilon z^2 + \varepsilon\beta y^2 \\ &= \begin{pmatrix} y \\ z \end{pmatrix}^T \begin{bmatrix} -2+\varepsilon\beta & \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2} & -\varepsilon \end{bmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + 2xy - \varepsilon xz \\ &\le \lambda_{\max}(Q)(y^2+z^2) + 2\max\{|L_1|,|L_2|\}|y| \\ &\quad + \varepsilon \max\{|L_1|,|L_2|\}|z| \\ &= \lambda_{\max}(Q)\left(|y| + \frac{L}{\lambda_{\max}(Q)}\right)^2 \\ &\quad + \lambda_{\max}(Q)\left(|z| + \frac{\varepsilon L}{2\lambda_{\max}(Q)}\right)^2 \\ &\quad - \frac{L^2}{\lambda_{\max}(Q)} - \frac{\varepsilon^2 L^2}{4\lambda_{\max}(Q)} \\ &< 0, \end{aligned}$$

when

$$\begin{aligned} -\lambda_{\max}(Q)\left(|y| + \frac{L}{\lambda_{\max}(Q)}\right)^2 &- \lambda_{\max}(Q)\left(|z| + \frac{\varepsilon L}{2\lambda_{\max}(Q)}\right)^2 \\ &> -\frac{L^2}{\lambda_{\max}(Q)} - \frac{\varepsilon^2 L^2}{4\lambda_{\max}(Q)}, \end{aligned}$$

where $L = \max\{|L_1|,|L_2|\}$, and $\lambda_{\max}(Q)$ denotes the largest eigenvalue of Q:

$$Q = \begin{bmatrix} -2+\varepsilon\beta & \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2} & -\varepsilon \end{bmatrix}.$$

Thus, $\lambda_{\max}(Q) < 0$. The above results show that there exists globally attractive and positive invariant set for system (1), Ω, as defined by (9).

The proof of Theorem 3 is complete. ■

Remark 1. Outside the global attractive set Ω_ε, no equilibrium point, periodic solution, almost periodic solution or other chaotic attractor exists.

(1) In fact, consider any equilibrium point, $(x^*, 0, z^*)$. If it is outside Ω_ε, then for any $t > t_0$, on one hand, $(x^*, 0, z^*)$ is the solution of the system, while on the other hand, $(x^*(t), 0, z^*(t)) \equiv (x^*(t_0), 0, z^*(t_0))$. So, $V(x^*(t), 0, z^*(t)) \equiv V(x^*(t_0), 0, z^*(t_0))$. However, $V(x^*(t), 0, z^*(t)) < V(x^*(t_0), 0, z^*(t_0))$, since $dV/dt < 0$ outside Ω_ε. This is a contradiction. Thus, there is no equilibrium point outside Ω_ε.

(2) We can apply the method by contradiction to prove that system (1) has no periodic solution outside Ω_ε. Otherwise, suppose that there exists a periodic solution, $X(t)$, outside Ω_ε with period T. Apparently, $X(t) = X(t+T)$, showing that $V(X(t)) = V(X(t+T))$. However, since $dV/dt < 0$, $V(X(t+T)) < V(X(t))$ outside Ω_ε, leading to a contradiction. Thus, there is no periodic solution outside Ω_ε.

(3) System (1) has no almost periodic solution outside Ω_ε. Otherwise, assume that there exists an almost periodic solution, $\tilde{X}(t)$, outside Ω_ε with almost period T. So there exists a compact set Q such that $\tilde{X}(t) \in Q$. Without loss of generality, assume $Q \bigcap \Omega_\varepsilon = \emptyset$. Let $\sup_{x\in Q} dV/dt = -l(l > 0)$. According to the definition of almost periodic function, $\forall\, \varepsilon > 0$, we have

$$\|\tilde{X}(t_0) - \tilde{X}(t_0+T)\| < \varepsilon.$$

Thus, since V is a continuous function,

$$|V(\tilde{X}(t_0)) - V(\tilde{X}(t_0+T))| \ll 1.$$

But, on the other hand, with the assumption, we have

$$V(\tilde{X}(t_0+T)) \le V(\tilde{X}(t_0)) - lT,$$

i.e.

$$V(\tilde{X}(t_0)) - V(\tilde{X}(t_0+T)) \ge lT.$$

This contradiction shows that system (1) has no almost periodic solution outside Ω_ε.

(4) There does not exist a chaotic attractor outside Ω_ε. If there is any chaotic attractor outside Ω_ε, $X(t)$ has to be chaotic outside Ω_ε and eventually enters Ω_ε as well, which is certainly impossible.

In summary, Theorem 1 assures that the trajectory is moving towards the direction in which V is monotonically decreasing, implying that the movement of trajectory is monotonic outside Ω_ε.

4. Analysis of Equilibrium Points

One of the most important tasks in chaos control is to stabilize the system's equilibrium points. In this section, we discuss the number of equilibrium points of system (1) and their stability.

Let

$$d = \begin{cases} \pi, & \text{when } n \text{ is odd,} \\ 0, & \text{when } n \text{ is even.} \end{cases}$$

Consider the second and third equations in system (1). Let $\dot{y} = 0$ and $\dot{z} = 0$. Solving the resulting equations yields $y = 0$ and $x = -z$. Due to the property of inverse-trigonometric function, x and z have multiple solutions.

(1) First consider $d = 0$, i.e. n is even, and so $n-1$ is odd. Let $\pi x_m/2a = m\pi$, where $m = -n+1, -n+2, \ldots, -1, 0, 1, \ldots, n-1$. The solutions of x and z are:

$$\begin{aligned} &x_0 = 0, && z_0 = 0, \\ &x_1 = 2a, && z_1 = -2a, \\ &x_2 = 4a, && z_2 = -4a, \\ &\ldots, && \ldots, \\ &x_{n-1} = 2(n-1)a, && z_{n-1} = 2(1-n)a; \\ &x_{-1} = -2a, && z_{-1} = 2a, \\ &x_{-2} = -4a, && z_{-2} = 4a, \\ &\ldots, && \ldots, \\ &x_{1-n} = 2(1-n)a, && z_{1-n} = 2(n-1)a. \end{aligned}$$

So system (1) has $2n-1$ equilibrium points, given by

$$\begin{array}{lll} (2(1-n)a, 0, 2(n-1)a), & (2(2-n)a, 0, 2(n-2)a), & \ldots, \\ (-2a, 0, 2a), & (0,0,0), & (2a, 0, -2a), \\ \ldots, & (2(n-2)a, 0, 2(2-n)a), & (2(n-1)a, 0, 2(1-n)a). \end{array}$$

For convenience, we divide the equilibrium points into two groups: even equilibrium points, denoted by even subscripts such as $(x_{-2}, 0, z_{-2})$, $(x_0, 0, z_0)$, $(x_2, 0, z_2)$, etc., and odd equilibrium points, denoted by odd subscripts such as $(x_{-1}, 0, z_{-1})$, $(x_1, 0, z_1)$, etc.

(2) When $d = \pi$, i.e. n is odd, let

$$\left(\frac{\pi x_m}{2a} + \pi\right) = m\pi,$$

where $m = -n+2, -n+3, \ldots, n-1, n$. The solutions of x and z are:

$$\begin{aligned} &x_0 = -2a, && z_0 = 2a, \\ &x_1 = 0, && z_1 = 0, \\ &\ldots, && \ldots, \\ &x_n = 2(n-1)a, && z_n = 2(1-n)a; \\ &x_{-1} = -4a, && z_{-1} = 4a, \\ &x_{-2} = -6a, && z_{-2} = 6a, \\ &\ldots, && \ldots, \\ &x_{2-n} = 2(1-n)a, && z_{2-n} = 2(n-1)a. \end{aligned}$$

Similarly, we can define even equilibrium points and odd equilibrium points for this case when $d = \pi$.

Theorem 2. *All odd equilibrium points of system (1) are locally asymptotically stable, while all even equilibrium points of the system are unstable.*

Proof. We only prove for $(x_0, 0, z_0)$ (the representative of even equilibrium points) and $(x_1, 0, z_1)$ (the representative of odd equilibrium points). Other cases can be similarly proved. Using Lyapunov first order approximation theorem, we can show that $(x_0, 0, z_0)$ is unstable. Since the second and third equations in system (1) are linear, we only need to linearize the first equation. Linearize the first equation in system (1) around $(x_0, 0, z_0)$, yielding

$$\begin{aligned} \dot{x} &= \alpha y + \alpha b \left(\sin\frac{\pi x}{2a}\right)'\Bigg|_{x=0} x \\ &= \alpha y + \alpha \frac{b\pi}{2a}\cos(0)x = \alpha y + \alpha\frac{b\pi}{2a}x, \qquad (14) \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned}$$

whose coefficient matrix is

$$B = \begin{bmatrix} \frac{\alpha b\pi}{2a} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}.$$

Since the trace of B is

$$\text{Trace}(B) = \frac{\alpha b\pi}{2a} - 1 + 0 > 0,$$

at least one of the eigenvalues of B has positive real part. Thus, the zero solution of linear system (14) is unstable, which shows that $(x_0, 0, z_0)$ is unstable.

Linearize the first equation in system (1) around $(x_1, 0, z_1)$, giving

$$\begin{aligned} \dot{x} &= \alpha y + \alpha b \left(\sin\frac{\pi x}{2a}\right)'\bigg|_{x=2a}(x-2a) \\ &= \alpha y + \alpha\frac{b\pi}{2a}\cos(\pi)(x-2a) \\ &= \alpha y - \alpha\frac{b\pi}{2a}(x-2a), \\ \dot{y} &= (x-2a) - y + (z+2a), \\ \dot{z} &= -\beta y, \end{aligned} \tag{15}$$

whose coefficient matrix is

$$B = \begin{bmatrix} -\frac{\alpha b\pi}{2a} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}.$$

Since B is a Hurwitz matrix, $(x_1, 0, z_1)$ is locally asymptotically stable. ■

5. Stabilization of the Modified Chua's Circuit

Chaos control includes two parts. One of them is to stabilize a given equilibrium point (not just the origin) by simple feedback control laws. That is, change the stability of a given equilibrium point, which is originally locally asymptotically stable or unstable, such that the system becomes globally asymptotically stable or globally exponentially stable. Since the function on the right-hand side of system (1) satisfies the global Lipschitz condition, we will show that, for any given equilibrium point, a very simple feedback control law can make the equilibrium point globally exponentially or globally asymptotically stable.

Theorem 3. *Let $(x^*, 0, z^*)$ be any given equilibrium point of system (1). Regardless of the stability of the equilibrium point, add the simple linear feedback control $-\alpha k(x - x^*)$ to the first equation of system (1). Then,*

when $k > (b\pi/2a) + 1$, the equilibrium point $(x^*, 0, z^*)$ is globally exponentially stabilized;

when $k \geq (b\pi/2a) + 1$, the equilibrium point $(x^*, 0, z^*)$ is globally asymptotically stabilized.

Proof. Let $\bar{x} = x - x^*$, $\bar{y} = y - 0$, $\bar{z} = z - z^*$. Then system (1) can be written as

$$\begin{aligned} \dot{\bar{x}} &= \alpha[y - f(x)] - \alpha[y^* - f(x^*)]x - \alpha k\bar{x} \\ &= \alpha\bar{y} - \alpha f_{(x,x^*)}\bar{x} - \alpha k\bar{x}, \\ \dot{\bar{y}} &= \bar{x} - \bar{y} + \bar{z}, \\ \dot{\bar{z}} &= -\beta\bar{y}. \end{aligned} \tag{16}$$

(1) When $k > (b\pi/2a) + 1$, let $\delta = k - (b\pi/2a) > 1$. Construct radially unbounded Lyapunov function for system (16):

$$\begin{aligned} V &= \frac{\bar{x}^2}{\alpha} + \bar{y}^2 + \frac{\bar{z}^2}{\beta} - \varepsilon\bar{y}\,\bar{z} \\ &= \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}^T \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & 1 & -\frac{\varepsilon}{2} \\ 0 & -\frac{\varepsilon}{2} & \frac{1}{\beta} \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} \\ &:= \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}^T Q \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}. \end{aligned}$$

For $0 < \varepsilon \ll 1$, V is positive definite. Then

$$\begin{aligned} \frac{dV}{dt}\bigg|_{(16)} &= \frac{2\bar{x}\dot{\bar{x}}}{\alpha} + 2\bar{y}\dot{\bar{y}} + \frac{2\bar{z}\dot{\bar{z}}}{\beta} - \varepsilon\dot{\bar{y}}\bar{z} - \varepsilon\bar{y}\dot{\bar{z}} \\ &= 2(-k - f_{(x,x^*)})\bar{x}^2 + 2\bar{x}\bar{y} + 2\bar{x}\bar{y} - 2\bar{y}^2 + 2\bar{y}\bar{z} - 2\bar{y}\bar{z} - \varepsilon\bar{x}\bar{z} + \varepsilon\bar{y}\bar{z} - \varepsilon\bar{z}^2 + \varepsilon\beta\bar{y}^2 \end{aligned}$$

$$= \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}^T \begin{bmatrix} -2(k+f_{(x,x^*)}) & 2 & -\frac{\varepsilon}{2} \\ 2 & -2+\varepsilon\beta & \frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \frac{\varepsilon}{2} & -\varepsilon \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

$$\leq \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}^T \begin{bmatrix} -2\left(k-\frac{b\pi}{2a}\right) & 2 & -\frac{\varepsilon}{2} \\ 2 & -2+\varepsilon\beta & \frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \frac{\varepsilon}{2} & -\varepsilon \end{bmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

$$:= \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}^T \Omega \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}, \tag{17}$$

where

$$\Delta_1 := -2\left(k-\frac{b\pi}{2a}\right) = -2\delta < 0,$$

$$\Delta_2 := \det\begin{bmatrix} -2\left(k-\frac{b\pi}{2a}\right) & 2 \\ 2 & -2+\varepsilon\beta \end{bmatrix} = 4\delta - 2\delta\varepsilon\beta - 4 > 0, \quad \text{when } 0<\varepsilon \ll 1,$$

and

$$\Delta_3 := \det(\Omega) = \det\begin{bmatrix} -2\left(k-\frac{b\pi}{2a}\right) & 2 & -\frac{\varepsilon}{2} \\ 2 & -2+\varepsilon\beta & \frac{\varepsilon}{2} \\ -\frac{\varepsilon}{2} & \frac{\varepsilon}{2} & -\varepsilon \end{bmatrix}$$
$$= \varepsilon\left(-4\delta + 2\delta\varepsilon\beta + \frac{\delta\varepsilon}{2} + 4 - \frac{\varepsilon}{2} - \frac{1}{4}\varepsilon^2\beta\right) < 0, \quad \text{when } 0<\varepsilon\ll 1.$$

Thus, Ω is negative definite. The above discussion shows that we can always choose $0 < \varepsilon \ll 1$ such that Q is positive definite while Ω is negative definite.

Further, let $\lambda_{\max}(\Omega)$ and $\lambda_{\max}(Q)$ be the maximum eigenvalues of Ω and Q, respectively, and $\lambda_{\min}(Q)$ the minimum eigenvalue of Q. We then obtain

$$\left.\frac{dV}{dt}\right|_{(16)} \leq \lambda_{\max}(\Omega)(\bar{x}^2(t)+\bar{y}^2(t)+\bar{z}^2(t)) \leq \lambda_{\max}(\Omega)\frac{1}{\lambda_{\max}(Q)}V. \tag{18}$$

Thus,

$$\lambda_{\min}(Q)(\bar{x}^2(t)+\bar{y}^2(t)+\bar{z}^2(t)) \leq V(t) \leq V(t_0)\exp\left(\frac{\lambda_{\max}(\Omega)}{\lambda_{\max}(Q)}(t-t_0)\right),$$

from which we have

$$\bar{x}^2(t)+\bar{y}^2(t)+\bar{z}^2(t) \leq \frac{1}{\lambda_{\min}(Q)}V(t_0)\exp\left(\frac{\lambda_{\max}(\Omega)}{\lambda_{\max}(Q)}(t-t_0)\right). \tag{19}$$

This shows that when $k > (b\pi/2a)+1$, $(x^*, 0, z^*)$ is globally exponentially stabilized.

(2) When $k \geq (b\pi/2a)+1$, let $\delta = k-(b\pi/2a) \geq 1$. Construct radially unbounded Lyapunov function for system (16):

$$V = \frac{\bar{x}^2}{\alpha} + \bar{y}^2 + \frac{\bar{z}^2}{\beta} \quad \text{when } |f_{(x,x^*)}| \neq \frac{b\pi}{2a}.$$

Thus,

$$\left.\frac{dV}{dt}\right|_{(16)} = 2(-k-f_{(x,x^*)})\bar{x}^2 + 4\bar{x}\bar{y} - 2\bar{y}^2 = 2\left(-k+\frac{b\pi}{2a}-\frac{b\pi}{2a}-f_{(x,x^*)}\right)\bar{x}^2 + 4\bar{x}\bar{y} - 2\bar{y}^2$$

$$\leq -2\bar{x}^2 + 4\bar{x}\,\bar{y} - 2\bar{y}^2 + 2\left(-\frac{b\pi}{2a} - f_{(x,x^*)}\right)\bar{x}^2$$

$$= -2(\bar{x} - \bar{y})^2 + 2\left(-\frac{b\pi}{2a} - f_{(x,x^*)}\right)\bar{x}^2$$

$$\begin{cases} < 0, & \text{when } \bar{x}^2 + \bar{y}^2 \neq 0, \\ = 0, & \text{when } \bar{x}^2 + \bar{y}^2 = 0. \end{cases}$$

Let $(dV/dt)|_{(16)} = 0$, resulting in $\bar{x} = 0$ and $\bar{y} = 0$. Substitute these equations into system (16), yielding $\bar{z} = 0$. Thus, there is no other nonzero solutions such that $(dV/dt)|_{(16)} = 0$, except $\bar{x} = \bar{y} = \bar{z} = 0$. From LaSalle invariant principle [Liao, 2001] we know that

$$\bar{x}(t) \to 0, \quad \bar{y}(t) \to 0, \quad \text{and} \quad \bar{z}(t) \to 0 \text{ as } t \to +\infty.$$

However, $|f_{(x,x^*)}| = b\pi/2a$ has only limited number of solutions. Let

$$E = \left\{ X \,\middle|\, \frac{dV}{dt} = 0 \right\}.$$

So E does not contain any nonzero positive trajectory of system (16) but $(0, 0, 0)$, or equivalently, except $(0, 0, 0)$, E does not contain any other invariant set. According to LaSalle invariant principle or Krosovskii–Barbashin theorem [Liao, 2001], $(x^*, 0, z^*)$ is globally asymptotically stable. ■

Theorem 4. *Assume that $(x^*, 0, z^*)$ is any given equilibrium point of system (1). Add the controls $-\alpha k_1(x - x^*)$ and $-k_2(y - y^*)$ to the first and second equations of system (16), respectively. Then,*

when $b\pi/2a < k_1 < (b\pi/2a) + 1$ and $k_2 > (1/(k_1 - (b\pi/2a))) - 1, (x^*, 0, z^*)$ is globally exponentially stabilized;

when $b\pi/2a < k_1 < (b\pi/2a) + 1$ and $k_2 \geq (1/(k_1 - (b\pi/2a))) - 1, (x^*, 0, z^*)$ is globally asymptotically stabilized.

Proof

(1) When $b\pi/2a < k_1 < (b\pi/2a) + 1$ and $k_2 > (1/(k_1 - (b\pi/2a))) - 1$, we have

$$(k_2 + 1)\left(k_1 - \frac{b\pi}{2a}\right) > 1.$$

Let

$$a_{11} = \frac{1}{2}\alpha(-k_1 - f_{(x,x^*)}) \leq \frac{1}{2}\alpha\left(-k_1 + \frac{b\pi}{2a}\right) < 0,$$

$$a_{12} = \frac{\alpha}{2},$$

$$a_{21} = 1,$$

$$a_{22} = -1 - k_2.$$

Since

$$a_{11}a_{22} > \frac{1}{2}\alpha(k_2 + 1)\left(k_1 - \frac{b\pi}{2a}\right) > \frac{\alpha}{2} = a_{12}a_{21},$$

from the Lyapunov–Lottor stability we know that there exists constant η such that the matrix,

$$\begin{bmatrix} 2a_{11} & a_{12} + \eta a_{21} \\ a_{12} + \eta a_{21} & 2\eta a_{22} \end{bmatrix}$$

is negative definite. Thus,

$$\det\begin{bmatrix} 2a_{11} & a_{12} + \eta a_{21} \\ a_{12} + \eta a_{21} & 2\eta a_{22} \end{bmatrix} > 0, \qquad (20)$$

i.e.

$$4a_{11}a_{22}\eta - (a_{12} + \eta a_{21})^2 = -a_{21}^2\eta^2 + (4a_{11}a_{22} - 2a_{12}a_{21})\eta - a_{12}^2 > 0.$$

Since

$$-(4a_{11}a_{22} - 2a_{12}a_{21}) < 0$$

and

$$\Delta := (4a_{11}a_{22} - 2a_{12}a_{21})^2 - 4a_{21}^2a_{12}^2 = 16a_{11}a_{22}(a_{11}a_{22} - a_{12}a_{21}) > 0,$$

the inequality (20) has positive solution for η.

Under the control law, system (16) can be written as:

$$\begin{aligned} \dot{\bar{x}} &= \alpha(\bar{y} - f_{(x,x^*)}\bar{x}) - \alpha k_1\bar{x}, \\ \dot{\bar{y}} &= \bar{x} - (1 + k_2)\bar{y} + \bar{z}, \\ \dot{\bar{z}} &= -\beta\bar{y}. \end{aligned} \qquad (21)$$

Construct radially unbounded Lyapunov function for system (21):

$$V = \frac{\bar{x}^2}{2} + \eta\bar{y}^2 + \frac{\eta\bar{z}^2}{\beta} - \varepsilon\bar{y}\,\bar{z}, \quad 0 < \varepsilon \ll 1.$$

Then,

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(21)} &= \bar{x}\dot{\bar{x}} + 2\eta\bar{y}\dot{\bar{y}} + \frac{2\eta\bar{z}\dot{\bar{z}}}{\beta} - \varepsilon\dot{\bar{y}}\bar{z} - \varepsilon\bar{y}\dot{\bar{z}} \\ &= \alpha(-k_1 - f_{(x,x^*)})\bar{x}^2 + \alpha\bar{x}\bar{y} + 2\eta\bar{x}\bar{y} - 2\eta(1+k_2)\bar{y}^2 + 2\eta\bar{y}\bar{z} - 2\eta\bar{y}\bar{z} \\ &\quad - \varepsilon\bar{x}\bar{z} + \varepsilon(1+k_2)\bar{y}\bar{z} - \varepsilon\bar{z}^2 + \varepsilon\beta\bar{y}^2 \\ &= \begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix}^T \begin{bmatrix} -\alpha(k_1 + f_{(x,x^*)}) & \dfrac{\alpha}{2} + \eta & -\dfrac{\varepsilon}{2} \\ \dfrac{\alpha}{2} + \eta & -2\eta(1+k_2) + \varepsilon\beta & \dfrac{\varepsilon(1+k_2)}{2} \\ -\dfrac{\varepsilon}{2} & \dfrac{\varepsilon(1+k_2)}{2} & -\varepsilon \end{bmatrix} \begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix}.\end{aligned}$$

Using a similar argument as that used in the proof of Theorem 3, we can show that

$$\begin{bmatrix} -\alpha(k_1 + f_{(x,x^*)}) & \dfrac{\alpha}{2} + \eta & -\dfrac{\varepsilon}{2} \\ \dfrac{\alpha}{2} + \eta & -2\eta(1+k_2) + \varepsilon\beta & \dfrac{\varepsilon(1+k_2)}{2} \\ -\dfrac{\varepsilon}{2} & \dfrac{\varepsilon(1+k_2)}{2} & -\varepsilon \end{bmatrix}$$

is negative definite. Thus, the zero solution of system (16) is globally exponentially stable, i.e. $(x^*, 0, z^*)$ is globally exponentially stable.

(2) When $b\pi/2a < k_1 < (b\pi/2a) + 1$ and $k_2 \geq (1/(k_1 - (b\pi/2a))) - 1$, we construct radially unbounded Lyapunov function for system (16), yielding

$$V = \frac{\bar{x}^2}{\alpha} + \bar{y}^2 + \frac{\bar{z}^2}{\beta}.$$

Similarly, we can prove that under the given control law, $(x^*, 0, z^*)$ is globally asymptotically stable. ∎

Theorem 5. *Assume that $(x^*, 0, z^*)$ is any given equilibrium point of system (1), apply the controls $-\alpha k_1(x - x^*)$ and $-(x - x^*)$ to the first and second equations of the system, respectively. Then, for $k_1 > b\pi/2a$, $(x^*, 0, z^*)$ is globally exponentially stabilized.*

Proof. Under the given control law, system (16) can be written as

$$\begin{aligned}\dot{\bar{x}} &= \alpha(\bar{y} - f_{(x,x^*)}\bar{x}) - \alpha k_1\bar{x}, \\ \dot{\bar{y}} &= -\bar{y} + \bar{z}, \\ \dot{\bar{z}} &= -\beta\bar{y}.\end{aligned} \tag{22}$$

The second and third equations in system (22) are decoupled from the first equation. So, first consider

$$\begin{aligned}\dot{\bar{y}} &= -\bar{y} + \bar{z}, \\ \dot{\bar{z}} &= -\beta\bar{y}.\end{aligned} \tag{23}$$

Obviously, the coefficient matrix of system (23) is a Hurwitz matrix, whose zero solution is globally exponentially stable. Particularly, there exist constants $M \geq 1$ and $\lambda > 0$ such that

$$|\bar{y}(t)| \leq Me^{-\lambda t}. \tag{24}$$

Assume $\lambda < \alpha(k_1 - (b\pi/2a))$. We construct positive definite and radially unbounded Lyapunov function for the first equation of system (22), $V = |\bar{x}|$. Substituting Eq. (24) into the first equation of system (22), we have

$$\begin{aligned}D^+|\bar{x}| &\leq -\alpha(k_1 + f_{(x,x^*)})|\bar{x}(t)| + \alpha|\bar{y}(t)| \\ &\leq -\alpha\left(k_1 - \frac{b\pi}{2a}\right)|\bar{x}(t)| + \alpha Me^{-\lambda t}.\end{aligned} \tag{25}$$

Consider the comparison equation to system (25), which can be written as

$$\begin{aligned}\frac{d\xi(t)}{dt} &= -\alpha\left(k_1 - \frac{b\pi}{2a}\right)\xi + \alpha Me^{-\lambda t}, \\ \xi(0) &= |\bar{x}(0)|.\end{aligned} \tag{26}$$

The solution of system (26) is

$$\begin{aligned}\xi(t) &= \xi(0)e^{-\alpha(k_1-\frac{b\pi}{2a})t} \\ &\quad + \int_0^t e^{-\alpha(k_1-\frac{b\pi}{2a})(t-\tau)}\alpha M e^{-\lambda\tau}d\tau \\ &= \xi(0)e^{-\alpha(k_1-\frac{b\pi}{2a})t} \\ &\quad + \alpha M e^{-\alpha(k_1-\frac{b\pi}{2a})t}\int_0^t e^{(\alpha(k_1-\frac{b\pi}{2a})-\lambda)\tau}d\tau \\ &< \xi(0)e^{-\alpha(k_1-\frac{b\pi}{2a})t} + \alpha M\frac{e^{-\lambda t}}{\alpha\left(k_1-\dfrac{b\pi}{2a}\right)-\lambda}.\end{aligned} \tag{27}$$

Thus,

$$|\bar{x}(t)| \le \xi(t) < \xi(0)e^{-\alpha(k_1-\frac{b\pi}{2a})t} + \alpha M\frac{e^{-\lambda t}}{\alpha\left(k_1-\dfrac{b\pi}{2a}\right)-\lambda}. \tag{28}$$

Inequality (28) shows that the conclusion of Theorem 5 is true. ■

Theorem 6. *Assume that* $(x^*, 0, z^*)$ *is any given equilibrium point of system (1), apply the feedback controls* $-\alpha k(x - x^*) - \alpha(y - y^*)$ *to the first equation of the system. Then, for* $k - (b\pi/2a) = \delta > 0$, $(x^*, 0, z^*)$ *is globally exponentially stabilized.*

Proof. Under the given control law, system (16) becomes

$$\begin{aligned}\dot{\bar{x}} &= \alpha(\bar{y} - f_{(x,x^*)}\bar{x}) - \alpha k\bar{x} - \alpha\bar{y} \\ &= -\alpha(k + f_{(x,x^*)})\bar{x}, \\ \dot{\bar{y}} &= \bar{x} - \bar{y} + \bar{z}, \\ \dot{\bar{z}} &= -\beta\bar{y}.\end{aligned} \tag{29}$$

Construct positive definite and radially unbounded Lyapunov function for the first equation of system (28):

$$V = |\bar{x}|.$$

Then, we have

$$\begin{aligned}D^+V &= D^+|\bar{x}| \\ &\le -\alpha(k + f_{(x,x^*)})|\bar{x}| \\ &\le -\alpha\left(k - \frac{b\pi}{2a}\right)|\bar{x}| \\ &\le -\alpha\delta|\bar{x}|.\end{aligned}$$

Thus,

$$|\bar{x}(t)| \le |\bar{x}(0)|e^{-\alpha\delta t}.$$

The coefficient matrix of the second and third equations of system (28) is

$$H = \begin{bmatrix} -1 & 1 \\ -\beta & 0 \end{bmatrix},$$

which is a Hurwitz matrix. The solution of the second and third equations of system (28) can be

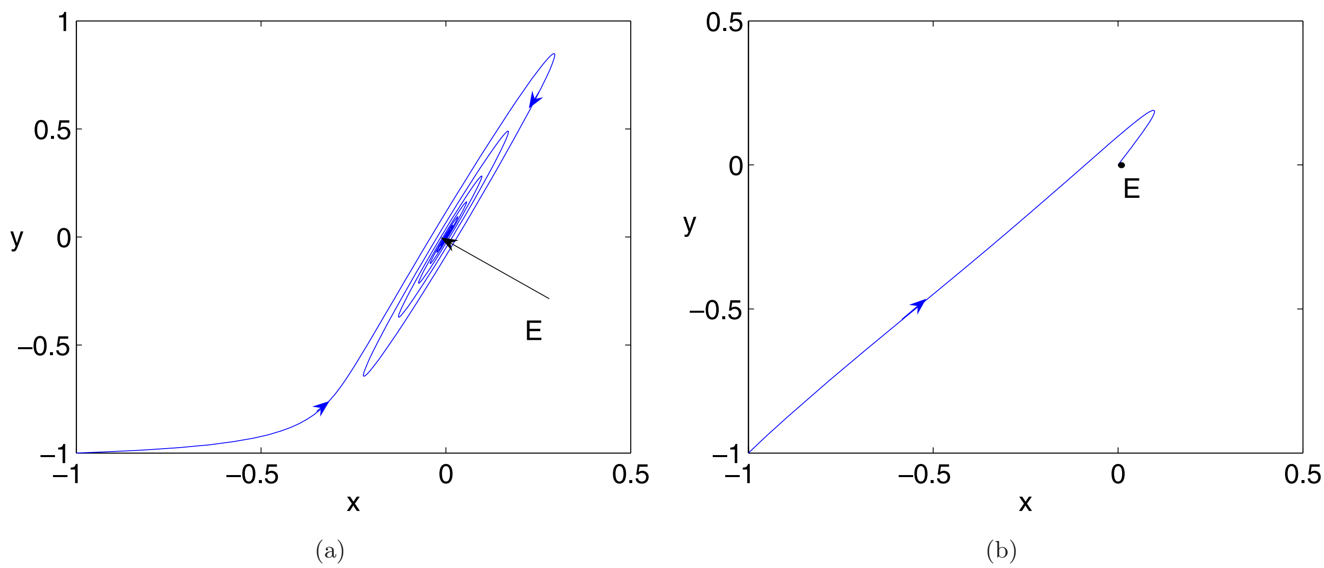


Fig. 2. (a) Trajectory of a five-scroll chaotic attractor for system (1) using the control law given in Theorem 3 for $k = 1.2$ with the initial condition $x(0) = -1,\ y(0) = -1,\ z(0) = 1$, convergent to the equilibrium point $E\colon (0, 0, 0)$; (b) trajectory of a five-scroll chaotic attractor for system (1) using the control law given in Theorem 4 for $k_1 = k_2 = 1$ with the initial condition $x(0) = -1,\ y(0) = -1,\ z(0) = 1$, convergent to the equilibrium point $E\colon (0, 0, 0)$.

written as

$$\begin{pmatrix}\bar{y}(t)\\ \bar{z}(t)\end{pmatrix} = \begin{pmatrix}\bar{y}(0)\\ \bar{z}(0)\end{pmatrix} e^{Ht} + \int_0^t e^{H(t-\tau)}\begin{pmatrix}\bar{x}(\tau)\\ 0\end{pmatrix} d\tau.$$

Due to H being Hurwitz, we have $|e^{Ht}| \leq Ke^{-\lambda t}$, where $K \geq 1$ and $\lambda > 0$. Further, assuming $\lambda \neq \alpha\delta$, we obtain

$$\begin{aligned}\left|\begin{pmatrix}\bar{y}(t)\\ \bar{z}(t)\end{pmatrix}\right| &\leq \left|\begin{pmatrix}\bar{y}(0)\\ \bar{z}(0)\end{pmatrix}\right| Ke^{-\lambda t}\\ &\quad + K|\bar{x}(0)|e^{-\lambda t}\left(\frac{e^{\lambda t-\alpha\delta t}}{|\lambda-\alpha\delta|} + \frac{1}{|\lambda-\alpha\delta|}\right)\\ &\leq K\Bigg(|(\bar{y}(0)+\bar{z}(0))^T|\\ &\quad + 2\frac{|\bar{x}(0)|}{|\lambda-\alpha\delta|}\Bigg)e^{-\min\{\alpha\delta,\lambda\}},\end{aligned}$$

which shows that $(x^*, 0, z^*)$ is globally exponentially stabilized. ■

To end this section, we present two numerical simulation examples to show that the control laws obtained above are efficient. These two examples use the control laws given in Theorems 3 and 4, and the results are depicted in Fig. 2. For the first example, k is chosen as $k = 1.2 > (b\pi/2a) + 1 \approx 1.1329$, satisfying the condition given in Theorem 3, so the trajectory of the controlled system globally exponentially converges to the designated equilibrium point $(0, 0, 0)$, as shown in Fig. 2(a). The second example takes $k_1 = k_2 = 1$. So $0.1329 \approx b\pi/2a < k_1 = 1 < (b\pi/2a) + 1 \approx 1.1329$ and $k_2 = 1 > (1/(k_1 - (b\pi/2a))) - 1 \approx 0.1533$. Thus, according to Theorem 4 the trajectory of the controlled system globally exponentially converges to the designated equilibrium point $(0, 0, 0)$. Other cases are similar to these two examples.

6. Global Synchronization of Two Modified Chua's Circuits

Since in 1990, Pecora and Carroll [Pecora, 1990] used electrical circuits to realize chaos synchronization, the traditional opinion that chaotic systems cannot be synchronized was changed.

In this section, we study the globally exponential synchronization of two modified Chua's circuits. The details about the method employed here can be found in [Liao, 1993].

Assume that the drive system is:

$$\begin{aligned}\dot{x}_d &= \alpha[y_d - x_d - g(x_d)],\\ \dot{y}_d &= x_d - y_d + z_d,\\ \dot{z}_d &= -\beta y_d,\end{aligned} \tag{30}$$

and the corresponding driven system is:

$$\begin{aligned}\dot{x}_r &= \alpha[y_r - x_r - g(x_r)]\\ &\quad + u_1((x_d - x_r), (y_d - y_r), (z_d - z_r)),\\ \dot{y}_r &= x_r - y_r + z_r\\ &\quad + u_2((x_d - x_r), (y_d - y_r), (z_d - z_r)),\\ \dot{z}_r &= -\beta y_r + u_3((x_d - x_r), (y_d - y_r), (z_d - z_r)).\end{aligned} \tag{31}$$

Let $e_x = x_d - x_r$, $e_y = y_d - y_r$ and $e_z = z_d - z_r$. Then the error system is obtained as

$$\begin{aligned}\dot{e}_x &= \alpha e_y - \alpha e_x - \alpha(g(x_d) - g(x_r))\\ &\quad - u_1(e_x, e_y, e_z)\\ &= -\alpha(1 + g_{(x_d,x_r)})e_x + \alpha e_y\\ &\quad - u_1(e_x, e_y, e_z),\\ \dot{e}_y &= e_x - e_y + e_z - u_2(e_x, e_y, e_z),\\ \dot{e}_z &= -\beta e_y - u_3(e_x, e_y, e_z),\end{aligned} \tag{32}$$

where u_1, u_2 and u_3 are feedback controls to be determined, satisfying $u_1(0) = u_2(0) = u_3(0) = 0$.

Definition 4. $\forall$ $(x_d(t_0), y_d(t_0), z_d(t_0)) \in R^3$, and the corresponding $(x_r(t_0), y_r(t_0), z_r(t_0)) \in R^3$, if the zero solution of (32) is globally exponentially stable, then systems (30) and (31) are said to be globally exponentially synchronized.

Here, our aim is to construct simple feedback controls u_1, u_2 and u_3 such that the zero solution of the error system (32) is globally exponentially stabilized. Thus, the drive system and its corresponding driven system are globally exponentially synchronized.

Theorem 7. *In the driven system (31), choose the following control law*:

$$u_1 = \alpha\delta_x(x_d - x_r) + \alpha(y_d - y_r), \quad u_2 = 0, \quad u_3 = 0.$$

If $\delta_x > b\pi/2a$, the zero solution of (32) is globally exponentially stabilized, and thus systems (30) and (31) are globally exponentially synchronized.

Proof. Let $\tilde{\delta}_x = \delta_x - (b\pi/2a)$, i.e. $\delta_x = \tilde{\delta}_x + (b\pi/2a)$. Under the given control law, we construct positive definite and radially unbounded Lyapunov function for the first equation of (32):

$$V = \frac{e_x^2}{2}.$$

Then we have

$$\begin{aligned}\frac{dV}{dt} &= e_x\dot{e}_x \\ &= -\alpha(1+g_{(x_d,x_r)}+\delta_x)e_x^2 \\ &\le -\alpha\left(1-\frac{b\pi}{2a}-1+\delta_x\right)e_x^2 \\ &= -\alpha\tilde{\delta}_x e_x^2 \\ &= -2\alpha\tilde{\delta}_x V,\end{aligned}$$

whose comparison equation is

$$\frac{d\xi}{dt} = -2\alpha\tilde{\delta}_x\xi(t).$$

Thus,

$$e_x^2 = 2V(t) \le 2\xi(t_0)e^{-2\alpha\tilde{\delta}_x(t-t_0)}, \tag{33}$$

which shows that the zero solution of system (32) is globally exponentially stabilized with respect to the partial variable e_x.

Next, consider the linear part of system (32), whose coefficient matrix is

$$A = \begin{bmatrix} -\alpha-\alpha\delta_x & 0 & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned}\det(\lambda I_3 - A) &= \det\begin{bmatrix} \lambda+\alpha+\alpha\delta_x & 0 & 0 \\ -1 & \lambda+1 & -1 \\ 0 & \beta & \lambda \end{bmatrix} \\ &= (\lambda+\alpha+\alpha\delta_x)(\lambda+1)\lambda \\ &\quad +\beta(\lambda+\alpha+\alpha\delta_x) \\ &= \lambda^3+(\alpha+\alpha\delta_x+1)\lambda^2 \\ &\quad +(\alpha+\alpha\delta_x+\beta)\lambda \\ &\quad +\beta(\alpha+\alpha\delta_x).\end{aligned}$$

We define

$$p := \alpha+\alpha\delta_x+1 > 0,$$
$$q := \alpha+\alpha\delta_x+\beta > 0,$$

and

$$r := \beta(\alpha+\alpha\delta_x) > 0.$$

Thus,

$$pq = (\alpha+\alpha\delta_x+1)(\alpha+\alpha\delta_x+\beta) > r,$$

which shows that A is a Hurwitz matrix.

The conclusion of Theorem 7 is true. ■

Theorem 8. *In the driven system (31), choose the following control law:*

$$u_1 = \alpha\delta_x(x_d-x_r)+\tilde{\alpha}(y_d-y_r),$$
$$u_2 = z_d-z_r, \quad u_3 = \delta_z(z_d-z_r).$$

If $\delta_x > b\pi/2a$, $\tilde{\alpha} > \alpha$ and $\delta_z > 0$, the zero solution of (32) is globally exponentially stabilized, and thus systems (30) and (31) are globally exponentially synchronized.

Proof. Under the given control law, system (32) can be written as

$$\begin{aligned}\dot{e}_x &= -\alpha(e_x-e_y)-\tilde{\alpha}e_y-\alpha\delta_x e_x \\ &\quad -\alpha(g(x_d)-g(x_r)), \\ \dot{e}_y &= e_x-e_y, \\ \dot{e}_z &= -\beta e_y-\delta_z e_z.\end{aligned} \tag{34}$$

Let $\tilde{\delta}_x = \delta_x-(b\pi/2a) > 0$ and $r = \tilde{\alpha}-\alpha > 0$. We construct positive definite and radially unbounded Lyapunov function for the first and second equations in system (34):

$$V = \frac{e_x^2}{2}+\frac{re_y^2}{2}.$$

Then we obtain

$$\begin{aligned}\frac{dV}{dt} &= e_x\dot{e}_x+re_y\dot{e}_y \\ &= -\alpha(1+g_{(x_d,x_r)}+\delta_x)e_x^2-re_y^2 \\ &\le -\alpha\left(-\frac{b\pi}{2a}+\delta_x\right)e_x^2-re_y^2 \\ &= -\alpha\tilde{\delta}_x e_x^2-re_y^2 \\ &\le -\min\{\alpha\tilde{\delta}_x,1\}(e_x^2+re_y^2) \\ &= -\min\{\alpha\tilde{\delta}_x,1\}2V,\end{aligned} \tag{35}$$

to which the comparison equation is

$$\frac{d\xi}{dt} = -2\min\{\alpha\tilde{\delta}_x,1\}\xi(t).$$

Thus,

$$e_x^2(t) \le 2V(t) \le 2\xi(t_0)e^{-2\min\{\alpha\tilde{\delta}_x,1\}(t-t_0)},$$

which shows that the zero solution of system (34) is globally exponentially stabilized with respect to e_x.

Consider the linear part of system (34), which has the coefficient matrix:

$$A = \begin{bmatrix} -\alpha-\alpha\delta_x & -r & 0 \\ 1 & -1 & 0 \\ 0 & -\beta & -\delta_z \end{bmatrix}.$$

Thus,

$$\det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda+\alpha+\alpha\delta_x & r & 0 \\ -1 & \lambda+1 & 0 \\ 0 & \beta & \lambda+\delta_z \end{bmatrix},$$

$$= (\lambda+\alpha+\alpha\delta_x)(\lambda+1)(\lambda+\delta_z) + r(\lambda+\delta_z),$$

$$= (\lambda+\delta_z)(\lambda^2+(\alpha+\alpha\delta_x+1)\lambda+r + \alpha+\alpha\delta_x).$$

Obviously,

$$\lambda_1 = -\delta_z < 0.$$

All coefficients of the polynomial

$$\lambda^2+(\alpha+\alpha\delta_x+1)\lambda+r+\alpha+\alpha\delta_x$$

are positive, which is the necessary and sufficient condition for a second degree polynomial to be a Hurwitz polynomial. Thus, the zero solution of system (34) is globally exponentially stable, implying that systems (30) and (31) are globally exponentially synchronized. ∎

Theorem 9. *In the driven system (31), choose the following control law:*

$$u_1 = \alpha\delta_x(x_d - x_r), \quad u_2 = l(x_d - x_r) + z_d - z_r,$$
$$u_3 = \delta_z(z_d - z_r).$$

If $\delta_x > b\pi/2a$, $l > 1$ and $\delta_z > 0$, the zero solution of (32) is globally exponentially stabilized, and thus systems (30) and (31) are globally exponentially synchronized.

Proof. Under the given control law, the error system (32) can be written as

$$\begin{aligned} \dot{e}_x &= -\alpha(e_x - e_y) - \alpha[\delta_x e_x + (g(x_d) - g(x_r))], \\ \dot{e}_y &= (1-l)e_x - e_y, \\ \dot{e}_z &= -\beta e_y - \delta_z e_z. \end{aligned} \tag{36}$$

Construct positive definite and radially unbounded Lyapunov function for the first and second equations in system (36):

$$V = \frac{e_x^2}{2\alpha} + \frac{e_y^2}{2(l-1)},$$

and then we have

$$\begin{aligned} \frac{dV}{dt} &= \frac{e_x\dot{e}_x}{\alpha} + \frac{e_y\dot{e}_y}{l-1} \\ &= -(1+g_{(x_d,x_r)}+\delta_x)e_x^2 - \frac{1}{l-1}e_y^2 \\ &\le -\left(-\frac{b\pi}{2a}+\delta_x\right)e_x^2 - \frac{1}{l-1}e_y^2. \end{aligned}$$

With the same method used in the proof of Theorems 7 and 8, it is easy to show that the zero solution of system (34) is globally exponentially stabilized with respect to the variable e_x.

Consider the linear part of system (36), whose coefficient matrix is

$$A = \begin{bmatrix} -\alpha-\alpha\delta_x & \alpha & 0 \\ 1-l & -1 & 0 \\ 0 & -\beta & -\delta_z \end{bmatrix}.$$

Thus,

$$\det(\lambda I_3 - A) = \det\begin{bmatrix} \lambda+\alpha+\alpha\delta_x & -\alpha & 0 \\ -1+l & \lambda+1 & 0 \\ 0 & \beta & \lambda+\delta_z \end{bmatrix}$$

$$= (\lambda+\delta_z)(\lambda^2+(\alpha+\alpha\delta_x+1)\lambda + \alpha+\alpha\delta_x+\alpha(l-1)).$$

Obviously,

$$\lambda_1 = -\delta_z < 0.$$

The coefficients of polynomial

$$\lambda^2+(\alpha+\alpha g_{(x_d,x_r)}+\alpha\delta_x+1)\lambda+\alpha+\alpha g_{(x_d,x_r)} + \alpha\delta_x+\alpha(l-1)$$

are all positive, indicating that the necessary and sufficient condition for a second degree polynomial to be a Hurwitz polynomial is satisfied. Thus, the zero solution of system (32) is globally exponentially stable, and so systems (30) and (31) are globally exponentially synchronized. ∎

Theorem 10. *In the driven system (31), choose the following control law:*

$$u_1 = \alpha\delta_x(x_d - x_r), \quad u_2 = \delta_y(x_d - x_r) + z_d - z_r,$$
$$u_3 = \delta_z(z_d - z_r).$$

If $\delta_x > b\pi/2a$ and $\delta_y > (1/(\delta_x - (b\pi/2a))) - 1$, the zero solution of (32) is globally exponentially stabilized, and thus systems (30) and (31) are globally exponentially synchronized.

Proof. Under the given control law, the error system (32) becomes

$$\begin{aligned}
\dot{e}_x &= \alpha(-e_x + e_y) - \alpha\delta_x e_x - \alpha(g(x_d) - g(x_r)),\\
\dot{e}_y &= e_x - e_y - \delta_y e_y,\\
\dot{e}_z &= -\beta e_y - \delta_z e_z.
\end{aligned} \tag{37}$$

From the given conditions we have

$$-\alpha - \alpha\delta_x - \alpha\left(-\frac{b\pi}{2a} - 1\right) < 0, \quad -1 - \delta_y < 0,$$

and

$$\left[\alpha + \alpha\delta_x + \alpha\left(-\frac{b\pi}{2a} - 1\right)\right](1 + \delta_y) > \alpha.$$

From the necessary and sufficient conditions for a second order real-matrix to be a Lyapunov–Volltor stable matrix we know that there exists $\xi > 0$ such that

$$\Omega = \begin{bmatrix} -\alpha\left(\delta_x - \dfrac{b\pi}{2a}\right) & \alpha + \xi \\ \alpha + \xi & -\alpha\xi(1 + \delta_y) \end{bmatrix}$$

is negative definite. Let $\lambda_{\max}(\Omega)$ denote the maximum eigenvalue of Ω.

Now, construct positive definite and radially unbounded Lyapunov function for the first and second equations in system (37):

$$V = e_x^2 + \xi e_y^2.$$

Then, we have

$$\begin{aligned}
\frac{dV}{dt} &= 2e_x\dot{e}_x + 2e_y\dot{e}_y\\
&\le \begin{pmatrix} e_x \\ e_y \end{pmatrix}^T \begin{bmatrix} -\alpha\left(\delta_x - \dfrac{b\pi}{2a}\right) & \alpha + \xi \\ \alpha + \xi & -\alpha\xi(1 + \delta_y) \end{bmatrix}\\
&\quad \times \begin{pmatrix} e_x \\ e_y \end{pmatrix}\\
&\le \lambda_{\max}(\Omega)(e_x^2 + e_y^2),
\end{aligned} \tag{38}$$

which shows that the zero solution of system (37) is globally exponentially stabilized with respect to the variables e_x and e_y, especially with respect to e_x.

Consider the linear part of system (37) with coefficient matrix,

$$A = \begin{bmatrix} -\alpha - \alpha\delta_x & \alpha & 0 \\ 1 & -1 - \delta_y & 0 \\ 0 & -\beta & -\delta_z \end{bmatrix}.$$

Using the similar argument as that for the proofs of Theorems 7 and 8 we can show that A is a Hurwitz matrix. Thus, the conclusion of Theorem 10 is true. ■

Theorem 11. *In the driven system (31), choose the following control law:*

$$u_1 = \alpha\delta_x(x_d - x_r), \quad u_2 = x_d - x_r, \quad u_3 = 0.$$

If $\delta_x > b\pi/2a$, the zero solution of (32) is globally exponentially stabilized, and so systems (30) and (31) are globally exponentially synchronized.

Proof. Under the given control law, the error system (32) can be written as

$$\begin{aligned}
\dot{e}_x &= \alpha(-e_x + e_y) - \alpha\delta_x e_x - \alpha(g(x_d) - g(x_r)),\\
\dot{e}_y &= -e_y + e_z,\\
\dot{e}_z &= -\beta e_y.
\end{aligned} \tag{39}$$

Construct radially unbounded Lyapunov function for the second and third equations in system (39):

$$\begin{aligned}
V &= e_y^2 + \frac{1}{\beta}e_z^2 - \varepsilon e_y e_z,\\
&= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^T \begin{bmatrix} 1 & -\dfrac{\varepsilon}{2} \\ -\dfrac{\varepsilon}{2} & \dfrac{1}{\beta} \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}\\
&:= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^T W \begin{pmatrix} e_y \\ e_z \end{pmatrix}.
\end{aligned}$$

Obviously, when $0 < \varepsilon \ll 1$, V is positive definite. Moreover, we have

$$\begin{aligned}
\frac{dV}{dt} &= 2e_y\dot{e}_y + \frac{2}{\beta}e_z\dot{e}_z - \varepsilon\dot{e}_y e_z - \varepsilon e_y\dot{e}_z\\
&= -2e_y^2 + \varepsilon(e_y - e_z)e_z + \varepsilon\beta e_y^2\\
&= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^T \begin{bmatrix} -2 + \varepsilon\beta & \dfrac{\varepsilon}{2} \\ \dfrac{\varepsilon}{2} & -\varepsilon \end{bmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}\\
&:= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^T Q \begin{pmatrix} e_x \\ e_y \end{pmatrix}\\
&< 0, \quad \text{when } e_y^2 + e_z^2 \ne 0 \text{ and } 0 < \varepsilon \ll 1.
\end{aligned}$$

Thus,

$$\frac{dV}{dt} \le \lambda_{\max}(Q)(e_y^2 + e_z^2) \le \frac{\lambda_{\max}(Q)}{\lambda_{\max}(W)}V,$$

and so

$$e_y^2(t) + e_z^2(t) \le \frac{V(t_0)}{\lambda_{\min}(W)}e^{\frac{\lambda_{\max}(Q)}{\lambda_{\max}(W)}(t - t_0)},$$

which shows that the zero solution of system (39) is globally exponentially stabilized with respect to the variables e_y and e_z.

Next, construct positive definite and radially unbounded Lyapunov function for the first equation in system (39):

$$V = \frac{e_x^2}{2}.$$

Choose $0 < \varepsilon \ll 1$ such that $(\varepsilon/2)^2 < \delta_x - (b\pi/2a)$. Then, we obtain

$$\begin{aligned}\frac{dV}{dt} &= -\alpha e_x^2 - \alpha\delta_x e_x^2 - \alpha g_{(x_d,x_r)} e_x^2 + \alpha e_x e_y \\ &\le -\alpha\delta_x e_x^2 + \alpha\frac{b\pi}{2a}e_x^2 + \alpha e_x e_y \\ &\le -\alpha\delta_x e_x^2 + \alpha\frac{b\pi}{2a}e_x^2 + \alpha\left(\frac{\varepsilon^2}{4}e_x^2 + \frac{1}{\varepsilon^2}e_y^2\right) \\ &= -\alpha e_x^2\left(\delta_x - \frac{b\pi}{2a} - \frac{\varepsilon^2}{4}\right) + \alpha\frac{e_y^2}{\varepsilon^2}, \\ V(t_0) &= V_0. \end{aligned} \tag{40}$$

Consider the comparison equation of (41), given by

$$\begin{aligned}\frac{du}{dt} &= -2\alpha u\left(\delta_x - \frac{b\pi}{2a} - \frac{\varepsilon^2}{4}\right) + \alpha\frac{e_y^2}{\varepsilon^2}, \\ u(t_0) &= V_0. \end{aligned} \tag{41}$$

Since e_y^2 has negative exponential decreasing estimation, $u(t)$ also has negative exponential decreasing estimation. Thus, the zero solution of system (39) is globally exponentially stabilized with respect to the variable e_x, and so systems (30) and (31) are globally exponentially synchronized. ■

Now, we present a general result for existence, and some algebraic criteria, which are convenient to use in applications.

Theorem 12. *For the driven system (31), there always exists the following control law:*

$$u_1 = \alpha\delta_x(x_d - x_r), \quad u_2 = \delta_y(y_d - y_r),$$
$$u_3 = \beta\delta_z(z_d - z_r),$$

where $\delta_x \ge$, $\delta_y \ge 0$ and $\delta_z \ge 0$, such that the zero solution of (32) is globally exponentially stabilized, and thus systems (30) and (31) are globally exponentially synchronized.

Proof. Construct positive definite and radially unbounded Lyapunov function for system (32):

$$V = \frac{e_x^2}{\alpha} + e_y^2 + \frac{e_z^2}{\beta}.$$

It is easy to obtain that

$$\begin{aligned}\left.\frac{dV}{dt}\right|_{(32)} &= 2e_x(e_y - e_x - g_{(x_d,x_r)}e_x) - 2\delta_x e_x^2 + 2e_y(e_x - e_y + e_z - \delta_y e_y) + 2e_z(-e_y - \delta_z e_z) \\ &= -2(1 + g_{(x_d,x_r)} + \delta_x)e_x^2 + 4e_x e_y - 2(1+\delta_y)e_y^2 - 2\delta_z e_z^2 \\ &= \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}^T \begin{bmatrix} -2\left(\delta_x - \frac{b\pi}{2a}\right) & 2 & 0 \\ 2 & -2(1+\delta_y) & 0 \\ 0 & 0 & -2\delta_z \end{bmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} \\ &< 0 \quad \text{when } e_x^2 + e_y^2 + e_z^2 \ne 0 \quad \text{and} \quad (1+\delta_y)\left(\delta_x - \frac{b\pi}{2a}\right) > 1. \end{aligned}$$

To satisfy the inequality, for example, one may choose $\delta_x > (b\pi/2a) + 1$, $\delta_y = 0$ and $\delta_z > 0$, or $\delta_x > b\pi/2a$, $\delta_y > (1/(\delta_x - (b\pi/2a))) - 1$ and $\delta_z > 0$.

Thus, the zero solution of system (32) is globally exponentially stabilized, and so systems (30) and (31) are globally exponentially synchronized. ■

Finally, we consider the use of only one variable of the drive system as the driving signal, i.e. one of the variables in the driven system is the same as the corresponding one in drive system. For example, assuming $x_r = x_d$, we discuss the synchronization between y_d and y_r, and z_d and z_r.

(I) When $x_r = x_d$, system (32) can be written as

$$\begin{aligned} \dot{e}_y &= -e_y + e_z - u_2(e_y, e_z), \\ \dot{e}_z &= -\beta e_y - u_3(e_y, e_z). \end{aligned} \tag{42}$$

Theorem 13. *In system (42), without applying controls (i.e. $u_2 = u_3 = 0$), the zero solution of system (42) is globally exponentially stabilized. Thus, for the drive-driven systems (30) and (31), when $x_r = x_d$, y_d and y_r, and z_d and z_r are globally exponentially synchronized.*

Proof. Let $0 < \varepsilon \ll 1$. Construct positive definite and radially unbounded Lyapunov function for system (42):

$$\begin{aligned} V &= e_y^2 + \frac{e_z^2}{\beta} - \varepsilon e_y e_z \\ &= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^T \begin{bmatrix} 1 & -\dfrac{\varepsilon}{2} \\ -\dfrac{\varepsilon}{2} & \dfrac{1}{\beta} \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}, \end{aligned}$$

and then we have

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(42)} &= -2e_y^2 + \varepsilon e_y - \varepsilon e_z^2 + \varepsilon\beta e_y^2 \\ &= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^T \begin{bmatrix} -2 + \varepsilon\beta & \dfrac{\varepsilon}{2} \\ \dfrac{\varepsilon}{2} & -\varepsilon \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix} \\ &< 0 \quad \text{when } e_y^2 + e_z^2 \neq 0. \end{aligned}$$

This shows that the conclusion of Theorem 13 is true. ■

(II) When $y_r = y_d$, system (32) can be written as

$$\begin{aligned} \dot{e}_x &= -\alpha(e_x + g_{(x_d,x_r)} e_x) - u_1(e_x, e_z), \\ \dot{e}_z &= 0 - u_3(e_x, e_z). \end{aligned} \tag{43}$$

Theorem 14. *In system (43), apply the control law: $u_1 = \alpha\delta_x e_x$, $u_3 = \delta_3 e_z$. When $\delta_x > b\pi/2a$ and $\delta_z > 0$, the zero solution of system (43) is globally exponentially stabilized. Thus, for the drive-driven systems (30) and (31), when $y_r = y_d$, x_d and x_r, and z_d and z_r are globally exponentially synchronized under the control law.*

Proof. Construct positive definite and radially unbounded Lyapunov function for system (43):

$$\begin{aligned} V &= \frac{e_x^2}{2} + \frac{e_z^2}{2} \\ &= \begin{pmatrix} e_y \\ e_z \end{pmatrix}^T \begin{bmatrix} \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} \end{bmatrix} \begin{pmatrix} e_y \\ e_z \end{pmatrix}, \end{aligned}$$

which, in turn, yields

$$\begin{aligned} \left.\frac{dV}{dt}\right|_{(43)} &= e_x \dot{e}_x + e_z \dot{e}_z \\ &= -\alpha e_x^2 (1 + g_{(x_d,x_r)} + \delta_x) - \delta_z e_z^2 \\ &\leq -\alpha e_x^2 \left(-\frac{b\pi}{2a} + \delta_x\right) - \delta_z e_z^2 \\ &< 0 \quad \text{when } e_x^2 + e_z^2 \neq 0. \end{aligned}$$

This clearly shows that the conclusion of Theorem 14 is true. ■

(III) When $z_r = z_d$, system (32) becomes

$$\begin{aligned} \dot{e}_x &= -\alpha(e_x + g_{(x_d,x_r)} e_x) + \alpha e_y - u_1(e_x, e_y), \\ \dot{e}_y &= e_x - e_y - u_2(e_x, e_y). \end{aligned} \tag{44}$$

It is easy to show that when we choose $u_1 = u_2 = 0$, i.e. no feedback control is applied, the zero solution of system (44) is unstable. Thus, in systems (30) and (31), when $z_r = z_d$, x_d and x_r, and y_d and y_r cannot be synchronized without applying controls.

Theorem 15. *In system (44), when the following control law,*

(1) $$u_1 = \alpha\delta_x e_x, \quad \left(\delta_x > \frac{b\pi}{2a} + 1\right), \quad u_2 = 0,$$

or

(2)

$$u_1 = \alpha\delta_x e_x, \quad \left(\delta_x > \frac{b\pi}{2a}\right), \quad u_2 = \delta_y e_y,$$
$$\left(\delta_y > \frac{1}{\delta_x - \dfrac{b\pi}{2a}} - 1\right),$$

is applied, the zero solution of system (44) is globally exponentially stabilized. Thus, for the drive-driven systems (30) and (31), when $z_r = z_d$, x_d and x_r, and y_d and y_r are globally exponentially synchronized under the above control law.

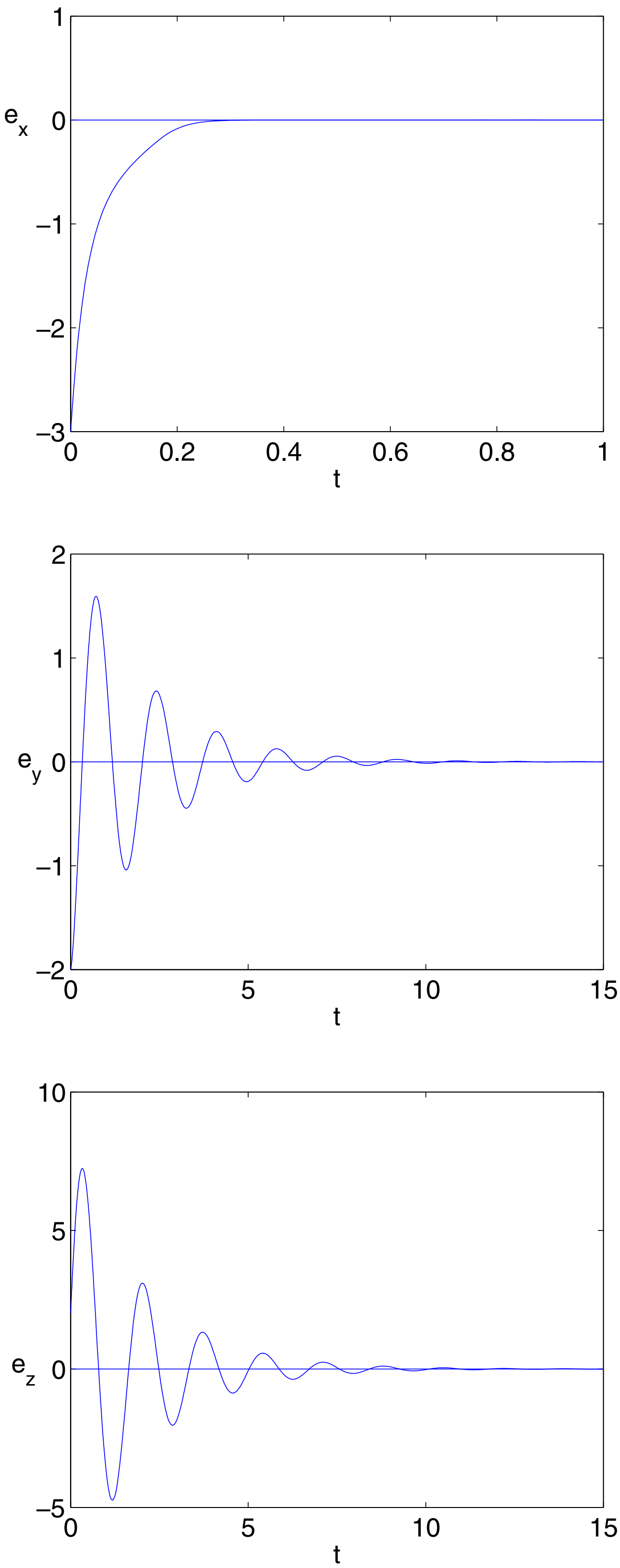


Fig. 3. Time history of the error system (32) for $c = 4$ using the control law given in Theorem 7, with the initial conditions: $x_d(0) = -2$, $y_d(0) = -1$, $z_d(0) = 1$, and $x_r(0) = 1$, $y_r(0) = 1$, $z_r(0) = -1$, when $\delta_x = 0.5$.

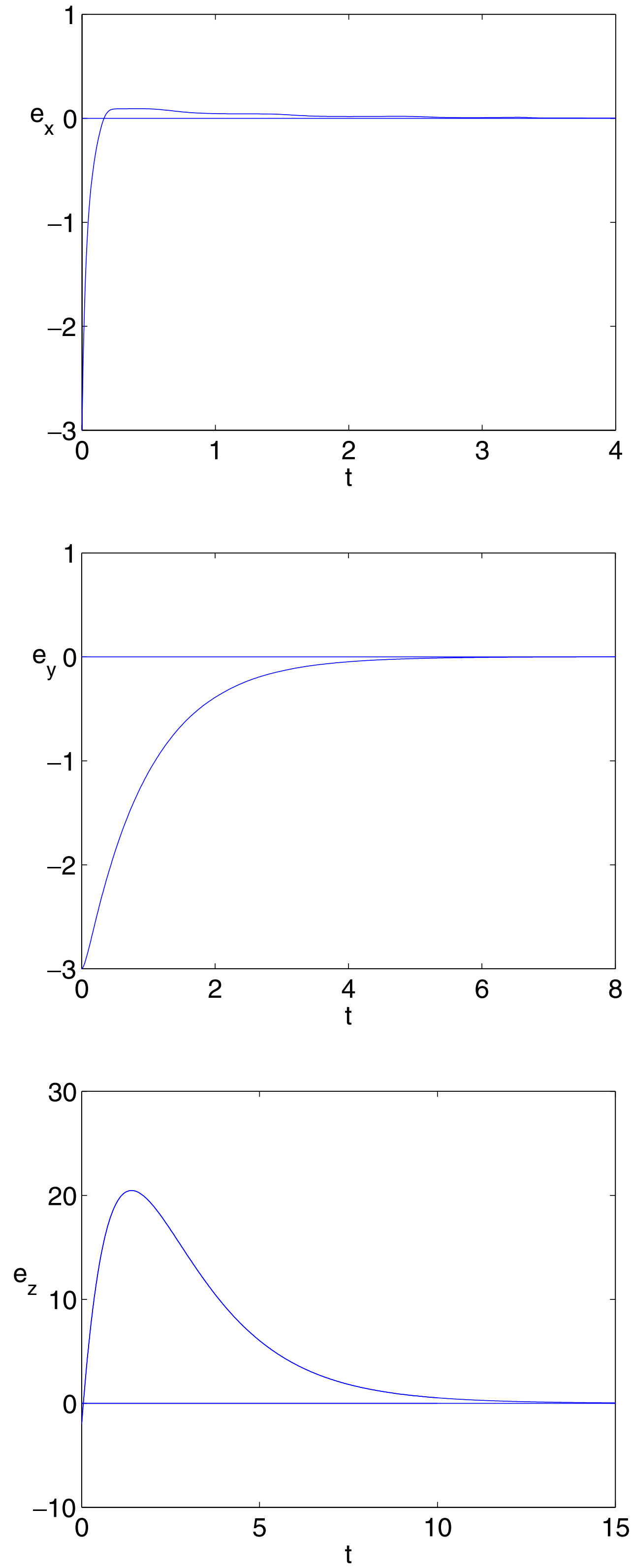


Fig. 4. Time history of the error system (32) for $c = 4$ using the control law given in Theorem 8, with the initial conditions, $x_d(0) = -2$, $y_d(0) = -1$, $z_d(0) = 1$, and $x_r(0) = 1$, $y_r(0) = 2$, $z_r(0) = 3$, when $\delta_x = 0.5$, $\delta_z = 0.5$ and $\tilde{\alpha} = 12$.

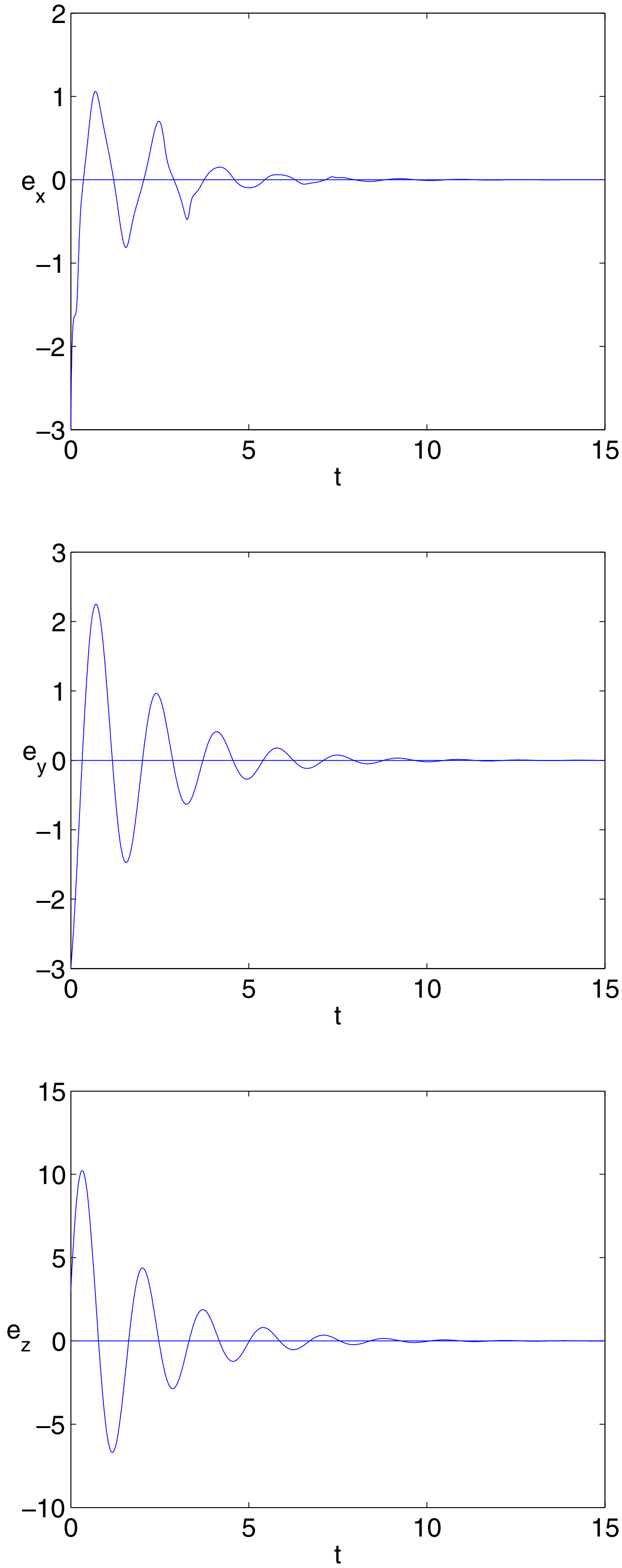


Fig. 5. Time history of the error system (32) for $c = 4$ using the control law given in Theorem 11, with the initial conditions: $x_d(0) = -2,\ y_d(0) = -1,\ z_d(0) = 1$, and $x_r(0) = 1,\ y_r(0) = 2,\ z_r(0) = -2$, when $\delta_x = 0.5$.

Proof. Construct positive definite and radially unbounded Lyapunov function for system (44):

$$V = \frac{e_x^2}{\alpha} + e_y^2.$$

In the first case, let $\tilde{\delta}_x = \delta_x - (b\pi/2a)$. We have

$$\begin{aligned}\frac{dV}{dt} &= \frac{2e_x\dot{e}_x}{\alpha} + 2e_y\dot{e}_y \\ &= -2e_x^2 - 2g_{(x_d,x_r)}e_x^2 - 2\delta_x e_x^2 + 4e_xe_y - 2e_y^2 \\ &\le \begin{pmatrix} e_x \\ e_y \end{pmatrix}^T \begin{bmatrix} -2\tilde{\delta}_x & 2 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix} \\ &< 0 \quad \text{when } e_x^2 + e_y^2 \neq 0.\end{aligned}$$

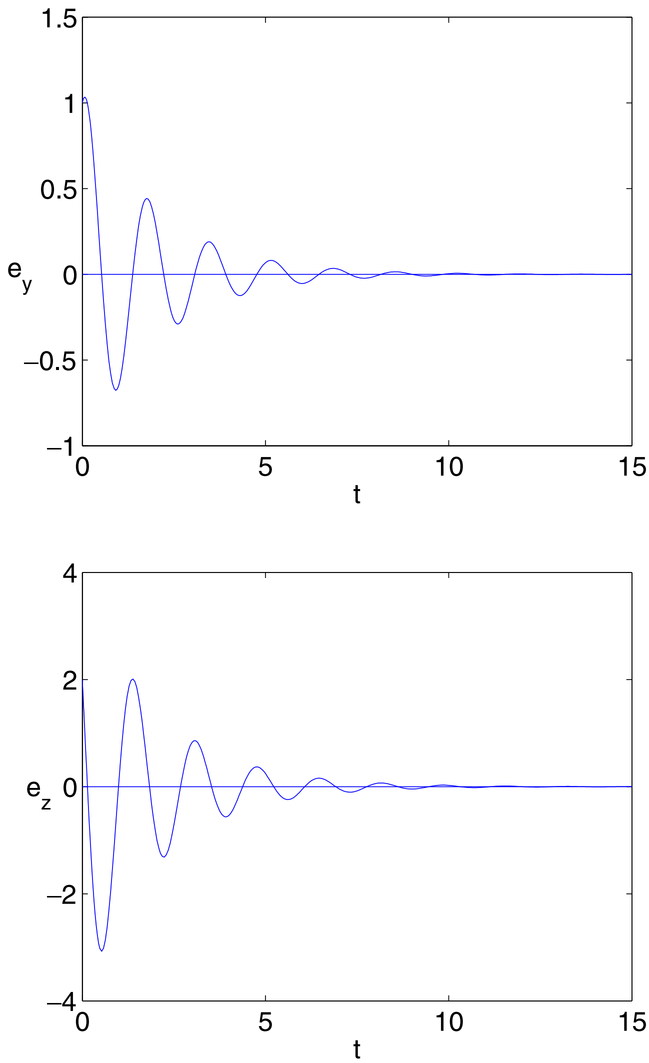


Fig. 6. Time history of the error system (32) for $c = 4$ using the control law given in Theorem 13, with the initial conditions: $x_d(0) = -2,\ y_d(0) = -1,\ z_d(0) = 1$, and $x_r(0) = -2,\ y_r(0) = -2,\ z_r(0) = -1$.

In the second case we obtain

$$\begin{aligned}\frac{dV}{dt} &= \frac{2e_x\dot{e}_x}{\alpha} + 2e_y\dot{e}_y \\ &= -2e_x^2 - 2g_{(x_d,x_r)}e_x^2 - 2\delta_x e_x^2 + 4e_xe_y \\ &\quad - 2e_y^2 - 2\delta_y e_y^2 \\ &\le \begin{pmatrix} e_x \\ e_y \end{pmatrix}^T \begin{bmatrix} -2\left(\delta_x - \dfrac{b\pi}{2a}\right) & 2 \\ 2 & -2-2\delta_y \end{bmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix} \\ &< 0 \quad \text{when } e_x^2 + e_y^2 \ne 0.\end{aligned}$$

The proof is complete. ■

The simulation results based on the error system (32), under the control laws given in Theorems 7, 8, 11 and 13 are shown in Figs. 3–6. It is seen from these figures that all the error components $e_x(t)$, $e_y(t)$ and $e_z(t)$ exponentially converge to zero as $t \to +\infty$, indicating that the drive-driven systems are globally exponentially synchronized. Note that the initial conditions chosen in the simulations for the drive and driven systems are different.

7. Conclusion

In this paper, we have studied a modified Chua's circuit and obtained a series of results. We have shown that the chaotic attractors of the modified Chua's circuit are globally attractive, and derived estimations for the globally attractive set and positive invariant set. We discussed the positions, number and local stability of equilibrium points, and designed simple feedback control laws to globally exponentially stabilize any given equilibrium point. We also applied the theory and methodology of absolute stability of Luré nonlinear control system and nonlinear feedback control to exponentially synchronize two modified Chua's circuits with the same structure. Moreover, we presented some numerical simulation results to show that the theories and conclusions obtained in this paper are applicable.

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