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GLOBAL STABILIZATION AND SYNCHRONIZATION OF N-SCROLL CHAOTIC ATTRACTORS IN A MODIFIED CHUA'S CIRCUIT WITH HYPERBOLIC TANGENT FUNCTION

FEI XU and PEI YU*
Department of Applied Mathematics,
The University of Western Ontario,
London, Ontario, Canada N6A 5B7
*pyu@uwo.ca

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In this paper, we study global stabilization and synchronization of n-scroll chaotic attractors for a modified Chua's circuit with hyperbolic tangent function using feedback control strategy. In particular, for any given equilibrium point of the modified Chua's circuit, we design simple and explicit controllers to globally exponentially stabilize the system. Simple controllers are also designed to globally exponentially synchronize two modified Chua's circuits. In addition to the theoretical analysis, numerical simulations are presented to illustrate the theoretical results.

Keywords: Modified Chua's circuit; n-scroll chaotic attractor; chaos control; chaos synchronization.

1. Introduction

The discovery of the Lorenz chaotic system [Lorenz, 1963] has led to a new era in the study of nonlinear dynamical systems. Since then, great progress has been made in the study of chaos theory. However, for a quite long period, chaos was considered unpredictable and uncontrollable. It is the OGY method [Ott *et al.*, 1990] that completely changed the situation. The aim of chaos control is to suppress or remove chaotic motion entirely and make the system convergent to one of its equilibrium points.

Recently, many efficient approaches have been proposed for controlling chaos, such as data sampling control [Rössler, 1976], structure true variation control [Yu, 1996], state feedback control [Chen & Lü, 2003], fuzzy control [Chang *et al.*, 2000; 2002], generalized OGY control [Yu *et al.*, 2001], inverse optimal control [Sanchez *et al.*, 2002], parameter identification control [Chen & Lü, 2003], digital control [Chen & Lü, 2003], etc. A basic idea used in these methodologies is to stabilize equilibrium points. First, these methods linearize the chaotic system using Jacobian matrix with respect to an equilibrium point of the system. Then, by analyzing the eigenvalues of the Jacobian matrix, control laws are designed to locally asymptotically stabilize the equilibrium point. However, lacking global property in these methods restricts the application of local asymptotic stability in practice. One way to achieve global stability and global synchronization is to design nonlinear feedback controls, which requires the construction of a Lyapunov function. However, to construct a proper Lyapunov function and its corresponding controller is difficult,

*Author for correspondence

for which special experiences and good skills are needed.

The well-known Chua's circuit [Chua *et al.*, 1986; Chua & Lin, 1990] is the first chaotic system realized by real electrical circuits. The circuit can generate a double-scroll chaotic attractor [Matsumoto *et al.*, 1985]. Recently, the study of double-scroll Chua's circuit has been extended to the study of the n-scroll modified Chua's circuit. In 2002, Özoguz *et al.* proposed a modified Chua's circuit model [Özoguz *et al.*, 2002; Elwakil *et al.*, 2002], in which the piecewise linear function was replaced by smooth hyperbolic tangent function. The modified Chua's circuit, which can generate chaotic attractors with arbitrary many scrolls, has promoted the study and application of chaos. However, further investigations are still needed to establish more fundamental mathematics for this system. More recently, another modified Chua's chaotic system using sinusoidal function [Tang *et al.*, 2001], which can also generate multiple-scroll chaotic attractors, was reinvestigated to show the existence of global attractive and positive invariant sets [Xu *et al.*, 2009].

In this paper, based on the properties of the modified Chua's circuit, by constructing proper Lyapunov functions, we design a number of controllers to globally exponentially stabilize the system and synchronize two such modified Chua's circuits. Besides the theoretical analysis, numerical simulation results are presented to illustrate the usefulness and effectiveness of the methodology. It should be pointed out that due to difficulty in constructing Lyapunov functions, the design of control laws based on Lyapunov function method may not necessarily lead to an "optimal" or the "simplest" controller.

The rest of the paper is organized as follows. In the next section, we briefly discuss the modified Chua's circuit and present the definitions of chaos control and chaos synchronization as well as a lemma which will be used in the following sections. In Sec. 3, a series of theorems and corollaries about chaos synchronization and chaos control of the modified Chua's circuit are given. Numerical simulations are presented in Sec. 4 to show the effectiveness of the designed controllers. The conclusion is drawn in Sec. 5.

2. Definition and Lemma

The modified Chua's circuit with n-scroll chaotic attractor, proposed by Özoguz *et al.* [2002], is described by the following equations:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= z,\\ \dot{z} &= -a\,(y + z + f(x)),\end{aligned} \tag{1}$$

where

$$f(x) = \sum_{j=-N}^{M} (-1)^{j-1} \tanh k(x - \sigma_j),$$

and $\sigma_j = 2j$. Here, M and N are odd integers, determining the number of scrolls in the chaotic attractor as $n = (M + N + 2)/2$. The values of other coefficients are chosen as:

$$a = 0.25 \quad \text{and} \quad k = 2.$$

It has been shown that system (1) can be represented by a Lurie's system [Suykens *et al.*, 1997a]. In fact, the classical Chua's circuit [Chua *et al.*, 1986] can also be represented by a standard Lurie's system (see [Liao & Yu, 2005] for more details). Therefore, absolute stability theory and methodology can be applied to analyze these Chua's systems. Since the tanh function belongs to section [0, 1], all criteria for synchronization of Lurie's systems can be directly applied here. The results for synchronization of Lurie's systems can be found in [Curran *et al.*, 1997; Suykens *et al.*, 1997b; Suykens *et al.*, 1997c; Suykens *et al.*, 1998; Suykens *et al.*, 1999].

System (1) can produce n-scroll chaotic attractors. For example, when $M = N = 1$, a double-scroll chaotic attractor is obtained, as shown in Fig. 1(a). To study the global and exponential synchronization of two modified Chua's circuits, consider system (1) as a drive system:

$$\begin{aligned}\dot{x}_d &= y_d,\\ \dot{y}_d &= z_d,\\ \dot{z}_d &= -a(y_d + z_d + f(x_d)),\end{aligned} \tag{2}$$

where the subscript d means "drive". The corresponding driven (receiving) system is:

$$\begin{aligned}\dot{x}_r &= y_r + u_1(x_d - x_r,\, y_d - y_r,\, z_d - z_r),\\ \dot{y}_r &= z_r + u_2(x_d - x_r,\, y_d - y_r,\, z_d - z_r),\\ \dot{z}_r &= -a\,(y_r + z_r + f(x_r))\\ &\quad + u_3(x_d - x_r,\, y_d - y_r,\, z_d - z_r),\end{aligned} \tag{3}$$

where r represents "receive" and u_i's are continuous, linear or nonlinear functions of their variables, satisfying $u_i(0, 0, 0) = 0$, $i = 1, 2, 3$.

Let $e_x = x_d - x_r$, $e_y = y_d - y_r$, $e_z = z_d - z_r$. Then, the error system can be written as

$$\begin{aligned} \dot{e}_x &= e_y - u_1(e_x, e_y, e_z), \\ \dot{e}_y &= e_z - u_2(e_x, e_y, e_z), \\ \dot{e}_z &= -a\,(e_y + e_z + f'(\xi)e_x) - u_3(e_x, e_y, e_z), \end{aligned} \tag{4}$$

where, by the intermediate value theorem, $f'(\xi)e_x = f(x_d) - f(x_r)$ with $\min(x_d, x_r) \le \xi \le \max(x_d, x_r)$. In Sec. 3, we will use the property $(\tanh(x))' = 1 - \tanh(x)^2 \le 1$ to study the globally exponential synchronization and globally exponential stability of the modified Chua's circuit.

Definition 1. For any given initial condition $(x_d(t_0), y_d(t_0), z_d(t_0)) \in R^3$ of the drive system and the corresponding initial condition $(x_r(t_0), y_r(t_0), z_r(t_0)) \in R^3$ of the driven system, if the zero solution of (4) satisfies the following inequality:

$$e_x^2(t) + e_y^2(t) + e_z^2(t) \le k(e(t_0))e^{-\alpha(t-t_0)}, \tag{5}$$

where $\alpha > 0$, and $k(e(t_0))$ is a constant depending on $e(t_0)$, then the zero solution of (4) is globally exponentially stable, and systems (2) and (3) are said to be globally exponentially synchronized.

Assume that $X^* = (x^*, y^*, z^*)$ is any given equilibrium point of system (1). Let $\overline{X} = X - X^* = (x - x^*, y - y^*, z - z^*)$. Then system (4) can be written as

$$\begin{aligned} \dot{\overline{x}} &= \overline{y} - u_1(\overline{x}, \overline{y}, \overline{z}), \\ \dot{\overline{y}} &= \overline{z} - u_2(\overline{x}, \overline{y}, \overline{z}), \\ \dot{\overline{z}} &= -a(\overline{y} + \overline{z} + f'(\eta)\overline{x}) - u_3(\overline{x}, \overline{y}, \overline{z}), \end{aligned} \tag{6}$$

where η is a value between x and x^*.

Definition 2. When a proper feedback control law, given by u_1, u_2, u_3, is chosen such that the zero solution of system (6) is globally exponentially stable, then $X^* = (x^*, y^*, z^*)$ is said to be globally exponentially stabilized.

Our basic idea is to construct simple feedback controllers, u_1, u_2 and u_3, such that the zero solution of the error system (4) is globally exponentially stabilized. Then, the drive system (2) and its corresponding driven system (3) are globally exponentially synchronized. Similarly, under the controllers, u_1, u_2 and u_3, if system (6) is globally exponentially stabilized, then the equilibrium point $X^* = (x^*, y^*, z^*)$ is globally exponentially stabilized.

In the following analysis, for convenience, we present a lemma about Hurwitz polynomial.

Lemma 1. *The necessary and sufficient conditions for the cubic-degree polynomial $\lambda^3 + p\lambda^2 + q\lambda + r$ with real coefficients being a Hurwitz polynomial are*: $p > 0$ *and* $pq > r > 0$ [*Liao, 2001*].

3. Chaos Synchronization and Chaos Control of Modified Chua's Circuit

In this section, we study globally exponential synchronization between systems (2) and (3), i.e. the globally exponential stability of the zero solution of error system (4). Moreover, using the same controllers, we will also study the globally exponential stability of system (6). A series of controller designs with proofs will be given.

Theorem 1. *The linear control law*, $u_1 = \delta_x e_x$, $u_2 = \delta_y e_y$, $u_3 = \delta_z e_z$, *where*

$$\delta_x > \frac{1}{2} + \frac{k}{2}(M + N + 1), \quad \delta_y > \frac{1}{2}, \quad \text{and}$$

$$\delta_z > \frac{ak}{2}(M + N + 1) - a,$$

can be applied to the driven system (3) such that the drive-driven systems (2) and (3) are globally exponentially synchronized, i.e. the zero solution of error system (4) is globally exponentially stabilized.

Proof. Construct a positive definite, radially unbounded Lyapunov function for system (4):

$$V = \frac{1}{2}\left(e_x^2 + e_y^2 + \frac{e_z^2}{a}\right) = \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}^T P \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix},$$

where

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2a} \end{bmatrix}.$$

For convenience, let $\lambda_m(P)$ and $\lambda_M(P)$ denote the minimum and maximum eigenvalues of P, respectively. Evaluating the derivative of V with respect to time t along the solution trajectory of system (4) yields the following inequality:

$$\left.\frac{dV}{dt}\right|_{(4)} \le |e_x e_y| - \delta_x e_x^2 - \delta_y e_y^2 - e_z^2 + k|e_x e_z|(M+N+1) - \frac{1}{a}\delta_z e_z^2$$

$$\le \begin{pmatrix}|e_x|\\|e_y|\\|e_z|\end{pmatrix}^T \begin{bmatrix} -\delta_x & \frac{1}{2} & \frac{k}{2}(M+N+1) \\ \frac{1}{2} & -\delta_y & 0 \\ \frac{k}{2}(M+N+1) & 0 & -\frac{\delta_z}{a} - 1 \end{bmatrix} \begin{pmatrix}|e_x|\\|e_y|\\|e_z|\end{pmatrix}$$

$$:= \begin{pmatrix}|e_x|\\|e_y|\\|e_z|\end{pmatrix}^T A_1 \begin{pmatrix}|e_x|\\|e_y|\\|e_z|\end{pmatrix}$$

$$\le \lambda_M(A_1)\,(e_x^2 + e_y^2 + e_z^2),$$

where $\lambda_M(A_1)$ is the maximum eigenvalue of A_1. It follows from the conditions of Theorem 1 that $\lambda_M(A_1) < 0$. Hence,

$$\frac{dV}{dt} \le \lambda_M(A_1)\,\frac{\lambda_M(P)}{\lambda_M(P)}\,(e_x^2 + e_y^2 + e_z^2) \le \frac{\lambda_M(A_1)}{\lambda_M(P)} V,$$

which implies that

$$V(X(t)) \le V(X(t_0)) e^{\frac{\lambda_M(A_1)}{\lambda_M(P)}(t-t_0)}.$$

Consequently,

$$(e_x^2(t) + e_y^2(t) + e_z^2(t)) \le \frac{V(X(t))}{\lambda_m(P)} \le \frac{V(X(t_0))}{\lambda_m(P)} e^{\frac{\lambda_M(A_1)}{\lambda_M(P)}(t-t_0)}. \quad (7)$$

Equation (7) shows that the zero solution of (4) is globally exponentially stabilized, and thus systems (2) and (3) are globally exponentially synchronized. ■

Corollary 1. *By choosing the linear feedback control law*:

$$u_1 = \delta_x \bar{x}, \quad u_2 = \delta_y \bar{y}, \quad u_3 = \delta_z \bar{z},$$

where

$$\delta_x > \frac{1}{2} + \frac{k}{2}(M+N+1), \quad \delta_y > \frac{1}{2}, \quad \textit{and}$$

$$\delta_z > \frac{ak}{2}(M+N+1) - a,$$

any given equilibrium point $X = X^*$ of system (1) can be globally exponentially stabilized.

Theorem 2. *Under the linear control law, $u_1 = \delta_x e_x$, $u_2 = \delta_y e_y$, $u_3 = \delta_z e_z$, if one of the following conditions holds*:

(1) $\delta_x > 1$, $\delta_y > 1$, $\delta_z > ak(M+N+1)$;
(2) $\delta_x > ak(M+N+1)$, $\delta_y > 1+a$, $\delta_z > 1-a$;

then the zero solution of the error system (4) is globally exponentially stabilized, and thus systems (2) and (3) are globally exponentially synchronized.

Proof. Construct a positive definite, radially unbounded vector Lyapunov function for system (4):

$$V = (|e_x|, |e_y|, |e_z|)^T.$$

Evaluating the Dini derivative of V with respect to time t along the solution trajectory of system (4) yields

$$\begin{pmatrix}D^+|e_x|\\D^+|e_y|\\D^+|e_z|\end{pmatrix}_{(4)} \le \begin{bmatrix} -\delta_x & 1 & 0 \\ 0 & -\delta_y & 1 \\ ak(M+N+1) & a & -a-\delta_z \end{bmatrix} \times \begin{pmatrix}|e_x|\\|e_y|\\|e_z|\end{pmatrix}. \quad (8)$$

To further discuss the property of the above inequality, we construct a comparison equation for inequality (8):

$$\begin{pmatrix}\dot{\eta}_x\\ \dot{\eta}_y\\ \dot{\eta}_z\end{pmatrix} = \begin{bmatrix}-\delta_x & 1 & 0\\ 0 & -\delta_y & 1\\ ak(M+N+1) & a & -a-\delta_z\end{bmatrix}\begin{pmatrix}\eta_x\\ \eta_y\\ \eta_z\end{pmatrix} := A_2\begin{pmatrix}\eta_x\\ \eta_y\\ \eta_z\end{pmatrix} \tag{9}$$

with the solution:

$$\begin{pmatrix}\eta_x(t)\\ \eta_y(t)\\ \eta_z(t)\end{pmatrix} = e^{A_2(t-t_0)}\begin{pmatrix}\eta_x(t_0)\\ \eta_y(t_0)\\ \eta_z(t_0)\end{pmatrix}.$$

By the conditions given in Theorem 2, A_2 is a Hurwitz matrix. Thus, we have

$$\|e^{A_2(t-t_0)}\| \le K_2 e^{-\alpha_2(t-t_0)},$$

where $K_2 \ge 1$ and $\alpha_2 > 0$. From the comparison principal, we have

$$\left\|\begin{pmatrix}|e_x(t)|\\ |e_y(t)|\\ |e_z(t)|\end{pmatrix}\right\| \le \left\|\begin{pmatrix}\eta_x(t)\\ \eta_y(t)\\ \eta_z(t)\end{pmatrix}\right\| \le K_2 e^{-\alpha_2(t-t_0)}\left\|\begin{pmatrix}\eta_x(t_0)\\ \eta_y(t_0)\\ \eta_z(t_0)\end{pmatrix}\right\|. \tag{10}$$

The above inequality shows that the zero solution of system (4) is globally exponentially stabilized and consequently, by Definition 1, systems (2) and (3) are globally exponentially synchronized. ■

Corollary 2. *For any given equilibrium point of system (1) with the linear feedback control law*:

$$u_1 = \delta_x \bar{x}, \quad u_2 = \delta_y \bar{y}, \quad u_3 = \delta_z \bar{z},$$

if one of the following conditions holds:

(1) $\delta_x > 1$, $\delta_y > 1$, $\delta_z > ak(M+N+1)$;
(2) $\delta_x > ak(M+N+1)$, $\delta_y > 1+a$, $\delta_z > 1-a$;

then the equilibrium point $X = X^*$ can be globally exponentially stabilized.

Theorem 3. *The linear control law, $u_1 = \delta_x e_x$, $u_2 = \delta_y e_y$, $u_3 = \delta_z e_z$, where*

$$\delta_x > ak(M+N+1), \quad \delta_y = 1+a, \quad \textit{and} \quad \delta_z = 1-a,$$

can be used to globally exponentially synchronize systems (2) and (3), i.e. globally exponentially stabilize the error system (4).

Proof. Construct positive definite, radially unbounded Lyapunov function for system (4):

$$V = |e_x| + |e_y| + |e_z|.$$

Then, we have

$$\begin{aligned} D^+V|_{(4)} &= \dot{e}_x\,\mathrm{sign}(e_x) + \dot{e}_y\,\mathrm{sign}(e_y) + \dot{e}_z\,\mathrm{sign}(e_z)\\ &\le (-\delta_x + a\,k(M+M+1))\,|e_x|\\ &\quad + (-\delta_y + a + 1)|e_y|\\ &\quad + (-\delta_z - a + 1)|e_z|\\ &= (-\delta_x + a\,k(M+N+1))\,|e_x|, \end{aligned} \tag{11}$$

which implies that the zero solution of (4) is globally exponentially stabilized with respect to the partial variable e_x, i.e.

$$|e_x(t)| \le h\,e^{-\tilde{\mu}(t-t_0)},$$

where $\tilde{\mu} > 0$.

Next, we consider the coefficient matrix of the linear part of system (4) with the feedback controller designed in Theorem 3:

$$A_3 = \begin{bmatrix}-\delta_x & 1 & 0\\ 0 & -a-1 & 1\\ 0 & -a & -1\end{bmatrix}$$

whose characteristic polynomial is

$$\begin{aligned}\det(\lambda I_3 - A_3) &= \det\begin{bmatrix}\lambda+\delta_x & -1 & 0\\ 0 & \lambda+a+1 & -1\\ 0 & a & \lambda+1\end{bmatrix}\\ &= \lambda^3 + (\delta_x + 2 + a)\lambda^2\\ &\quad + (2\delta_x + a\delta_x + 1 + 2a)\lambda + \delta_x(2a+1)\\ &:= \lambda^3 + p\lambda^2 + q\lambda + r,\end{aligned}$$

where the coefficients p, q and r satisfy $p > 0$ and $pq > r > 0$. According to Lemma (1) we know that A_3 is a Hurwitz matrix. Thus, there exist constants $K_3 \ge 1$ and $\mu > 0$, such that

$$\|e^{A_3(t-t_0)}\| \le K_3 e^{-\mu(t-t_0)}.$$

The solution of (4) can be written as

$$\begin{pmatrix} e_x(t) \\ e_y(t) \\ e_z(t) \end{pmatrix} = e^{A_3(t-t_0)} \begin{pmatrix} e_x(t_0) \\ e_y(t_0) \\ e_z(t_0) \end{pmatrix} + \int_{t_0}^{t} e^{A_3(t-\tau)} \begin{pmatrix} 0 \\ 0 \\ \sum_{j=-N}^{M} -a\,(-1)^{j-1} \tanh' k(x-\sigma_j)\, e_x(\tau) \end{pmatrix} d\tau. \tag{12}$$

Thus,

$$\left\| \begin{pmatrix} e_x(t) \\ e_y(t) \\ e_z(t) \end{pmatrix} \right\| \le K_3 \left\| \begin{pmatrix} e_x(t_0) \\ e_y(t_0) \\ e_z(t_0) \end{pmatrix} \right\| e^{-\mu(t-t_0)} + \int_{t_0}^{t} K_3\, e^{-\mu(t-\tau)} ak(M+N+1) h e^{-\tilde{\mu}(\tau-t_0)}\, d\tau. \tag{13}$$

Apparently, the first term of (13) has negative exponential form, which implies that we only need to prove that the second term also has negative exponential form. For simplicity, let

$$K_3\, akh(M+N+1) := L.$$

Without loss of generality, assume $\mu \ne \tilde{\mu}$. Otherwise one may decrease $\tilde{\mu}$ to reach $\mu \ne \tilde{\mu}$. Then,

$$\begin{aligned} &\int_{t_0}^{t} Le^{-\mu(t-\tau)}\, e^{-\tilde{\mu}(\tau-t_0)} d\tau \\ &\quad = Le^{-\mu t} \int_{t_0}^{t} e^{(\mu-\tilde{\mu})\,\tau} e^{\tilde{\mu} t_0} d\tau \\ &\quad = Le^{-\mu t} \frac{e^{(\mu-\tilde{\mu})t} - e^{(\mu-\tilde{\mu})t_0}}{\mu-\tilde{\mu}} e^{\tilde{\mu} t_0} \\ &\quad \le \begin{cases} \dfrac{L}{\mu-\tilde{\mu}} e^{-\tilde{\mu}(t-t_0)} & \text{when } \mu > \tilde{\mu}, \\ \dfrac{L}{\tilde{\mu}-\mu} e^{-\mu(t-t_0)} & \text{when } \mu < \tilde{\mu}, \end{cases} \end{aligned} \tag{14}$$

which clearly indicates that the second term of (13) also has a negative exponential estimate. Thus, the zero solution of system (4) is globally exponentially stabilized and so the drive-driven systems (2) and (3) are globally exponentially synchronized. ■

Corollary 3. *By choosing the following linear feedback control law*:

$$u_1 = \delta_x \bar{x}, \quad u_2 = \delta_y \bar{y}, \quad u_3 = \delta_z \bar{z},$$

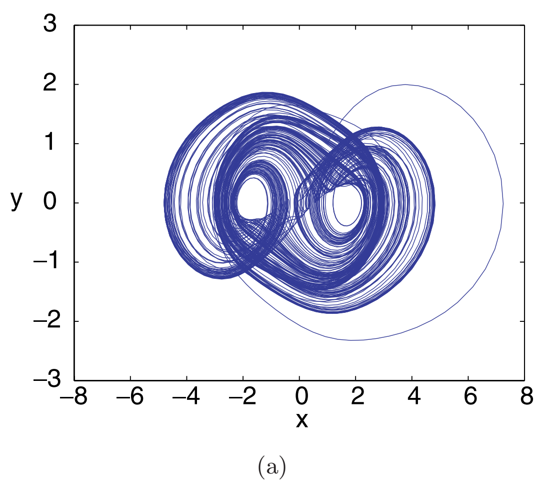
where

$$\delta_x > ak(M+N+1), \quad \delta_y = 1+a, \quad \text{and} \quad \delta_z = 1-a,$$

any given equilibrium point $X = X^*$ *of system (1) can be globally exponentially stabilized.*

4. Numerical Simulations

In this section, to show the applicability of the controllers discussed in the previous section, we present some numerical simulation results for globally exponential stabilization of system (6) as well



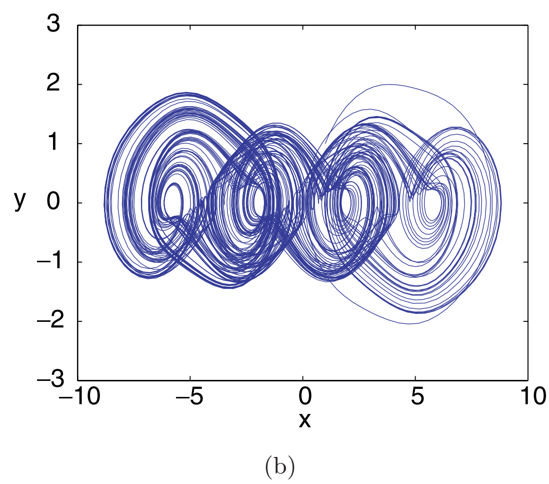


Fig. 1. (a) Simulated phase portrait of a double-scroll chaotic attractor for system (1) when $M = N = 1$, $a = 0.25$ and $k = 2$, projected on the x–y plane; (b) simulated phase portrait of a four-scroll chaotic attractor for system (1) when $M = N = 3$, $a = 0.25$ and $k = 2$, projected on the x–y plane.

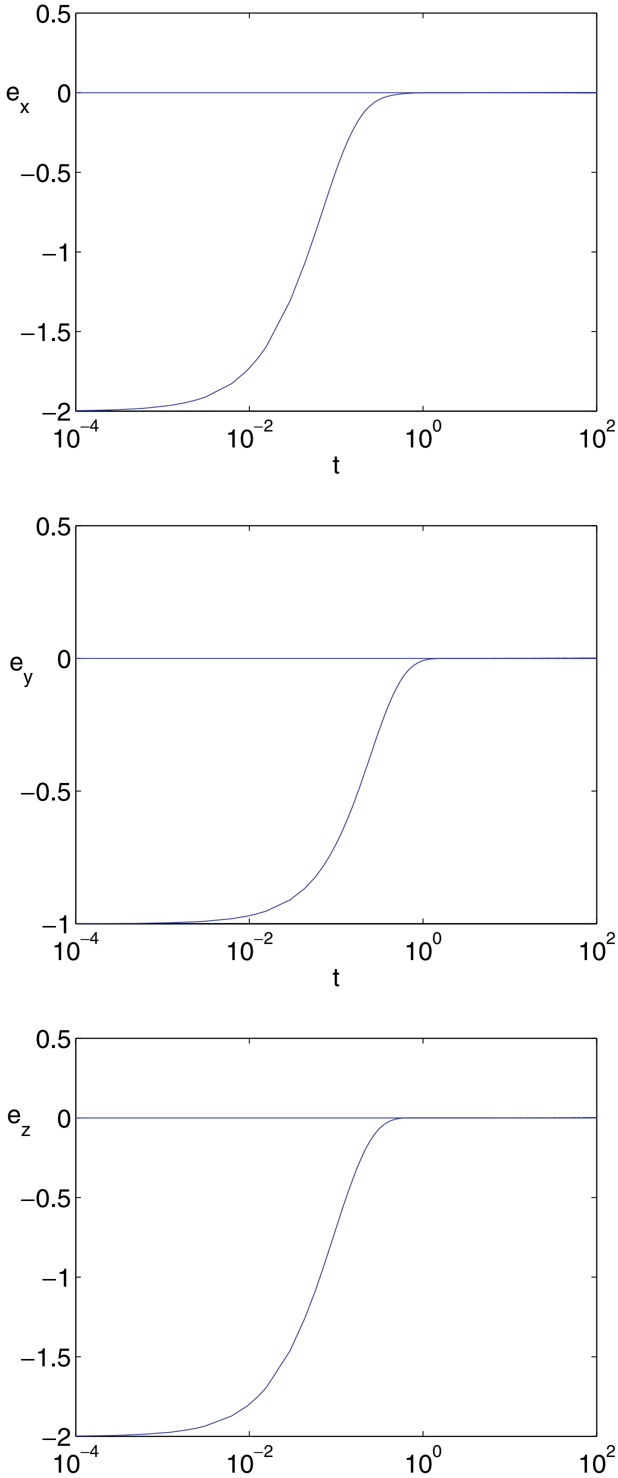


Fig. 2. Time history of the error system (4) for a double-scroll chaotic attractor when $a = 0.25$, $k = 2$ and $M = N = 1$ using the control law given in Theorem 1, with the initial conditions: $x_d(0) = 1,\ y_d(0) = 1,\ z_d(0) = 1$, and $x_r(0) = 3,\ y_r(0) = 2,\ z_r(0) = 3$, when $\delta_x = 15$, $\delta_y = 5$ and $\delta_z = 9$.

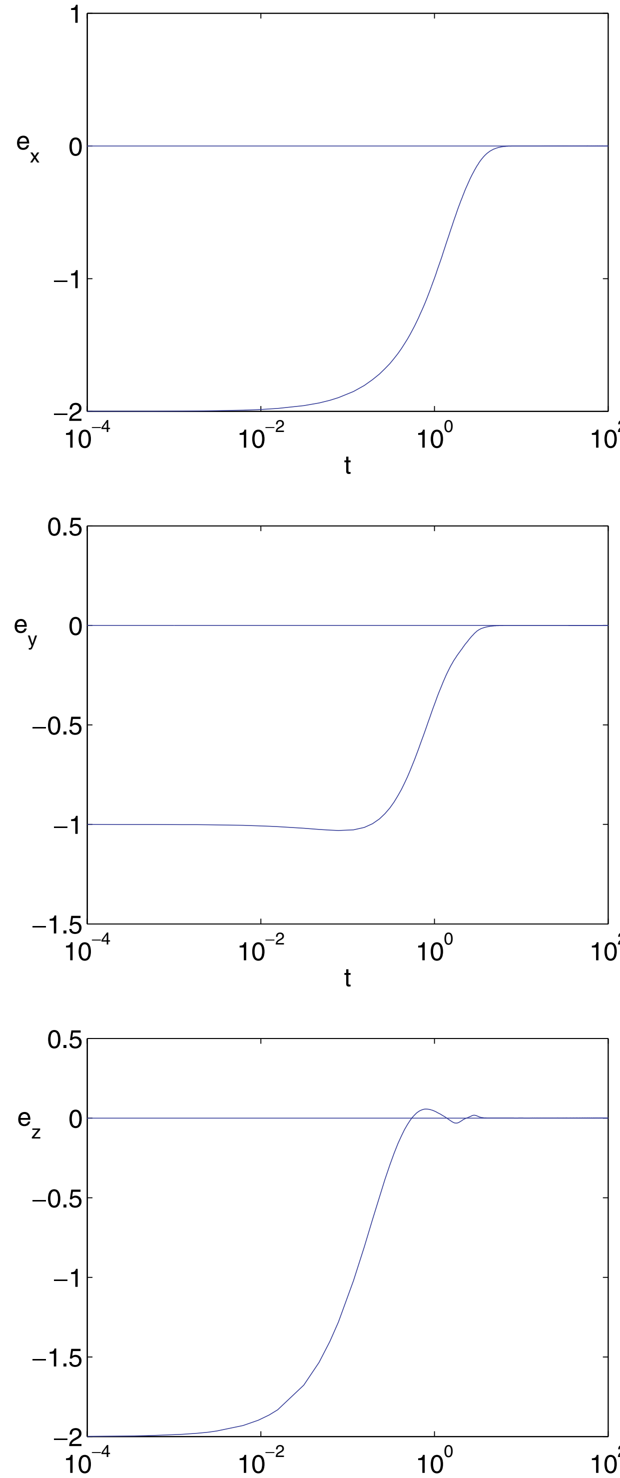


Fig. 3. Time history of the error system (4) for a four-scroll chaotic attractor when $a = 0.25$, $k = 2$ and $M = N = 3$ using the control law given in Theorem 2(1), with the initial conditions: $x_d(0) = 1,\ y_d(0) = 1,\ z_d(0) = 1$, and $x_r(0) = 3,\ y_r(0) = 2,\ z_r(0) = 3$, when $\delta_x = 1.2$, $\delta_y = 1.2$ and $\delta_z = 5$.

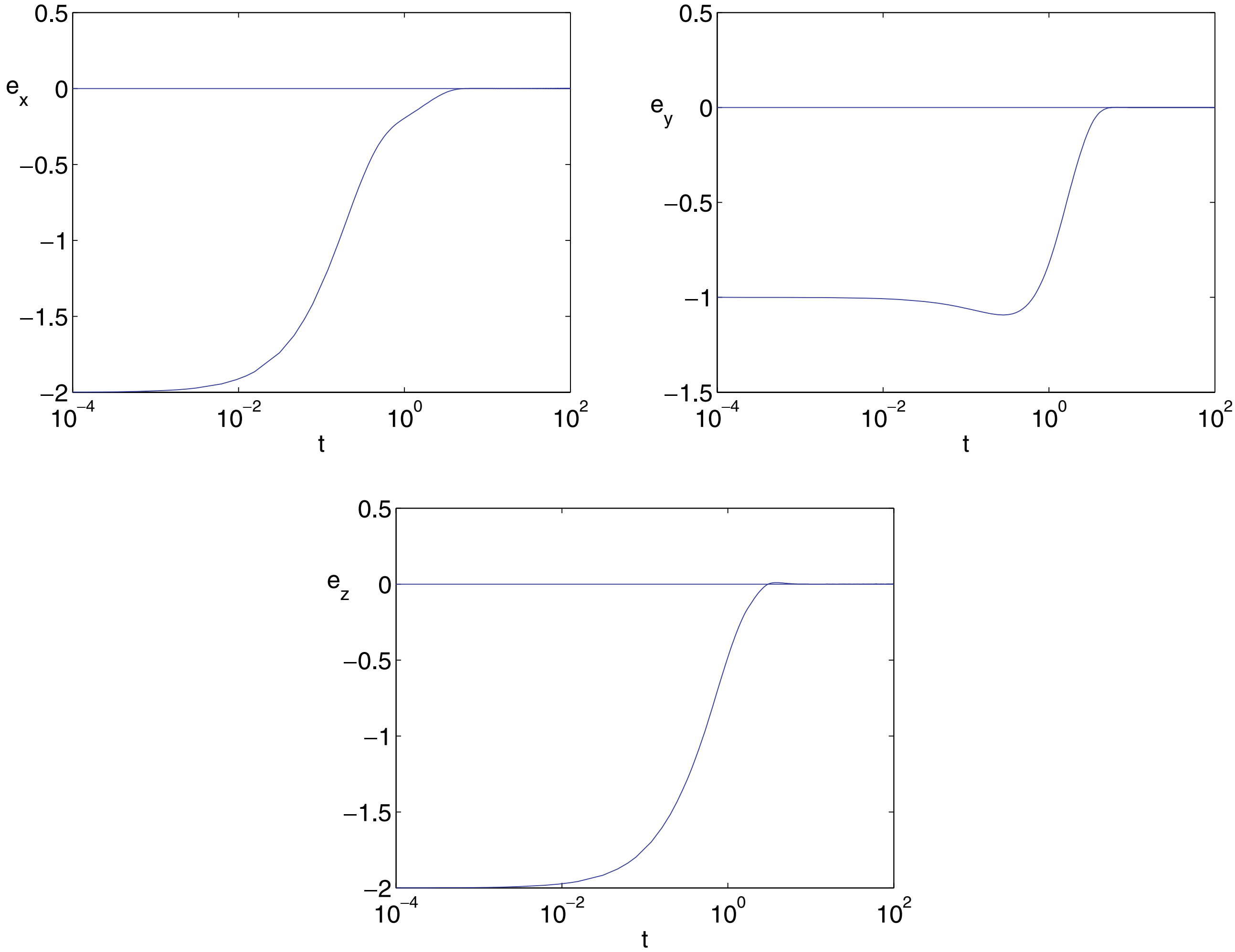


Fig. 4. Time history of the error system (4) for a four-scroll chaotic attractor for $a = 0.25$, $k = 2$ and $M = N = 3$ using the control law given in Theorem 3, with the initial conditions: $x_d(0) = 1,\ y_d(0) = 1,\ z_d(0) = 1$, and $x_r(0) = 3,\ y_r(0) = 2,\ z_r(0) = 3$, when $\delta_x = 5$, $\delta_y = 1.25$ and $\delta_z = 0.75$.

as globally exponential synchronization between systems (2) and (3).

When there is no control law applied to the chaotic system (1), the phase portraits, depicted in Figs. 1(a) and 1(b), show that the solution trajectories of the system are chaotic. For $M = N = 1$, system (1) generates a double-scroll chaotic attractor, as shown in Fig. 1(a). When $M = N = 3$, system (1) generates a four-scroll chaotic attractor (see in Fig. 1(b)).

Numerical simulations for chaos synchronization are generated with the control laws given in Theorems 1–3, as shown in Figs. 2–4, respectively. Here, we investigate the differences between the solution trajectory of the drive system and driven system, i.e. the zero solution of the error system. To show that the convergent speed of synchronization is exponential, we use logarithmic scale instead of linear scale. In these examples, the initial conditions for the drive system and driven system are chosen as:

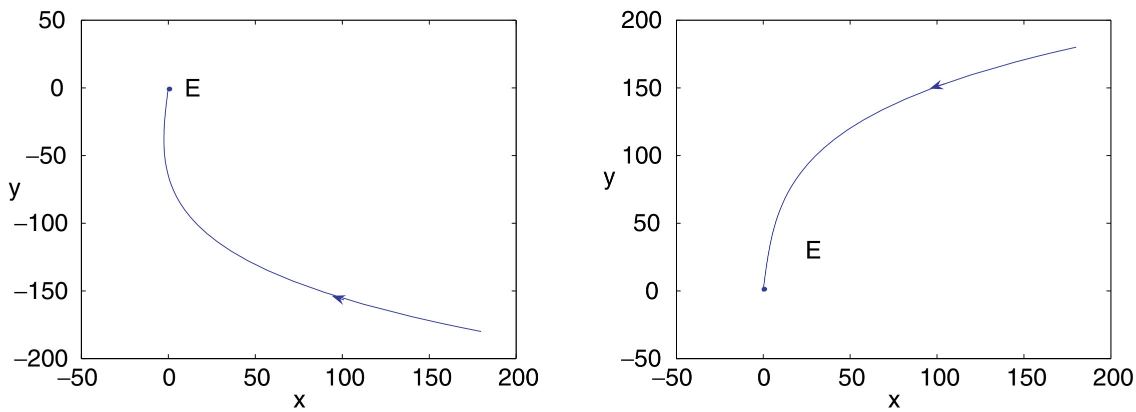
$$x_d(0) = 1, \quad y_d(0) = 1, \quad z_d(0) = 1,$$

and

$$x_r(0) = 3, \quad y_r(0) = 2, \quad z_r(0) = 3,$$

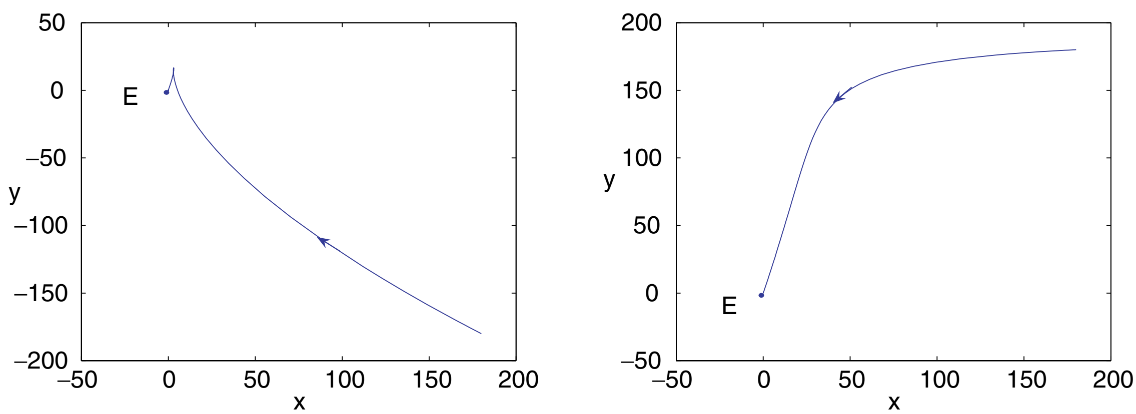
respectively.

Numerical simulations are also presented to show the effectiveness of the control laws given in Corollaries 1–3. Under the control law given in Corollary 1, the trajectory of the system is no longer chaotic but convergent to the equilibrium E, as shown in Figs. 5(a) and 5(b). Similar numerical simulation results are obtained for the control laws of Corollaries 2 and 3, as shown in Figs. 6(a) and 6(b), respectively.



(a) (b)

Fig. 5. (a) Trajectory of a double-scroll chaotic attractor for system (1) using the control law given in Corollary 1 for $\delta_x = 15$, $\delta_y = 5$ and $\delta_z = 9$ with the initial condition $x(0) = 180,\ y(0) = -180,\ z(0) = -180$, convergent to the equilibrium point $E:\ (0,\ 0,\ 0)$; (b) trajectory of a four-scroll chaotic attractor for system (1) using the control law given in Corollary 1 for $\delta_x = 15$, $\delta_y = 5$ and $\delta_z = 9$ with the initial condition $x(0) = 180,\ y(0) = 180,\ z(0) = 180$, convergent to the equilibrium point E: (0, 0, 0).



(a) (b)

Fig. 6. (a) Trajectory of a four-scroll chaotic attractor for system (1) using the control law given in Corollary 2(2) for $\delta_x = 5$, $\delta_y = 3$ and $\delta_z = 1$ with the initial condition $x(0) = 180,\ y(0) = -180,\ z(0) = 180$, convergent to the equilibrium point $E:\ (0,\ 0,\ 0)$; (b) trajectory of a four-scroll chaotic attractor for system (1) using the control law given in Corollary 3 for $\delta_x = 5$, $\delta_y = 1.25$ and $\delta_z = 0.75$ with the initial condition $x(0) = 180,\ y(0) = 180,\ z(0) = 180$, convergent to the equilibrium point E: (0, 0, 0).

5. Conclusion

In this paper, we have studied the globally exponential synchronization and globally exponential stabilization for a modified Chua's circuit via feedback control strategy. In particular, for a drive-driven chaotic system pair, a simple, linear control law was designed to globally exponentially synchronize them. Simple linear controllers were also derived to globally exponentially stabilize any given equilibrium point of the chaotic system. Finally, we used numerical simulations to show the effectiveness of the theoretical results. The controllers presented in this paper, designed using Lyapunov function method, are not necessarily the "simplest" or the "best" controllers. How to construct a optimal Lyapunov function to obtain the "simplest" controller is a challenging task and is still open for future research.

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