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Symbolic computation of limit cycles associated with Hilbert's 16th problem

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ABSTRACT

This paper is concerned with the practical complexity of the symbolic computation of limit cycles associated with Hilbert's 16th problem. In particular, in determining the number of small-amplitude limit cycles of a non-linear dynamical system, one often faces computing the focus values of Hopf-type critical points and solving lengthy coupled polynomial equations. These computations must be carried out through symbolic computation with the aid of a computer algebra system such as Maple or Mathematica, and thus usually gives rise to very large algebraic expressions. In this paper, efficient computations for the focus values and polynomial equations are discussed, showing how to deal with the complexity in the computation of non-linear dynamical systems.

1. Introduction

Limit cycles are common solutions for almost all non-linear dynamical systems. They model systems that exhibit self-sustained oscillations. Due to the wide occurrence of limit cycles in science and technology, limit cycle theory has also been extensively studied by physicists, and more recently by chemists, biologists and economists. Limit cycles are generated through bifurcations (perturbations). From the point of view of dynamical system theory, there are four principal bifurcations in producing limit cycles: (i) Multiple Hopf bifurcations from a center or focus; (ii) Separatrix cycle bifurcations from homoclinic or heteroclinic orbits; (iii) global center bifurcation from a periodic annuli; and (iv) limit cycle bifurcations from multiple limit cycles. Limit cycles bifurcated from a focus, center or limit cycles are called local bifurcations of limit cycles or small limit cycles, which are usually studied by normal form and other local bifurcation theories [1–3]. The limit cycles generated from separatrix cycles or global period annuli are called global bifurcations of limit cycles, which are usually investigated by global bifurcation theories, such as the Poincaré–Pontryagin–Andronov theorem or higher order Melnikov function analysis [4,5].

One well-known problem closely related to limit cycle theory is Hilbert's 16th problem, which is one of the 23 mathematical problems proposed by D. Hilbert at the Second International Congress of Mathematics in 1900 [6]. Recently, a modern version of the second part of Hilbert's 16th problem was formulated by S. Smale, and chosen as one of his 18 most challenging mathematical problems for the 21st century [7]. To be more specific, consider the following planar system:

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$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \tag{1}$$

where the dot denotes differentiation with respect to time t, and $P_n(x,y)$ and $Q_n(x,y)$ represent nth-degree polynomials of x and y. The second part of Hilbert's 16th problem is to find an upper bound of the type $K = H(n) \leqslant n^q$ on the number of limit cycles that the system can have, where q is a universal constant. If the problem is restricted to the vicinity of isolated fixed points, it is equivalent to studying degenerate Hopf bifurcations, and the main tasks become computing the so-called *focus values* of the point and determining centre conditions. In the past half-century, many researchers have investigated the local problem and obtained many results (e.g., see [8–15]). For a quadratic system, it is now known that the maximum number of small limit cycles is three [8]. However, globally, the problem is unsolved even for quadratic systems. For cubic-order systems, on the other hand, the best results published so far are twelve limit cycles [1–3,16,17]. In order to find the number of limit cycles of a system in the neighbourhood of a fixed point (which is a linear center), one must compute the focus values of the point with the aid of a computer algebra system such as Maple [18], or Mathematica [19]. In fact, many researchers have recently paid attention to developing efficient computational methods for the computation of focus values (e.g., see [20–26]). There is another method of finding limit cycles, called the method of stability-changing of homoclinic loops (e.g., see [27–31]) as well as the reference cited therein.

Symbolic computation plays an important role in the study of limit cycles associated with Hopf critical points. Three main tasks are involved in determining the number of limit cycles. First of all, one must compute the focus values, and then solve a system of polynomial equations to determine parameter values such that as many focus values become zero as possible. Finally, one needs to give appropriate perturbations to prove the existence of the exact number of limit cycles. The first two tasks must use symbolic computations, while the last task can be carried out with numerical computation. The symbolic computations usually result in very large expressions for polynomial equations, and one cannot avoid this using a pure numerical computation. Therefore, efficient computation is essential in the study of multiple limit cycles.

In this paper, we will use Hilbert's 16th problem, as an example, to demonstrate the efficient computation of limit cycles. The rest of the paper is organized as follows. In the next section, we shall present some basic concepts and lemmas which are needed in the following sections. We also discuss some methods for computing focus values. Symbolic computation with examples are given in Section 3, and finally, the conclusion is drawn in Section 4.

2. Preliminaries

In this section, we first present some basic concepts and lemmas which will be used in the next two sections, and give a brief discussion on the methods for computing focus values.

Definition 1. A limit cycle is an attracting set to which orbits or trajectories converge and upon which trajectories are periodic. A stable limit cycle is usually called a periodic attractor.

Definition 2. A singular point of a planar vector field is called elementary if the linearization of the field at this point has at least one non-zero eigenvalue. A polycycle is called elementary if it contains elementary singular points only.

Definition 3. Hilbert's 16th problem is to estimate $H(n)$ for any $n \in \mathbf{Z}_+$, where $H(n)$ denotes the uniform bound for the number of limit cycles of (1).

Hilbert's problem is still open even for $n = 2$. In fact, so far only a lower bound (4) is known for quadratic systems, i.e., $H(2) \geqslant 4$ [32,33]. For more detailed discussion on Hilbert's 16th problem, the reader is referred to the review articles [34–36].

Although, it has not been possible to obtain a uniform upper bound for $H(n)$, various efforts have been made in finding the maximal number of limit cycles and raising the lower bound of Hilbert number $H(n)$ for general planar polynomial systems or for individual degree of systems, through which people hope to get better estimates of the upper bound of $H(n)$. Even just estimating a good lower bound of $H(n)$ is, in general, a very difficult problem.

Since the main attention of this paper is given to studying small-amplitude limit cycles, in the following we give sufficient conditions for the existence of small limit cycles. We suppose that the normal form of system (1) has been obtained in the polar coordinates up to the $(2k+1)$th order term (interested readers can find the details of normal form computation in [20]):

$$\dot{r} = r(v_1 + v_3 r^2 + v_5 r^4 + \cdots + v_{2k+1} r^{2k}), \tag{2}$$
$$\dot{\theta} = \omega + t_3 r^2 + t_5 r^4 + \cdots + t_{2k+1} r^{2k}, \tag{3}$$

where r and θ denote the amplitude and phase of motion, respectively. Both v_k and t_k are explicitly expressed in terms of the original system's coefficients. v_{2k+1} is called the kth-order focus value of the Hopf-type critical point (the origin). Note that here v_1 is the term obtained from linear perturbation.

The basic idea of finding k small limit cycles of system (1) around the origin is as follows: First, find the conditions such that $v_3 = v_5 = \cdots = v_{2k-1} = 0$ (note that $v_1 = 0$ is automatically satisfied at the critical point), but $v_{2k+1} \neq 0$, and then perform appropriate small perturbations to prove the existence of k limit cycles. This indicates that the procedure for finding multiple limit cycles involves two steps: Computing the focus values (i.e., computing the normal form) and solving the coupled non-linear equations: $v_3 = v_5 = \ldots = v_{2k-1} = 0$. In the following two lemmas, we give sufficient conditions for the existence of small limit cycles. (The proofs can be found in [1–3].)

Lemma 1. *If the system parameters are chosen such that the focus values v_{2i+1} in Eq. (2) satisfy the following conditions:*

$$v_{2i+1} v_{2i+3} < 0 \quad \text{and} \quad |v_{2i+1}| \ll |v_{2i+3}| \ll 1, \quad \text{for} \quad i = 0, 1, 2, \ldots, k-1,$$

then the polynomial equation given by $\dot{r} = 0$ in Eq. (2) has k positive real roots of r^2, and thus the original system (1) has k limit cycles in the vicinity of the origin.

However, in many cases, v_{2j+1} depends on k parameters:

$$v_{2j+1} = v_{2j+1}(\epsilon_1, \epsilon_2, \ldots, \epsilon_k), \quad j = 0, 1, \ldots, k. \tag{4}$$

In this case, the following lemma is more convenient in applications.

Lemma 2. *Suppose that condition (4) holds, and further assume that*

$$\begin{aligned} & v_{2k+1}(0, \ldots, 0) \neq 0, \\ & v_{2j+1}(0, \ldots, 0) = 0, \quad j = 0, 1, \ldots, k-1, \\ & \text{and} \quad \det\left[\frac{\partial(v_1, v_3, \ldots, v_{2k-1})}{\partial(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)}(0, \ldots, 0)\right] \neq 0. \end{aligned} \tag{5}$$

Then for any given $\epsilon_0 > 0$, there exist $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$ and $\delta > 0$ with $|\epsilon_j| < \epsilon_0,\ j = 1, 2, \ldots, k$ such that the equation $\dot{r} = 0$ has exactly k real positive roots r^2 (i.e., system (2) has exactly k limit cycles) in a δ-ball with the center at the origin.

In this paper, we will consider examples chosen from quadratic and cubic systems to demonstrate computation of small limit cycles. A general cubic system with a fixed point at the origin can be written as

$$\begin{aligned} \dot{x} &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{6}$$

where a_{ij}'s and b_{ij}'s are real constant coefficients (parameters). It is obvious that the origin $(x, y) = (0, 0)$ is a fixed point. The system has a total of eighteen parameters. However, not all of them are independent. First, note that we may use a linear transformation such that system (6) can be rewritten as

$$\begin{aligned} \dot{x} &= \alpha x + \beta y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} &= \pm\beta x + \alpha y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{7}$$

where α and $\beta > 0$ are used to represent the eigenvalues of the linearized system of (6). Note that the other coefficients in (7) should be different from that of system (6), but we use the same notation for convenience. Here, when the negative sign is taken, the origin is a focus point or a centre (if $\alpha = 0$); otherwise, it is a saddle point or node.

Now, suppose we are interested in the small limit cycles in the neighborhood of the origin. So the negative sign is taken in (7), and the eigenvalues are now given by $\lambda_{1,2} = \alpha \pm \beta i$, where i is the imaginary unit, satisfying $i^2 = -1$. Then we can apply a time scale, $\tau = \beta t$, into system (7) to obtain

$$\begin{aligned} \frac{dx}{d\tau} &= \alpha x + y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \frac{dy}{d\tau} &= -x + \alpha y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{8}$$

where again the same notations for the parameters are used. Henceforth, we assume that the leading β has been scaled to 1, and rename $\tau = t$. Now, system (8) has only fifteen parameters. Further, by a rotation we can remove one parameter [8,37] from system (8), which can be written in the general form:

$$\begin{aligned} \dot{x} &= \alpha x + y + Ax^2 + (B + 2D)xy + Cy^2 + Fx^3 + Gx^2y + (H - 3P)xy^2 + Ky^3, \\ \dot{y} &= -x + \alpha y + Dx^2 + (E - 2A)xy - Dy^2 + Lx^3 + (M - H - 3F)x^2y + (N - G)xy^2 + Py^3. \end{aligned} \tag{9}$$

This form is perhaps the simplest form for cubic systems in the literature [37]. The system has fourteen parameters. However, since the same order terms on the right-hand side of (9) are homogeneous, we can remove one more parameter. Suppose $A \neq 0$ (in case $A = 0$ one may use another non-zero parameter in the scaling), we let

$$\begin{aligned} & B = bA, C = cA, D = dA, E = eA, F = fA^2, G = gA^2, \\ & H = hA^2, K = kA^2, L = \ell A^2, M = mA^2, N = nA^2, P = pA^2, \end{aligned} \tag{10}$$

and apply a spatial scaling $x \to x/A$, $y \to y/A$ to system (9) to obtain

$$\begin{aligned} \dot{x} &= \alpha x + y + x^2 + (b + 2d)xy + cy^2 + fx^3 + gx^2y + (h - 3p)xy^2 + ky^3, \\ \dot{y} &= -x + \alpha y + dx^2 + (e - 2)xy - dy^2 + \ell x^3 + (m - h - 3f)x^2y + (n - g)xy^2 = py^3, \end{aligned} \tag{11}$$

which has only thirteen independent parameters. It is easy to see that the zeroth-order focus value is $v_0 = \alpha$. Other focus values are given in terms of the remaining twelve parameters. Let

$$\mathscr{S} = \{b, c, d, e, f, g, h, k, \ell, m, n, p\}. \tag{12}$$

Then, $v_i = v_i(\mathscr{S})$. In general, the maximum number of small limit cycles which exist in the vicinity of the origin is not greater than the number of independent parameters. Here, it is 13. In other words, the best possibility one can have is

$$v_i = 0, \quad i = 0, 1, \ldots, 12, \quad \text{but} \quad v_{13} \neq 0.$$

Then according to the lemmas given above, the maximum number of small limit cycles which can be obtained by appropriate perturbations is 13. Of course, this conclusion is obtained under the assumption that the origin is a linear centre (i.e., the origin is a Hopf-type critical point). If the origin is a saddle point or a node, then the situation is different, which will be discussed in Section 3.

In order to find the number of small limit cycles around a focus point, one needs to compute the focus values of the point. There are a number of methods which can be used to compute the focus values. In this section, we briefly describe two efficient methods for computing the focus values. A perturbation technique based on the normal form theory associated with Hopf singularity was developed early [20]. The approach can be employed to a general n-dimensional system associated with Hopf bifurcation to yield the normal form given by Eqs. (2) and (3). Another well-known method, called singular point method, is to compute the singular point quantities (see [24–26,38] for details). However, this method is only applicable to two-dimensional systems described on center manifold. We have the following results for the singular point method and the relation between the focus value and the singular point method [24,25,39],

Theorem 1. *For any positive integer m, the following assertion holds:*

$$v_{2k+1}(2\pi) = i\pi\left(\mu_k + \sum_{j=1}^{k-1} \xi_m^{(j)} \mu_j\right), \quad k = 1, 2, \ldots, \tag{13}$$

where $\xi_m^{(j)}$ $(j = 1, 2, \ldots, k-1)$ *are polynomial functions, and the singular point quantity* μ_k *is given by*

$$\mu_m = \sum_{k+j=3}^{2m+4} [(m-k+2)a_{k,j-1} - (m-j+2)b_{j,k-1}]C_{m-k+2,m-j+2}, \tag{14}$$

where $C_{11} = 1, C_{20} = C_{02} = C_{kk} = 0, k = 2, 3, \ldots$, *and* $\forall(\alpha, \beta), \alpha \neq \beta$, $m \geqslant 1$, *and*

$$C_{\alpha\beta} = \frac{1}{\beta - \alpha} \sum_{k+j=3}^{\alpha+\beta+2} [(\alpha-k+1)a_{k,j-1} - (\beta-j+1)b_{j,k-1}]C_{\alpha-k+1,\beta-j+1}, \tag{15}$$

where $a_{kj} = b_{kj} = C_{kj} = 0$ *for* $k < 0$ *or* $j < 0$.

It is clearly seen from Eq. (13) that

$$\mu_1 = \mu_2 = \cdots = \mu_{k-1} \iff v_3 = v_5 = \cdots = v_{2k-1}.$$

Therefore, when determining the conditions such that $v_1 = v_2 = \cdots = v_{k-1} = 0$, one can instead use the equations: $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$. If the μ_k's are simpler than the v_k's then this method is better than the method of directly computing v_k. However, in general such μ_k are not necessarily simpler than v_k. We shall see this in the next section.

3. Symbolic computation with examples

The formulas given in the previous section for computing the focus values (normal form) or the singular point quantities can be coded using a computer algebra system such as Maple or Mathematica. In fact, Maple has been used to code the perturbation method (the source code and sample inputs can be found from the website: pyu1.apmaths.uwo.ca/pyu/pub/software). The formulas for computing the singular point quantities have been coded using both Maple and Mathematica. Both methods have been used in computing limit cycles. In this section, we will present a number of examples to demonstrate the complexity of symbolic computation.

3.1. Three small limit cycles in quadratic systems

We start from the simplest case, namely, consider the limit cycles bifurcating from the origin of quadratic vector fields. Thus, we take Eq. (11) only up to the second-order terms to obtain the following system:

$$\dot{x} = \alpha x + y + x^2 + (b + 2d)xy + cy^2 \quad \dot{y} = -x + \alpha y + dx^2 + (e - 2)xy - dy^2, \tag{16}$$

which has five independent parameters: α, b, c, d, e. It is clear that $v_1 = \alpha$. When one calculates v_{2k+1} or μ_k $(k \geqslant 1)$, one sets $\alpha = 0$. Therefore, there are only four independent parameters which appear in the focus values or the singular point quantities.

To find the focus values of system (16), we apply the Maple program given in [20]. The execution of the program is straightforward: put Eq. (16) into a Maple input file and give an order, say, $n = 9$, and then execute the program to produce an output containing the focus values as well as non-linear transformation. Source codes and guideline can be found from the website: pyu1.apmaths.uwo.ca/p~yu/pub/software (names: program1 and input1). The focus values are given by

$$\begin{aligned}
v_3 &= -\frac{1}{8}(c+1)b, \\
v_5 &= -\frac{1}{288}(c+1)[6de(5c-e+5) + b(18+3e-18c+19ce+34bd+5b^2+20c^2+56d^2-e^2)], \\
v_7 &= -\frac{1}{663552}(c+1)\bar{v}_7(b,c,d,e), \\
v_9 &= -\frac{1}{238878720}(c+1)\bar{v}_9(b,c,d,e), \\
&\vdots
\end{aligned} \tag{17}$$

where $\bar{v}_i(b,c,d,e)$ $(i = 5,7,9)$ are polynomials of b,c,d,e. It is seen that $c = -1$ yields a centre [8]. Hence, in order to have $v_3 = 0$, one must choose $b = 0$. When $b = 0$, $v_5 = -\frac{1}{48}de(c+1)(5c-e+5)$, which indicates that one must choose $e = 5(c+1)$ to obtain $v_5 = 0$, under which $v_7 = -\frac{25}{64}d(c+1)^3(c+2c^2+d^2)$. If we set $v_7 = 0$, then $v_9 = 0$ too. Actually, Bautin showed that setting v_7 zero leads to a centre. Therefore, one can only choose the four parameters such that $v_3 = v_5 = 0$, but $v_7 \neq 0$, implying that the maximum number of the small limit cycles surrounding the origin is three.

To prove the existence of exact three small limit cycles, we apply appropriate perturbations such that the perturbed focus values satisfy the sufficient conditions given in Lemma 1. There are infinitely many choices for the parameter values. Note that due to the scaling given in (10), the focus values for the original system (9) can be adjusted to any small values using the free parameter A. Under the critical conditions:

$$b = 0, \quad e = 5(c+1),$$

we have $v_3 = v_5 = 0$, $v_7 = -\frac{25}{64}d(c+1)^3(c+2c^2+d^2)$. Since exactly one parameter is used for each of the two focus values, v_3 and v_5, the perturbations for the quadratic system is straightforward, as shown below.

For convenience, suppose $d(c+1)(c+2c^2+d^2) > 0$, and thus $v_7 < 0$. Further, for definiteness, we may assume that $d > 0$ and $c > 0$, since we are not interested in finding all solutions (which we are certainly be able to obtain) but only in the existence of the three small limit cycles. Then we want to give a perturbation to $e = 5(c+1)$ such that $v_5 > 0$ and $0 < v_5 \ll -v_7$. By (17), we have the derivative of v_5 with respect to e, evaluated at the critical values: $\frac{dv_5}{de} = \frac{5}{48}d(c+1)^2 > 0$. So we may select $\epsilon_1 > 0$ such that $e = 5(c+1) + \epsilon_1$. Then the perturbed v_5 is

$$v_5 = \frac{d}{48}[5(c+1)^2\epsilon_1 - (c+1)\epsilon_1^2] \approx \frac{5}{48}d(c+1)^2\epsilon_1 > 0,$$

and thus $0 < v_5 \ll -v_7$ as long as $0 < \epsilon_1 \ll 1$.

Next, we want to perturb v_3 such that the perturbed values satisfy $0 < -v_3 \ll v_5 \ll -v_7$. By (17), we have $\frac{dv_3}{db} = -\frac{1}{8}(c+1) < 0$, implying that we should perturb b from $b = 0$ to $b = 0 + \epsilon_2$. Thus, the perturbed value of v_3 is given by $v_3 = -\frac{1}{8}(c+1)\epsilon_2$, where $0 < \epsilon_2 \ll \epsilon_1 \ll 1$ which guarantees that $0 < -v_3 \ll v_5$.

Finally, we need a perturbation to $v_1 = \alpha = 0$, which must be positive. Simply let $\alpha = \epsilon_3$. Then $v_1 = \epsilon_3$ with $0 < \epsilon_3 \ll \epsilon_2$ yields

$$0 < v_1 \ll -v_3 \ll v_5 \ll -v_7,$$

provided that $0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1$.

Summarizing the above results gives the following theorem.

Theorem 2. *Given the quadratic system (16), suppose $c > 0$, $d > 0$, and $b = 0$, $e = 5(c+1)$. Then, under the perturbations: $e5(c+1) + \epsilon_1$, $b = \epsilon_2$ and $\alpha = \epsilon_3$, where $0 < \epsilon_3 \ll \epsilon_2 \ll \epsilon_1 \ll 1$, the system (16) has exactly three small limit cycles in the neighborhood of the origin.*

As a numerical example, let $c = d = 1/2$, and exactly choose the perturbations as $\epsilon_1 = \frac{1}{10}, \epsilon_2 = \frac{1}{5000}, \epsilon_3 = \frac{1}{50000000}$. Then the perturbed focus values are given by

$$v_1 = \frac{1}{50000000}, \quad v_3 = -\frac{3}{80000}, \quad v_5 = \frac{56673782999}{4800000000000}, \quad v_7 = -\frac{1099541240782350199352293}{138240000000000000000000}.$$

Thus, the normal form given up to term r^7 is

$$\dot{r} = r\left(\frac{1}{50000000} - \frac{3}{80000}r^2 + \frac{56673782999}{4800000000000}r^4 - \frac{1099541240782350199352293}{138240000000000000000000}r^6\right),$$

which yields three positive roots for r: $r_1 = 0.0258299909, r_2 = 0.0595361101, r_3 = 0.1031148918$, representing the approximate solutions for the amplitudes of the three small limit cycles bifurcating from the origin.

The perturbed quadratic system is given by

$$\dot{x} = 0.00000002x + y + x^2 + 1.0002xy + 0.5y^2,$$
$$\dot{y} = -x + 0.00000002y + 0.5x2 + 5.6xy - 0.5y^2.$$

The system has two fixed points: $C_0 = (0,0)$, a third-order unstable focus point, and $C_1 = (-0.1368474832, -1.7042735041)$, a saddle point. The phase portrait for this perturbed system is shown in Fig. 1, where the three limit cycles are shown in the vicinity of the origin. It should be noted that trajectories of the system near the origin shows behaviour similar to that around a centre due to the degeneracy of the singular point. Therefore, it is impossible in this fashion to verify the multiple limit cycles bifurcating from a high order singular point. It can be shown that a quadratic system cannot simultaneously have two linear centres or two fine focus points, but may have one fine focus point and one saddle point (as in the case depicted in Fig. 1), or one fine focus point and an unstable focus point [32]. Note that the trajectories shown in this figure are obtained by solving the above perturbed quadratic equations based on a 4th-order Runge–Kutta method. Figs. 2 and 3 are obtained in the same way.

If we use the formulas given in (15) and (14) and execute the Maple program, we obtain the following singular point quantities:

$$\mu_1 = \frac{1}{4} i(c+1)b,$$
$$\mu_2 = -\frac{1}{48} i(c+1)[-2de(5c-e+5) + b(15+7e+20c+11ce-2bd+4b^2+5c^2+6e^2)],$$
$$\vdots$$

Comparing (17) with the above formulas shows that $v_3 = \frac{1}{2} i\mu_1$, but v_5 and μ_2 are quite different. Nevertheless, μ_2 does not simplify the expression (only reducing one term d^2). This also happens to v_7 and μ_3. If set $b = 0$ (so that $v_3 = \mu_1 = 0$), then $v_5 = \frac{1}{2} i\mu_2$. Further, letting $e = 5(c+1)$ (so that $v_5 = \mu_2 = 0$) yields $v_7 = \frac{1}{2} i\mu_3$.

The above example indicates that the two methods described in the previous section have comparable computing efficiency.

3.2. Twelve small limit cycles in Z_2-equivariant cubic systems

In this section, we turn to study the small limit cycles bifurcating from cubic systems. First, we consider the case when the origin is not a centre, and then investigate the case that the origin is a centre. Since the 1980's many researchers have studied the limit cycles of cubic systems, and, in particular, the main attention has been focused on local bifurcations (e.g., see [37,40–44]). The maximum number of limit cycles obtained so far for cubic systems is twelve [1–3,16,17,25].

When the origin is not a centre, we may consider Z_2-equivariant vector fields in order to simplify computation. It has been proved [17,2,3] that the cubic-order system (9) can have twelve small limit cycles. This system has a saddle point, and two

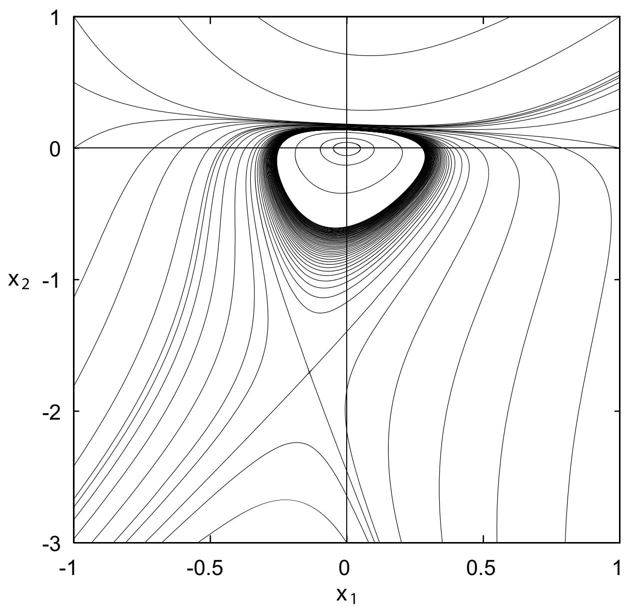

Fig. 1. The phase portrait of system (16) having 3 small limit cycles around the origin, for $\alpha = 0.00000002$, $b = 0.0002$, $c = 0.5$, $d = 0.5$, $e = 7.6$.

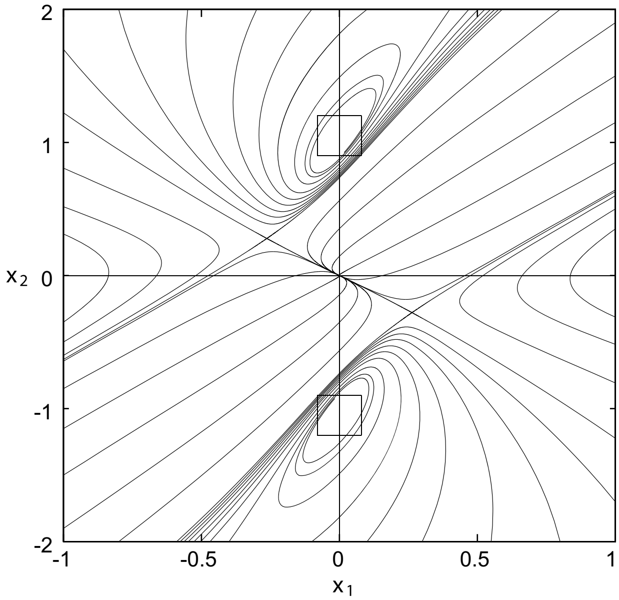

Fig. 2. The phase portrait of system (21) having 12 limit cycles, when the origin is a node for $a = a_{12} = -b_{03} = -0.7, b = -a_{03} = -0.3336019980$, $a_{30} = 4.5658610164$, $a_{21} = -5.0539492766, b_{30} = -1.3447323014, b_{21} = -0.8333681006, b_{12} = -4.1028222711$.

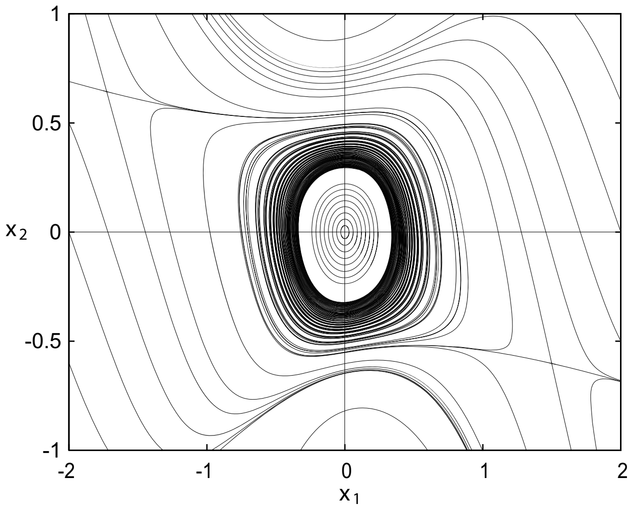

Fig. 3. The phase portrait of system (26) having 8 limit cycles around the origin, for $\lambda = -0.4 \times 10^{-47}$, $a_3 = 0.5$, $a_4 = 0.2935258759$, $a_5 = 3.3759641940$, $a_7 = -2.5483319254$, $b_4 = -0.1454777790$, $b_5 = 0.1 \times 10^{-37}$, $b_6 = 3.6440555615$, $b_7 = 0.2935258759$.

weak focus points which are symmetric about the origin. Six small limit cycles exist in the neighborhood of each of the two weak focus points. More generally, consider the following general cubic-order Z_2-equivariant vector field [35]:

$$\dot{z} = F_2(z, \bar{z}), \quad \dot{\bar{z}} = \bar{F}_2(z, \bar{z}), \tag{18}$$

where $F_2(z, \bar{z}) = P(w_1, w_2) + iQ(w_1, w_2)$, $w_1 = \frac{1}{2}(z + \bar{z}), w_2 = \frac{1}{2i}(z - \bar{z})$, P, Q, w_1 and w_2 are all real, and

$$F_2(z, \bar{z}) = (A_0 + A_1|z|^2)z + (A_2 + A_3|z|^2)\bar{z} + A_4 z^3 + A_5 \bar{z}^3. \tag{19}$$

Let $A_j = a_j + ib_j$ where a_j, b_j are real. Then we obtain the following real Z_2-equivariant vector field:

$$\begin{aligned}
\dot{w}_1 &= (a_0 + a_2)w_1 - (b_0 - b_2)w_2 + (a_1 + a_3 + a_4 + a_5)w_1^3 - (b_1 - b_3 + 3b_4 - 3b_5)w_1^2 w_2 \\
&\quad + (a_1 + a_3 - 3a_4 - 3a_5)w_1 w_2^2 - (b_1 - b_3 - b_4 + b_5)w_2^3, \\
\dot{w}_2 &= (b_0 + b_2)w_1 + (a_0 - a_2)w_2 + (b_1 + b_3 + b_4 + b_5)w_1^3 + (a_1 - a_3 + 3a_4 - 3a_5)w_1^2 w_2 \\
&\quad + (b_1 + b_3 - 3b_4 - 3b_5)w_1 w_2^2 + (a_1 - a_3 - a_4 + a_5)w_2^3.
\end{aligned} \tag{20}$$

The eigenvalues of the Jacobian of system (20) evaluated at the origin are $\lambda_{1,2} = a_0 \pm \sqrt{a_2^2 + b_2^2 - b_0^2}$. There are two cases:

(I) when $a_2^2 + b_2^2 - b_0^2 \geqslant 0$, the origin is either a saddle point or a node; and
(II) when $a_2^2 + b_2^2 - b_0^2 < 0$, the origin is either a focus point or a center.

In order to take advantage of the Z_2-symmetry, we consider, instead of the origin, two non-zero weak focus points which are symmetric about the origin. Therefore, if one finds a certain number of limit cycles around one of the focus points, the total number of limit cycles of the system is doubled. To find the limit cycles around each of the weak focus points, introduce the following linear transformation $(w_1,\ w_2)^T = T(x,\ y)^T$, where

$$T = \begin{cases} \begin{bmatrix} 0 & b_0 - b_3 \\ -b & a_3 \end{bmatrix} & \left(b = \pm\sqrt{a_3^2 + b_3^2 - b_0^2}\right) \quad \text{for Case (I)}, \\ \begin{bmatrix} 0 & b_0 - b_3 \\ b & a_3 \end{bmatrix} & \left(b = \sqrt{-(a_3^2 + b_3^2 - b_0^2)}\right) \quad \text{for Case (II)}, \end{cases}$$

into system (20) with renamed coefficients gives the normalized equations:

$$\begin{aligned} \dot{x} &= ax + by + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ \dot{y} &= \pm bx + ay + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3; \end{aligned} \tag{21}$$

where the positive sign is taken for Case (I), the negative sign for Case (II), and $a = a_0$. Without loss of generality, one may assume that the two weak focus points are located at $(0, \pm 1)$, which yields $a_{03} = -b, b_{03} = -a$. Further, applying the following scalings: $a, b, a_{30}, a_{21}, b_{30}, b_{21} \Rightarrow \omega a, \omega b, \omega a_{30}, \omega a_{21}, \omega b_{30}, \omega b_{21}$, with $b_{12} = (4a^2 - 2b^2 + \omega^2)/(2b)$, and the time scaling: $\tau = \omega t$, together with introducing the following transformation:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ \pm 1 \end{bmatrix} + \begin{bmatrix} 2b & 0 \\ 2a & -1 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

to the above normalized equations yields the following equations for computing the normal form of system (20) associated with the Hopf critical points $(0, \pm 1)$:

$$\begin{aligned} \frac{du}{d\tau} &= v + 2\bar{a}_{21}u^2 + 4auv - \frac{3}{2}v^2 + 4b\bar{a}_{30}u^3 - 2\bar{a}_{21}u^2v - 2auv^2 + \frac{1}{2}v^3, \\ \frac{dv}{d\tau} &= -u - 4\bar{b}_{21}u^2 + 2(2a^2 \mp 2b^2 + 1)uv - 8\bar{b}_{30}u^3 + 4\bar{b}_{21}u^2v - (2a^2 \mp b^2 + 1)uv^2, \end{aligned} \tag{22}$$

in which

$$\begin{aligned} \bar{a}_{21} &= a_{21}b - a^2, \\ \bar{a}_{30} &= a_{30}b + a_{21}a, \\ \bar{b}_{21} &= b_{21}b^2 - a_{21}ab \mp 2ab^2 + 2a^3 + a, \\ \bar{b}_{30} &= b_{30}b^3 - a_{30}ab^2 - a_{21}a^2b + b_{21}ab^2 \mp a^2b^2 + a^4 + \tfrac{1}{2}a^2, \end{aligned} \tag{23}$$

where the '−' sign is for case (I), while the '+' sign for case (II). It is noted that $\bar{a}_{21}, \bar{a}_{30}, \bar{b}_{21}, \bar{b}_{30}$ and $a_{21}, a_{30}, b_{21}, b_{30}$ are mutually uniquely determined.

The above normalizing procedure shows that one of the parameters: a, b and b_{12} can be chosen arbitrarily since the frequency ω can be normalized to 1 using a time scaling. Therefore, we may, instead of the time scaling, let $b_{12} = \frac{1+4a^2}{2b} - b$, under which $\omega = 1$ and so $\tau = t$.

It is clearly seen from Eq. (22) that the normal forms for the two cases (I) and (II) will be in the same formulas. Therefore, one only needs to consider one case. In the following, we briefly summarize the computation of focus values. For the generic case, letting $v_1 = 0$ yields the solution for $\bar{a}_{30}$. Then, setting $v_2 = 0$ result in

$$\bar{b}_{30} = \bar{b}_{30}(a, b, \bar{a}_{21}, \bar{b}_{21}) = -\frac{\bar{b}_N(a, b, \bar{a}_{21}, \bar{b}_{21})}{6\bar{b}_D(a, b, \bar{a}_{21}, \bar{b}_{21})},$$

where $\bar{b}_N$ and $\bar{b}_D$ are polynomials of $a, b, \bar{a}_{21}$ and $\bar{b}_{21}$. The remaining focus values are then simplified as

$$\begin{aligned} v_3 &= \frac{4FF_1}{9[a(8\bar{a}_{21} - 1) - 10\bar{b}_{21}(a\bar{a}_{21} + 2a^2 - 2a^2 + 1) + 8a(a^2 - b^2)]^2}, \\ v_4 &= \frac{FF_2}{405[a(8\bar{a}_{21} - 1) - 10\bar{b}_{21}(a\bar{a}_{21} + 2a^2 - 2a^2 + 1) + 8a(a^2 - b^2)]^3}, \\ v_5 &= \frac{FF_3}{233280[a(8\bar{a}_{21} - 1) - 10\bar{b}_{21}(a\bar{a}_{21} + 2a^2 - 2a^2 + 1) + 8a(a^2 - b^2)]^4}, \\ v_6 &= \frac{FF_4}{146966400[a(8\bar{a}_{21} - 1) - 10\bar{b}_{21}(a\bar{a}_{21} + 2a^2 - 2b^2 + 1) + 8a(a^2 - b^2)]^5}, \end{aligned} \tag{24}$$

where F, is a common factor, and $F_i = F_i(a, b, \bar{a}_{21}, \bar{b}_{21}), i = 1, 2, 3, 4$. Since $F = 0$ leads to centers, we should find solutions from $F_i = 0$. Further, note that F_1 is linear in $\bar{a}_{21}$, so one can explicitly solve $\bar{a}_{21}$ from $F_1 = 0$ to obtain

$$\bar{a}_{21} = \bar{a}_{21}(a, b, \bar{b}_{21}) = \frac{-\bar{a}_{21N}(a, b, \bar{b}_{21})}{20\bar{a}_{21D}(a, b, \bar{b}_{21})}.$$

Then, substituting the above $\bar{a}_{21}$ into F_i's yields three polynomial equations:

$$F_2 = -\frac{672(10\bar{b}_{21} - a + 3b)^2(10\bar{b}_{21} - a - 3b)^2[5\bar{b}_{21}(4\bar{b}_{21} - 4a^3 + 4ab^2 - 5a) + a^2(8a^2 - 8b^2 + 5)]^2}{25[280\bar{b}_{21}^4 - 112a\bar{b}_{21}^3 + 6(11a^2 - b^2)\bar{b}_{21}^2 - 2a(5a^2 - 3b^2)\bar{b}_{21} + a^4 - 9a^2b^2]^2}F_2^* = 0,$$
$$F_3 = -\frac{21504(3125\bar{b}_{21} - a + 3b)^3(10\bar{b}_{21} - a - 3b)^3[5\bar{b}_{21}(4\bar{b}_{21} - 4a^3 + 4ab^2 - 5a) + a^2(8a^2 - 8b^2 + 5)]^3}{25[280\bar{b}_{21}^4 - 112a\bar{b}_{21}^3 + 6(11a^2 - b^2)\bar{b}_{21}^2 - 2a(5a^2 - 3b^2)\bar{b}_{21} + a^4 - 9a^2b^2]^5}F_3^* = 0,$$
$$F_4 = -\frac{2016(78125\bar{b}_{21} - a + 3b)^4(10\bar{b}_{21} - a - 3b)^4[5\bar{b}_{21}(4\bar{b}_{21} - 4a^3 + 4ab^2 - 5a) + a^2(8a^2 - 8b^2 + 5)]^4}{25[280\bar{b}_{21}^4 - 112a\bar{b}_{21}^3 + 6(11a^2 - b^2)\bar{b}_{21}^2 - 2a(5a^2 - 3b^2)\bar{b}_{21} + a^4 - 9a^2b^2]^8}F_4^* = 0,$$

where $F_i^* = F_i^*(a, b, \bar{b}_{21}) = 0,\ i = 2, 3, 4$. These three polynomial equations are coupled and have to be solved simultaneously. Similarly, we need to eliminate one of the coefficients a, b and $\bar{b}_{21}$ from the three equations. Eliminating b from the first two equations (using the Maple `resultant` command), $F_2 = 0$ and $F_3 = 0$, yields the resultant equation:

$$F_5 = 160\bar{b}_{21}^3 + 140a\bar{b}_{21}^2 - 40a^2\bar{b}_{21} + a^3. \tag{25}$$

Similarly, eliminating b from another two equations, $F_2 = 0$ and $F_4 = 0$, yields another resultant equation: $F_6 = 0$, where F_6 is a lengthy expression, omitted here. We must solve $F_5 = F_6 = 0$ for a and $\bar{b}_{21}$, which may yield possible solutions such that $v_i = 0,\ i = 1, 2, \ldots, 6$. It should be pointed out that the above non-linear, variable elimination process does not miss any possible solutions, but as is well-known, it introduce extra, spurious, solutions, and thus one has to verify all solutions using the original expressions of v_i.

Finally, eliminating $\bar{b}_{21}$ from the two equations, $F_5 = F_6 = 0$, results in the final equation:

$$\begin{aligned}F_7 = (&112783323904064139036783273943 04a^{12} + 943926835793226811152157855484 9280a^{10}\\ &+ 15955549149699417710133152016699200a^8 + 78067338319585667946062243018556 00a^6\\ &+ 26896701242331499420963123077015 00a^4 + 13179071965242968131829121466706 25a^2\\ &+ 248225087542552407942139964497500)a = 0.\end{aligned}$$

It is obvious that the above equation has only one real solution $a = 0$. But if $a = 0$, then $\bar{b}_{21} = 0$ resulting in $\bar{a}_{21} = \bar{b}_{30} = \bar{a}_{30} = 0$, which can be shown to give a center. Thus, a must be non-zero. This implies that one cannot find possible non-zero values of a and $\bar{b}_{21}$ such that $F_5 = F_6 = 0$, indicating that there is no solution for $F_2 = F_3 = F_4 = 0$. Therefore, there do not exist possible non-trivial solutions for $a, b, \bar{a}_{21}, \bar{b}_{21}, \bar{a}_{30}$ and $\bar{b}_{30}$ such that $v_i = 0, i = 1, 2, \ldots, 6$. Hence, *fourteen small limit cycles are not possible* for a cubic system with Z_2 symmetry.

Since there is one free parameter (i.e., a), we have infinitely many solutions. All parametric solutions can be found as follows. Finding these solutions only requires $F_2 = F_3 = 0$ ($F_4 \neq 0$). Numerically solving these equations for $\bar{b}_{21}$ in terms of a yields three solutions: $\bar{b}_{21} = 0.2033343806a, -1.1061229255a$ and $0.2778854492a$. Then, for each of the above solutions, solving Eq. (25) gives two solutions for b^2. By checking the equations $F_2 = F_3 = 0$, four solutions are obtained: two for the case when the origin is a saddle point and two for the case when the origin is a node. In the following, we consider one of the two cases when the origin is a node. Let the critical values be denoted by

$$\begin{aligned}b^* &= 0.4765747114a,\\ \bar{b}_{21}^* &= 0.2033343806a,\\ \bar{a}_{21}^* &= 0.7000000000 + 1.0149654014a^2,\\ \bar{b}_{30}^* &= \frac{a^2(0.0481488581 + 65.9546167690a^2 - 9379.2591506305a^4)}{0.0008286738 - 0.1076372236a^2},\\ \bar{a}_{30}^* &= -(0.8202076319 + 2.4368685248a^2).\end{aligned}$$

Then, we have the following theorem for the generic case.

Theorem 3. *Given the cubic system (21) which is assumed to have a saddle point or a node at the origin and a pair of symmetric fine focus points at $(x, y) = (0, 1)$ and $(0, -1)$. Further suppose $a_{12} = -b_{03} = a, a_{03} = -b, b_{12} = \frac{1+4a^2}{2b} - b$. Then, for an arbitrarily given $a \neq 0$, if $b, \bar{b}_{21}, \bar{a}_{21}(b, \bar{b}_{21}), \bar{b}_{30}(b, \bar{a}_{21}, \bar{b}_{21})$ and $\bar{a}_{30}(b, \bar{a}_{21}, \bar{b}_{21})$ are perturbed as*

$$
\begin{aligned}
b &= b^* + \epsilon_1, \\
\bar{b}_{21} &= \bar{b}^*_{21} + \epsilon_2, \\
\bar{a}_{21} &= \bar{a}_{21}(b^* + \epsilon_1, \bar{b}^*_{21} + \epsilon_2) + \epsilon_3, \\
\bar{b}_{30} &= \bar{b}_{30}(b^* + \epsilon_1, \bar{b}^*_{21} + \epsilon_2, \bar{a}_{21}(b^* + \epsilon_1, \bar{b}^*_{21} + \epsilon_2) + \epsilon_3) + \epsilon_4, \\
\bar{a}_{30} &= \bar{a}_{30}(b^* + \epsilon_1, \bar{b}^*_{21} + \epsilon_2, \bar{a}_{21}(b^* + \epsilon_1, \bar{b}^*_{21} + \epsilon_2) + \epsilon_3) + \epsilon_5, \\
a_{12} &= a + \epsilon_6,
\end{aligned}
$$

where $0 < |\epsilon_6| \ll |\epsilon_5| \ll |\epsilon_4| \ll |\epsilon_3| \ll (|\epsilon_2|, |\epsilon_1|) \ll 1$, *system (21) has exactly twelve small limit cycles. The notation* $(|\epsilon_2|, |\epsilon_1|)$ *means that* ϵ_2 *and* ϵ_1 *are in the same order, with* $\epsilon_2 = (\delta + \bar{\epsilon})\epsilon_1$ *for some* $\delta > 0$ *and some small* $\bar{\epsilon} > 0$. *(Note that here the perturbations,* ϵ_i*'s, can take positive or negative values since here a is not specified.)*

To end with a case in which the origin is not a centre, we present a numerical example for the case when the origin is a node. We choose $a = -0.7$, and then $b^* = -0.3336022980, \bar{b}^*_{21} = -0.1423340664, \bar{a}^*_{21} = 1.1973330467, \bar{b}^*_{30} = 0.0744658372$, $\bar{a}^*_{30} = 2.0142732091$, for which $b_{12} = -4.1028179815$. Further, we take the following perturbations: $\epsilon_1 = 0.3 \times 10^{-6}$, $\epsilon_2 = -0.4 \times 10^{-3}, \epsilon_3 = -0.7 \times 10^{-7}, \epsilon_4 = 0.2 \times 10^{-11}, \epsilon_5 = 0.1 \times 10^{-14}, \epsilon_6 = 0.3 \times 10^{-19}$, under which the system has five real fixed points: $C_0 = (0, 0)$, $C_{1,2} = (0, \pm 1)$ and $C_{3,4} = (\pm 0.2398840466, \mp 0.1797759006)$. A linear analysis shows that C_0 is a stable node, $C_{1,2}$ are two weakly unstable focus points, and $C_{3,4}$ are saddle points. Computing the focus values for the perturbed system finally yields the six amplitudes for the small limit cycles: $r = 0.0072744336, 0.0100774933, 0.0150111942$, $0.0824725838, 0.2066015944, 0.3591142108$. The phase portrait for the above perturbed system is shown in Fig. 2, where the two boxes contain the twelve small limit cycles near the focus points $(0, \pm 1)$. The stabilities of these limit cycles can be easily determined from the signs of the focus values.

3.3. Eight small limit cycles in a simple cubic system

In the previous subsection we have shown that a cubic system with Z_2 symmetry exhibits twelve small limit cycles, but distributed in the neighborhood of two fine focus points. Now we turn to the case that the origin is a centre and want to investigate the small limit cycles around the origin. Many results have been obtained [44] showed six limit cycles bifurcating from one critical point. Another such an example can be found in [37]. Later, seven limit cycles were found (e.g., see [40,42]). In [42], eight limit cycles were obtained. All the results were based on the symbolic computation of focus values. Recently, it was claimed that a cubic system can have eleven limit cycles around one critical point [43], which, however, does not provide detailed computation of focus values.

In the remaining of the section, we will first show that the simplest cubic system given in [42] which has seven limit cycles can actually have eight limit cycles, and then present a cubic system which has nine limit cycles around the origin. These new results are based on symbolic computation with the aid of Maple.

First, consider the simple cubic system given in [42]:

$$
\begin{aligned}
\dot{x} &= \lambda x + y + a_3x^2 + a_4x^3 + a_5x^2y - 3b_7xy^2 + a_7y^3, \\
\dot{y} &= -x + \lambda y + (b_4 - a_7)x^3 + (b_5 - 3a_4)x^2y + (b_6 - a_5)xy^2 + b_7y^3.
\end{aligned}
\tag{26}
$$

In [42], an extra condition $3a_5 = -(10a_3^2 + 11a_7)$ is imposed in order to simplify computation. In fact, lifting this restriction leads to eight limit cycles, as we will see. First, choose $a_3 \neq 0$ as a scaling parameter and apply the following scalings:

$$
a_i \to A_i a_i^2, \quad b_i \to B_i a_i^2, \quad x \to x/a_3, \quad y \to y/a_3, \tag{27}
$$

into system (26) to obtain

$$
\begin{aligned}
\dot{x} &= \lambda x + y + x^2 + A_4x^3 + A_5x^2y - 3B_7xy^2 + A_7y^3, \\
\dot{y} &= -x + \lambda y + (B_4 - A_7)x^3 + (B_5 - 3A_4)x^2y + (B_6 - A_5)xy^2 + B_7y^3.
\end{aligned}
\tag{28}
$$

Executing the Maple program [20] yields (with $\lambda = 0$) $v_3 = \frac{1}{8}B_5$. Setting $v_3 = 0$ gives $B_5 = 0$. Then computing v_5 results in

$$
v_5 = -\frac{1}{8}B_6(A_4 - B_7).
$$

There are two choices satisfying $v_5 = 0$: either $B_6 = 0$ or $B_7 = A_4$. Setting $B_6 = 0$ leads to $v_5 = v_7 = \cdots = 0$, a centre. So let $B_7 = A_4$, under which v_7 is to be

$$
v_7 = -\frac{1}{192}A_4B_6(-35 + 3B_6 + 15B_4).
$$

For the same reason, we must choose $B_4 = \frac{1}{15}(35 - 3B_6)$, in order to have $v_7 = 0$.

Continuing the calculation shows that

$$v_9 = \frac{A_4 B_6}{9600}[5A_7(18B_6 - 385) - 1750 - 525A_5 - 140B_6 - 6B_6^2 + 30A_5B_6)],$$

from which we obtain $A_7 = \frac{1750+525A_5+140B_6+6B_6^2-30A_5B_6}{5(18B_6-385)}$.

Having determined the values of four parameters: B_5, B_7, B_4 and A_7, executing the Maple program gives v_{11} from which one solves for A_4^2 to obtain

$$\begin{aligned} A_4^2 = -\frac{1}{150}\{&6912B_6^5 - 1131660B_6^4 + 13935075B_6^3 - 269206875B_6^2 + 5571820625B_6 + 6458046875 \\ &+ 50A_5[A_5(864B_6^3 - 114660B_6^2 + 3557400B_6 - 32413500) - 864B_6^4 + 115470B_6^3 - 3149265B_6^2 \\ &+ 27878550B_6 - 62811875]\} \end{aligned} \tag{29}$$

under which $v_{11} = 0$, and then v_{13} and v_{15} are simplified. Now eliminating A_5 from $v_{13} = 0$ and $v_{15} = 0$ results in a solution for A_5:

```
A5 := 1/10 * (28661513482345328640 * B6^18 - 38320168058532606394368 * B6^17
    + 94353181171684211171902240 * B6^16 - 57001690267990101526478670 0 * B6^15
```

$$\begin{aligned}
\mathtt{A5} := \mathtt{1/10*(}&\mathtt{28661513482345328640*B6^{\wedge}18 - 38320168058532606394368*B6^{\wedge}17} \\
&\mathtt{+ 94353181171684211171902240*B6^{\wedge}16 - 5700169026799010152647867 00*B6^{\wedge}15}
\end{aligned}$$

(30)

and a resultant equation:

$$F = (18B_6 - 385)(8B_6 - 735)F_1(B_6), \tag{31}$$

where F_1 is a 25th-degree polynomial of B_6, given by

$$\begin{aligned}
\mathrm{F1} := {} & 25517942739795723889201643520 \ast \mathrm{B6}^{\wedge}25 - 15961557129783711309683786317824 \ast \mathrm{B6}^{\wedge}24 \\
& + 192625552310827002440803128023040 \ast \mathrm{B6}^{\wedge}23 \\
& + 195550386732260521210324495097798400 \ast \mathrm{B6}^{\wedge}22 \\
& - 28158815198433584833713402047338176000 \ast \mathrm{B6}^{\wedge}21 \\
& + 1033734722674616222724983132524050525000 \ast \mathrm{B6}^{\wedge}20 \\
& - 59379935935449378171276843185977097343750 \ast \mathrm{B6}^{\wedge}19 \\
& + 9590292320511086071943531542881338566546875 \ast \mathrm{B6}^{\wedge}18 \\
& - 451193899934739158581843476544588721670937500 \ast \mathrm{B6}^{\wedge}17 \\
& + 11207826884672338273965968245267373870987109375 \ast \mathrm{B6}^{\wedge}16 \\
& - 135696149188209357581589508528862248038023437500 \ast \mathrm{B6}^{\wedge}15 \\
& - 2513618748935572240155021795318134123658105468750 \ast \mathrm{B6}^{\wedge}14 \\
& + 66289843074171791813769089263414112799266015625000 \ast \mathrm{B6}^{\wedge}13 \\
& + 977604060330518550753995048095833893466941894531250 \ast \mathrm{B6}^{\wedge}12 \\
& - 24802878891314023830241132965042843004940749511718750 \ast \mathrm{B6}^{\wedge}11 \\
& + 135355422329906288642447252991195213905651605224609375 \ast \mathrm{B6}^{\wedge}10 \\
& - 6332203795984067960375159713574273013757487915039062500 \ast \mathrm{B6}^{\wedge}9 \\
& + 53780987949344129764991805868491683170968479461669921875 \ast \mathrm{B6}^{\wedge}8 \\
& + 1353866539349998679702350808860664326424912896728515625000 \ast \mathrm{B6}^{\wedge}7 \\
& - 13860624815869973501744680650905940025539405349731445312500 \ast \mathrm{B6}^{\wedge}6 \\
& + 27031619976231087198815226613734972443015888824462890625000 \ast \mathrm{B6}^{\wedge}5 \\
& - 535003645332341656553005034190528839048580353927612304687500 \ast \mathrm{B6}^{\wedge}4 \\
& + 2764135604231277672299533748358798801483962497711181640625000 \ast \mathrm{B6}^{\wedge}3 \\
& - 1449744782642143940578968415021056130318871974945068359375000 \ast \mathrm{B6}^{\wedge}2 \\
& + 1186466561216918800834924597669148316265182495117187500000000 \ast \mathrm{B6} \\
& - 647397584756496918245103660411259896479415893554687500000000 :
\end{aligned} \tag{32}$$

Now, we can employ a numerical approach to solve the single variable polynomial F_1. For example, we may use the Maple built-in solver *fsolve* to find all the real roots of the polynomial. The first two real solutions: $B_6 = 385/18$ and $B_6 = 735/8$ yield two centres. The remaining given by $F_1 = 0$ have seven real roots. To verify these roots, for each solution B_6, in backing order, first use Eq. (30) to obtain A_5, and then A_4. Substituting the three solutions to verify v_{11}, v_{13} and v_{15} while $v_i = 0,\ i = 1, 3, 5, 7, 9$ are automatically satisfied since they are solved one by one using one parameter at each step. Back to the original system (26), these two solutions are given by (say, up to 50 decimal places):

$$\begin{aligned}
a_4 &= 11.72677636147392252103108673156625861130540134632217 a_3^2, \\
a_5 &= -96.28308723695332837277347016717443395057677635891191 a_3^2, \\
a_7 &= 23.87070969364661297334553081391162008758996623623660 a_3^2, \\
b_4 &= 32.30434891349705952261668588556605555819029249428315 a_3^2, \\
b_5 &= 0, \\
b_6 &= -149.85507790081863094641676276116361112428479580474909 a_3^2, \\
b_7 &= 11.72677636147392252103108673156625861130540134632217 a_3^2,
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
a_4 &= 1.16364223535095645608539458464523114309856393525890 a_3^2, \\
a_5 &= 13.51273677593795918678747785044498166551946137068013 a_3^2, \\
a_7 &= -10.18920587951907640059427939895810711880480806331355 a_3^2, \\
b_4 &= -0.58111111583561668080376402285512367825820241344231 a_3^2, \\
b_5 &= 0, \\
b_6 &= 14.57222224584475007068548678094228505795767873387819 a_3^2, \\
b_7 &= 1.16364223535095645608539458464523114309856393525890 a_3^2.
\end{aligned} \tag{34}$$

Using the above second group of parameter values yields the following focus values $v_i \approx 0, i = 3, 5, \ldots, 15$, and

$$v_{17} = -3860.66413547116908775589728360466124422760155003742678 a_3^{16}.$$

One can choose an appropriate value of a_3 to make v_{17} smaller. For example, choosing $a_3 = 1/2$ gives

$$v_{17} = -0.058909059684313493160337788140940265567437767 79231.$$

Thus, system (26) can at most have eight small limit cycles around the origin. Further, we may apply Lemma 2 to prove that the system indeed has eight limit cycles. Since the five parameters A_4, A_7, B_4, B_7 and B_5 can be used one by one to perturb the focus values: v_{11}, v_9, v_7, v_5 and v_3, we only need to verify the Jacobian matrix obtained from the equations v_{13} and v_{15}. Evaluating this Jacobian matrix at the second group of parameter values results in

$$\begin{aligned}\det(J_c) &= \det\begin{bmatrix} 374.61334770404070439946a_3^{10} & -151.19857162131752129186a_3^{10} \\ 6805.71872946034421955644a_3^{12} & -2775.84271190336979973928a_3^{12}\end{bmatrix} \\ &= -10852.78025513242068667926a_3^{22} \neq 0 \quad (\text{because } a_3 \neq 0).\end{aligned} \tag{35}$$

This shows that for the given second group of parameter values with proper perturbations, system (26) has exactly eight limit cycles.

Summarizing the above results gives the following theorem.

Theorem 4. *For the cubic system (26), when the system parameters are properly perturbed to the critical values:* $\lambda = 0$, $b_5 = 0$, $b_7 = a_4$, $b_4 = \frac{1}{15}(35a_3^2 - 3b_6)$, $a_7 = \frac{1750a_3^4 + 525a_5a_3^2 + 140b_6a_3^2 + 6b_6^2 - 30a_5b_6}{5(18b_6 - 385a_3^2)}$, *$a_4$ and a_5 are given by (29) and (30), respectively, through the back scaling (27), and $B_6 = b_6/a_3^2$ is one of the two real roots of the polynomial equation $F_1 = 0$ (see Eq. (32)) satisfying the Jacobian condition (35), then system (26) has exactly eight small limit cycles around the origin.*

Before moving on to the next case, we present a numerical example with the second group of critical values given in Eq. (34) at $a_3 = 1/2$. We take the following perturbations:

$$\begin{aligned}&\epsilon_1 = 0.222 \times 10^{-2}, \quad \epsilon_2 = 0.1 \times 10^{-2}, \quad \epsilon_3 = 0.1 \times 10^{-10}, \quad \epsilon_4 = 0.1 \times 10^{-15}, \\ &\epsilon_5 = 0.1 \times 10^{-22}, \quad \epsilon_6 = 0.2 \times 10^{-30}, \quad \epsilon_7 = 0.1 \times 10^{-37}, \quad \epsilon_8 = 0.4 \times 10^{-47},\end{aligned}$$

under which the perturbed focus values are: $v_0 = -0.4 \times 10^{-47}$, $v_1 = 0.125 \times 10^{-38}$, $v_2 = -0.9110138803 \times 10^{-31}$, $v_3 = 0.8356442057 \times 10^{-24}$, $v_4 = -0.1707886684 \times 10^{-17}$, $v_5 = 0.5055921200 \times 10^{-12}$, $v_6 = -0.2408082000 \times 10^{-7}$, $v_7 = 0.1707722232 \times 10^{-3}$, $v_8 = -0.5829996790 \times 10^{-1}$, which results in the eight positive roots of r for the amplitudes of the small limit cycles:

$$\begin{aligned}&r_1 = 0.0000694069, \; r_2 = 0.0001028080, \; r_3 = 0.0003639564, \; r_4 = 0.0006441196, \\ &r_5 = 0.0018656976, \; r_6 = 0.0045431825, \; r_7 = 0.0110759988, \; r_8 = 0.0527431270.\end{aligned}$$

The perturbed system has three fixed points: $C_0 = (0, 0)$, $C_1 = (1.4993481472, -0.6020762119)$ and $C_3 = (-1.7265198511, 0.6436456497)$. A linear analysis shows that C_0 is a fine stable focus point, while C_1 and C_3 are saddle points. The phase portrait of this example is shown in Fig. 3. It should be noted that since the origin is a high order focus point, the dynamical behaviour of the system in the vicinity of the origin is similar to that of a center. The limit cycles shown in this figure are not exact trajectories, but used to demonstrate what it may look like.

3.4. Nine small limit cycles in a cubic system

So far, in the literature, the maximum number of limit cycles in the neighborhood of one singular point obtained using symbolic computation is eight [42]. Although it was shown eleven small limit cycles might exist around one singular point [43], the result has not been verified by computation. The difficulty in computing higher order focus values are obvious. Also, solving coupled higher degree multivariate polynomials is difficult.

In the remaining part of this section, we present a cubic system which exhibits nine small limit cycles around one critical point. Consider the general normalized cubic system (11). For convenience, we rewrite it here:

$$\begin{aligned}\dot{x} &= \alpha x + y + x^2 + (b + 2d)xy + cy^2 + fx^3 + gx^2y + (h - 3p)xy^2 + ky^3, \\ \dot{y} &= -x + \alpha y + dx^2 + (e - 2)xy - dy^2 + \ell x^3 + (m - h - 3f)x^2y + (n - g)xy^2 + py^3.\end{aligned}$$

The system has thirteen independent parameters. It is easy to see that $v_1 = \alpha$, and other focus values are given in terms of the remaining twelve parameters. It has been shown that the origin is a centre if $\alpha = b = e = h = n = m = 0$. We let

$$\alpha = d = e = h = 0, \tag{36}$$

and then compute the focus values. Here, we use the recursion formulas to compute the singular point quantities. The first-order quantity is given by $\mu_1 = \frac{1}{4}i[b(c + 1) - m]$. setting $\mu_1 = 0$ results in

$$m = b(c + 1). \tag{37}$$

For simplicity, letting $b = 0$, and so $m = 0$. Therefore, n must be non-zero, otherwise it is a centre. Thus, 7 free parameters are remained: c, f, g, k, ℓ, n, p.

Under the above choices of parameters, the second-order singular point quantity becomes $\mu_2 = -\frac{1}{4} in(p-f)$. In order to have $\mu_2 = 0$, the only choice is

$$p = f, \tag{38}$$

since $n \neq 0$. Then $\mu_3 = \frac{1}{96} ifn(45 - 30c - 35c^2 + 15\ell + 15k + 3n)$. One can choose n to set $\mu_3 = 0$, yielding

$$n = \frac{1}{3}(35c^2 + 30c - 15\ell - 15k - 45). \tag{39}$$

The next singular point quantity can be found as

$$(40)\mu_4 = \frac{1}{192} ifn[g(7c^2 + 30c + 6k + 6\ell - 45) - 648 + 162c - 81k - 72\ell + 30k\ell - 60c\ell - 54ck + 24k^2 + 6\ell^2 + 516c^2 - 56c^2\ell - 21c^2k + 434c^3 + 168c^4],$$

from which we obtain

$$g = (648 - 162c + 81k + 72\ell - 30k\ell + 60c\ell + 54ck - 24k^2 - 6\ell^2 - 516c^2 + 56c^2\ell + 21c^2k - 434c^3 - 168c^4)/(7c^2 + 30c + 6k + 6\ell - 45). \tag{40}$$

Then μ_5 is given by $\mu_5 = -\frac{ifn}{10368(30c+7c^2+6k+6\ell-45)^2} \bar{\mu}_5(f,c,k,\ell)$, where $\bar{\mu}_5$ is a polynomial of f,c,k,ℓ, from which we can solve for f^2 to obtain (here `f2` $= f^2$)

```
f2 := -(-54584604*c*l - 39188124*c*k - 27032049*c^2*l + 48200751*c^2*k - 110968380*c
    + 19497105*k + 59650425*l + 1029976236*c^4 - 15231726*k^2 - 21311586*l^2
    + 159042042*c^3 - 850276440*c^2 - 37330632*l*k - 843453*l^2*k + 6160698*c^2*k^2
    + 47523483*l*c^4 - 13941396*c*l^2 - 9106668*c*k^2 + 88773246*l*c^3
    + 173490822*k*c^3 - 3849120*c*l^3 + 2978208*c*k^3 - 2719710*c^2*l^3
    - 6922962*c^2*k^3 + 34758558*c^4*k^2 + 31216050*c^4*l^2 - 132783063*l*c^6
    - 85492743*k*c^6 + 477900*k*l^3 + 538164*k^3*l + 11818224*c^3*k^2 + 44393184*c^3*l^2
    - 46656*k^3*l^2 - 46656*k^2*l^3 - 15552*k^4*l - 15552*l^4*k + 2024946*c^6*l^2
    + 19211742*c^2*l^2 + 1234197*k^2*l - 84526605*k*c^4 + 42444*c^2*k^4
    + 83592*c^2*l^4 - 9672012*l*c^8 - 2526636*k*c^8 - 284445*l^3*c^4 - 34587*k^3*c^4
    + 23328*c*k^4 + 127656*c*l^4 - 802656*l^3*c^3 - 3672000*k^3*c^3 + 957420*k^2*l^2
    + 29794716*c^5*k^2 - 195603336*k*c^5 - 105820344*l*c^5 + 11002068*c^5*l^2
    - 58934442*l*c^7 + 754029*k^3 + 44388*k^4 + 14256*l^4 - 522775800*c^6
    - 1323621*l^3 + 57931686*c^5 + 38574144*c^5*k*l + 145800*c*l^3*k
    - 68040*c*l^2*k^2 - 62856*c*l*k^3 - 7236864*l^2*c^3*k - 10106208*l*c^3*k^2
    + 7135128*c^6*k*l - 1242675*k^2*c^4*l - 1492533*k*c^4*l^2 - 10692*c^2*l^3*k
    - 229716*c^2*k^2*l^2 - 92988*c^2*k^3*l + 68219928*c^4*k*l + 707616*c*k^2*l
    - 6119712*c*k*l^2 + 59710608*c^3*k*l - 13159746*c^2*k^2*l - 8956494*c^2*k*l^2
    + 21873240*c^2*l*k - 21473424*c*l*k - 105232554*c^7 + 154411628*c^8
    + 12910520*c^10 + 84322630*c^9 - 18758754*k*c^7 + 5851062*c^6*k^2 + 239345280)/
     (24*k + 24*l - 495 + 330*c + 385*c^2)/(30*c + 7*c^2 + 6*k + 6*l - 45)^2/18:
```

(41)

Then the next three singular point quantities are

$$\mu_6 = \mu_6(c,k,\ell), \quad \mu_7 = \mu_7(c,k,\ell), \quad \mu_8 = \mu_8(c,k,\ell),$$

which are polynomials of c, k and ℓ, with leading (combined) degrees of 14, 22 and 26 for μ_6, μ_7 and μ_8, respectively. The Maple output files for the three polynomial equations have roughly 100, 315 and 588 lines, respectively. The three singular point quantities are coupled, we thus have to simultaneously solve the three polynomial equations: $\mu_6 = \mu_7 = \mu_8 = 0$. To find the solutions of these equations, we might first eliminate one parameter from the three equations to obtain two resultant equations, and then further eliminate one more parameter from the two equation to get a final resultant equation which is a univariate polynomial and thus we could find all possible solutions. This elimination method has been used in our other publications (e.g., [1–3]). The method may increase the degree of the final resultant polynomial substantially.

In this paper, we use a numerical approach to simultaneously solve the three equations. The solution of the system of equations may be approached in several ways. A simple numerical approach, using *solve* in Maple directly, is successful in many cases. By examining its source code, we find that this routine uses a variant of damped multivariate Newton iteration (there appears to be no published paper describing the routine), and by judicious choices of our initial guess we obtained a result (with $f_2 > 0$) after only two tries.

In general, this in not completely satisfactory, because we would like to find all real positive roots. Advanced techniques such as those described in [45] will be pursued for extensions of the work described in this paper.

Finally, for system (11), by taking $a = 0.3$, we have used the numerical method to obtain the critical parameter values (results from computer output using Maple up to 1000 digits, but list only 50 digits):

$$\begin{aligned}
&b = d = e = h = 0,\\
&m^* = 0,\\
&p^* = f^* = 0.57205513738312947531208972107913988960903196760780,\\
&n^* = 1.5587366234496833705957910715181908410140471444159 8,\\
&g^* = 0.32875002652319114041354789797311142571533487466423,\\
&f^* = 0.57205513738312947531208972107913988960903196760780,\\
&\ell^* = -0.28703189754662589381676063046084638581078384981160,\\
&c^* = -0.52179929787663453432829806398960426486646485537068,\\
&k^* = 0.02751217774798897933976121289219741452562430620627.
\end{aligned} \tag{42}$$

Under the above critical parameter values and conditions, we execute the Maple program (Yu, 1998) to find the following focus values (again up to 1000 decimal digits, but v_9 is only given up to 50 digits here): $v_1 = 0, v_3 = 0.1 \times 10^{-999}$, $v_5 = -0.5 \times 10^{-1000}, v_7 = -0.23 \times 10^{-1000}, v_9 = 0.62 \times 10^{-999}, v_{11} = -0.405 \times 10^{-999}, v_{13} = 0.135 \times 10^{-998}, v_{15} = 0.14 \times 10^{-998}$, $v_{17} = 0.9 \times 10^{-999}$, and

$$v_{19} = -0.0014315972268236725754647008537844355295621179315106.$$

The above result indeed indicates that an excellent approximate solution has been obtained, and the values of $v_i, i = 1, 3, \ldots, 17$ can be considered as being very close to a true real zero. Thus, the maximum number of small limit cycles which can be obtained in the vicinity of the origin of system (11) is nine. In order to prove that the nine small limit cycles indeed exist, we need to check the Jacobian matrix obtained from the three equations:

$$v_{13}(\ell, c, k) = v_{15}(\ell, c, k) = v_{17}(\ell, c, k) = 0,$$

with respect to ℓ, c and k, since the other singular point quantities, $\mu_5, \mu_4, \mu_3, \mu_2$ and μ_1 can be perturbed one after another by the parameters, f, g, n, p and m (or b). Further, it can be shown that due to the relation between v_{2k+1} and μ_k (given in Eq. (13)), the determinant of the Jacobian evaluated at the critical point, based on the μ_i formulas is equivalent to that based on v_i expressions. In other words, the determinant of the Jacobian based on the former is non-zero if and only if that based on the latter is non-zero. Thus, we can use the following real equations:

$$i\mu_6(\ell, c, k) = i\mu_7(\ell, c, k) = i\mu_8(\ell, c, k) = 0,$$

to compute the determinant, where $i = \sqrt{-1}$. It should be noted that the values of these two determinants are not equal since the constant coefficients between the relations are ignored, which does not change the non-zero property of the determinant. Therefore,

$$\begin{aligned}
\det(J_c) &= \det \begin{bmatrix} \frac{\partial i\mu_6}{\partial \ell} & \frac{\partial i\mu_6}{\partial c} & \frac{\partial i\mu_6}{\partial k} \\ \frac{\partial i\mu_7}{\partial \ell} & \frac{\partial i\mu_7}{\partial c} & \frac{\partial i\mu_7}{\partial k} \\ \frac{\partial i\mu_8}{\partial \ell} & \frac{\partial i\mu_8}{\partial c} & \frac{\partial i\mu_8}{\partial k} \end{bmatrix}_{(\ell,c,k)=(\ell^*,c^*,k^*)} = \det \begin{bmatrix} 0.0451319561 & 0.2439914595 & 0.0220645269 \\ 0.1223727073 & -0.0022077660 & 0.1580040454 \\ 0.0055179038 & 0.6737398583 & -0.0271869307 \end{bmatrix}\\
&= -0.0019578515 \neq 0.
\end{aligned} \tag{43}$$

Then, according to Lemma 2 we know that nine small-amplitude limit cycles bifurcating from the origin of system (11) can be obtained by properly perturbing the critical values given in Eq. (42). The above results are summarized in the following theorem:

Theorem 5. *For the cubic system (11), suppose $b = d = e = h = 0$. When the remaining parameters are properly perturbed to the critical values: $\alpha = 0$, $m^* = 0$, $p^* = f$, $n^* = \frac{1}{3}(35c^2 + 30c - 15\ell - 15k - 45)$, g^*, f^{*2} is given by (41), while ℓ^*, c^* and k^* are given in (42) satisfying the Jacobian condition (43), then the system (11) has exactly nine small limit cycles around the origin.*

4. Conclusion

In this paper, we have considered the existence of small-amplitude limit cycles of nonlinear dynamical systems. Particular attention is given to planar quadratic and cubic systems, associated with Hilbert's 16th problem. Up to now, all the studies of small limit cycles in Hilbert's 16th problem are almost based on computations of focus values and solving of polynomial equations. This paper presents a computational method which are used to obtain some new results about small limit cycles. Several cases are studied to show that efficient symbolic computation is crucial in finding the limit cycles of dynamical systems. All the results presented in this paper were obtained using Maple. This may help motivate developing new methods for the research in limit cycles of Hilbert's 16th problem. On the other hand, we see in this problem a potentially interesting avenue for application of accurate real solving of large systems of multivariate polynomial equations. This may stimulate research interest in developing more efficient symbolic computational methods for solving polynomial equations.

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